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Trees of manifolds and boundaries of systolic groups

by

Paweł Zawiślak (Wrocław)

Abstract. We prove that the Pontryagin sphere and the Pontryagin nonorientable surface occur as the Gromov boundary of a 7-systolic group acting geometrically on a 7-systolic normal pseudomanifold of dimension 3.

1. Introduction. k-systolic simplicial complexes ($k \geq 6$ is a natural number) were introduced by T. Januszkiewicz and J. Świątkowski in [JS] and independently by F. Haglund in [H]. These are simplicial analogues of metric spaces of nonpositive curvature. The idea of systolicity leads to an answer to the question posed by M. Gromov about a simple, easily checkable combinatorial condition for a simplicial complex implying hyperbolicity of this complex for the standard piecewise euclidean metric on it. In [JS] Januszkiewicz and Świątkowski have shown that a 7-systolic simplicial complex is hyperbolic.

Gromov boundaries of 7-systolic complexes were investigated by D. Osajda in [O]. He showed that the ideal boundary $\partial_G X$ of a (locally finite) 7-systolic simplicial complex X is a strongly hereditarily aspherical compactum. He also showed that the Gromov boundary of a normal 7-systolic pseudomanifold of finite dimension at least 3 is connected and has no local cutpoints. In the present paper we study in detail the case of such pseudomanifolds in dimension 3.

Trees of manifolds are inverse limits of certain inverse systems of manifolds. The most common examples of such spaces are the Pontryagin sphere and the nonorientable Pontryagin surface, which are trees of 2-tori and of projective planes, respectively. Trees of manifolds were defined and investigated by W. Jakobsche (see [J]) and by P. R. Stallings (see [S]). These spaces occur as CAT(0) boundaries of right-angled Coxeter groups (see [F]). In the

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case when these groups are hyperbolic their CAT(0) boundaries coincide with their Gromov boundaries.

The main result of this paper is:

Main Theorem. *Let X be a 7-systolic normal pseudomanifold of dimension 3. Let a group G act geometrically on X. Then:*

(a) (Theorem 7.2) *if X is orientable, then $\partial_G G$ is homeomorphic to the Pontryagin sphere,*
(b) (Theorem 9.5) *if X is nonorientable, then $\partial_G G$ is homeomorphic to the nonorientable Pontryagin surface.*

Let us make a few remarks on the Main Theorem. First, the existence of n-dimensional, k-systolic normal (orientable) pseudomanifolds with geometric group actions was shown in [JS]. Second, note that in this paper we prove stronger results. In the orientable case we show that the Gromov boundary of a pseudomanifold is homeomorphic to the Pontryagin sphere. In the nonorientable case we use only the cocompactness of the group action. However, the above formulation seems to be more natural.

This paper is organized as follows. In Section 2 we recall some terminology related to simplicial complexes and systolic complexes. We also recall some facts about systolic complexes. In Section 3 we thoroughly examine the properties of combinatorial spheres S_n in 3-dimensional 7-systolic normal pseudomanifolds and properties of natural projections $\Pi_n : S_n \to S_{n-1}$ between them. D. Osajda showed that in the case of a locally finite 7-systolic simplicial complex X of finite dimension the inverse limit $\varprojlim (S_n, \Pi_n)$ of the system of these spheres and projections is homeomorphic to the Gromov boundary $\partial_G X$. In our case, we show that every sphere S_n is a surface. Moreover, we show that up to a homeomorphism the sphere S_{n+1} is a connected sum of S_n and links of vertices $w \in S_n$. In Section 4 we recall the results of Jakobsche from [J] on inverse systems of compact orientable manifolds. The proof of the first statement of the Main Theorem is given in Sections 5, 6 and 7. In Section 5 we modify the maps $\Pi_n : S_n \to S_{n-1}$ (without changing the inverse limit $\varprojlim (S_n, \Pi_n)$). These maps become injective on some appropriate parts of their domains, which is one of the conditions in the definition of a Jakobsche inverse system (which is in turn an object used to define a tree of manifolds). Properties of such modified maps $\Pi'_n : S_n \to S_{n-1}$ allow us, in Section 6, to further modify the inverse system (S_n, Π'_n). We call these modifications a refinement. Every element of the refined system is a connected sum of its predecessor and some finite number of tori. This is one of the conditions in the definition of the Pontryagin sphere. In Section 7 we define families $\mathcal{D}_{n,k}$ of pairwise disjoint discs in surfaces $S_{n,k}$, which turns the refined system $(S_{n,k}, \Pi'_{n,k})$ into a Jakobsche inverse system of tori, thus finishing the proof of part (a) of the Main Theorem. In Section 8 we examine the properties of trees of nonorientable surfaces. In Section 9 we prove the second statement of the Main Theorem.

2. Definitions and properties of systolic complexes. In this section we recall the notion of a systolic complex and some of its basic properties.

Let X be a simplicial complex and let $\sigma \subset X$ be a simplex. The *link* of X at σ (denoted by X_σ) is the subcomplex of X consisting of all simplices disjoint from σ and spanning together with σ a simplex in X. The *residue* of σ in X (denoted by $\mathrm{Res}(\sigma, X)$) is the union of all simplices in X that contain σ.

For simplices σ_1 and σ_2 in X we denote by $\sigma_1 * \sigma_2$ the *simplicial join* of σ_1 and σ_2 (if it exists); this means that σ_1 and σ_2 are disjoint and $\sigma_1 * \sigma_2$ is the smallest simplex in X containing both of them. X is *flag* if every set of vertices $v_1, \ldots, v_n \in X$ pairwise connected by edges in X spans a simplex $v_1 * \cdots * v_n$ in X. A subcomplex $K \subset X$ is *full* if for every set of vertices $v_1, \ldots, v_n \in K$ spanning a simplex $v_1 * \cdots * v_n$ in X this simplex is a simplex in K.

A simplicial complex X is a *pseudomanifold* of dimension n if it is locally finite, it is a union of its n-simplices and each $(n-1)$-simplex is contained in exactly two n-simplices. A pseudomanifold is *orientable* if it admits a choice of orientations on top-dimensional simplices in a consistent way, i.e. such that the orientations on each simplex of codimension 1 inherited from two top-dimensional simplices containing it are opposite. An n-dimensional pseudomanifold is *normal* if for every nonempty simplex σ in X of dimension $\dim(\sigma) < n-1$ the link X_σ is connected.

REMARK 2.1. Note that if a pseudomanifold is orientable then all its links are also orientable. The converse is not true in general. However, for a simply-connected normal pseudomanifold of dimension 3 its orientability is equivalent to the orientability of its vertex links.

A *cycle* in X is a subcomplex $\gamma \subset X$ isomorphic to some triangulation of the circle S^1. The *length* of a cycle γ (denoted by $|\gamma|$) is the number of its 1-simplices.

DEFINITION 2.2.

1. Let X be a flag simplicial complex and let $k \geq 4$ be a natural number.
   - X is *k-large* if no cycle γ in X of length $3 < |\gamma| < k$ is full in X.
   - X is *locally k-large* if for every nonempty simplex σ in X the link X_σ is k-large.
   - X is *k-systolic* if it is connected, simply-connected and locally k-large.

2. A group G is *k-systolic* if it acts geometrically (i.e. properly discontinuously and cocompactly) by simplicial automorphisms on some k-systolic simplicial complex X.

For brevity a 6-systolic complex or group is called *systolic*.

REMARK 2.3. Note that a full subcomplex of a k-large simplicial complex is k-large itself.

Now we recall some basic facts about systolic complexes. For the proofs see [JS] and [O]. We start with a theorem relating the notions of systolicity and Gromov hyperbolicity.

THEOREM 2.4 ([JS, Theorem 2.1]). *The 1-skeleton of a 7-systolic simplicial complex is hyperbolic.*

For a subset $A \subset X$ which is a union of some simplices in X we denote by $\mathrm{span}_X(A)$ the full subcomplex of X spanned by A (i.e. the intersection of all full subcomplexes of X containing A). Now we recall the definition of combinatorial *balls* and *spheres* in a simplicial complex X centered at a simplex $\sigma \subset X$:

- $B_0(\sigma, X) = \sigma$, $B_{n+1}(\sigma, X) = \mathrm{span}_X(\{\tau \subset X : \tau \cap B_n(\sigma, X) \neq \emptyset\})$,
- $S_n(\sigma, X) = \mathrm{span}_X(\{w \in X^{(0)} : d(w, \sigma) = n\})$, where $d(w, \sigma)$ denotes the distance in the 1-skeleton $X^{(1)}$.

In the following proposition we recall some natural properties of balls and spheres in systolic complexes.

FACT 2.5 ([JS, Lemma 7.9]). *Let X be a systolic simplicial complex and let $v \in X$ be a vertex. Then for every natural number $n > 0$ and for every simplex $\tau \subset S_n(v, X)$ the intersection $\rho = B_{n-1}(v, X) \cap X_\tau$ is a single simplex. Moreover, $X_\tau \cap B_n(v, X) = B_1(\rho, X_\tau)$ and $X_\tau \cap S_n(v, X) = S_1(\rho, X_\tau)$.*

Let b_τ denote the barycenter of a simplex τ and let X' denote the first barycentric subdivision of the simplicial complex X. We view the barycenters b_τ of simplices $\tau \subset X$ as the vertices of X'. The combinatorial properties of balls and spheres mentioned in Fact 2.5 are crucial in the definition of projections

$$\Pi_n : S_n(v, X) \to [S_{n-1}(v, X)]'$$

that we recall now.

For a systolic complex X and a vertex $v \in X$ let S_n denote the sphere $S_n(v, X)$ and let B_n denote the ball $B_n(v, X)$. Let $Y^{(0)}$ denote the 0-skeleton of Y, i.e. the vertex set of the simplicial complex Y. Define the map

$$\Pi_n : S_n^{(0)} \to (S'_{n-1})^{(0)}$$

by the equalities $\Pi_n(w) = b_\tau$ for all vertices $w \in S_n^{(0)}$, where the simplex τ is the intersection $B_{n-1} \cap X_w$.

Spheres and balls in 7-systolic complexes have stronger properties than those recalled above. The following fact allows us to extend the map $\Pi_n : S_n^{(0)} \to (S'_{n-1})^{(0)}$ to a simplicial map

$$\Pi_n : S_n \to S'_{n-1}.$$

FACT 2.6 ([O, Lemma 3.1]). *If X is 7-systolic then, for any vertices $v_1, v_2 \in S_n$ connected by an edge in S_n, $\Pi_n(v_1)$ and $\Pi_n(v_2)$ are either equal or span a 1-simplex in S'_{n-1}.*

Define $\Pi_n : S_n \to S'_{n-1}$ as a simplicial extension of the map $\Pi_n : S_n^{(0)} \to (S'_{n-1})^{(0)}$ defined above.

We now comment on the notation used in this paper. We use the same symbol Π_n for the simplicial map $\Pi_n : S_n \to S'_{n-1}$ and the related continuous map $\Pi_n : S_n \to S_{n-1}$ (when we forget the simplicial structure and treat the complexes S_n and S'_{n-1} just as metric spaces). For example, this is the case in the following fact describing the metric properties of the maps Π_n. We denote by d_X the standard piecewise euclidean metric on X.

FACT 2.7 ([O, Lemma 3.3]). *Let X be a 7-systolic complex with finite dimension. Then there is a positive constant $C < 1$ depending only on $\dim X$ such that for all natural numbers n and for all $x, y \in S_n$, $d_{S_{n-1}}(\Pi_n(x), \Pi_n(y)) \le C \cdot d_{S_n}(x, y)$.*

The next theorem shows that the inverse system (S_n, Π_n) can be used to describe the Gromov boundary of a 7-systolic complex X.

THEOREM 2.8 ([O, Lemma 4.1]). *Let X be a 7-systolic locally finite simplicial complex of finite dimension. For a vertex $v \in X$ let S_n denote the sphere $S_n(v, X)$ and let the maps $\Pi_n : S_n \to S_{n-1}$ be defined as before. Then the inverse limit $\varprojlim (S_n, \Pi_n)$ is homeomorphic to the Gromov boundary of X.*

3. Spheres and projections in 7-systolic normal pseudomanifolds of dimension 3. In this section, X is a 7-systolic, normal pseudomanifold of dimension 3. We thoroughly examine the properties of the combinatorial spheres S_n in such pseudomanifolds and of the projections Π_n defined in Section 2.

In Lemmas 3.1, 3.2 and 3.3 we describe the links of X at simplices of dimensions 2, 1 and 0 respectively.

LEMMA 3.1. *Let $\sigma \subset X$ be a 2-simplex. Then the link X_σ consists of two vertices.*

Proof. σ is contained in exactly two simplices of dimension 3. ∎

Lemma 3.2. *Let $\varepsilon \subset X$ be a* 1*-simplex (i.e. an edge). Then the link X_ε is a* 7*-large triangulation of the circle S^1 (i.e. a triangulation consisting of at least seven edges).*

Proof. Let ε be a join $v_1 * v_2$. Let $u \in X_\varepsilon$ be a vertex. There are exactly two vertices $w_1, w_2 \in X_\varepsilon$ adjacent to u (since $u * v_1 * v_2$ is a 2-simplex lying in exactly two 3-simplices). Thus, since X is locally finite, the link X_ε is a disjoint union of copies of triangulated circles. But since X is normal, the link X_ε is connected, so there must be exactly one copy. Since X is locally 7-large, it follows that this triangulation of X_ε must be 7-large. ∎

Lemma 3.3. *Let $u \in X$ be a* 0*-simplex (i.e. a vertex). Then the link X_u is topologically a closed connected surface with a* 7*-large triangulation. Moreover, if X is orientable then X_u is also orientable.*

Proof. For a vertex $w \in X_u$ we have $(X_u)_w = X_{u*w}$. Thus the link $(X_u)_w$ is a triangulated circle (see Lemma 3.2). A simplicial complex all of whose vertex links are triangulated circles is itself a triangulated surface. Since X is 7-systolic, this triangulation is 7-large. The connectedness of X_u follows from the normality of X. The last assertion follows from Remark 2.1. ∎

Combinatorial properties of 7-large complexes imply the following:

Remark 3.4. Let Σ be a 7-large triangulated closed surface and let $\sigma \subset \Sigma$ be a simplex. Then:

1. the balls $B_1(\sigma, \Sigma)$ and $B_2(\sigma, \Sigma)$ are triangulated 2-discs; moreover, their topological boundaries in Σ are the spheres $S_1(\sigma, \Sigma)$ and $S_2(\sigma, \Sigma)$ respectively,
2. the ball $B_3(\sigma, \Sigma)$ can contain a loop in the 1-skeleton $\Sigma^{(1)}$ homotopically nontrivial in Σ (if $\dim \sigma > 0$).

In the next lemma we describe the combinatorial and topological properties of the spheres $S_n(v, X)$.

Lemma 3.5. *Let $v \in X$ be a vertex and let $S_n = S_n(v, X)$ be the combinatorial sphere of radius n centered at v. Then S_n is a connected surface with a* 7*-large triangulation.*

Proof. For $n = 1$ we have $S_1 = X_v$. Thus the assertion follows from Lemma 3.3.

Let $w \in S_n$ be a vertex and let $\rho := X_w \cap S_{n-1}$. Since X_w is a 7-large surface and ρ is a simplex (see Fact 2.5), Remark 3.4 shows that $B_1(\rho, X_w)$ is a triangulated 2-disc. As

$$(S_n)_w = X_w \cap S_n = S_1(\rho, X_w) = \mathrm{bd}(B_1(\rho, X_w)) = S^1$$

(see Fact 2.5) it follows that the vertex links of S_n are triangulated circles. Thus S_n is a triangulated surface.

Since S_n is full in X (by definition) and X is 7-large, it follows that this triangulation of S_n is 7-large (see Remark 2.3).

Connectedness of S_n can be shown using an inductive argument and Corollary 3.18 below. ∎

Lemmas 3.6, 3.7, 3.8 and 3.9 describe the local properties of the projections $\Pi_{n+1} : S_{n+1} \to S_n$.

LEMMA 3.6. *For a 2-simplex $\sigma \subset S_n$ there is exactly one vertex $w_\sigma \in S_{n+1}$ such that the join $w_\sigma * \sigma$ is a simplex in X. This vertex coincides with the preimage $\Pi_{n+1}^{-1}(b_\sigma)$.*

Proof. By Fact 2.5, $X_\sigma \cap S_{n-1}$ is a single simplex. Thus, for dimensional reasons, it is a vertex. By Lemma 3.1, the link X_σ consists of two vertices. Moreover, $X_\sigma \cap S_n = \emptyset$, since S_n is a surface and a full subcomplex. It follows that $X_\sigma \cap S_{n+1}$ must be the other vertex of X_σ, say w_σ. From the definition of the projections it is easy to see that $\Pi_{n+1}^{-1}[b_\sigma] = X_\sigma \cap S_{n+1}$ (σ is a 2-simplex). It follows that $\Pi_{n+1}^{-1}[b_\sigma] = w_\sigma$. ∎

LEMMA 3.7. *For an edge $\varepsilon \subset S_n$ the intersection $\alpha_\varepsilon = X_\varepsilon \cap S_{n+1}$ is an arc (triangulated). If σ_1 and σ_2 are two 2-simplices in S_n containing ε, then the endpoints of this arc coincide with the preimage vertices $\Pi_{n+1}^{-1}(b_{\sigma_1})$ and $\Pi_{n+1}^{-1}(b_{\sigma_2})$.*

Proof. Since S_n is a surface, there are exactly two 2-simplices in S_n, say $\sigma_1 = v_1 * \varepsilon$ and $\sigma_2 = v_2 * \varepsilon$, that contain ε. For these two simplices there are two vertices w_{σ_1} and w_{σ_2} in S_{n+1} such that for $i = 1, 2$ the joins $w_{\sigma_i} * \sigma_i$ are simplices in X.

First we show that w_{σ_1} and w_{σ_2} do not lie in a common simplex in X. To see this suppose that $w_{\sigma_1} * w_{\sigma_2}$ is a simplex in X. By Fact 2.6, $\Pi_{n+1}(w_{\sigma_1})$ and $\Pi_{n+1}(w_{\sigma_2})$ lie in a common simplex in the barycentric subdivision S_n'. Now Π_{n+1} maps w_{σ_i} to the barycenter b_{σ_i} for $i = 1, 2$. But b_{σ_1} and b_{σ_2} do not span a simplex in S_n', a contradiction.

Now for $i = 1, 2$ let a vertex u_i be the intersection $X_{\sigma_i} \cap S_{n-1}$. Note that since u_1 and u_2 belong to $X_\varepsilon \cap S_{n-1}$, they are equal or span a simplex in S_{n-1} (see Fact 2.5). Since the link X_ε is a triangulated circle, and u_1, u_2, v_1, v_2 are all vertices of X_ε lying in $B_n(v, X)$, it follows that w_{σ_1} and w_{σ_2} are connected by an arc $\alpha_\varepsilon = (w_{\sigma_1} = w_0, w_1, \ldots, w_m = w_{\sigma_2})$ in S_{n+1} (for some $m > 1$). Lemma 3.6 implies that the vertices w_{σ_i} are exactly the preimages $\Pi_{n+1}^{-1}(b_{\sigma_i})$ for $i = 1, 2$. ∎

LEMMA 3.8. *Let $\varepsilon \subset S_n$ be an edge, let σ_1 and σ_2 be two different 2-simplices in S_n containig ε, and let $\alpha_\epsilon = (w_0, w_1, \ldots, w_m)$ be the arc in S_{n+1} given by Lemma 3.7. Then the projection Π_{n+1} maps the edges $w_0 * w_1$ and $w_{m-1} * w_m$ homeomorphically onto the edges $b_{\sigma_1} * b_\varepsilon$ and $b_{\sigma_2} * b_\varepsilon$ in S_n'*

respectively, and collapses the subarc $(w_1, w_2, \dots, w_{m-2}, w_{m-1})$ *to the barycenter* b_ε.

Proof. By Lemma 3.7, Π_{n+1} maps w_0 to b_{σ_1} and w_m to b_{σ_2}. We show that Π_{n+1} maps w_i to b_ε for $i = 1, \dots, m-1$. It is enough to show that $X_{w_i} \cap S_n$ is exactly the edge ε.

For this, note that w_i and ε span a simplex in X. Thus ε is a simplex in $X_{w_i} \cap S_n$. If this intersection contains a vertex u not contained in ε, then u and ε span a simplex in S_n. But 2-simplices in S_n containing ε are exactly σ_1 and σ_2. It follows that w_i is w_0 or w_m, a contradiction. ∎

LEMMA 3.9. *Let* $w \in S_n$ *be a vertex. Then there exists a cycle (i.e. a triangulated circle)* α_w *in the* 1*-skeleton of* $X_w \cap S_{n+1}$ *such that the image* $\Pi_{n+1}[\alpha_w]$ *is equal to the sphere* $S_1(w, S_n')$ *(which is a cycle in the barycentric subdivision* S_n'*) and the preimage* $\Pi_{n+1}^{-1}[S_1(w, S_n')]$ *is equal to* α_w.

Proof. As S_n is a triangulated surface, the residue $\operatorname{Res}(w, S_n)$ is a triangulated 2-disc. Let it consist of the 2-simplices $\sigma_i = w * w_i * w_{i+1}$ for $i = 0, 1, \dots, k-1$, where $k = |X_w \cap S_n| \geq 7$ is the length of the link $(S_n)_w$ (indices taken modulo k). Let $\alpha_i = X_{w * w_i} \cap S_{n+1}$ be the arc in S_{n+1} given by Lemma 3.7. Let $\alpha_w := \alpha_0 \cup \dots \cup \alpha_{k-1}$. We claim that α_w is a cycle.

It is enough to show that $\alpha_i \cap \alpha_j \neq \emptyset$ only for $|i-j| \leq 1$, and for $|i-j| = 1$ it consists of one point. Suppose that $\alpha_i \cap \alpha_j \neq \emptyset$ for some $i < j \in \{0, 1, \dots, k-1\}$ and let $u \in \alpha_i \cap \alpha_j$ be a vertex. Since the arcs α_i and α_j are contained in the links X_{w*w_i} and X_{w*w_j} respectively, the simplices $w * w_i$ and $w * w_j$ lie in $X_u \cap S_n$. By Fact 2.5 the join $w_i * w_j * w$ is a simplex in $S_n \cap X_u$. It follows that $j = i+1$. Since α_w is connected ($\alpha_i \cap \alpha_{i+1}$ is exactly the single vertex equal to $X_{\sigma_i} \cap S_{n+1}$), it must be a cycle.

By Lemma 3.7 and the definition of the cycle α_w it follows that Π_{n+1} maps α_w onto $S_1(w, S_n')$. Since $\Pi_{n+1}^{-1}[B_1(w, S_n')] \subseteq X_w \cap S_{n+1}$, it is enough to show that for all vertices $u \in X_w \cap S_{n+1}$ not contained in α_w the projection Π_{n+1} maps u to w. We show that $X_u \cap S_n$ is exactly w. Suppose that there is another vertex, say w', lying in $X_u \cap S_n$. Then $w' = w_i$ for some $i = 0, 1, \dots, k-1$. Thus u lies in the arc α_i, a contradiction. ∎

From the proof of Lemma 3.9 we get the following additional information:

FACT 3.10. *Let* $w \in S_n$ *be a vertex and let* $\{\varepsilon_i : i = 1, \dots, k\}$ *be the set of all edges in* S_n *that contain* w. *Then the cycle* α_w *is equal to* $\bigcup_{i=1}^k \alpha_{\varepsilon_i}$.

In the next lemma we show that the cycle α_w given by Lemma 3.9 bounds some 2-disc $B_w \subset X_w \cap B_{n+1}$.

LEMMA 3.11. *Each cycle* α_w *bounds a* 2*-disc* $B_w = B_2(\sigma_w, X_w)$ *in* $X_w \cap B_{n+1}$*, for some simplex* $\sigma_w \subset X_w$.

Proof. For a vertex $w \in S_n$ giving the arc α_w let $\sigma_w := X_w \cap S_{n-1}$ (this intersection is a single simplex). We show that $\alpha_w = S_2(\sigma_w, X_w)$. It is obvious that $\alpha_w \subseteq S_2(\sigma_w, X_w)$. For the opposite inclusion let $u \in S_2(\sigma_w, X_w)$ be a vertex. There is a vertex $u' \in S_n \cap X_w$ connected by edges to u and to some vertex of σ_w. It follows that u is a vertex in the arc $\alpha_{w*u'}$. Thus, by Fact 3.10, u is a vertex in α_w.

By Remark 3.4, the cycle α_w is the boundary of $B_2(\sigma_w, X_w)$. Since the link X_w is a surface (with a 7-large triangulation), it follows that $B_2(\sigma_w, X_w)$ is a 2-disc (see Remark 3.4 again). ∎

For a vertex $w \in S_n$ let $P_w := \mathrm{cl}(X_w \setminus B_w)$. Clearly, we have the following:

FACT 3.12. *The set P_w is a subcomplex of S_{n+1}. Topologically it is a connected surface with boundary α_w.*

The next lemma describes the map Π_{n+1} restricted to the subcomplex $P_w \subset S_{n+1}$.

LEMMA 3.13. *For every vertex $w \in S_n$ the projection $\Pi_{n+1} : S_{n+1} \to S_n$ maps the subcomplex P_w onto the ball $B_1(w, S'_n)$. Moreover, the preimage $\Pi_{n+1}^{-1}[w]$ is the union of the simplices in P_w disjoint from the cycle α_w.*

Before proving Lemma 3.13 note the following:

REMARK 3.14.

- The ball $B_1(w, S'_n)$ is topologically a 2-disc with boundary $S_1(w, S'_n)$.
- Lemma 3.13 together with previous results (Lemmas and Facts 3.8–3.12) fully describe the restricted map $\Pi_{n+1}\lceil_{P_w}$.

Proof of Lemma 3.13. By Lemma 3.9, $\Pi_{n+1}^{-1}[S_1(w, S'_n)] = \alpha_w$. Let $u \in P_w$ be a vertex not contained in α_w. Since P_w is a subcomplex of the link X_w, the vertices w and u span an edge in X. We show that $X_u \cap S_n$ is exactly the vertex w. It follows that Π_{n+1} maps u to w. It is enough to show that $\dim(X_u \cap S_n) = 0$ (since w a vertex in this intersection, which is a single simplex).

Assume the opposite and let $\sigma = X_u \cap S_n$. Then Π_{n+1} maps u to b_σ. Since σ contains w and has dimension at least 1, b_σ is contained in $S_1(w, S'_n)$. Thus $u \in \Pi_{n+1}^{-1}[S_1(w, S'_n)]$. This contradicts the equality $\Pi_{n+1}^{-1}[S_1(w, S'_n)] = \alpha_w$. ∎

For a better understanding of the map Π_{n+1} we introduce another cell structure on the sphere S_n. We call this cell structure *dual*.

- The set of dual 0-cells (denoted by e^0_σ) consists of the barycenters b_σ of all 2-simplices $\sigma \subset S_n$.
- The set of dual 1-cells (denoted by e^1_ε) consists of the unions $b_{\sigma_1} * b_\varepsilon \cup b_{\sigma_2} * b_\varepsilon$, where ε is an edge in S_n while σ_1 and σ_2 are the two 2-simplices in S_n containing ε.

- The set of dual 2-cells (denoted by e_w^2) consists of the balls $B_1(w, S'_n)$ around all vertices $w \in S_n$.

We denote by S_n^d the cell complex related to this cell structure, and by $(S_n^d)^{(k)}$ its k-skeleton, i.e. the cell subcomplex consisting of all cells of dimension at most k.

Using this dual cell structure, as a consequence of previous lemmas we get:

Lemma 3.15.

1. $\Pi_{n+1}^{-1}[e_\sigma^0]$ *is the vertex* $w_\sigma = X_\sigma \cap S_{n+1}$.
2. $\Pi_{n+1}^{-1}[e_\varepsilon^1]$ *is the arc* α_ε.
3. $\Pi_{n+1}^{-1}[e_w^2]$ *is the subcomplex* P_w.

Proof. Assertion 1 follows from Lemma 3.6.

By Lemma 3.8, Π_{n+1} maps α_ε onto e_ε^1. By Lemma 3.7, the preimages of the endpoints of e_ε^1 are exactly the endpoints of α_ε. By Lemma 3.9, $\Pi_{n+1}^{-1}[e_\varepsilon^1]$ is contained in the cycle α_u for every endpoint u of ε. Let u and u' be the two endpoints of ε. Since $\alpha_u \cap \alpha_{u'} = \alpha_\varepsilon$, we get assertion 2.

Assertion 3 follows from Lemma 3.13. ∎

The next lemma describes the relationship between the 1-skeleton $(S_n^d)^{(1)}$ of the dual cell structure on the sphere S_n and its preimage under Π_{n+1}.

Lemma 3.16. *The preimage* $\Pi_{n+1}^{-1}[(S_n^d)^{(1)}]$ *of the* 1*-skeleton of the dual cell structure is naturally homeomorphic to this* 1*-skeleton.*

Proof. The 1-skeleton $(S_n^d)^{(1)}$ is the union $\bigcup e_\varepsilon^1$ of 1-cells. By Lemmas 3.8 and 3.15, Π_{n+1} gives a one-to-one correspondence between the arcs α_ε and the dual 1-cells e_ε^1. Namely, α_ε is mapped onto e_ε^1. Moreover, this correspondence is consistent with the incidence relation, i.e. $e_u^2 \cap e_{u'}^2 \neq \emptyset$ if and only if $\alpha_u \cap \alpha_{u'} \neq \emptyset$, and the same holds for triples of vertices. ∎

Remark 3.17.

- Note that the restriction of Π_{n+1} to $\Pi_{n+1}^{-1}[(S_n^d)^{(1)}]$ is not a homeomorphism onto $(S_n^d)^{(1)}$. However, it can be approximated by homeomorphisms of the form described later in Lemma 5.2. More precisely, the map $w_\sigma \to e_\sigma^0$ can be extended to a map $S_{n+1} \to S_n$ such that every arc α_ε is homeomorphically mapped onto the dual 1-cell e_ε^1. As a consequence, the cycle α_w is mapped homeomorphically onto the boundary $\mathrm{bd}(e_w^2)$ of the dual 2-cell e_w^2.
- The sphere S_{n+1}, up to homeomorphism, can be thought of as obtained from S_n by cutting out the interiors of all dual 2-cells e_w^2 and replacing them by surfaces P_w such that each boundary $\mathrm{bd}(P_w) = \alpha_w$ is glued homeomorphically to $\mathrm{bd}(e_w^2)$.

Recall that a *connected sum of the manifolds M and N of dimension n (with or without boundaries) along n-discs $D \subset \mathrm{int}(M)$* and $D' \subset \mathrm{int}(N)$ is the quotient space

$$\big((M \setminus \mathrm{int}(D)) \cup (N \setminus \mathrm{int}(D'))\big)/_{x \sim f(x)}$$

where $f : \mathrm{bd}(D) \to \mathrm{bd}(D')$ is a homeomorphism.

As a consequence of the second part of Remark 3.17 we have:

COROLLARY 3.18. *The sphere S_{n+1} is topologically a connected sum of the sphere S_n and the links X_w of the vertices $w \in S_n$ along the discs $B_w \subset X_w$ and $e^2_w \subset S_n$.*

4. Inverse limits, Jakobsche spaces and outline of proof of the Main Theorem. In this section we recall the result of Jakobsche from [J] concerning inverse systems of appropriately iterated connected sums of compact orientable manifolds. This result justifies the notion of tree of manifolds.

Recall that a family $\mathcal{A}$ of subsets of a metric space X is a *null family* if for every $\epsilon > 0$ only finitely many elements $A \in \mathcal{A}$ have diameter greater than ϵ. The family $\mathcal{A}$ is *dense* if $\bigcup \mathcal{A}$ is a dense subset of X.

THEOREM 4.1 ([J, Theorem 4.6]). *Let $(L_0 \xleftarrow{\alpha_1} L_1 \xleftarrow{\alpha_2} L_2 \xleftarrow{\alpha_3} \cdots)$ be an inverse system of connected closed orientable m-manifolds ($m \geq 2$) and for each $k \geq 0$ let $\mathcal{D}_k$ be a finite collection of pairwise disjoint bicollared discs in L_k such that:*

1. *each L_k is a connected sum of finitely many copies of L_0,*
2. *every map α_{k+1} restricted to $\alpha^{-1}_{k+1}[L_k \setminus \bigcup\{\mathrm{int}(D) : D \in \mathcal{D}_k\}]$ is a homeomorphism onto $L_k \setminus \bigcup\{\mathrm{int}(D) : D \in \mathcal{D}_k\}$,*
3. *every preimage $\alpha^{-1}_{k+1}[D]$ (for $D \in \mathcal{D}_k$) is homeomorphic to a copy of L_0 with the interior of a bicollared disc removed,*
4. *the family $\{\alpha_{j,i}[D] : i \geq j, D \in \mathcal{D}_i\}$ (1) is null and dense in L_j for all j,*
5. *$\alpha_{j,i}[D] \cap \mathrm{bd}(D')$ is empty for all $D \in \mathcal{D}_i$, $D' \in \mathcal{D}_j$ and all $i > j$.*

Then the inverse limit $\varprojlim(L_0 \xleftarrow{\alpha_1} L_1 \xleftarrow{\alpha_2} L_2 \xleftarrow{\alpha_3} \cdots)$ depends only on L_0.

We denote this inverse limit by $X(L_0)$ and call it the *Jakobsche space* for L_0, or the *Jakobsche tree of manifolds* L_0. We call $(L_k, \alpha_k, \mathcal{D}_k)_{k \geq 0}$ satisfying assumptions 1–5 of Theorem 4.1 a *Jakobsche inverse system* for L_0. If $(L_k, \alpha_k, \mathcal{D}_k)_{k \geq 0}$ satisfies assumptions 2, 4, 5 and the condition:

3a. every preimage $\alpha^{-1}_{k+1}[D]$ (for $D \in \mathcal{D}_k$) is homeomorphic to a connected closed (orientable) m-manifold (not necessarily fixed) with the interior of a bicollared disc removed,

(1) For $i > j$ we denote by $\alpha_{j,i}$ the composition $\alpha_{j+1} \circ \cdots \circ \alpha_i$, whereas $\alpha_{i,i}$ denotes the identity on L_i.

then we call it a *Jakobsche inverse system* of (orientable) m-manifolds. We call the corresponding inverse limit a *tree of (orientable) manifolds.*

Remark 4.2.

1. Note that we have not stated the result of Jakobsche in its full generality.
2. For $L_0 = \mathbb{T}^2$, the 2-dimensional torus, the space $X(\mathbb{T}^2)$ is known as the *Pontryagin sphere.*
3. For $m = 2$ and $L_0 = \Sigma_g$, the orientable surface of genus $g > 1$, the space $X(\Sigma_g)$ is homeomorphic to the Pontryagin sphere. Actually, every tree of orientable surfaces of genera greater than 0 is homeomorphic to the Pontryagin sphere. We sketch some details of this in Section 8 (see Remark 8.6(2)).

If X is a locally finite 7-systolic simplicial complex of finite dimension, then by Theorem 2.8, the Gromov boundary $\partial_G X$ is homeomorphic to the inverse limit $\varprojlim(S_n, \Pi_n)$. The results of Section 3 imply that the inverse system (S_n, Π_n) of spheres and projections in a 7-systolic orientable normal pseudomanifold X of dimension 3 is close to satisfying assumptions 1–5 of the Jakobsche theorem. In the next remark we make this observation more precise.

Remark 4.3. The maps Π_k are natural candidates for projections α_k, and the families $\mathcal{D}_k = \{e^2_w : w \in S_k^{(0)}\}$ of dual 2-cells in the spheres S_k are natural candidates for families of discs as in a Jakobsche inverse system. More precisely, Fact 2.7 implies that for such a choice of families $\mathcal{D}_k$ the family $\{\Pi_{j,i}[D] : i \geq j,\ D \in \mathcal{D}_i\}$ is null in every sphere S_j. Moreover, since $\bigcup \mathcal{D}_j$ covers the sphere S_j, it follows that the families $\{\Pi_{j,i}[D] : i \geq j,\ D \in \mathcal{D}_i\}$ are dense in every S_j. If the links of all vertices of X are triangulations of the same surface Σ_0, then assumptions 4.1 and 4.1 are satisfied with $L_0 = \Sigma_0$ by Lemma 3.15(3).

On the other hand, the maps Π_k and the families $\mathcal{D}_k$ defined as above fail to satisfy some other assumptions of the Jakobsche theorem. In particular:

- elements of $\mathcal{D}_k$ are not pairwise disjoint,
- even though Π_{k+1} maps $\Pi_{k+1}^{-1}[S_k \setminus \bigcup\{\operatorname{int}(e^2_w) : w \in S_k^{(0)}\}]$ onto $S_k \setminus \bigcup\{\operatorname{int}(e^2_w) : w \in S_k^{(0)}\}$ the restriction of Π_{k+1} to this preimage is not a homeomorphism, and
- assumption 5 of Theorem 4.1 fails.

The strategy of the proof of part (a) of the Main Theorem is as follows. In Section 5 we modify the inverse system (S_n, Π_n), without affecting the inverse limit, by changing the bonding maps appropriately. This modification will make the inverse system satisfy assumptions 2, 4 and 5 of Theorem 4.1 (after choosing the families of discs appropriately). The modified inverse system (S_n, Π'_n) (with families of discs chosen appropriately) will be a Jakobsche inverse system of orientable surfaces. In Section 6 we refine this new system without changing the inverse limit either. The refined system will consist of orientable surfaces $S_{n,k}$ for $k = 0, 1, \dots, g_n$ (for some natural numbers g_n) and maps $\Pi'_{n,k+1\to k} : S_{n,k+1} \to S_{n,k}$ satisfying $S_{n,0} = S_n$, $S_{n,g_n} = S_{n+1}$ and $\Pi'_{n,1\to 0} \circ \Pi'_{n,2\to 1} \circ \dots \circ \Pi'_{n,g_n\to g_n-1} = \Pi'_{n+1}$. The refinement is necessary to get a connected sum with tori, rather than with higher genera surfaces. In Section 7 we define a family of discs in every surface of the refined system to match all the assumptions of Theorem 4.1.

5. Modification of the inverse system. We denote by d_{sup} the uniform metric on the set of continuous maps between two compact spaces. We perform small (with respect to the uniform distance) modifications of the maps Π_n in the inverse system (S_n, Π_n) keeping the inverse limit unchanged. To do this we use the following result due to M. Brown.

THEOREM 5.1 ([B, Theorem 3]). *Let $Y = \varprojlim (Y_i, f_i)$, where Y_i are compact metric spaces. For $i > 0$ let K_i be a nonempty collection of maps from Y_i to Y_{i-1}. Suppose that for each $i > 0$ and $\epsilon > 0$ there is $g_i \in K_i$ such that $d_{sup}(f_i, g_i) < \epsilon$. Then there is a sequence (g_i) where $g_i \in K_i$ such that Y is homeomorphic to $\varprojlim (Y_i, g_i)$.*

The next lemma shows that it is possible to approximate the projections $\Pi_{n+1} : S_{n+1} \to S_n$ arbitrarily closely by maps $\Pi_{n+1,\epsilon} : S_{n+1} \to S_n$ having much better properties (from the point of view of fulfilling the requirements of Jakobsche inverse system).

LEMMA 5.2. *For any number $\epsilon > 0$ and any integer $n > 0$ there is a continuous map $\Pi_{n+1,\epsilon} : S_{n+1} \to S_n$ satisfying the following:*

1. $d_{\mathrm{sup}}(\Pi_{n+1}, \Pi_{n+1,\epsilon}) < \epsilon$,
2. $\Pi_{n+1}^{-1}[w] = (\Pi_{n+1,\epsilon})^{-1}[w] = (X_w \setminus B_3(\sigma_w, X_w)) \cup S_3(\sigma_w, X_w)$ *for all vertices $w \in S_n^{(0)}$ (where $\sigma_w = X_w \cap S_{n-1}$),*
3. *the restriction of $\Pi_{n+1,\epsilon}$ to $S_{n+1} \setminus \bigcup \{(\Pi_{n+1,\epsilon})^{-1}[w] : w \in S_n^{(0)}\}$ is a homeomorphism onto $S_n \setminus \{w : w \in S_n^{(0)}\}$,*
4. $\Pi_{n+1}[S_{n+1}^{(0)}] \subseteq \Pi_{n+1,\epsilon}[S_{n+1}^{(0)}]$.

Proof. Let $w \in S_n$ be a vertex. Let l_w denote the number of 2-simplices in S_n that contain w. For $i = 0, 1, \dots, l_w - 1$ let $\sigma_i = w * w_i * w_{i+1}$ be all these 2-simplices (with indices taken modulo l_w).

Consider the cycle $\alpha_w \subset S_{n+1}$ as described in Lemma 3.9. Denote the vertices of α_w in the following way:

$$w_{0,0}, w_{0,1}, \ldots, w_{0,k_0} = w_{1,0}, w_{1,1}, \ldots, w_{1,k_1} = w_{2,0}, \ldots,$$
$$w_{l_w-1,0}, \ldots, w_{l_w-1,k_{l_w-1}} = w_{0,0}$$

(for some natural numbers $k_0 > 1, \ldots, k_{l_w-1} > 1$). We choose the indices in such a way that $\Pi_{n+1}(w_{i,0}) = b_{\sigma_i}$, $\Pi_{n+1}(w_{i,j}) = b_{\sigma_i \cap \sigma_{i+1}}$ for $0 < j < k_i$ and successive vertices are connected by an edge.

Consider the 2-simplices in P_w intersecting α_w. Denote them in the following way (see Figure 1; note that this figure does not reflect the geometry

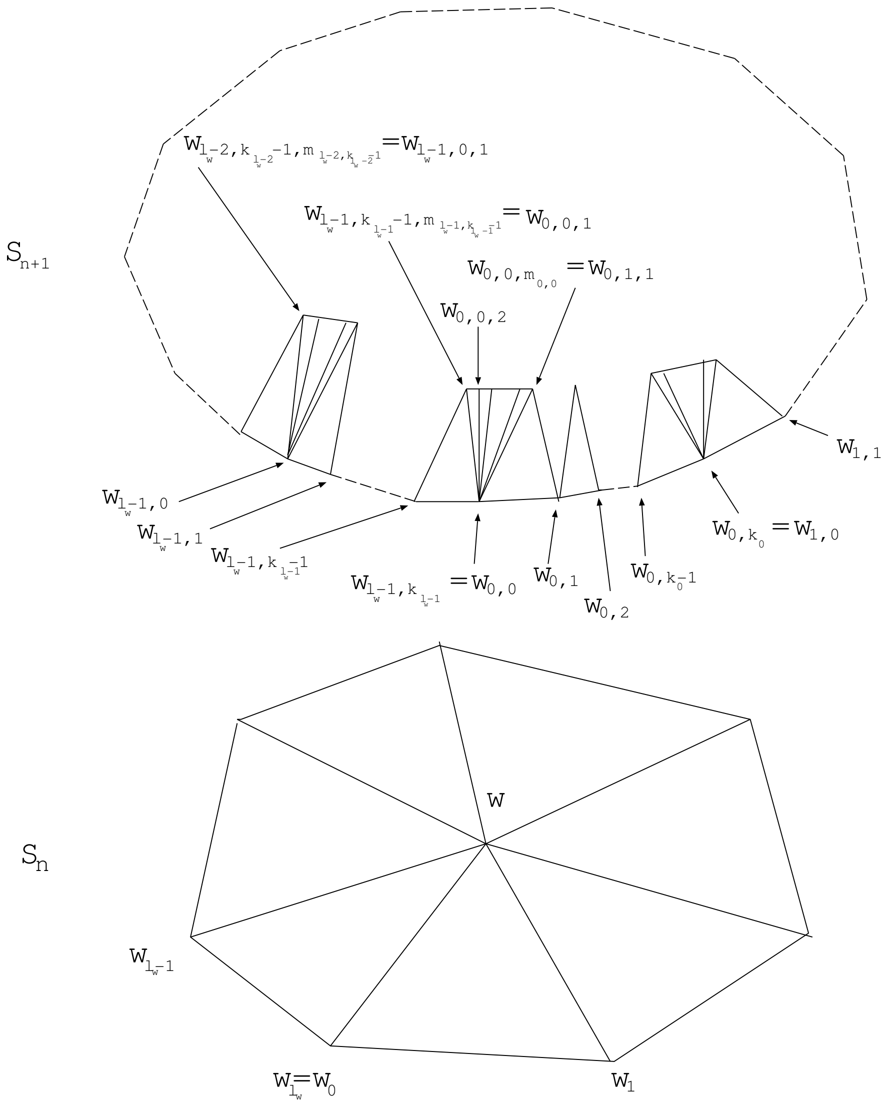

Fig. 1. Proof of Lemma 5.2

of the subcomplex P_w, in fact all simplices have sides of length 1):

$$w_{0,0} * w_{0,0,1} * w_{0,0,2},\ w_{0,0} * w_{0,0,2} * w_{0,0,3},\ \ldots,\ w_{0,0} * w_{0,0,m_{0,0}-1} * w_{0,0,m_{0,0}},$$
$$w_{0,0} * w_{0,1} * w_{0,0,m_{0,0}}, w_{0,1} * w_{0,1,1} * w_{0,1,2} \text{ (where } w_{0,1,1} = w_{0,0,m_{0,0}}), \ldots,$$
$$w_{0,k_0-1} * w_{0,k_0} * w_{0,k_0-1,m_{0,k_0-1}},\ w_{1,0} * w_{1,0,1} * w_{1,0,2}$$
$$\text{(where } w_{1,0} = w_{0,k_0} \text{ and } w_{1,0,1} = w_{0,k_0-1,m_{0,k_0-1}}),\ \ldots,$$
$$w_{l_w-1,k_{(l_w-1)}-1} * w_{l_w-1,k_{(l_w-1)}-1,0} * w_{l_w-1,k_{(l_w-1)}-1,1},\ \ldots,$$
$$w_{l_w-1,k_{(l_w-1)}-1} * w_{l_w-1,k_{(l_w-1)}-1,m_{[l_w-1,k_{(l_w-1)}-1]}-1} * w_{l_w-1,k_{(l_w-1)}-1,m_{[l_w-1,k_{(l_w-1)}-1]}},$$
$$w_{l_w-1,k_{(l_w-1)}} * w_{0,0} * w_{0,0,1}$$
$$\text{(where } w_{l_w,k_{l_w}} = w_{0,0} \text{ and } w_{l_w-1,k_{(l_w-1)}-1,m_{[l_w-1,k_{(l_w-1)}-1]}} = w_{0,0,1}).$$

For two points x and y lying in a single simplex we denote by $[x,y]$ the interval connecting them.

For $i = 0,1,\ldots,l_w-1$ choose points a_i, b_i, c_i, d_i in the following way: $a_i \in [w_{i,0}, w_{i-1,k_{(i-1)}-1}]$ with $d(a_i, w_{i-1,k_{(i-1)}-1}) = \epsilon$, $b_i \in [w_{i,0}, w_{i-1,k_{(i-1)}-1}]$ with $d(a_i, w_{i,0}) = \epsilon$, $c_i \in [w_{i,0}, w_{i,1}]$ with $d(c_i, w_{i,0}) = \epsilon$, $d_i \in [w_{i,0}, w_{i,1}]$ with $d(d_i, w_{i,1}) = \epsilon$. For $s \in [0, \sqrt{3}/2]$ and $i = 0,1,\ldots,l_w-1$ choose $a_i^s, b_i^s, c_i^s, d_i^s$ in the following way: $a_i^s \in [w_{i,0,1}, a_i]$ with $d(a_i^s, [w_{i,0}, w_{i-1,k_{(i-1)}-1}]) = s$, $b_i^s \in [w_{i,0,1}, b_i]$ with $d(b_i^s, [w_{i,0}, w_{i-1,k_{(i-1)}-1}]) = s$, $c_i^s \in [w_{i,0,m_{i,0}}, c_i]$ with $d(c_i^s, [w_{i,0}, w_{i,1}]) = s$, $d_i^s \in [w_{i,0,m_{i,0}}, d_i]$ with $d(d_i^s, [w_{i,0}, w_{i,1}]) = s$. For $s \in [0, \sqrt{3}/2]$, $i = 0,1,\ldots,l_w-1$, $j = 0,1,\ldots,k_i-1$ and $k = 1,\ldots,m_{i,j}$ choose $e_{i,j,k}^s \in [w_{i,j}, w_{i,j,k}]$ with $d(e_{i,j,k}^s, [w_{i,j,k}, w_{i,j,k+1}]) = \sqrt{3}/2 - s$.

For $i = 0,1,\ldots,l_w-1$ let $a_i' = \Pi_{n+1}(a_i)$, $b_i' = \Pi_{n+1}(b_i)$, $c_i' = \Pi_{n+1}(c_i)$ and $d_i' = \Pi_{n+1}(d_i)$. For $i = 0,1,\ldots,l_w-1$ and $s \in [0,\sqrt{3}/2]$ let $a_i'^s = \Pi_{n+1}(a_i^s)$, $b_i'^s = \Pi_{n+1}(b_i^s)$, $c_i'^s = \Pi_{n+1}(c_i^s)$, $d_i'^s = \Pi_{n+1}(d_i^s)$, $e_i'^s = \Pi_{n+1}(e_{i,0,k}^s)$ and $e_{i,i+1}'^s = \Pi_{n+1}(e_{i,j,k}^s)$ (for $j > 0$).

For each $i = 0,1,\ldots,l_w-1$ choose vertices $w_{i_i,j_i} \in \{w_{i,1}, \ldots, w_{i,k_i-1}\}$ and $w_{i_i,j_i,k_i} \in \{w_{i_i,j_i,1}, \ldots, w_{i_i,j_i,m_{i_i,j_i}}\}$ (the vertices w_{i_i,j_i} will be mapped onto the barycenters $b_{w*w_{i+1}}$ in order to fulfill condition 4).

Define the map $\Pi_{n+1}^{w,\epsilon} : P_w \to S_n$ as follows:

- $\Pi_{n+1}^{w,\epsilon}(x) = \Pi_{n+1}(w)$ for $w \in [w_{i,0,1}, a_i, b_i] \cup [w_{i,0,m_{i,0}}, c_i, d_i]$ and for $x \in \Pi_{n+1}^{-1}[w]$,
- for $s \in [0,\sqrt{3}/2]$ and $i = 0,1,\ldots,l_w-1$ let $\Pi_{n+1}^{w,\epsilon} : [b_i^s, e_{i,0,1}^s] \cup [e_{i,0,1}^s, e_{i,0,2}^s] \cup \cdots \cup [e_{i,0,m_{i,0}}^s, c_i^s] \to [b_i'^s, e_i'^s] \cup [e_i'^s, c_i'^s]$ be linear (with respect to the length of segments),
- for $s \in [0,\sqrt{3}/2]$ and $i = 0,1,\ldots,l_w-1$ let
$$\Pi_{n+1}^{w,\epsilon} : [d_i^s, e_{i,1,1}^s] \cup [e_{i,1,1}^s, e_{i,1,2}^s] \cup \cdots \cup [e_{i_i,j_i,k_i-1}^s, e_{i_i,j_i,k_i}^s] \to [d_i'^s, e_{i,i+1}'^s]$$

and

$$\Pi_{n+1}^{w,\epsilon} : [e^s_{i_i,j_i,k_i}, e^s_{i_i,j_i,k_i+1}] \cup \cdots \cup [e^s_{i,k_i-1,m_{i,k_i-1}}, a^s_{i+1}] \to [e'^s_{i,i+1}, a'^s_{i+1}]$$

be linear.

Note that $\Pi_{n+1}^{w,\epsilon}$ is a well defined continuous map. Note also that with vertices $w_{i_{i_w},j_{i_w}}$ chosen in a coherent way (i.e. for two adjacent vertices $w, w' \in S_n$ the chosen vertices $w_{i_{i_w},j_{i_w}}$ and $w_{i_{i_{w'}},j_{i_{w'}}}$ lying on the arc $\alpha_{w*w'}$ must coincide) the map $\Pi^\epsilon_{n+1} = \bigcup_{w\in S_n^{(0)}} \Pi_{n+1}^{w,\epsilon} : S_{n+1} \to S_n$ is well defined and satisfies the required conditions. We omit further details. ∎

The following lemma is an obvious corollary of Theorem 5.1 and Lemma 5.2. We define a sequence of maps $(\Pi'_{n+1} : S_{n+1} \to S_n)_{n\geq 1}$ such that the inverse limits $\varprojlim (S_n, \Pi_n)$ and $\varprojlim (S_n, \Pi'_n)$ are homeomorphic. The new inverse system (S_n, Π'_n) satisfies the conditions mentioned at the beginning of this section. As we will show later, after refinement of this new system, we will be able to define the families of discs $\mathcal{D}_{n,k}$ such that the refined system $(S_{n,k}, \Pi'_{n,k}, \mathcal{D}_{n,k})$ will become a Jakobsche inverse system for the torus.

LEMMA 5.3. *There is a sequence of continuous maps $(\Pi'_{n+1} : S_{n+1} \to S_n)_{n\geq 1}$ and a decreasing sequence of positive real numbers $(\epsilon_n)_{n>1}$ such that:*

1. *the inverse limits $\varprojlim (S_n, \Pi_n)$ and $\varprojlim (S_n, \Pi'_n)$ are homeomorphic,*
2. *$\Pi_{n+1}^{-1}[w] = (\Pi'_{n+1})^{-1}[w] = (X_w \setminus B_3(\sigma_w, X_w)) \cup S_3(\sigma_w, X_w)$ for all vertices $w \in S_n^{(0)}$ (where $\sigma_w = X_w \cap S_{n-1}$),*
3. *the restriction of Π'_{n+1} to $S_{n+1} \setminus \bigcup\{(\Pi'_{n+1})^{-1}[w] : w \in S_n^{(0)}\}$ is a homeomorphism onto $S_n \setminus \{w : w \in S_n^{(0)}\}$,*
4. *$\Pi_{n+1}[S_{n+1}^{(0)}] \subseteq \Pi'_{n+1}[S_{n+1}^{(0)}]$,*
5. *$d_{\sup}(\Pi_{n+1}, \Pi'_{n+1}) < \epsilon_n$,*
6. *$\epsilon_n/(1-C) < 1$, where C is the constant given by Fact* 2.7 (this property will be used in the proof of Lemma 9.4).

6. Refinement of the inverse system. In this section X is orientable. Recall that the vertex links X_u are orientable surfaces (see Remark 2.1). We refine the inverse system (S_n, Π'_n) without changing the inverse limit. The refined system $(S_{n,k}, \Pi'_{n,k})$ will have the property that every surface $S_{n,k+1}$ is a connected sum of its predecessor $S_{n,k}$ and a finite number of tori.

As will be made clear in Section 7, the inverse system (S_n, Π'_n), after appropriate choice of families $\mathcal{D}_n$ of discs in S_n, fulfills assumptions 2, 4 and 5 of Theorem 4.1. Preimages of the chosen discs under the bonding maps Π'_{n+1} will correspond to surfaces that are contained in links of X at vertices of S_n.

Two phenomena may appear that prevent the system (S_n, Π'_n) from satisfying assumptions 1 and 3 of the Jakobsche theorem for $L_0 = \mathbb{T}^2$. The first is that links at vertices do not have to be surfaces of the same genus. The second is that even if all vertex links are homeomorphic, they may be surfaces of genus greater than 1.

Using the following two lemmas we will be able to refine the system (S_n, Π'_n) to overcome these difficulties. We start with some terminology.

Definition 6.1. Let $f : \Sigma \to \Sigma'$ be a map between compact orientable surfaces and let $\mathcal{D} = \{D_1, \ldots, D_l\}$ be a family of pairwise disjoint discs in Σ'. We say that f *collapses* Σ to Σ' along the family $\mathcal{D}$ if:

- Σ is a connected sum of Σ' and a finite number of surfaces $\Sigma_{g_1}, \ldots, \Sigma_{g_l}$ of genera $g_1 > 0, \ldots, g_l > 0$ respectively (for some $l > 0$) along discs $D_i \subset \Sigma'$ and $D'_i \subset \Sigma_{g_i}$ for $i = 1, \ldots, l$,
- $f(x) = x$ for all $x \in \Sigma' \setminus \bigcup_{i=1}^{l} \mathrm{int}(D_i)$,
- there are open neighbourhoods U_i of $\mathrm{bd}(D'_i)$ in $\Sigma_{g_i} \setminus \mathrm{int}(D'_i)$ and points $x_i \in \mathrm{int}(D_i)$ such that f maps U_i homeomorphically onto $D_i \setminus \{x_i\}$ and collapses $(\Sigma_{g_i} \setminus \mathrm{int}(D'_i)) \setminus U_i$ to x_i.

We call such a map a *collapsing map*. If it is clear which family $\mathcal{D}$ we mean, we say that f *collapses* Σ to Σ'.

Note that the maps $\Pi_{n,\epsilon}$ from Lemma 5.2, and hence the maps Π'_n from Lemma 5.3, are examples of collapsing maps.

We state without proof two obvious lemmas which we use in the refinement procedure.

Lemma 6.2. *Let Σ be an orientable surface of genus $g > 1$. Then there exist orientable surfaces $\Sigma_1, \ldots, \Sigma_g = \Sigma$, discs $D_i \subset \Sigma_i$ (for $i = 1, \ldots, g-1$) and maps $f_i : \Sigma_i \to \Sigma_{i-1}$ (for $i = 2, \ldots, g$) such that:*

- *Σ_i is an orientable surface of genus i for $i = 1, \ldots, g$,*
- *Σ_i is a connected sum of Σ_{i-1} and a torus T^2_{i-1} along the disc D_{i-1} and some disc $D'_{i-1} \subset T^2_{i-1}$ such that $D_i \subset T^2_{i-1} \setminus D'_{i-1}$ and f_i collapses Σ_i to Σ_{i-1} along $\mathcal{D}_{i-1} = \{D_{i-1}\}$.*

Lemma 6.3. *Let $f : \Sigma \to \Sigma'$ collapse an orientable surface Σ to an orientable surface Σ' along a family $\mathcal{D} = \{D_1, \ldots, D_l\}$. Let $\Sigma_{g_1}, \ldots, \Sigma_{g_l}$ be orientable surfaces as in Definition* 6.1. *For $i = 1, \ldots, l$ let $D'_i \subset \Sigma_{g_i}$ be discs as in Definition* 6.1. *Let the genus g_j of Σ_{g_j} be greater than* 1 *for some $j \in \{1, \ldots, l\}$. Then there exist:*

- *a decomposition of Σ_{g_j} to a connected sum of two orientable surfaces $\Sigma_{g'_j}$ and $\Sigma_{g''_j}$ of genera $g'_j = 1$ and $g''_j = g_j - 1$ respectively (i.e. $\Sigma_{g'_j}$ is a torus) along discs $D''_j \subset \Sigma_{g'_j}$ and $D'''_j \subset \Sigma_{g''_j}$ such that $D'_j \subset \Sigma_{g'_j}$ and $D''_j \cap D'_j = \emptyset$,*

- *a surface Σ'', which is a connected sum of Σ' and $\Sigma_{g_1}, \dots, \Sigma_{g_{j-1}}, \Sigma_{g'_j}$, $\Sigma_{g_{j+1}}, \dots, \Sigma_{g_l}$ along discs D_i and D'_i respectively,*
- *maps $f_1 : \Sigma \to \Sigma''$ and $f_2 : \Sigma'' \to \Sigma'$ such that f_1 collapses Σ to Σ'' along $\{D''_j\}$, f_2 collapses Σ'' to Σ' along $\mathcal{D}$, and $f = f_2 \circ f_1$.*

REMARK 6.4. Note that Lemmas 6.2 and 6.3 are also true for nonorientable surfaces. The only difference is that collapsing maps involve connected sums with projective planes rather than tori.

As an immediate consequence we get the following:

COROLLARY 6.5. *For $n \geq 1$ let $\Pi'_{n+1} : S_{n+1} \to S_n$ be defined as before. For each vertex $w \in (S_n)^{(0)}$ denote by g_w the genus of the link X_w (which is a closed orientable surface). Let $g_n = \max\{g_w : w \in (S_n)^{(0)}\}$. Then there exist surfaces*

$$S_n = S_{n,0},\ S_{n,1},\ \dots,\ S_{n,g_n} = S_{n+1}$$

and maps

$$S_{n,0} \xleftarrow{\Pi'_{n,1\to 0}} S_{n,1} \xleftarrow{\Pi'_{n,2\to 1}} S_{n,2} \xleftarrow{\Pi'_{n,3\to 2}} \cdots \xleftarrow{\Pi'_{n,g_n-1\to g_n-2}} S_{n,g_n-1} \xleftarrow{\Pi'_{n,g_n\to g_n-1}} S_{n,g_n}$$

such that:

- *$S_{n,k}$ is a connected sum of $S_{n,k-1}$ and some tori $T_{n,k,1}, T_{n,k,2}, \dots, T_{n,k,m_{n,k}}$ (for some natural number $m_{n,k} \geq 1$) along pairwise disjoint discs*

$$D_{n,k,1} \subset S_{n,k-1},\quad D_{n,k,2} \subset S_{n,k-1},\ \dots,\ D_{n,k,m_{n,k}} \subset S_{n,k-1}$$

and

$$D'_{n,k,1} \subset T_{n,k,1},\quad D'_{n,k,2} \subset T_{n,k,2},\ \dots,\ D'_{n,k,m_{n,k}} \subset T_{n,k,m_{n,k}}$$

respectively,

- *every disc $D_{n,k+1,i}$ is contained in some torus $T_{n,k,j}$ and is disjoint from the disc $D'_{n,k,j}$,*
- *for each $n > 0$ and each $k = 1, \dots, g_n$ the map $\Pi'_{n+1,k\to k-1}$ collapses $S_{n,k}$ to $S_{n,k-1}$ along the family $\{D_{n,k,i} : i = 1, \dots, m_{n,k}\}$,*
- *$\Pi'_{n+1} = \Pi'_{n,g_n\to 0}$ (where $\Pi'_{n,g_n\to 0}$ denotes the composition $\Pi'_{n,1\to 0} \circ \Pi'_{n,2\to 1} \circ \dots \circ \Pi'_{n,g_n\to g_n-1}$).*

Corollary 6.5 gives the refined inverse system of orientable surfaces

$$(S_1 \xleftarrow{\Pi'_{1,1\to 0}} S_{1,1} \xleftarrow{\Pi'_{1,2\to 1}} \cdots \xleftarrow{\Pi'_{1,g_1\to g_1-1}} S_2 \xleftarrow{\Pi'_{2,1\to 0}} \cdots).$$

In this inverse system every surface is a connected sum of its predecessor and a finite number of tori (possibly only one). If the genus of S_1 is greater than 1, we use Lemma 6.2 for S_1 to get the inverse system

$$(S_0 \xleftarrow{\Pi'_{0,1\to 0}} S_{0,1} \xleftarrow{\Pi'_{0,2\to 1}} \cdots \xleftarrow{\Pi'_{0,g_0\to g_0-1}} S_1 \xleftarrow{\Pi'_{1,1\to 0}} \cdots)$$

with S_0 a torus. We do not do this if S_1 is a torus.

The last condition of Corollary 6.5 implies that refining the inverse system does not change the inverse limit. Thus we get the following:

COROLLARY 6.6. *Suppose that X is a 7-systolic normal orientable pseudomanifold of dimension 3. Then the Gromov boundary $\partial_G X$ is homeomorphic to the inverse limit of the refined inverse system*

$$\varprojlim(S_0 \xleftarrow{\Pi'_{0,1\to 0}} S_{0,1} \xleftarrow{\Pi'_{0,2\to 1}} \cdots \xleftarrow{\Pi'_{0,g_0\to g_0-1}} S_1 \xleftarrow{\Pi'_{1,1\to 0}} S_{1,1} \xleftarrow{\Pi'_{1,2\to 1}} \cdots)$$

$$(\varprojlim(S_1 \xleftarrow{\Pi'_{1,1\to 0}} S_{1,1} \xleftarrow{\Pi'_{1,2\to 0}} \cdots) \textit{ if } S_1 \textit{ is a torus}).$$

7. Getting the structure of a Jakobsche inverse system. In this section we continue the previous considerations, under the same assumption that X is orientable. We define some finite families $\mathcal{D}_{n,k}$ of pairwise disjoint discs in every surface $S_{n,k}$. The inverse system $(S_{n,k}, \Pi'_{n,k})$ with families $\mathcal{D}_{n,k}$ will satisfy all assumptions of the Jakobsche theorem, with $L_0 = \mathbb{T}^2$.

To define these families we need some preparation. For $n = 0, 1, \ldots$ let $A_n = \{\Pi'_{n,l}(w) : l \geq n,\ w \in S_l^{(0)}\} \subset S_n$ and let $A_{n,k} = \Pi'_{n,g_n\to k}[A_{n+1}] \subset S_{n,k}$, where $\Pi'_{n,l} : S_n \to S_l$ and $\Pi'_{n,g_n\to k} : S_{n+1} \to S_{n,k}$ are the compositions $\Pi'_{l+1} \circ \cdots \circ \Pi'_n$ and $\Pi'_{n,k+1\to k} \circ \cdots \circ \Pi'_{n,g_n\to g_n-1}$ respectively.

LEMMA 7.1. *A_n is a countable dense subset of S_n for all $n \geq 1$, and $A_{n,k}$ is a countable dense subset of $S_{n,k}$ for all $n \geq 0$ and all $k = 0, 1, \ldots, g_n$.*

Proof. Recall that every map Π_i is a C-contraction and the 0-skeleton $S_i^{(0)}$ is a finite 1-net in S_i for all i. It follows that $\{\Pi_{n,l}(w) : w \in S_l^{(0)}\}$ is a finite C^{l-n}-net in S_n. Since the maps Π'_i satisfy the condition $\Pi_i[S_i^{(0)}] \subset \Pi'_i[S_i^{(0)}]$ (see assertion 4 of Lemma 5.3) it follows that the assertion holds for the sets A_n. Since $A_{n,k}$ is the image of the countable dense set A_{n+1} under a surjection, it is itself countable and dense. ∎

Now we define inductively families of discs $\mathcal{D}_n$ and $\mathcal{D}_{n,k}$ in the spheres S_n and $S_{n,k}$ respectively to match all assumptions of Theorem 4.1.

Suppose the genus of the sphere S_1 is equal to 1 (i.e. S_1 is a torus). Let

$$\mathcal{D}_1 = \{D_w : w \in (S_1)^{(0)}\}$$

be a family of pairwise disjoint discs with $w \in \mathrm{int}(D_w)$, $\mathrm{diam}(D_w) < 1/2$ and $\mathrm{bd}(D_w) \cap A_1 = \emptyset$. Note that since every disc is the union of an uncountable family of disjoint circles and a point (just take a homeomorphism to the unit plane disc and circles of radius $r \in (0, 1]$ centered at 0), it follows that such discs exist.

If S_1 has genus greater than 1, then as in Corollary 6.6 we start with the surface S_0. Let $\mathcal{D}_0$ be a family consisting of one small 2-disc D in S_0 (contained in the disc $D_{0,0}$ as in Lemma 6.2 and satisfying the inequality $\mathrm{diam}(D) < 1$), with $x \in \mathrm{int}(D)$ (where a point $x \in \mathrm{int}(D_{0,0})$ and a disc $D_{0,0}$ are given by the fact that the map $\Pi'_{0,1\to 0} : S_{0,1} \to S_0$ is a collapsing map). Again we can assume that $\mathrm{bd}(D) \cap A_0 = \emptyset$.

Now suppose we have defined the families $\mathcal{D}_0, \dots, \mathcal{D}_{n-1}$ and $\mathcal{D}_{i,j}$ for all $i \le n-1$ and $j = 0, 1, \dots, g_i$. We define the family $\mathcal{D}_n$ as follows. For every vertex $u \in S_n^{(0)}$ we choose a small 2-disc D_u containing u in its interior such that:

- the discs D_u are pairwise disjoint,
- $\mathrm{bd}(D_u) \cap A_n = \emptyset$ for all vertices $u \in S_n^{(0)}$,
- $\Pi'_{i,n}[D_u] \cap \mathrm{bd}(D') =$ for all $i < n$ and all $D' \in \mathcal{D}_i$,
- $\Pi'_{i,g_i\to j} \circ \Pi'_{i+1,n}[D_u] \cap \mathrm{bd}(D') = \emptyset$ for all $i < n$, all $j = 0, 1, \dots, g_i$ and all $D' \in \mathcal{D}_{i,j}$,
- $\mathrm{diam}_{S_i}(\Pi'_{i,n}[D_u]) < 1/2^n$ for all $i < n$,
- $\mathrm{diam}_{S_{i,j}}(\Pi'_{i,g_i\to j} \circ \Pi'_{i+1,n}[D_u]) < 1/2^n$ for all $i < n$ and all $j = 1, \dots, g_i$.

To choose such a family $\mathcal{D}_n$, note that no point $a = \Pi'_{i,n}(u) \in A_i$ lies in the boundary $\mathrm{bd}(D')$ of any disc $D' \in \mathcal{D}_i$. Analogously, no $a = \Pi'_{i,g_i\to j} \circ \Pi'_{i+1,n}(u) \in A_{i,j}$ is in $\mathrm{bd}(D')$ for any $D' \in \mathcal{D}_{i,j}$. Thus for small enough $\epsilon > 0$, $\Pi'_{i,n}[B_{S_n}(u,\epsilon)] \cap \mathrm{bd}(D') = \emptyset$ for all $D' \in \mathcal{D}_i$ and $\Pi'_{i,g_i\to j} \circ \Pi'_{i+1,n}[B_{S_n}(u,\epsilon)] \cap \mathrm{bd}(D') = \emptyset$ for all $D' \in \mathcal{D}_{i,j}$ (there are only finitely many such discs D'). Since S_n is a surface, the metric ball $B_{S_n}(u,\epsilon)$ contains a 2-disc D_u containing u in its interior. Again we can assume that $\mathrm{bd}(D_u) \cap A_n = \emptyset$.

Now suppose we have defined the families $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_n$, the families $\mathcal{D}_{i,j}$ for all $i < n$ and $j = 0, 1, \dots, g_i$ and the families $\mathcal{D}_{n,j}$ for all $j < k$. We define the family $\mathcal{D}_{n,k}$ as follows. For all points $x_{n,k,l}$ given by the fact that the map $\Pi'_{n,k+1\to k} : S_{n,k+1} \to S_{n,k}$ is a collapsing map let $D_{n,k,l}$ be a small disc in $S_{n,k}$ containing $x_{n,k,l}$ in its interior such that:

- the discs $D_{n,k,l}$ are pairwise disjoint,
- $\mathrm{bd}(D_{n,k,l}) \cap A_{n,k} = \emptyset$ for all l,
- $\Pi'_{i,n} \circ \Pi'_{n,k\to 0}[D_{n,k,l}] \cap \mathrm{bd}(D') = \emptyset$ for all $i < n$ and all $D' \in \mathcal{D}_i$,
- $\Pi'_{i,g_i\to j} \circ \Pi'_{i+1,n} \circ \Pi'_{n,k\to 0}[D_{n,k,l}] \cap \mathrm{bd}(D') = \emptyset$ for all $i < n$, all $j = 0, 1, \dots, g_i$ and all $D' \in \mathcal{D}_{i,j}$,
- $\mathrm{diam}_{S_i}(\Pi'_{i,n} \circ \Pi'_{n,k\to 0}[D_{n,k,l}]) < 1/2^n$ for all $i < n$,
- $\mathrm{diam}_{S_{i,j}}(\Pi'_{i,g_i\to j} \circ \Pi'_{i+1,n} \circ \Pi'_{n,k\to 0}[D_{n,k,l}]) < 1/2^n$ for all $i < n$ and all $j = 0, 1, \dots, g_i$.

Such discs can be chosen analogously to the way we have defined the elements of the families $\mathcal{D}_n$.

Now the families of discs $\mathcal{D}_n$ and $\mathcal{D}_{n,k}$ together with the maps Π'_n and $\Pi'_{n,k\to k-1}$ satisfy the assumptions of Theorem 4.1 (note that A_n is dense in S_n and $A_{n,k}$ is dense in $S_{n,k}$). Since the maps Π'_n were chosen close enough to the maps Π_n to preserve the inverse limit, as a corollary we get:

THEOREM 7.2. *The Gromov boundary of a* 7*-systolic normal orientable pseudomanifold of dimension* 3 *is a Jakobsche tree of tori, i.e. the Pontryagin sphere.*

8. Nonorientable trees of surfaces. In this section we examine the properties of Jakobsche inverse systems of nonorientable surfaces. An extension of Jakobsche's construction for nonorientable case was considered in [S]. In dimension 2, i.e. for nonorientable surfaces, it is possible and more convenient to follow Jakobsche's approach rather than that of Stallings. We sketch here some details of this.

We call a family $\mathcal{D}$ of pairwise disjoint closed discs contained in the interior of a manifold M a *good family* if it is a null family and $\{\mathrm{int}(D) : D \in \mathcal{D}\}$ is a dense family in M.

The following lemma is a simple extension of Toruńczyk's lemma (see [J, Lemma 3.1]).

LEMMA 8.1. *Let Σ and Σ' be nonorientable surfaces (with or without boundaries) and let $f : \Sigma \to \Sigma'$ be a homeomorphism. Let $\mathcal{Z}$ and $\mathcal{Z}'$ be two good families of closed* 2*-discs in Σ and Σ' respectively. Then there exist a bijective function $p : \mathcal{Z} \to \mathcal{Z}'$ and a homeomorphism*

$$f' : \Sigma \setminus \bigcup_{D\in\mathcal{Z}} \mathrm{int}(D) \to \Sigma' \setminus \bigcup_{D'\in\mathcal{Z}'} \mathrm{int}(D')$$

such that

$$f'{\restriction_{\mathrm{bd}(\Sigma)}} = f{\restriction_{\mathrm{bd}(\Sigma)}} \quad \textit{and} \quad f'[\mathrm{bd}(D)] = \mathrm{bd}(p(D)) \textit{ for each } D \in \mathcal{Z}.$$

The proof of this lemma is the same as in [J], thus we omit it.

Using Lemma 8.1 and the fact that every homeomorphism of the boundary of a closed nonorientable surface with the interior of a disc removed can be extended to a homeomorphism of this surface, by the same argument as in the proof of Theorem 4.6 in [J], we get the following:

THEOREM 8.2. *Let $(L_0 \xleftarrow{\alpha_1} L_1 \xleftarrow{\alpha_2} L_2 \xleftarrow{\alpha_3} \cdots)$ be an inverse system of connected closed nonorientable surfaces and for each $k \geq 0$ let $\mathcal{D}_k$ be a finite collection of pairwise disjoint discs in L_k such that:*

1. *each L_k is a connected sum of finitely many copies of L_0,*

2. *every map* α_{k+1} *restricted to* $\alpha_{k+1}^{-1}[L_k \setminus \bigcup\{\mathrm{int}(D) : D \in \mathcal{D}_k\}]$ *is a homeomorphism onto* $L_k \setminus \bigcup\{\mathrm{int}(D) : D \in \mathcal{D}_k\}$,
3. *every preimage* $\alpha_{k+1}^{-1}[D]$ *(for* $D \in \mathcal{D}_k$*) is homeomorphic to a copy of* L_0 *with the interior of a disc removed,*
4. *the family* $\{\alpha_{j,i}[D] : i \geq j, D \in \mathcal{D}_i\}$ *is null and dense in* L_j *for all* j,
5. $\alpha_{j,i}[D] \cap \mathrm{bd}(D') = \emptyset$ *for all* $i > j$ *and all* $D \in \mathcal{D}_i$ *and* $D' \in \mathcal{D}_j$.

Then the inverse limit $\varprojlim(L_0 \xleftarrow{\alpha_1} L_1 \xleftarrow{\alpha_2} L_2 \xleftarrow{\alpha_3} \cdots)$ *depends only on* L_0.

As in the orientable case we denote this space by $X(L_0)$ and call it a *Jakobsche tree of nonorientable surfaces* L_0. Just as in the orientable case, we call the system $(L_k, \alpha_k, \mathcal{D}_k)_{k\geq 0}$ satisfying assumptions 1–5 of Theorem 8.2 a *Jakobsche inverse system* for L_0. If $(L_k, \alpha_k, \mathcal{D}_k)_{k\geq 0}$ satisfies assumptions 2, 4, 5 and the condition:

3a. every preimage $\alpha_{k+1}^{-1}[D]$ (for $D \in \mathcal{D}_k$) is homeomorphic to a connected closed nonorientable surface with the interior of a disc removed,

then we call it a *Jakobsche inverse system* of nonorientable surfaces. We call the corresponding inverse limit a *tree of nonorientable surfaces.*

Remark 8.3.

1. For $L_0 = \mathbb{RP}^2$, the projective plane, the space $X(\mathbb{RP}^2)$ is known as the nonorientable Pontryagin surface.
2. For $L_0 = \Sigma_g$, the nonorientable surface of genus $g > 1$, the space $X(\Sigma_g)$ is homeomorphic to the nonorientable Pontryagin surface. That is, every tree of nonorientable surfaces is homeomorphic to the nonorientable Pontryagin surface (see Remark 8.6).

The next two lemmas show that if nonorientable surfaces occur densely enough in a tree of surfaces, then this tree is homeomorphic to the nonorientable Pontryagin surface. The proof of the following lemma is similar to the proof of Theorem 2.1.49 from [M], thus we omit it.

Lemma 8.4. *Let* $(X_0 \xleftarrow{s_1} X_1 \xleftarrow{s_2} X_2 \xleftarrow{s_3} \cdots)$ *and* $(Y_0 \xleftarrow{t_1} Y_1 \xleftarrow{t_2} Y_2 \xleftarrow{t_3} \cdots)$ *be two inverse systems of topological spaces such that the maps* s_i *and* t_i *are continuous and onto for all natural numbers* i *and there exist:*

- *increasing sequences* $\{n_k\}$, $\{m_k\}$, $\{n'_k\}$ *and* $\{m'_k\}$ *of natural numbers satisfying* $n_{k-1} \leq n'_k \leq n_k$ *and* $m_{k-1} \leq m'_k \leq m_k$,
- *continuous onto maps* $f_k : X_{n_k} \to Y_{m_k}$ *and* $g_k : Y_{m'_k} \to X_{n'_k}$,

such that the following diagrams are commutative:

$$\begin{array}{ccc} X_{n'_k} & \xleftarrow{s_{n'_k,n_k}} & X_{n_k} \\ {\scriptstyle g_k}\big\uparrow & & \big\downarrow{\scriptstyle f_k} \\ Y_{m'_k} & \xleftarrow{t_{m'_k,m_k}} & Y_{m_k} \end{array} \qquad \begin{array}{ccc} X_{n_{k-1}} & \xleftarrow{s_{n_{k-1},n'_k}} & X_{n'_k} \\ {\scriptstyle f_{k-1}}\big\downarrow & & \big\uparrow{\scriptstyle g_k} \\ Y_{m_{k-1}} & \xleftarrow{t_{m_{k-1},m'_k}} & Y_{m'_k} \end{array}$$

Then the inverse limits $\varprojlim(X_k, s_k)$ *and* $\varprojlim(Y_k, t_k)$ *are homeomorphic.*

LEMMA 8.5. *Let* $(L_0 \xleftarrow{\alpha_1} L_1 \xleftarrow{\alpha_2} L_2 \xleftarrow{\alpha_3} \cdots)$ *be an inverse system of connected closed nonorientable surfaces and for each* $k \geq 0$ *let* $\mathcal{D}_k$ *be a finite collection of pairwise disjoint discs in* L_k *such that:*

1. $(L_k, \alpha_k, \mathcal{D}_k)$ *is a Jakobsche inverse system of surfaces* ([2]),
2. *for every natural number* k *and for every disc* $D \in \mathcal{D}_k$ *there is a natural number* $l_D > k$ *such that* $(\alpha_{k,l_D})^{-1}[D]$ *is a nonorientable surface with the interior of a disc removed,*
3. *every map* α_{k+1} *collapses* L_{k+1} *to* L_k *along* $\mathcal{D}_k$.

Then the inverse limit $\varprojlim(L_0 \xleftarrow{\alpha_1} L_1 \xleftarrow{\alpha_2} L_2 \xleftarrow{\alpha_3} \cdots)$ *is homeomorphic to the nonorientable Pontryagin surface.*

Proof. We shall define the following collection of data:

- an infinite increasing sequence $\{n_k\}$ of natural numbers,
- a sequence $\{L'_k\}$ of nonorientable closed surfaces,
- a sequence $\{\mathcal{D}'_k\}$ of finite families of pairwise disjoint discs in every surface L'_k,
- sequences of maps $\{f_k : L_{n_k} \to L'_{k-1}\}$, $\{g_k : L'_k \to L_{n_k}\}$ and $\{\alpha'_k : L'_k \to L'_{k-1}\}$

satisfying the following:

(a) the diagrams:

$$\begin{tikzcd} L_{n_k} & & L_{n_{k+1}} \arrow[ll, "\alpha_{n_k,n_{k+1}}"'] \arrow[dl, "f_{k+1}"] \\ & L'_k \arrow[ul, "g_k"] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & L_{n_k} \arrow[dl, "f_k"'] & \\ L'_{k-1} & & L'_k \arrow[ll, "\alpha'_k"] \arrow[ul, "g_k"'] \end{tikzcd}$$

are commutative,

(b) g_k maps $L'_k \setminus \bigcup_{D\in\mathcal{D}'_{k-1}} (\alpha'_k)^{-1}[\mathrm{int}(D)]$ homeomorphically onto $L_{n_k} \setminus \bigcup_{D\in\mathcal{D}'_{k-1}} f_k^{-1}[\mathrm{int}(D)]$ and maps $(\alpha'_k)^{-1}[D]$ onto $f_k^{-1}[D]$ for all discs $D \in \mathcal{D}'_{k-1}$,

([2]) In particular, we require that $\alpha_{k+1}^{-1}[D]$ (for $D \in \mathcal{D}_k$) is a closed surface (orientable or not) with the interior of a disc removed.

(c) f_{k+1} maps $L_{n_{k+1}} \setminus \bigcup_{D\in\mathcal{D}'_k} f_{k+1}^{-1}[\mathrm{int}(D)]$ homeomorphically onto $L'_k \setminus \bigcup_{D\in\mathcal{D}'_k} \mathrm{int}(D)$,

(d) α'_k collapses L'_k to L'_{k-1} along $\mathcal{D}'_{k-1}$,

(e) $(L'_k, \alpha'_k, \mathcal{D}'_k)$ is a Jakobsche inverse system of nonorientable surfaces.

Note that by Lemma 8.4 the inverse limits $\varprojlim(L_k, \alpha_k)$ and $\varprojlim(L'_k, \alpha'_k)$ are homeomorphic. By the nonorientable analogues of Lemmas 6.2 and 6.3 the inverse limit $\varprojlim(L'_k, \alpha'_k)$ is homeomorphic to the Jakobsche tree of projective planes. Thus, by Theorem 8.2, both these inverse limits are homeomorphic to the nonorientable Pontryagin surface.

It remains to construct the desired data. We do this inductively. Let $n_0 = 0$, $L'_0 = L_0$, $g_0 = \mathrm{Id}_{L_0}$, $\mathcal{D}'_0 = \mathcal{D}_0$. Let $n_1 = 1$, $f_1 = \alpha_1$.

Suppose that we have defined the following:

- natural numbers n_j for $j = 0, 1, \ldots, k$ satisfying $n_j < n_{j+1}$ for $j = 0, 1, \ldots, k-1$,
- nonorientable closed surfaces L'_j for $j = 0, 1, \ldots, k-1$,
- finite families $\mathcal{D}'_j$ (for $j = 0, 1, \ldots, k-1$) of pairwise disjoint discs in every surface L'_j respectively,
- maps $f_j : L_{n_j} \to L'_{j-1}$ for $j = 0, 1, \ldots, k$, $g_j : L'_j \to L_{n_j}$ for $j = 0, 1, \ldots, k-1$ and $\alpha'_j : L'_j \to L'_{j-1}$ for $j = 0, 1, \ldots, k-1$ (if $k > 0$)

satisfying (a), (b), (c), (d) and an additional condition:

(f) every preimage $(\alpha'_j)^{-1}[D]$ (for $D \in \mathcal{D}'_{j-1}$) is homeomorphic to a nonorientable closed surface with the interior of a disc removed.

Let $n_{k+1} > n_k$ be the smallest integer such that $(f_k \circ \alpha_{n_k, n_{k+1}})^{-1}[D]$ is a nonorientable surface with the interior of a disc removed for all discs $D \in \mathcal{D}'_{k-1}$ (such a number exists due to assumption 2).

Let

$$L'_k = \Big(L'_{k-1} \setminus \bigcup_{D\in\mathcal{D}'_{k-1}} \mathrm{int}(D)\Big) \cup \bigcup_{D\in\mathcal{D}'_{k-1}} (f_k \circ \alpha_{n_k, n_{k+1}})^{-1}[D]$$

where points $x \in \mathrm{bd}(D)$ are identified with their preimages $(f_k \circ \alpha_{n_k, n_{k+1}})^{-1}[x]$ due to (c) and assumption 1.

Define $g_k : L'_k \to L_{n_k}$, $f_{k+1} : L_{n_{k+1}} \to L'_k$ and $\alpha'_k : L'_k \to L'_{k-1}$ by

$$g_k(x) = \begin{cases} f_k^{-1}(x) & \text{if } x \in L'_{k-1} \setminus \bigcup_{D\in\mathcal{D}'_{k-1}} \mathrm{int}(D) \text{ (by (c) for } f_k\text{)}, \\ \alpha_{n_k, n_{k+1}}(x) & \text{if } x \in \bigcup_{D\in\mathcal{D}'_{k-1}} (f_k \circ \alpha_{n_k, n_{k+1}})^{-1}[D], \end{cases}$$

$$f_{k+1}(x) = \begin{cases} x & \text{if } x \in \bigcup_{D \in \mathcal{D}'_{k-1}} (f_k \circ \alpha_{n_k,n_{k+1}})^{-1}[D], \\ f_k \circ \alpha_{n_k,n_{k+1}}(x) & \text{otherwise,} \end{cases}$$

$$\alpha'_k(x) = \begin{cases} x & \text{if } x \in L'_{k-1} \setminus \bigcup_{D \in \mathcal{D}'_{k-1}} \operatorname{int}(D), \\ f_k \circ \alpha_{n_k,n_{k+1}}(x) & \text{otherwise.} \end{cases}$$

These maps are of course well defined and continuous. They satisfy (a), (b) and (d) in the obvious way.

To define $\mathcal{D}'_k$ we need some technical definition. For $n_k \le j < n_{k+1}$ let

$$\mathcal{D}_j^+ = \{D \in \mathcal{D}_j : D \cap (f_k \circ \alpha_{n_k,j})^{-1}[D'] = \emptyset \text{ for } D' \in \mathcal{D}'_{k-1} \text{ and } D \cap \alpha_{s,j}^{-1}[D''] = \emptyset \text{ for } n_k \le s < j \text{ and } D'' \in \mathcal{D}_s\}.$$

Define

$$\mathcal{D}'_k = \{f_{k+1}[\alpha^{-1}_{j,n_{k+1}}[D]] : n_k \le j < n_{k+1},\, D \in \mathcal{D}_j^+\}.$$

We skip the straightforward check of conditions (c) and (e). ∎

Remark 8.6.

1. Note that assumption 3 in Lemma 8.5 is not necessary. Indeed, as in the proof of Theorem 4.6 in [J], it is possible to change $\alpha_k : L_k \to L_{k-1}$ on $\alpha_k^{-1}[D]$ (for $D \in \mathcal{D}_{k-1}$), keeping the inverse limit unchanged, to get collapsing maps.
2. The same argument shows that every tree of orientable surfaces of genera greater than 0 is homeomorphic to the Pontryagin sphere.

9. Proof of part (b) of the Main Theorem. In this section we extend Theorem 7.2 to the nonorientable case. We need some preparations. We start with the following property of group actions on metric spaces, the proof of which we skip.

Lemma 9.1. *Let X be a proper metric space and let a group G act on X cocompactly by isometries. Then there is a positive constant $R > 0$ such that for all $x \in X$ the translates of the metric ball $B_X(x,R)$ under elements of G cover X, i.e. $G \cdot B_X(x,R) = X$.*

Consider now a 3-dimensional 7-systolic normal pseudomanifold X with a cocompact action of a group G by simplicial automorphisms. For a vertex $w \in X$ and a simplex $\sigma \subset X_w$ consider the subcomplex

$$X_{w,\sigma} = (X_w \setminus B_2(\sigma, X_w)) \cup S_2(\sigma, X_w).$$

Let $K_{w,\sigma} = \operatorname{diam}(X^{(1)}_{w,\sigma})$ (in the intrinsic metric $d_{X^{(1)}_{w,\sigma}}$). Note that $K = \max\{K_{w,\sigma} : w \in X^{(0)},\, \sigma \subset X_w\}$ is finite.

The next lemma describes the relationship between distances in successive spheres in a 7-systolic normal pseudomanifold of dimension 3.

LEMMA 9.2. *Let X be a 7-systolic 3-dimensional normal pseudomanifold with a cocompact action of a group G by simplicial automorphisms. Let K be as above. Let p and q be two vertices in the sphere S_k and let p' and q' be two vertices in S_{k+1} connected by an edge to p and q respectively. Then*

$$d_{S_{k+1}^{(1)}}(p',q') \le K(d_{S_k^{(1)}}(p,q)+1).$$

Proof. Let $p=p_0,p_1,\dots,p_n=q$ be a geodesic in the 1-skeleton $S_k^{(1)}$. For $i=1,\dots,n$ let p_i' be a vertex in $X_{p_{i-1}*p_i}\cap S_{k+1}$. Note that $\operatorname{diam}((X_{p_i}\cap S_{k+1})^{(1)})\le K$, since $X_{p_i}\cap S_{k+1}=X_{p_i,\rho}$, where $\rho=\Pi_k(p_i)$. Thus

$$d_{S_{k+1}^{(1)}}(p',q') \le d_{S_{k+1}^{(1)}}(p',p_1') + \sum_{j=1}^{n-1} d_{S_{k+1}^{(1)}}(p_i',p_{i+1}') + d_{S_{k+1}^{(1)}}(p_n',q') \le K(n+1). \ \blacksquare$$

The next lemma shows that if a nonorientable complex X is as in Lemma 9.2 then there are enough vertices with nonorientable links in X, in a certain precise sense.

LEMMA 9.3. *Let X be a 7-systolic normal nonorientable pseudomanifold of dimension 3 with a cocompact action of a group G by simplicial isometries. Let $v\in X$ be a vertex. Let $w\in S_k=S_k(v,X)$ be a vertex. Then for every $\epsilon>0$ there is a $k'>k$ and a vertex $u\in S_{k'}$ such that the link X_u is a nonorientable surface and $\Pi_{k,k'}(u)\in B_{S_k}(w,\epsilon)$.*

Proof. Let $\epsilon>0$. By Lemma 9.1 there is $R>0$ such that for all $x\in X$ the translates of the metric ball $B_X(x,R)$ under elements of G cover X, i.e. $G\cdot B_X(x,R)=X$. Thus there is a positive integer N such that $G\cdot B_N(w,X)=X$ for all vertices $w\in X$ (where $B_N(w,X)$ denotes the combinatorial N-ball).

For a natural number $l>0$ and for $i=0,1,\dots,2N$ consider the combinatorial spheres $S_{k+l+i}=S_{k+l+i}(v,X)$. Let $u_i\in S_{k+l+i}^{(0)}$ be such that $\Pi_{k+l+i+1}(u_{i+1})=u_i$ for $i=0,1,\dots,2N-1$ and $\Pi_{k,k+l}(u_0)=w$. There is a vertex $u\in B_N(u_N,X)$ such that the link X_u is a nonorientable surface. Since $B_N(u_N,X)\subseteq B_{k+l+2N}(v,X)\setminus B_{k+l-1}(v,X)$, it is enough to show that for l large enough, for all vertices $z\in B_N(u_N,X)$, we have $d_{S_k}(\Pi_{k,k+l+i}(z),w)<\epsilon$, where $k+l+i=d_{X^{(1)}}(v,z)$ (here we use the convention that $\Pi_{k,k}=\mathrm{Id}_{S_k}$).

For this let z be a vertex in $B_N(u_N,X)\cap S_{k+l+i}$ for some $i=0,1,\dots,2N$. Let

$$u_0=z_{0,1},z_{0,2},\dots,z_{0,j_0},z_{1,1},\dots,z_{1,j_1},\dots,z_{i-1,1},\dots,z_{i-1,j_{i-1}},z_{i,1},z_{i,j_i}=z$$

(for some natural numbers $j_0,j_1,\dots,j_i$) be a geodesic in the 1-skeleton

$X^{(1)}$ satisfying $z_{m,n} \in B_N(u_N, X) \cap S_{k+l+m}(v, X)$ for $m = 0, 1, \dots, i$ and $n = 1, \dots, j_m$ (actually all geodesics between z and u_0 have this form, since combinatorial balls are convex (see [JS, Corollary 7.5]) and thus geodesically convex (see [HS, Proposition 4.9])).

Let K be the constant of Lemma 9.2 and let $L = \max\{K, 2N + 2\}$. We will show that

$$d_{S_{k+l+i}}(z, u_i) < L^{2N+3}.$$

Using this inequality we get

$$d_{S_k}(\Pi_{k,k+l+i}(z), w) < C^{l+i} L^{2N+3} < C^l L^{2N+3}$$

where C is the constant given by Fact 2.7. Thus for l large enough the assertion holds.

To prove the above inequality, we inductively show that for all $t = 0, 1, \dots, i$,

$$d_{S^{(1)}_{k+l+t}}(z_{t,j_t}, u_t) < tL^{t+1} + 2R \quad \text{and} \quad d_{S^{(1)}_{k+l+t}}(z_{t,0}, u_t) < tL^{t+1}.$$

Since z_{0,j_0} and u_0 are vertices in the intersection $B_N(u_N, X) \cap B_{k+l}(v, X)$, it follows that

$$d_{S^{(1)}_{k+l}}(z_{0,j_0}, u_0) = d_{X^{(1)}}(z_{0,j_0}, u_0) \le 2N.$$

Suppose that

$$d_{S^{(1)}_{k+l+t}}(z_{t,j_t}, u_t) < tL^{t+1} + 2N$$

By Lemma 9.2,

$$d_{S^{(1)}_{k+l+t+1}}(z_{t+1,0}, u_{t+1}) < K(tL^{t+1} + 2N + 1) < L(t+1)L^{t+1} = (t+1)L^{t+2}$$

and thus

$$d_{S^{(1)}_{k+l+t+1}}(z_{t+1,j_{t+1}}, u_{t+1}) < (t+1)L^{t+2} + 2N.$$

It follows that

$$d_{S^{(1)}_{k+l+i}}(z, u_i) < iL^{i+1} + 2N < L^{2N+3},$$

and the lemma follows. ∎

Lemma 9.4. *Let X and G be as in Lemma 9.3. Let $v \in X$ be a vertex. Let $w \in S_k = S_k(v, X)$ be a vertex. Then there is a $k' > k$ and a vertex $u \in S_{k'}$ such that the link X_u is a nonorientable surface and $\Pi'_{k,k'}(u) = w$.*

Proof. Let $w' \in S_{k+1}$ be a vertex such that $\Pi_{k+1}[\mathrm{Res}(w', S_{k+1})] = w$. Let $\epsilon > 0$ be such that $\epsilon + \epsilon_l/(1 - C) < 1$ for $l = 2, 3, \dots$ (where the ϵ_l are given by Lemma 5.3). Due to Lemma 9.3 there is a $k' > k + 1$ and a vertex $u \in S_{k'}$ such that X_u is a nonorientable surface and $\Pi_{k+1,k'}(u) \in B_{S_{k+1}}(w', \epsilon)$.

Note that if $d_{S_l}(x,y) \le \delta$, then

$$\begin{aligned} d_{S_{l-1}}(\Pi_l'(x),\Pi_l(y)) &\le d_{S_{l-1}}(\Pi_l'(x),\Pi_l(x)) + d_{S_{l-1}}(\Pi_l(x),\Pi_l(y)) \\ &\le C\delta + \epsilon_l. \end{aligned}$$

It follows that

$$\begin{aligned} d_{S_{k+1}}(\Pi'_{k+1,k'}(u),\Pi_{k+1,k'}(u)) &\le \epsilon_{k+1} + C\epsilon_{k+2} + \cdots + C^{k'-k-1}\epsilon_{k'} \\ &< \epsilon_{k+1}\frac{1}{1-C} < 1-\epsilon. \end{aligned}$$

Thus $\Pi'_{k+1,k'}(u) \in B_{S_{k+1}}(w',1) \subset \operatorname{Res}(w',S_{k+1})$, so $\Pi_{k,k'}(u) = w$. ∎

Now we can prove part (b) of the Main Theorem.

THEOREM 9.5. *Let X be a 7-systolic nonorientable pseudomanifold of dimension 3. Let a group G act cocompactly on X by simplicial automorphisms. Then the Gromov boundary $\partial_G X$ is homeomorphic to the nonorientable Pontryagin surface.*

Proof. By Sections 3, 5, 6 and 7 we can assume that $\partial_G X$ is homeomorphic to the inverse limit of a system of nonorientable surfaces satisfying assumptions 1 and 3 of Lemma 8.5. By Lemma 9.4 we can assume that assumption 2 is also satisfied. Thus the assertion holds by Lemma 8.5. ∎

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References

[B] M. Brown, *Some applications of an approximation theorem for inverse limits*, Proc. Amer. Math. Soc. 11 (1960), 478–483.

[F] H. Fischer, *Boundaries of right-angled Coxeter groups with manifold nerves*, Topology 42 (2003), 423–446.

[H] F. Haglund, *Complexes simpliciaux hyperboliques de grande dimension*, preprint, Prépublication Orsay 71, 2003.

[HS] F. Haglund and J. Świątkowski, *Separating quasi-convex subgroups in 7-systolic groups*, Groups Geom. Dynam. 2 (2008), 223–244.

[J] W. Jakobsche, *Homogeneous cohomology manifolds which are inverse limits*, Fund. Math. 137 (1991), 81–95.

[JS] T. Januszkiewicz and J. Świątkowski, *Simplicial nonpositive curvature*, Publ. Math. IHES 104 (2006), 1–85.

[M] S. Macias, *Topics on Continua*, Chapman & Hall/CRC, Boca Raton, 2005.

[O] D. Osajda, *Ideal boundary of 7-systolic complexes and groups*, Algebr. Geom. Topology 8 (2008), 81–99.

[S] P. R. Stallings, *An extension of Jakobsche's construction of n-homogeneous continua to the nonorientable case*, in: Continua (Cincinnati, OH, 1994), Lecture Notes in Pure Appl. Math. 170, Dekker, New York, 1995, 347–361.

Paweł Zawiślak
Mathematical Institute
University of Wrocław
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: Pawel.Zawislak@math.uni.wroc.pl

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