Bounded geometry of quadrilaterals
and variation of multipliers for rational maps

by

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Abstract. Let Q be the unit square in the plane and $h : Q \to h(Q)$ a quasiconformal map. When h is conformal off a certain self-similar set, the modulus of $h(Q)$ is bounded independent of h. We apply this observation to give explicit estimates for the variation of multipliers of repelling fixed points under a "spinning" quasiconformal deformation of a particular cubic polynomial.

1. Introduction. Grötzsch's length-area argument shows that if $h : Q \to h(Q)$ is a side-preserving K-quasiconformal map between Euclidean rectangles, then

$$\frac{\mathrm{mod}(h(Q))}{\mathrm{mod}(Q)} \leq \frac{\int_Q K_h(z)\, dA}{\int_Q dA} = \mathrm{average}_Q(K_h(z)),$$

where

$$K_h(z) = \frac{1 + |\mu_h(z)|}{1 - |\mu_h(z)|}$$

and $\mu_h = h_{\bar{z}}/h_z$ is the complex dilatation of h (see e.g. [GL, Prop. 1.4.3]). Thus, $\mathrm{mod}(h(Q))/\mathrm{mod}(Q) \leq K$. The distortion of moduli may also be controlled if $\mathrm{area}(\mathrm{supp}(\mu_h))/\mathrm{area}(Q)$ is small relative to K. Furthermore, if h is conformal off a fixed compact subset $\mathcal{M}$ of Q, then $\mathrm{mod}(h(Q))/\mathrm{mod}(Q)$ is bounded above by a constant depending only on $\mathcal{M}$. To see this, simply compute lower bounds on the extremal lengths of horizontal and vertical path families using the Euclidean metric.

The following result, however, is somewhat surprising. It was used by Yoccoz in unpublished work proving local connectivity of the Mandelbrot set at certain parameters, and seems to have been common knowledge also to Douady, Hubbard, and McMullen at the same time. Thus, we shall refer to it as the DHMY bound.

Let

$$Q = \{z \in \mathbb{C} : 0 < \mathrm{Im}(z), \mathrm{Re}(z) < 1\}$$

denote the unit square in the complex plane. Subdivide Q into nine congruent squares of side $1/3$, and denote the bottom three squares L, M, R, from left to right; see the upper-left image of Figure 1 (in §2). Let $F : \overline{L} \sqcup \overline{R} \to \overline{Q}$ be given by $F(z) = 3z$ if $z \in \overline{L}$ and $F(z) = 3(z-1) + 1$ if $z \in \overline{R}$. Let $M_n = F^{-n}(M)$ and $\mathcal{M} = \bigcup_n M_n$, $n \geq 0$.

THEOREM 1.1 (Douady–Hubbard–McMullen–Yoccoz (DHMY) bound). *Let $h : \overline{Q} \to \mathbb{C}$ be a homeomorphism onto its image which is quasiconformal on Q and conformal on $Q - \mathcal{M}$. Then the modulus of the quadrilateral $h(Q)$ is bounded above and below by positive constants independent of h.*

In §2 below we give a proof of the DHMY bound following an idea of A. Douady, and of a more general version suitable for application to dynamics. In §3 we apply this to a concrete problem concerning variation of multipliers under certain quasiconformal deformations called *spins*. Computer pictures aid significantly in visualizing the decomposition of the dynamical plane needed in order to apply the bound; see Figure 3.

The problem we consider is a very special case of the following general situation. For relevant definitions, see [MS]. Let Rat_d denote the space of degree d rational maps from the complex projective line $\mathbb{P}^1$ to itself. Let $f \in \mathrm{Rat}_d$ be generic and hyperbolic. Then the *Teichmüller space* $\mathrm{Teich}(f)$ is isomorphic to a product $\prod_i \mathrm{Teich}(T_i)$ of classical Teichmüller spaces of punctured tori T_i arising from attracting basins. There is a natural holomorphic map $\eta : \mathrm{Teich}(f) \to \mathrm{Rat}_d / \mathrm{Aut}(\mathbb{P}^1)$ obtained via the Measurable Riemann Mapping Theorem. The map η is in fact the universal orbifold covering of the open subspace of Möbius conjugacy classes of maps which are quasiconformally conjugate to f on all of $\mathbb{P}^1$. The *deck group* $\mathrm{Mod}(f)$ is an infinite index subgroup of the product of classical modular groups $\mathrm{Mod}(T_i)$ which acts on $\mathrm{Teich}(f)$ by holomorphic isometries via precomposition of markings. By a result of Makienko ([Mak, Thm. 1]) $\mathrm{Mod}(f)$ is infinitely generated if and only if some basin attracts at least two critical points. In this case, the map η is necessarily very wild. It is natural to ask e.g. what happens to a sequence of the form $f_n = \eta([\omega]^n . f_0)$, $n = 0, 1, \ldots$, where $[\omega] \in \prod_i \mathrm{Mod}(T_i) - \mathrm{Mod}(f)$.

Spinning. Consider the one-parameter family

$$f_a(z) = \frac{1}{2}z + az^2 - \frac{1}{3}z^3, \quad a \in \mathbb{C},$$

of normalized cubic polynomials for which the origin is an attracting fixed point of constant multiplier $1/2$. Here, $1/2$ is chosen merely for concreteness. Note that the maps $f_{\pm a}$ are conjugate via $z \mapsto -z$.

The basepoint $f_0(z) = z/2 - z^3/3$ is odd, and both critical points $c_{\pm} = \pm 1/\sqrt{2}$ lie in the immediate basin Ω_0 of the origin. By Königs' Theorem, there is a unique holomorphic map $\phi : \Omega_0 \to \mathbb{C}$ such that $\phi \circ f(z) = \phi(z)/2$ and normalized so that $\phi'(0) = 1$. Let T be the torus $(\mathbb{C} - \{0\})/\langle w \mapsto w/2\rangle$, let $p : \mathbb{C} - \{0\} \to T$ denote the projection, and put $S = T - (p \circ \phi)(c_{\pm})$. Then S is a twice-punctured torus. Let $A^{\pm}$ be the image of the domain $\{w \in \mathbb{C} - \{0\} : \pm \operatorname{Arg}(w) \in (\pi/4, 3\pi/4)\}$ under the projection p. Then $A^{\pm}$ is a pair of disjoint, symmetric, open annuli compactly contained in S.

Let A be the annulus $[0, 2\pi] \times [0, 1]$ where the first coordinate is read modulo 2π, equipped with the counterclockwise orientation inherited from $\mathbb{R}^2$. The *standard affine positive/negative Dehn twist* is given by $(\theta, t) \mapsto (\theta \pm 2\pi t, t)$. Thus, the image of a curve of the form $\{\theta_0\} \times [0, 1]$ under a positive/negative Dehn twist bends right/left as the curve is traversed, independent of orientation of the curve.

Let $\omega : S \to S$ be the homeomorphism which is the identity off $A^{\pm}$, and is the standard affine positive/negative Dehn twist on $A^{\pm}$. The homeomorphism ω is called a *spin*. See [PT], [GK], [Cui] for spinning in the context of rational maps. We consider the sequence $\eta([\omega]^n.f_0)$. Recalling that the action is by *precomposition* of markings, note that the image of the basepoint S under the action of $[\omega]^n$ is represented by $(\omega^{-n} : S \to S)$ where ω^{-1} is now a *negative* Dehn twist on A^+ and a *positive* Dehn twist on A^-. Let ν_n denote the complex dilatation of $\omega_n = \omega^{\circ -n}$ on S, and let μ_n be the Beltrami differential on $\mathbb{C}$ given by $(p \circ \phi)^*(\nu_n)$ on Ω_0 and zero elsewhere.

By the Measurable Riemann Mapping Theorem, there is a unique quasiconformal map $H_n : \mathbb{C} \to \mathbb{C}$ whose complex dilatation is μ_n a.e., normalized by $H_n(0) = 0$ and $H(z) = z + O(1)$ near infinity. Since the maps ω_n extend to maps of T, $H_n \circ f_0 \circ H_n^{-1} = f_{a_n}$ for some $a_n \in \mathbb{C}$. Since $\mu_n(\bar{z}) = \overline{\mu_n(z)}$, we have $a_n \in \mathbb{R}$, and in fact $a_n > 0$. For each n, the map f_{a_n} has a pair v_n^+, v_n^- of repelling fixed points in the upper and lower half-planes, respectively. Let $\lambda_n^{\pm} = f'_{a_n}(v_n^{\pm})$. When $n = 0$, $\lambda_0^{\pm} = 2$. There are two unique components $\widetilde{A}^{\pm}$ of $(p \circ \phi)^{-1}(A^{\pm})$ which join the origin to $v_0^{\pm}$.

THEOREM 1.2. *As $n \to \infty$,*

$$\lambda_n^{\pm} = \exp\left(-2\pi i \frac{\tau_0}{\pm n\tau_0 + 1}\right) + O\left(\frac{1}{n^2}\right) = 1 \mp 2\pi i \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

where $\tau_0 = (i \ln 2)/2\pi$. More precisely, for sufficiently large n, the indices $\iota_n^{\pm} = 1/(1 - \lambda_n^{\pm})$ satisfy

$$\iota_n^{\pm} = \pm \frac{n}{2\pi i} + \left(\frac{1}{2} - \frac{1}{\ln 2}\right) + O(1)$$

where the real part of the constant is strictly less than $1/\ln 2$.

An easy application of the Holomorphic Fixed Point Formula ([Mil, Thm. 12.4]) and a bit of algebra yields

COROLLARY 1.1. *As* $n \to \infty$, $f_{a_n} \to f_a$, *where* $a = \sqrt{6}/3$. *More precisely,*

- $a_n = \dfrac{\sqrt{6}}{3} - \dfrac{\pi\sqrt{6}}{3}\dfrac{1}{n^2} + O\left(\dfrac{1}{n^3}\right)$,
- *the repelling fixed points* $v_n^{\pm}$ *collide to a common parabolic fixed point of* f_a *of multiplier 1.*

In particular, $f_n \nrightarrow f$ horocyclically in the sense of McMullen [McM], since $\tau_0/(\pm n\tau_0 + 1) \to 0$ along a horocycle in the upper half-plane model of the hyperbolic plane. A non-quantitative version of this corollary was obtained in much greater generality in [PT], using indirect methods.

REMARKS. Versions of the DHMY bound appear in several other places. In [Haï, Prop. 3.6] it is stated more or less as above; the proof given there is very different. There, the DHMY bound is used to prove uniform moduli of continuity of quasiconformal conjugacies under "pinching" deformations. Spinning, however, produces parabolic implosions and equicontinuity fails. G. Z. Cui [Cui], J. Kahn [K], and J. Rivera-Letelier [R] have more general rigidity statements. Cui's proof uses some novel estimates for conformal distortion, while the latter two rely on certain Sobolev estimates.

2. Proof of DHMY bound. We prove the DHMY bound first, followed by a quasiconformally distorted version suitable for our application.

Let $\Gamma_{\mathrm{hor/ver}}$ denote the families of rectifiable paths joining respectively the vertical and horizontal sides of Q. It is enough to establish lower bounds on the extremal lengths $\Lambda(h(\Gamma_{\mathrm{hor/ver}}))$ of the horizontal and vertical path families $h(\Gamma_{\mathrm{hor/ver}})$ for $h(Q)$, since these quantities are reciprocals of one another. Let $U_0 = Q \cap \{\mathrm{Im}(z) > 1/3\}$, let $U_n = F^{-n}(U_0)$, $n = 1, 2, \ldots$, and let $\mathcal{U} = \mathrm{int}(\overline{\bigcup_{n \geq 0} U_n})$. Since h is conformal on $\mathcal{U}$, $\Lambda(h(\Gamma_{\mathrm{hor/ver}}))$ is bounded below by

$$\sup_{\varrho} \frac{(\inf_{\gamma \in \Gamma_{\mathrm{hor/ver}}} \int_{\gamma} |dz|)^2}{\int_Q \varrho^2 |dz|^2},$$

where $\varrho \geq 0$ is square-integrable and supported on $\mathcal{U}$. Letting $\varrho \equiv 1$ on $\mathcal{U}$ shows that $\Lambda(h(\Gamma_{\mathrm{ver}})) \geq 2/3$.
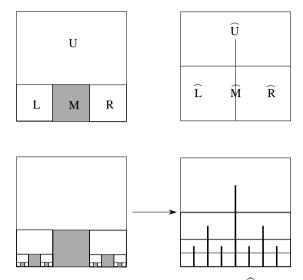
Fig. 1. $\mathcal{U}$ is the complement of the gray region at lower left. $\widehat{\mathcal{U}}$ is the complement of the vertical segments at the lower right. The map ψ sends the unit square continuously to itself so that $\psi(\mathcal{U}) = \widehat{\mathcal{U}}$ and $\psi|_{\mathcal{U}}$ is K-qc.

To find a lower bound for $\Lambda(h(\Gamma_{\mathrm{hor}}))$, we use an idea of A. Douady; see Figure 1. By postcomposing h with a Riemann map to a rectangle followed by a conformal affine map, we may assume h is a side-preserving homeomorphism from $\overline{Q}$ to the rectangle $[0, l] \times [0, 1]$, so that $\Lambda(h(\Gamma_{\mathrm{hor}})) = l = \mathrm{area}(h(Q))$. To bound the area from below, we claim there exists a continuous map $\psi : \overline{Q} \to \overline{Q}$ such that ψ collapses each component of M_n to a vertical segment of the form $\{i/2^{-(n+1)}\} \times \left[0, \frac{3}{4} \cdot 2^{-n}\right]$ and which is K-quasiconformal on the complement of $\mathcal{U}$ of the closure of $\mathcal{M}$; see Figure 1. Assuming this claim for the moment, we reason as follows. Let $\widehat{\mathcal{U}} = \psi(\mathcal{U})$.

Let $H = h \circ \psi^{-1}|_{\widehat{\mathcal{U}}}$. Then H is K-quasiconformal on $\widehat{U}$. Thus

$$\mathrm{area}(h(Q)) \geq \mathrm{area}(h(\mathcal{U})) = \mathrm{area}(H(\widehat{\mathcal{U}})) = \int_{\widehat{\mathcal{U}}} \mathrm{Jac}(H) \, dA$$

$$\geq \frac{1}{K} \iint_{\widehat{\mathcal{U}}} \left| \frac{\partial H}{\partial y} \right|^2 dy \, dx \quad \text{since } H \text{ is } K\text{-qc}$$

$$\geq \frac{1}{K} \left(\iint_{\widehat{\mathcal{U}}} \left| \frac{\partial H}{\partial y} \right| dy \, dx \right)^2 \bigg/ \iint_{\widehat{\mathcal{U}}} 1 \, dy \, dx \quad \text{by Schwarz' } \leq.$$

Each vertical segment of the form $\{x\} \times [0, 1]$, where $x \neq i/2^k$, $i, k \in \mathbb{N}$, is mapped by H to a continuous path joining the top and bottom edges of the rectangle $h(\overline{Q}) = [0, l] \times [0, 1]$ and thus has length ≥ 1. By Fubini's Theorem the integral in the numerator is therefore at least one. The domain $\widehat{\mathcal{U}}$ has area 1, so $\mathrm{area}(h(Q)) \geq 1/K$.

To find ψ is straightforward but tedious; we merely sketch the idea. Let $\widehat{L}, \widehat{R}$ be as shown in Figure 1. Combinatorially, the map $\widehat{F} : \overline{\widehat{L}} \sqcup \overline{\widehat{R}} \to \overline{Q}$ given by $\widehat{F}(z) = 2z$ on $\overline{\widehat{L}}$ and by $2(z-1)+1$ on $\overline{\widehat{R}}$ is the same as F.

Let $\widehat{U}$ be the slit rectangle at the top of Figure 1. Let ψ send the top, left, and right sides of ∂U to the corresponding sides of $\partial \widehat{U}$ via affine maps. Fold the middle third of the bottom to the slit, and send the bottom left third and right third via pulling back so that ψ commutes with the dynamics there. Extend ψ to a continuous map $\psi : \overline{U} \to \overline{\widehat{U}}$ via piecewise-linear maps on U. The maps on the sides of M will be determined by the dynamics; extend to the interior of M by a continuous collapsing map. By insisting that ψ be continuous and conjugate F to $\widehat{F}$, ψ is determined on all of $\overline{Q}$. Since both F and $\widehat{F}$ are conformal, ψ will be K-qc on $\mathcal{U}$. ∎

In practice, the setup in Theorem 1.1 must be applied in a slightly distorted context.

ASSUMPTIONS.

(1) Q' is a quadrilateral bounded by a quasicircle. Suppose Q' is subdivided into nine quadrilaterals, each bounded by quasicircles, in a manner combinatorially the same as that described in §1. Let L', M', R', U' be defined similarly.

(2) Suppose U'_L, U'_R, V' are open Jordan domains such that $\overline{L'} \subset U'_L$, $\overline{R'} \subset U'_R$, $\overline{Q'} \subset V'$, and that U'_L and U'_R are compactly contained in V'. Assume we are given a proper holomorphic map $F' : U'_L \sqcup U'_R \to V'$, univalent on each component, such that $F(\overline{L'}) = \overline{Q'} = F(\overline{R'})$. It follows that the bottom corners of Q' are repelling fixed points of F'. Let $\mathcal{M}' = \bigcup_{n \geq 0} (F')^{-n}(M')$. We also assume that there exists an anticonformal involution of V' sending Q' to itself which interchanges L' and R', sends U', M' to themselves, and commutes with F' (this is the analog of reflection in the line $x = 1/2$ for Q).

(3) The four arcs comprising ∂Q are analytic.

THEOREM 2.1. *Under assumptions* (1)–(3) *above, let* $h' : \overline{Q'} \to \mathbb{C}$ *be a homeomorphism onto its image which is quasiconformal on* Q' *and conformal on* $Q' - \mathcal{M}'$. *Then the modulus of* $h'(Q')$ *is bounded above and below by positive constants independent of* h'.

Proof. It suffices to show the existence of a side-preserving quasiconformal map $\phi : Q \to Q'$ sending $\mathcal{M}$ to $\mathcal{M}'$. The only difficult point is that we may no longer use a piecewise-affine extension to the interior to get a qc map—we must be careful, since the gluing of two quasisymmetric maps at a point need not be quasisymmetric.

Assume first that there exists a side-preserving K-quasiconformal map $\phi_0 : Q \to Q'$ with the following properties:

- ϕ_0 sends U, L, M, R to U', L', M', R',
- $\phi_0 \circ F = F' \circ \phi_0$ on the left, right, and top sides of L and R.

For $n \geq 0$, let $U_n = F^{-n}(U)$, $M_n = F^{-n}(M)$, and define U_n', etc. similarly. Inductively define $\phi_{n+1} : Q \to Q'$ by setting $\phi_{n+1} = \phi_n$ on $\overline{U_n \bigcup M_n}$ and by $(F')^{-1} \circ \phi_n \circ F$ on $\overline{Q' - U_n \bigcup M_n}$. For each n, ϕ_n is a side-preserving K-quasiconformal map from Q to Q'. By compactness properties of K-quasiconformal mappings, after passing to a subsequence, there is a limit $\phi : Q \to Q'$ which conjugates F to F', hence sends $\mathcal{M}$ to $\mathcal{M}'$.

We now show the existence of ϕ_0. We shall use repeatedly the fact that quasisymmetric maps between quasicircles extend over the complementary components to quasiconformal maps.

Consider the quotient tori T_L, T_R, T_L', T_R' for the repelling fixed points of F and F'; see Figure 2.
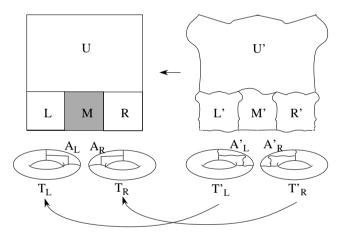

Fig. 2. Finding ϕ_0. Send $A_{L/R}'$ to $A_{L/R}$ via a qc map which sends the crosscut curves to curves homotopic to crosscut curves.

The anticonformal involutions of Q, Q' descend to anticonformal isomorphisms $T_L \leftrightarrow T_R$ and $T_L' \leftrightarrow T_R'$. The domains L, R, L', R' project to annuli A_L, A_R, etc. bounded by real analytic closed curves. Each such annulus contains a distinguished piecewise-analytic crosscut curve (the projection of the union of the top and side of L, R) which joins the two boundary components. Moreover, the projection of the bottom of M' (and of M) project to distinguished segments on the boundary components of A_L, A_R, etc.

Condition (3) implies that there exists a C^1, hence qc map $T_L' \to T_L$ sending $A_L' \to A_L$, mapping the distinguished crosscut curves to curves homotopic to the corresponding crosscut curves, and sending the distinguished

segments to the corresponding segments. By conjugating with the anticonformal involution, we get a similar map $T'_R \to T_R$.

Lifting to the dynamical plane, we obtain a continuous map defined on the bottom, left, and right sides of Q' to the corresponding sides of Q, which commutes with the dynamics on the left and bottom (respectively, right and bottom) of L' and R', and which sends the corners of L', R' lying on $\partial Q'$ to the corresponding corners in ∂Q. The resulting map is quasisymmetric since it arises as the boundary values of a locally defined qc map. This map extends to a quasisymmetric map $\partial Q' \to \partial Q$, which in turn extends to a quasiconformal map $Q' \to Q'$ which we denote by ϕ_{-1}.

Set $\phi_0|_{L'/R'} = F_{L/R}^{-1} \circ \phi_{-1} \circ F'_{L'/R'}$. The resulting map extends, using the boundary values of ϕ_{-1}, continuously to $L' \sqcup R' \bigcup \partial Q'$ such that the restriction to $\partial Q'$ is quasisymmetric. Extend ϕ_0 over the complement of $\overline{Q'}$ in the Riemann sphere to a quasiconformal map of the quasidisk $(\widehat{\mathbb{C}} - \overline{Q'}) \cup \overline{L'} \cup \overline{R'}$ onto $(\widehat{\mathbb{C}} - \overline{Q}) \cup \overline{L} \cup \overline{R}$. It remains only to extend ϕ_0 over $U' \cup M'$ so that $\phi_0(U') = U$ and $\phi_0(M') = M$. The Corollary to Lemma 2.1 below says precisely that this is possible, and the proof is complete. ∎

LEMMA 2.1. *Suppose two Jordan domains $V_\pm \subset \mathbb{C}$ share a common boundary arc, and that each component of $\widehat{\mathbb{C}} - (\partial V_+ \cup \partial V_-)$ is a quasicircle. Let $f_+ : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiconformal homeomorphism for which $f(\Delta_+) = V_+$. Then there is a quasiconformal homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $f(\Delta_\pm) = V_\pm$ and $f = f_+$ on Δ_+.*

Proof. Recall that Δ denotes the open unit disk in $\mathbb{C}$, and $\Delta_\pm$ the upper and lower halves. Then $\Delta_\pm$ are both quasidisks, i.e. each is the image of Δ under a quasiconformal homeomorphism $g_\pm$ of all of $\mathbb{C}$, and without loss of generality we may assume that g_+ and g_- send the lower and upper semicircles, respectively, onto the interval $[-1, 1]$.

Let f_+ be a quasiconformal map of the plane sending Δ_+ to V_+. The composition $f_-^{-1} \circ f_+|_{[-1,1]}$ is a homeomorphism of the interval $[-1, 1]$ to itself. The composition $g_-^{-1} \circ (f_-^{-1} \circ f_+|_{[-1,1]}) \circ g_-$ is a quasisymmetric map of the upper semicircle to itself. Extend this map over the lower semicircle by complex conjugation. Lemma 5.1 of [Leh] implies that the resulting map $h : \partial \Delta \to \partial \Delta$ is quasisymmetric. Applying e.g. the Ahlfors–Beurling extension, we obtain a quasiconformal extension $H : \Delta \to \Delta$.

Let $f : \Delta \to \mathbb{C}$ be given by f_+ on $\Delta_+ \cup (-1, 1)$ and by $f_- \circ g_- \circ H \circ g_-^{-1}$ on $\Delta_+ \cup (-1, 1)$. The map f is a homeomorphism on Δ, and quasiconformal on Δ_+ and Δ_-, hence is quasiconformal on Δ by the removability properties of analytic arcs for qc maps. The map f extends to a homeomorphism of $\overline{\Delta}$ sending the quasicircle $\partial \Delta$ to a quasicircle, by assumption. It follows from [LV, Thm. II.8.2] that f extends to a quasiconformal map on a neighborhood

of $\overline{\Delta}$, which then by [LV, Thm. II.8.1] extends to a quasiconformal map on the entire plane. ∎

COROLLARY 2.1. *Suppose two Jordan domains $V_\pm \subset \mathbb{C}$ share a common boundary arc, and that each component of $\widehat{\mathbb{C}} - (\partial V_+ \cup \partial V_-)$ is a quasicircle. Let $W = \widehat{\mathbb{C}} - (\overline{V}_+ \cup \overline{V}_-)$, and suppose $f : \widehat{\mathbb{C}} - \overline{\Delta} \to W$ is a quasiconformal map. Then there exists a quasiconformal extension of f to the whole sphere such that $f(\Delta_\pm) = V_\pm$.*

The proof follows by letting W correspond to V_+ in Lemma 2.1.

3. Proof of Theorem 1.2. We prove the bound for λ_n^+; the other is similar. Let $\phi^+ : (\mathbb{C}, 0) \to (\mathbb{P}^1, v_0^+)$ denote the Königs linearizing map for the repelling fixed point v_0^+, and let $p^+ : \mathbb{C} - \{0\} \to T_0^+$ be projection to the quotient torus T_0^+. Identify T_0^+ with $\mathbb{C}/\langle z \mapsto z + 1, z \mapsto z + \tau_0 \rangle$ where $\tau_0 = (\ln 2)i/2\pi$. Let α, β be generators of $\pi_1(T_0^+, 0)$ corresponding to translation by 1 and translation by τ_0, respectively. Thus, a lift of α under $p^+ \circ \phi^+$ goes once around v_0^+ counterclockwise, while a lift of β points in toward v_0^+.

Let $\mathbb{H}$ denote the upper half-plane and $\mathcal{T}$ the Teichmüller space of complex tori modeled on T_0^+. A point of $\mathcal{T}$ is represented by a complex torus $T^+ = \mathbb{C}/\langle 1, \tau \rangle, \tau \in \mathbb{H}$, equipped with a quasiconformal homeomorphism $h : T_0^+ \to T^+$ such that a lift of h to $\mathbb{C}$ conjugates translation by 1 and by τ_0 to translation by 1 and by τ, respectively. The map sending a marked torus $(h : T_0^+ \to T^+)$ to the point τ is an isomorphism between $\mathcal{T}$ and the upper half-plane $\mathbb{H}$ and is an isometry with respect to the Teichmüller and hyperbolic metrics.

The mapping class group acts on $\mathcal{T}$ by *precomposition* of markings. Suppose $\omega : T_0^+ \to T_0^+$ is an arbitrary quasiconformal map. Let $[\omega]$ denote the corresponding mapping class element and ω_* the induced linear map on $H_1(T_0^+, \mathbb{Z})$, equipped with the ordered basis $[\alpha, \beta]$. Then

$$[\omega].\tau = \frac{a\tau + b}{c\tau + d}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ is the matrix for ω_*^{-1}. If ω is a *positive* Dehn twist in an annulus with core curve β, then ω_*^{-1} sends $\beta \mapsto \beta$ and $\alpha \mapsto \alpha + \beta$ on homology and so $[\omega].\tau = \tau/(\tau + 1)$.

Now let $\omega : S \to S$ be a spin on the quotient twice-punctured torus S corresponding to the attractor at the origin, $\omega_n = \omega^{-n}$, and let μ_n be the Beltrami differential of ω_n lifted to the sphere (recall the discussion in §1). The Beltrami differentials μ_n are f_0-invariant, hence descend to Beltrami differentials on T_0^+ which we denote by μ_n^+. Then the μ_n^+ determine tori

$$T_n^+ = \mathbb{C}/\langle z \mapsto z + 1, z \mapsto z + \tau_n^+ \rangle, \quad \tau_n^+ \in \mathbb{H},$$

and quasiconformal maps $h_n^+ : T_0^+ \to T_n^+$, i.e. points τ_n^+ in Teichmüller space. The multipliers satisfy $\lambda_n^+ = \exp(-2\pi i \tau_n^+)$.

The map $(p^+ \circ \phi^+) \circ (p \circ \phi|_{A^+})^{-1} : A^+ \to A$ is a conformal isomorphism from the annulus A^+ in the quotient torus S for the attractor at the origin to an annulus A in the quotient torus T_0^+ for the repellor at v_0^+. Write $\mu_n^+ = \mu_n^A + \kappa_n$, where $\kappa_n \equiv 0$ on A. The key observation is that the Beltrami differential μ_n^A agrees with that of $\omega_n^+ = \omega^{\circ -n} : T_0^+ \to T_0^+$, where ω is a positive Dehn twist on A. Let $t_n^+ = [\omega]^{-n}.\tau_0 = \tau_0/(n\tau_0 + 1) \in \mathbb{H}$.

LEMMA 3.1. *There is a constant C_0 such that for all $n \in \mathbb{N}$,*

$$d(t_n^+, \tau_n^+) \leq C_0,$$

where d is the hyperbolic (Teichmüller) distance.

Given the lemma, let us now prove the theorem. The composition

$$\mathbb{H} \xrightarrow{\lambda = \exp(-2\pi i \tau)} \{\lambda : |\lambda| > 1\} \xrightarrow{\iota = 1/(1-\lambda)} \{\iota : \operatorname{Re}(\iota) < 1/2\} - \{0\}$$

is an isometry with respect to the hyperbolic metrics. On the last domain, for large imaginary values, the hyperbolic metric is comparable to the metric $dx/(x-1/2)$. This observation and a routine power series computation yields the second conclusion.

To prove the first estimate, note that the point on a hyperbolic circle in $\mathbb{H}$ which is Euclidean furthest away from the center lies directly above the center. Thus

$$d(z, w) \leq C \implies |z - w| \leq (e^C - 1)\operatorname{Im}(z)$$

and so

$$|t_n^+ - \tau_n^+| \leq (e^{C_0} - 1)\operatorname{Im}(t_n^+).$$

Putting $C_1 = e^{C_0} - 1$ and substituting $t_n^+ = \tau_0/(n\tau_0 + 1)$ yields

$$|t_n^+ - \tau_n^+| \leq C_1 \frac{\frac{\ln 2}{2\pi}}{n^2\left(\frac{\ln 2}{2\pi}\right)^2 + 1}.$$

As $n \to \infty$, both t_n^+ and τ_n^+ tend to zero, where the map $\tau \mapsto \exp(-2\pi i \tau)$ is nearly a Euclidean affine map with derivative $-2\pi i$. Since $\lambda_n^+ = \exp(-2\pi i \tau_n)$, this yields after cancelling 2π's

$$|\exp(-2\pi i t_n^+) - \lambda_n^+| < C_2 \frac{\ln 2}{n^2\left(\frac{\ln 2}{2\pi}\right)^2 + 1} = O\left(\frac{1}{n^2}\right). \quad \blacksquare$$

Proof of Lemma 3.1. In $\mathcal{T}$, the point τ_n^+ is represented by $h_n^+ : T_0^+ \to T_n^+$, while t_n^+ is represented by $\omega_n^+ : T_0^+ \to T_0^+$. By construction, $h_n^+ \circ \omega_n^{-1} : T_0^+ \to T_n^+$ is a quasiconformal map which is holomorphic off a subset $\mathcal{K}$ which is independent of n.

Let $h : T_0^+ \to T^+$ be an arbitary quasiconformal mapping whose dilatation μ is supported on $\mathcal{K}$. By the definition of Teichmüller distance, it suffices to show that there are lower bounds, independent of h, on the moduli of annuli in $h(T_0^+)$ whose core curves are homotopic to $h(\alpha)$ and to $h(\beta)$. The lower bound for $h(\beta)$ is immediate from the Grötzsch inequality, since $\mu = 0$ on A by hypothesis.

Let $\widetilde{\mu}$ be the Beltrami differential given by $(p^+ \circ \phi^+)^*(\mu)$ on Ω_0 and zero elsewhere. Then $\widetilde{\mu}$ is supported on $\widetilde{\mathcal{K}} = (p^+ \circ \phi^+)^{-1}(\mathcal{K})$, which is the grand orbit of $\widetilde{A}^{-1}$. Let $H_{\widetilde{\mu}} : \mathbb{P}^1 \to \mathbb{P}^1$ be a straightening map for $\widetilde{\mu}$.

Let $\mathcal{D}$ be an annulus in the dynamical plane which is a fundamental domain for the action of f_0 near v_0^+. Finding a lower bound on the modulus of an annulus in $h(T_0^+)$ about $h(\alpha)$ is equivalent to finding a lower bound on the extremal length of the path family joining the boundary components of $H_{\widetilde{\mu}}(\mathcal{D})$.

It will be convenient to deal with a more symmetric object. Let $s(z) = 1/\overline{z}$ denote inversion in the unit circle. Let $\Phi : (\Delta, 0) \to (\Omega_0, 0)$ be the unique Riemann map for which $\Phi'(0) > 0$. Then Φ conjugates f_0 on Ω_0 to the Blaschke product

$$B_0(z) = -z \, \frac{z^2 - a^2}{1 - a^2 z^2}, \qquad a = \frac{1}{\sqrt{2}}.$$

By reflection, Φ extends to a quasiconformal conjugacy in a neighborhood of the Julia set. It suffices to prove the analogous result for B_0, where now the Beltrami differential μ is supported on the s-symmetric set $\widetilde{\mathcal{K}} \cup s(\widetilde{\mathcal{K}})$, but where we do not assume that $\widetilde{\mu} = s^*\widetilde{\mu}$.
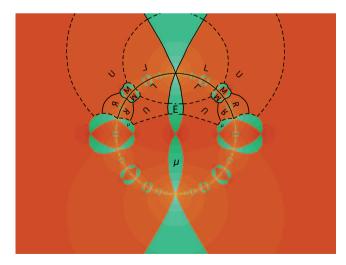

Fig. 3. Four copies of Q near the repelling fixed point v_0^+. The Beltrami differential $\widetilde{\mu}$ is supported on the grand orbit of $\widetilde{A}^-$, indicated by the letter μ.

Consider the dynamical plane for B_0; see Figure 3. It is easy to prove the existence of a quadrilateral Q' and subdomains U', L', M', R' as in Theorem 2.1 in e.g. the upper right-hand corner of the figure. (The anticonformal involution comes from the fact that $B_0|Q'$ is univalent, and the image $B_0(Q')$ is symmetric with respect to the real axis.) The domains will be quasicircles: since the annuli $A^\pm$ can be fattened slightly, arcs which meet at the Julia set or at a point in the grand orbit of the origin do so in definite angles, while the remaining corners are intersections of analytic arcs meeting transversely.

Let
$$\mathcal{R} = \text{int}(\text{closure}(Q' \cup s(Q') \cup r(Q') \cup rs(Q') \cup E \cup s(E)))$$
where E is the fundamental domain for the restriction of B_0 to the lift $\widetilde{A}^+$ indicated in the figure. Then $\mathcal{R}$ is an annulus surrounding v_0^+. A fundamental domain $\mathcal{D}$ for the action of B_0 near v_0^+ is the region between the dashed curves in Figure 3. Combinatorially, the situation is as depicted in Figure 4.
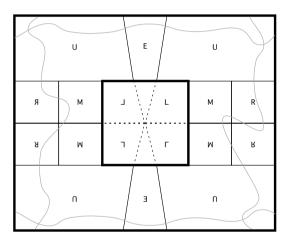

Fig. 4. An annulus surrounding the repelling fixed point is decomposed combinatorially as shown above. The grey curves are typical elements of the path families $\Gamma_{h/v}^\pm$.

To complete the argument, we use an observation of G. Cui. As suggested in Figure 4, let Γ_h^+ consist of those paths lying in the rectangle which is the union of the top two copies of U and the top copy of E, and which join the horizontal sides of this rectangle. Let Γ_h^- be the corresponding family in the bottom half. Let Γ_v^+ consist of those paths lying in the rectangle which is the union of the right copies of U, M, R and which join the extreme top and bottom sides of this rectangle. Let Γ_v^- be the corresponding family in the left half.

Since $\widetilde{\mu} = 0$ on $U \cup E$, the extremal lengths of $h(\Gamma_h^\pm)$ in the annulus $h(\mathcal{D})$ are independent of μ. By the DHMY bound and the subadditivity properties

of extremal length, the extremal lengths of $h(\Gamma_{\mathrm{v}}^{\pm})$ are bounded from above independent of μ. Thus

$$\max\{\Lambda(h(\Gamma_{\mathrm{h}}^{\pm})),\ \Lambda(h(\Gamma_{\mathrm{v}}^{\pm}))\} \leq k$$

for some constant k.

Let Γ now be the family of paths γ in $\mathcal{D}$ joining the inner and outer boundary components of $\mathcal{D}$. Given any γ, among the four families $\Gamma_{\mathrm{v}}^{\pm}, \Gamma_{\mathrm{h}}^{\pm}$, there is at least one such family with the property that γ intersects every path in this family. Since extremal lengths of horizontal and vertical path families in rectangles are reciprocal quantities, we have

$$\Lambda(h(\Gamma)) \geq \frac{1}{k}$$

and the proof of the bound is complete. ∎

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