FUNDAMENTA
MATHEMATICAE
182 (2004)

On stability of 3-manifolds

by

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Abstract. We address the following question: *How different can closed, oriented 3-manifolds be if they become homeomorphic after taking a product with a sphere?*

For geometric 3-manifolds this paper provides a complete answer to this question. For possibly non-geometric 3-manifolds, we establish results which concern 3-manifolds with finite fundamental group (i.e., 3-dimensional fake spherical space forms) and compare these results with results involving fake spherical space forms of higher dimensions.

The purpose of this paper is to study the following stability problem:

Problem. *Let M^3, N^3 be closed oriented 3-manifolds. Assume that $M^3 \times S^n$ and $N^3 \times S^n$ are homeomorphic (i.e., $M^3 \times S^n \approx N^3 \times S^n$) for some $n \geq 1$. Are M^3 and N^3 homeomorphic?*

It is not difficult to show that if Σ^3 is a homotopy 3-sphere, then $\Sigma^3 \times S^n \approx S^3 \times S^n$ for any $n \geq 1$. Therefore, to avoid the unresolved status of the Poincaré conjecture, until the final section we assume first that 3-manifolds considered in this paper are geometric in the sense of Thurston (cf. [T]). The well-known conjecture of Thurston asserts that all 3-manifolds are geometric. Also, for simplicity, we assume all 3-manifolds to be prime (cf. [H]). Given that, the results of this paper give a complete solution to the stability problem.

Stability by S^1. It turns out that in terms of the behavior under stabilization by S^1 one class of 3-manifolds stands out. It is a subclass of Seifert fibered 3-manifolds. This was already noticed by V. Turaev in [Tu], namely

Theorem 1. *Let M^3, N^3 be closed oriented geometric 3-manifolds. Then $M^3 \times S^1 \approx N^3 \times S^1$ is equivalent to $M^3 \approx N^3$ unless M^3 and N^3 are*

2000 *Mathematics Subject Classification*: Primary 58R80, 57S25.

Research supported by the NSF Grant INT–9811590.

Supported by KBN:127/E–335/5/99 and Gdańsk University Grant BW/5100-S-0201-0.

Seifert fibered 3*-manifolds which are surface bundles over* S^1 *with periodic monodromy and surface genus* > 1.

A short sketch of proof of the above theorem was presented in [Tu]. For the convenience of the reader, we insert a detailed proof of Theorem 1, different from Turaev's sketch, in the appendix of this paper. Our proof contains a discussion of the case of "small" Seifert manifolds, which was not discussed in Turaev's paper, and a proof that the surface genus must be greater than 1. This condition is essential in the discussion of the next natural problem.

PROBLEM. *Let* F_g *be a fixed surface of genus* $g \geq 2$. *How many different* (*i.e. nonhomeomorphic*) *closed oriented* 3*-manifolds which fiber over* S^1 *with* F_g *as fiber become homeomorphic after crossing with* S^1*?*

Our main result concerning the above problem is the following:

THEOREM 2. *Let* F_g *be a fixed closed oriented surface of genus* $g \geq 2$. *Then there are at least* $\phi(4g+2)$ *nonhomeomorphic* 3*-manifolds which fiber over* S^1 *with* F_g *as fiber and which become homeomorphic after crossing with* S^1.

Here $\phi(-)$ is the Euler function, i.e., $\phi(n)$ is the number of integers q, $0 \leq q \leq n$, such that q and n are relatively prime.

Proof of Theorem 2. Let F_g be a closed oriented surface of genus $g \geq 2$. By [H1], the maximal cyclic group which can act (nontrivially) on F_g has order $4g+2$. Let $f : F_g \to F_g$ be a periodic homeomorphism generating the action of Z_{4g+2} on F_g (i.e., $f^{4g+2} = \mathrm{id}$).

It is known (cf. [H1], [BC]) that the quotient space F_g/Z_{4g+2} is a sphere S^2 and the branched covering $p : F_g \to F_g/Z_{4g+2}$ has indices $(2, 2g+1, 4g+2)$ (cf. [BC, p. 582]). To be more specific, the corresponding triangle group Γ, which determines the action of Z_{4g+2}, has signature $(0; 2, 2g+1, 4g+2)$ (cf. [H1]). This gives the corresponding branched data as observed in [H2, p. 393]. Let m be an integer relatively prime to $4g+2$, i.e., $(m, 4g+2) = 1$. Consider the homeomorphism $f^m : F_g \to F_g$. Put $f^m = h$. Clearly h is periodic of period $4g+2$.

CLAIM 1. *The homeomorphisms* f *and* h *determine topologically inequivalent actions of* Z_{4g+2} *on* F_g.

Proof of Claim 1. We will show that f and h are not conjugate in $\mathrm{Homeo}_+(F_g)$. The equivalence of the actions of Z_{4g+2} on F_g can be analyzed in terms of generating vectors (cf. [Br]). In our case, given the presentation $Z_{4g+2} = \langle x \mid x^{4g+2} = 1\rangle$, the generating vectors for the action determined by f are given by (x^{2q+1}, x^{2q}, x).

For the action determined by h the generating vectors are $(x^{(2q+1)m}, x^{2qm}, x^m)$. Since our equivalence of group actions is simply conjugation (the identity automorphism of Z_{4q+2}), the generating vectors are different and the actions are inequivalent. On the other hand, it is well known (cf. [N], [S]) that the actions of cyclic groups on F_g are classified by the "fixed point data" (or equivalently, by the branched point data).

Let $\varphi : F_q \to F_q$ be a homeomorphism and let M_φ be the mapping torus of φ, i.e.,

$$M_\varphi = F_g \times [0,1]/\sim,$$

where $\sim$ is the equivalence relation identifying $F_g \times \{0\}$ with $F_g \times \{1\}$ via the homeomorphism φ.

Now, if φ is periodic, then M_φ is a Seifert fibered space with the Seifert invariants uniquely determined by the fixed point data of φ (cf. [S, pp. 390–391]). In particular, inequivalent actions, generated by f and f^m, lead to different manifolds M_f and M_{f^m}. Indeed, the manifolds M_f and M_{f^m} are distinguished by their Seifert invariants.

Claim 2. *$M_f \times S^1 \approx M_{f^m} \times S^1$ for each m with $(m, 4g+2) = 1$.*

Proof of Claim 2. This claim is definitely known. Its proof (in a different notation) is given in [CR, p. 258]. However, for completeness, we outline here the argument from [CR].

Let S^1 be the circle equipped with the standard free action of a finite cyclic group. Then the manifolds M_f and M_h are homeomorphic to the twisted products

$$M_f = S^1 \underset{\langle f \rangle}{\times} F_q, \qquad M_h = S^1 \underset{\langle h \rangle}{\times} F_q,$$

where $\langle f \rangle$ and $\langle h \rangle$ are the cyclic groups of order $4q+2$ generated by f and h, respectively. Both M_f and M_h admit natural actions of Z_{4g+2}. Namely, if $qm = 1$, then the action of Z_{4g+2} on M_f is generated by f^m and the action on M_h is generated by f^q. Both of these actions extend to actions of S^1. In particular, these actions are generated by homeomorphisms isotopic to the identity. As a consequence, we have

$$S^1 \underset{\langle f^m \rangle}{\times} M_f = S^1 \times M_f, \qquad S^1 \underset{\langle f^q \rangle}{\times} M_h = S^1 \times M_h.$$

Consider now the action of $Z_{4g+2} \times Z_{4g+2}$ on the manifold $S^1 \times S^1 \times F_g$. This action is generated by f and f^m. Namely, $\langle f \rangle$ acts on the second and third factor, and $\langle f^m \rangle$ on the first and third factor. Clearly these actions commute. Now factoring out these actions one gets

$$[(S^1 \times S^1 \times F_q)/\langle f\rangle]/\langle f^q\rangle = S^1 \underset{\langle f^m\rangle}{\times} M_f = S^1 \times M_f,$$
$$[(S^1 \times S^1 \times F_q)/\langle f^q\rangle]/\langle f\rangle = S^1 \underset{\langle f^q\rangle}{\times} M_h = S^1 \times M_h,$$

and hence $S^1 \times M_f = S^1 \times M_h$ as claimed.

Remark. By using the results in [Br] it is probably possible to determine all nonhomeomorphic 3-manifolds which fiber over S^1 with fiber F_g, $g = 2, 3$, and which become homeomorphic after stabilization by S^1.

Stabilization by even-dimensional spheres. Our main result is the following

Theorem 3. *Let M^3, N^3 be closed, oriented geometric 3-manifolds. Then $M^3 \times S^{2k} \approx N^3 \times S^{2k}$, $k \geq 1$, is equivalent to $M^3 \approx N^3$.*

Proof. Assume first that $\pi_1(M^3)$ is infinite. The existence of a homeomorphism $M^3 \times S^{2k} \approx N^3 \times S^{2k}$ implies that $\pi_1(M^3) = \pi_1(N^3)$ and hence M^3 is homotopy equivalent to N^3 (i.e., $M^3 \simeq N^3$). This is because we can take M^3 and N^3 to be irreducible and therefore aspherical. Geometric aspherical 3-manifolds are determined by their fundamental groups (cf. [KS1, p. 738]) and hence $M^3 \approx N^3$ in this case.

Now assume that $\pi_1(M^3)$ is finite. Again the isomorphism $\pi_1(M^3) = \pi_1(N^3)$ implies $M^3 \approx N^3$ unless M^3 and N^3 are lens spaces. Suppose then that $M^3 = L$ is a lens space. We consider the case $k = 1$ first. A simple computation of $H_3(L \times S^2; \mathbb{Z})$ gives

$$H_3(L \times S^2; \mathbb{Z}) \cong H_1(L; \mathbb{Z}) \otimes H_2(S^2) \oplus H_3(L; \mathbb{Z}) \otimes H_0(S^2; \mathbb{Z}) \cong Z_k \oplus \mathbb{Z}.$$

It follows that the map

$$f : M^3 \xrightarrow{i} M^3 \times S^2 \xrightarrow[\approx]{h} N^3 \times S^2 \xrightarrow{p} N^3$$

has degree ± 1, where i is the natural inclusion and p is the projection. Since $\deg(f) = \pm 1$, f is a homotopy equivalence (cf. [C, p. 95]).

Let $\tau_0 = \tau(f) \in \mathrm{Wh}(\pi_1(N^3))$ be the Whitehead torsion of f. We will show that $\tau_0 = 0$, which implies $M^3 \approx N^3$ (cf. [C, p. 100]). Let $j : N^3 \hookrightarrow N^3 \times D^3$ be the standard embedding $j(x) = (x, 0)$. Consider the homotopy equivalence $\bar{f} = j \circ f : M^3 \to N^3 \to N^3 \times D^3$. It can be approximated by an embedding i (cf. [W1, Cor. 11.34]) and in fact by a smooth one (cf. [M1, p. 579]). A simple calculation (cf. [M1, p. 579]) shows that the normal bundle of this embedding is trivial. In particular, the closed disk bundle (its total space) is given by $M^3 \times D^3$. The complement of the disk bundle gives an h-cobordism $(W; W_0; W_1)$ between $W_0 = M^3 \times S^2$ and $W_1 = N^3 \times S^2$. The Whitehead torsion $\tau(W; W_0)$ of this h-cobordism is equal to τ_0.

Let $\Delta(W_0)$, $\Delta(W_1)$ be the Reidemeister torsions of W_0 and W_1, respectively, as defined in [M2, pp. 404–405]. By Theorem 12.8 in [M2] we have

$$\Delta(W_0) \sim u^2\Delta(W_1),$$

where u is some unit in the group ring $\mathbb{Z}[\pi_1(M^3)]$. Since $W_1 = N^3 \times S^2 \approx W_0 = M^3 \times S^2$, we have $\Delta(W_0) \sim \Delta(W_1)$, which implies that $u^2 \sim 1$. But $u^2 \sim 1$ is equivalent to the triviality of the Whitehead torsion $\tau(W; W_0)$, again by Theorem 12.8 in [M2]. This means that $\tau_0 = 0$ and hence $\tau(f) = 0$ and $M^3 \approx N^3$ as claimed.

Now we consider the case $k \geq 2$, i.e., we have a homeomorphism

$$M^3 \times S^{2k} \approx N^3 \times S^{2k},$$

where M^3, N^3 are lens spaces. As in the case $k = 1$, there is a homotopy equivalence $f : M^3 \to N^3$. Again we approximate $i \circ f : M^3 \to N^3 \times D^3$ by a smooth embedding with trivial normal bundle. Let $k : M^3 \to N^3 \times D^3$ be such an embedding. Consider the embedding $k' : M^3 \to N^3 \times D^3 \times D^{2k-2}$ given by $k'(x) = (k(x), 0)$. The normal bundle for k' is trivial, which leads to an h-cobordism $(W; W_0, W_1)$ with $W_0 = M^3 \times S^{2k}$, $W_1 = N^3 \times S^{2k}$ and $\tau(W; W_0) = \tau(f)$. The same argument as in the case $k = 1$ shows $\tau(f) = 0$ and completes the proof of Theorem 3.

Stabilization by odd-dimensional spheres

Theorem 4. *Let M^3, N^3 be closed, oriented geometric 3-manifolds. Then $M^3 \times S^{2k+1} \approx N^3 \times S^{2k+1}$, $k \geq 1$, is equivalent to*:

(a) *$M^3 \approx N^3$ if M^3 is not a lens space.*
(b) *$M^3 \simeq N^3$ if M^3 is a lens space and $k > 1$, and $\pi_1(N^3) \cong \pi_1(M^3)$ if M^3 is a lens space and $k = 1$.*

Proof. Part (a) follows directly from the isomorphism $\pi_1(M^3) \cong \pi_1(N^3)$. The second part of (b) is a consequence of result in [Mℓ], more precisely, [Mℓ, (41), p. 19]. We are then left with the first part of (b). Given a homeomorphism $M^3 \times S^{2k+1} \approx N^3 \times S^{2k+1}$, $k > 1$, we obtain a homotopy equivalence $f : M^3 \to N^3$ between two lens spaces (as in the proof of Theorem 3). Conversely, assume $f : M^3 \to N^3$ is a homotopy equivalence. Then

$$f \times \mathrm{id}_{S^{2k+1}} : M^3 \times S^{2k+1} \to N^3 \times S^{2k+1}$$

is a *simple* homotopy equivalence because

$$\tau(f \times \mathrm{id}_{S^{2k+1}}) = \chi(S^{2k+1})i_*\tau(f) + \chi(M^3)j_*\tau(\mathrm{id}_{S^{2k+1}})$$

(cf. [C, 23.2, p. 77]).

Now the Sullivan–Wall surgery exact sequence (cf. [W1, p. 107])

$$\to [\Sigma(M^3 \times S^{2k+1}); G/\mathrm{Top}] \xrightarrow{\theta_{2k+1}} L^s_{2k+1}(\pi_1(M^3)) \xrightarrow{\gamma} S_{\mathrm{Top}}(N^3 \times S^{2k+1})$$
$$\xrightarrow{\eta} [N^3 \times S^{2k+1}; G/\mathrm{Top}] \xrightarrow{\theta_{2k}} L^s_{2k}(\pi_1(N^3))$$

easily implies that the class of $f \times \mathrm{id}_{S^{2k+1}}$ is trivial in the structure set $S_{\mathrm{Top}}(N^3 \times S^{2k+1})$. This is because:

(i) f being a homotopy equivalence is normally cobordant to the identity (cf. [KS2], and hence $f \times \mathrm{id}_{S^{2k+1}}$ is normally cobordant to the identity as well (i.e., $\eta(f \times \mathrm{id}_{S^{2k+1}}) = 0$).

(ii) The group $L^s_{2k+1}(\pi_1(N^3))$ is either 0 or Z_2 (cf. [W2, p. 33]). In the case $L^s_{2k+1}(\pi_1(N^3)) \cong Z_2$, it turns out that $\mathrm{im}\,\theta_{2k+1} \cong Z_2$. This leads to the triviality of $f \times \mathrm{id}_{S^{2k+1}}$ in $S_{\mathrm{Top}}(N^3 \times S^{2k+1})$ as claimed. But the triviality of $f \times \mathrm{id}_{S^{2k+1}}$ in $S_{\mathrm{Top}}(N^3 \times S^{2k+1})$ simply means that $f \times \mathrm{id}_{S^{2k+1}}$ is homotopic to a homeomorphism. In particular, $N^3 \times S^{2k+1} \approx M^3 \times S^{2k+1}$.

Dropping the geometricity assumption for 3-manifolds with finite fundamental groups. In this section we look closer at 3-manifolds with finite fundamental groups. To be more specific, we consider 3-manifolds of the form S^3/G, where G is a finite group acting freely (not necessarily linearly) on S^3. It turns out, as announced in [R] (cf. [L]), that G has to be a subgroup of SO(4). Many of these groups have to act linearly (cf. [R]), but it is still a conjecture that all of them do. A complete list of corresponding groups G (up to isomorphism with the direct product with a cyclic group of coprime order) is given in Table 1 of [Th]. The manifold $M^3 = S^3/G$ with an arbitrary free action of the group G from the list in [Th] will be called a *fake spherical space form.*

The following (cf. [JK, Theorem 1]) will be useful in our considerations.

Fact 5. *Let $f : M^3 \to N^3$ be a homotopy equivalence of oriented* 3-*manifolds. Then the normal invariant $\eta(f) \in [N^3; G/\mathrm{Top}]$ is trivial.*

Remark. It has been proved in [Th] that every fake spherical space form S^3/G is homotopy equivalent to a linear one.

Let us consider first the stabilization of fake spherical space forms by odd-dimensional spheres. Let M^3, N^3 be fake spherical space forms. Then clearly $M^3 \times S^1 \approx N^3 \times S^1 \Leftrightarrow M^3$ is h-cobordant to N^3. Therefore, we assume $k > 1$ in S^{2k+1}.

Theorem 6. *Let M^3, N^3 be closed oriented* (*prime*) 3-*manifolds.*

(a) *If $|\pi_1(M^3)| = \infty$ or M^3, N^3 are fake spherical space forms*, *then $M^3 \times S^3 \approx N^3 \times S^3 \Leftrightarrow \pi_1(M^3) \cong \pi_1(N^3)$.*

(b) $M^3 \times S^{2k+1} \approx N^3 \times S^{2k+1}$, $k \geq 2 \Leftrightarrow M^3 \simeq N^3$.

Proof. (a) Let us consider first manifolds with finite fundamental groups. Write $M^3 = S^3/G_1$, $N^3 = S^3/G_2$.

CASE 1: G_1 *nonabelian.* In this case there is a unique linear space form $X^3 = S^3/G_1$ and homotopy equivalences $f_1 : X^3 \to M^3$ and $f_2 : X^3 \to N^3$. In particular we have a homotopy equivalence $f : M^3 \to N^3$. Let $F : W \to N^3 \times I$ be a normal cobordism between f and the identity $\mathrm{id}_{N^3} : N^3 \to N^3$. Crossing F with id_{S^3} gives a normal map $F \times \mathrm{id}_{S^3}$ with a surgery obstruction $\Theta(F \times \mathrm{id}_{S^3}) \in L_3^s(\pi_1(N^3))$ (we recall that $\chi(S^3) = 0$ so indeed we have an obstruction in the simple surgery group $L^s(-)$). It turns out however that this obstruction is trivial by the surgery product formula in [Mo, p. VI]. This means that $f \times \mathrm{id}_{S^3} : M^3 \times S^3 \to N^3 \times S^3$ is homotopic to a homeomorphism, in particular $M^3 \times S^3 \approx N^3 \times S^3$.

CASE 2: G_1 *abelian.* Let $f_1 : X^3 \to M^3$ and $f_2 : Y^3 \to N^3$ be homotopy equivalences with X^3, Y^3 linear lens spaces. As before, we have $X^3 \times S^3 \approx M^3 \times S^3$ and $Y^3 \times S^3 \approx N^3 \times S^3$. But $X^3 \times S^3 \approx Y^3 \times S^3$ by Theorem 4(b) and hence $M^3 \times S^3 \approx N^3 \times S^3$ as well. The converse $M^3 \times S^3 \approx N^3 \times S^3 \Rightarrow G_1 \cong G_2$ is obvious.

If $|\pi_1(M^3)| = \infty$, then the isomorphism $\pi_1(M^3) \cong \pi_1(N^3)$ is equivalent to existence of a homotopy equivalence $f : M^3 \to N^3$. Since f is normally cobordant to the identity, $f \times \mathrm{id}_{S^3}$ is homotopic to a homeomorphism by the surgery product formula, as in the finite fundamental group case.

(b) We simply observe that the homeomorphism $M^3 \times S^{2k+1} \approx N^3 \times S^{2k+1}$, $k \geq 2$, gives a degree one map $f : M^3 \to N^3$ inducing an isomorphism $f_\# : \pi_1(M^3) \to \pi_1(N^3)$. By Lemma 1.1 in [S] this gives a homotopy equivalence $M^3 \simeq N^3$. As before, a homotopy equialence $M^3 \simeq N^3$ leads to the existence of a homeomorphism $M^3 \times S^{2k+1} \approx N^3 \times S^{2k+1}$ via the surgery product formula from [Mo].

REMARK 7. By using Fact 5 one can show that Theorem 6(b) holds for *all* closed (not necessarily prime) oriented 3-manifolds. In Theorem 6(a) for many groups, for example: quaternionic 2-groups D^*_{2q}, tetrahedral groups T^*_v, $v \geq 2$, cyclic groups Z_k, one can replace S^3 by a homotopy 3-sphere Σ^3 in the definition of M^3 and N^3, i.e., $M^3 = \Sigma^3/G_1$, $N^3 = \Sigma^3/G_2$.

We now consider the following families of 4-periodic finite groups:

1. Generalized quaternionic groups $Q(8p, q, r)$,
2. Cyclic groups Z_k,
3. Platonic groups T^*_1, O^* and I^*.

The particular interest in these groups is because:

1. The groups $Q(8p, q, r)$ are not subgroups of SO(4) but potentially can act freely on some homotopy S^3.

2. The groups Z_k for some $k > 3$ can potentially act nonlinearly on S^3 (cf. [R]).
3. There are homotopically exotic actions of T_1^*, O^* and I^* (cf. [Th, Corollary on p. 293]).

THEOREM 8. *Let $M^3 = \Sigma^3/G_1$, $N^3 = \Sigma^3/G_2$ be fake spherical space forms, where Σ^3 is a homotopy 3-sphere and G_1, G_2 are groups from the families 1–3 above. Then $M^3 \times S^{2k} \approx N^3 \times S^{2k}$, $k \geq 1 \Leftrightarrow M^3$ and N^3 are simply homotopy equivalent.*

Proof. Assume $M^3 \times S^{2k} \approx N^3 \times S^{2k}$. Then there is a homotopy equivalence $f : M^3 \to N^3$. We will show that the Whitehead torsion $\tau(f)$ is trivial. As in the proof of Theorem 3, we obtain an h-cobordism W between $M^3 \times S^{2k}$ and $N^3 \times S^{2k}$ whose torsion is that of f. To conclude that W is in fact an s-cobordism we argue as in the proof of Theorem 3. The only new ingredients needed here are:

(a) The involution on the Whitehead group $Wh(\pi_1(N^3))$ is trivial (cf. [KS1, Theorem 2.1]).
(b) The Whitehead group $\mathrm{Wh}(\pi_1(N^3))$ is torsion free for the groups in question (cf. [KS1]).

On the other hand, if $f : M^3 \to N^3$ is a simple homotopy equivalence, then

$$f \times \mathrm{id}_{S^{2k}} : M^3 \times S^{2k} \to N^3 \times S^{2k}, \quad k \geq 1,$$

is homotopic to a homeomorphism by the surgery product formula in [Mo].

We end this paper with an example which shows that Theorem 3 cannot be extended to manifolds of dimension higher than 3 and that the assertion of Theorem 8 is the best one can hope for.

EXAMPLE 9. *There are simple homotopy equivalent but nonhomeomorphic fake lens spaces L_1 and L_2 (in every dimension $2n+1$, $n \geq 2$) such that $L_1 \times S^{2k} \approx L_2 \times S^{2k}$ for each $k \geq 1$.*

Proof. Let Z_r, $r \geq 2$, be the finite cyclic group. Let $L_0^s(Z_r)$ be the surgery obstruction group and let $\widetilde{L}_0^s(Z_r)$ be the reduced group $\widetilde{L}_0^s(Z_r) = L_0^s(Z_r)/L_0^s(0)$. The group $\widetilde{L}_0^s(Z_r)$ is torsion free and is detected by the multisignature (cf. [W2]). Let L_2 be a seven-dimensional linear lens space with $\pi_1(L_2) \cong Z_r$.

Let $0 \neq \alpha \in \widetilde{L}_0^s(Z_r)$ be a nontrivial element. Realize α by a normal map

$$F : (W; \partial_0 W, \partial_1 W) \to (L_2 \times I, L_2 \times \{0\}, L_2 \times \{1\})$$

with $\Theta(F) = \alpha \in \widetilde{L}_0^s(Z_k)$ and $f = F|_{\partial_1 W} : \partial_1 W \to L_1 \times \{1\}$ a simple homotopy equivalence. Put $\partial_1 W = L_1$. Clearly, L_1 is a fake lens space.

CLAIM. $L_1 \not\approx L_1$.

Suppose $L_1 \approx L_2$. Then their ϱ-invariants are the same (cf. [W1]). But this would imply $\alpha = 0$, a contradiction.

Now applying the surgery product formula, we see that $f \times \mathrm{id}_{S^{2k}}$ is homotopic to a homeomorphism and hence $L_1 \times S^{2k} \approx L_2 \times S^{2k}$. By replacing $\widetilde{L}_0^s(Z_r)$ by $\widetilde{L}_2^s(Z_r)$ and using the fact that the torsion free part of $\widetilde{L}_2^s(Z_r)$ is determined by the multisignature (cf. [W2]) we infer that the analogous construction for five-dimensional lens spaces leads to the desired result in all odd dimensions.

Appendix: Proof of Theorem 1. There are two cases to be considered:

CASE 1: *$\pi_1(M^3)$ is finite.*

CASE 2: *$\pi_1(M^3)$ is infinite.*

Moreover, for clarity of our argument, Case 2 is split further into:

CASE 2.A: *M^3 is a "large" Seifert manifold.*

CASE 2.B: *M^3 is a "small" Seifert manifold.*

Proof of Case 1. The homeomorphism $M^3 \times S^1 \approx N^3 \times S^1$ implies that $\pi_1(N^3)$ is finite as well. If $\pi_1(M^3)$ is nonabelian, then so is $\pi_1(N^3)$ and both groups are isomorphic. Now, 3-manifolds with finite nonabelian fundamental groups are classified by those groups (cf. [KS1, p. 737], [TS, p. 567]). If $\pi_1(M^3)$ is abelian, then $\pi_1(N^3)$ is also abelian and we have two lens spaces (cf. [O]). The homeomorphism $h : M^3 \times S^1 \to N^3 \times S^1$ lifts to a homeomorphism $\widetilde{h} : M^3 \times \mathbb{R} \to N^3 \times \mathbb{R}$. Indeed, if $h_\# : Z_k \times \mathbb{Z} \to Z_k \times \mathbb{Z}$ is the induced homomorphism on the fundamental groups, then $h_\#(Z_k) = Z_k$ and $h_\#(\mathbb{Z}) = \mathbb{Z}$. The existence of a homeomorphism $\widetilde{h}$ implies the existence of an h-cobordism between M^3 and N^3. Now the Atiyah–Bott fixed point theorem (cf. [M2, Corollary 12.12]) implies that M^3 and N^3 are homeomorphic.

Proof of Case 2 (general comments). To simplify the notation, let $G = \pi_1(M^3)$, $H = \pi_1(N^3)$ and let $h_\# : G \times \mathbb{Z} \to H \times \mathbb{Z}$ be the induced isomorphism. Let $Z(G)$ be the center of G.

CLAIM. *If $Z(G) = \{1\}$, then $Z(H) = \{1\}$.*

Proof of claim. The center of $G \times \mathbb{Z}$ consists of a copy of $\mathbb{Z}$ (i.e., $\{1\} \times \mathbb{Z}$ in $G \times \mathbb{Z}$). Consider the image $h_\#(\{1\} \times \mathbb{Z})$ of the center. If $h_\#(\{1\} \times \mathbb{Z}) = \{1\} \times \mathbb{Z}$ in $H \times \mathbb{Z}$, then we are done. Suppose then that $h_\#(\{1\} \times t) = (p, s)$, $p \in H$, where t is a generator of $\mathbb{Z}$ and $p \neq \{1\}$. The manifold N^3, being prime, is either a $K(\pi_1(N^3), 1)$-manifold or $S^1 \times S^2$ (cf. [H]). In both cases the fundamental group of N^3, i.e., H, is torsion free (cf. [H]). Let C be the cyclic subgroup of H generated by p. It follows that the center of $H \times \mathbb{Z}$

contains $\mathbb{Z}\times\mathbb{Z}$. The inverse isomorphism $h_{\#}^{-1}$ sends the center of $H\times\mathbb{Z}$ to the center of $G\times\mathbb{Z}$. This implies that the center of G is nontrivial, which is a contradiction. As a consequence, $h_{\#}(\{1\}\times\mathbb{Z})=\{1\}\times\mathbb{Z}\subset H\times\mathbb{Z}$, which proves the claim.

Now given the claim, in the absence of the center of G, the homeomorphism $h: M^3\times S^1\to N^3\times S^1$ lifts to a homeomorphism $\widetilde{h}: M^3\times\mathbb{R}\to N^3\times\mathbb{R}$. In particular, we have an isomorphism $h_{\#}:\pi_1(M^3)\to\pi_1(N^3)$. This in turn implies the existence of a homeomorphism between M^3 and N^3 (cf. [KS1]). Consider now the case of nontrivial center of $\pi_1(M^3)$. Without loss of generality, we can assume that M^3 is irreducible (the only reducible case of $S^1\times S^2$ is fully understood). Now M^3 being closed, oriented, irreducible with infinite $\pi_1(M^3)$ and nontrivial center is Seifert fibered by the algebraic characterization of Seifert manifolds (cf. [CJ], [G]).

Proof of Case 2.A. We consider the case when M and N are "large" Seifert manifolds (cf. [O, pp. 91–92]). Let $\pi=\pi_1(M)$ and $\pi'=\pi_1(N)$. These groups are given by central extensions

$$0\to C\xrightarrow{i}\pi\xrightarrow{p}\Gamma\to 0,\qquad 0\to C'\xrightarrow{i'}\pi'\xrightarrow{p'}\Gamma'\to 0,$$

where $C\approx C'\approx\mathbb{Z}$. The homomorphisms i, i' are the natural inclusions and p, p' the projections. Now the Hopf formula (cf. [B, p. 41]) applied to these extensions leads to the following exact sequences in homology of groups (cf. [B, p. 47]):

$$H_2(\pi)\to H_2(\Gamma)\xrightarrow{\beta_*}H_1(C)\xrightarrow{i_*}H_1(\pi)\xrightarrow{p_*}H_1(\Gamma)\to 0,$$
$$H_2(\pi')\to H_2(\Gamma')\xrightarrow{\beta'_*}H_1(C)\xrightarrow{i'_*}H_1(\pi')\xrightarrow{p'_*}H_1(\Gamma')\to 0.$$

Suppose that, say, M does not fiber over S^1 with periodic monodromy. This is equivalent to the condition that the class $\langle h\rangle$ represented by the generator h of C is of finite order in $H_1(\pi)$ (cf. [O, p. 122] and [J, VI.31, p. 106]). This is then equivalent to the condition $\operatorname{im}\beta_*\neq 0$ in the exact homology sequence.

Let $f: M\times S^1\to N\times S^1$ be a homeomorphism. The natural extensions

$$0\to C\times\mathbb{Z}\xrightarrow{i\times\mathrm{id}}\pi\times\mathbb{Z}\xrightarrow{\overline{p}}\Gamma\to 0,$$
$$0\to C'\times\mathbb{Z}\xrightarrow{i'\times\mathrm{id}}\pi'\times\mathbb{Z}\xrightarrow{\overline{p'}}\Gamma'\to 0$$

lead to the commutative diagram

$$\begin{tikzcd}
H_2(\pi\times\mathbb{Z}) \arrow[r] \arrow[d, "\cong"', "f_*"] & H_2(\Gamma) \arrow[r, "\overline{\beta}_*"] \arrow[d, "\cong"] & H_1(C\times\mathbb{Z}) \arrow[r, "(i\times\mathrm{id})_*"] \arrow[d, "\cong"] & H_1(\pi\times\mathbb{Z}) \arrow[r, "\overline{p}_*"] \arrow[d, "\cong"', "f_*"] & H_1(\Gamma) \arrow[r] \arrow[d, "\cong"] & 0 \\
H_2(\pi'\times\mathbb{Z}) \arrow[r] & H_2(\Gamma') \arrow[r, "\overline{\beta'_*}"] & H_1(C'\times\mathbb{Z}) \arrow[r, "(i'\times\mathrm{id})_*"] & H_1(\pi'\times\mathbb{Z}) \arrow[r, "\overline{p'_*}"] & H_1(\Gamma') \arrow[r] & 0
\end{tikzcd}$$

Since $\operatorname{im}\overline{\beta}_*$ can be naturally identified with $\operatorname{im}\beta_*$ (and the same for $\overline{\beta'}_*$), it follows that $\operatorname{im}\beta_* \neq 0$ implies $\operatorname{im}\beta'_* \neq 0$. This shows that the isomorphism f on $C\times\mathbb{Z} \to C'\times\mathbb{Z}$ must be a product isomorphism. As a consequence, the homeomorphism $f : M \times S^1 \to N \times S^1$ can be lifted to a homeomorphism $\widetilde{f} : M \times \mathbb{R} \to N \times \mathbb{R}$. This, however, means that $\pi \approx \pi'$ and hence $M \approx N$.

Proof of Case 2.B. We now assume that M (or N) is a "small" Seifert manifold (cf. [O, pp. 91–92]). In the notation of [O] the possibilities for these manifolds are classes (iii), (iv) or (ix) in [O, pp. 124–125] (we recall that our manifolds are orientable). Class (iii) consists of a single manifold X which is a torus bundle over S^1 with periodic monodromy of period 2. Class (ix) consists of manifolds which either have a fundamental group with trivial center or are homeomorphic to X. Since the case of trivial center has already been considered, we are left with the manifold X and class (iv). For the manifolds in class (iv), the class $\langle h\rangle$ has finite order in $H_1(\pi)$ (cf. [O, p. 124]). In view of [O, Theorem 12.10, p. 131], the argument used for the case of "large" Seifert manifolds shows that if M, N are in class (iv) and $M \times S^1 \approx N \times S^1$, then $M \approx N$. For the manifold X in (iii), the corresponding class $\langle h\rangle$ is of infinite order in $H_1(\pi)$ and hence $X \not\approx M$ for any M in class (iv). As a consequence, if M is a "small" Siefert manifold and $M \times S^1 \approx N \times S^1$, then $M \approx N$.

Finally, if M is a T^2 bundle over S^1 with periodic monodromy and N is a surface bundle over S^1 with periodic monodromy with $M \times S^1 \approx N \times S^1$, then it follows that N is a T^2 bundle over S^1 as well. Given the classification of T^2 bundles over S^1 with periodic monodromy in [H, Ex. 12.3, p. 122], one concludes that if M, N are such bundles with $M \times S^1 \approx N \times S^1$, then $M \approx N$. Indeed, if $M \not\approx N$, then $H_1(M \times S^1;\mathbb{Z}) \not\cong H_1(N \times S^1;\mathbb{Z})$.

References

[Br] S. A. Broughton, *Classifying finite group actions on surfaces of low genus*, J. Pure Appl. Algebra 69 (1990), 233–270.

[B] K. S. Brown, *Cohomology of Groups*, Grad. Texts in Math. 87, Springer, 1982.

[BC] F. Bujolance and M. Conder, *On cyclic groups of automorphisms of Riemann surfaces*, J. London Math. Soc. (2) 59 (1999), 573–584.

[CJ] A. Casson and D. Jungries, *Convergence groups and Seifert fibered* 3*-manifolds*, Invent. Math. 118 (1994), 441–456.

[C] M. M. Cohen, *A Course in Simple Homotopy Theory*, Springer, New York, 1973.

[CR] P. Conner and F. Raymond, *Derived actions*, in: Proc. 2nd Conf. on Compact Transformation Groups (Amherst, MA, 1971), Lecture Notes in Math. 299, Springer, Berlin, 1972, 237–310.

[G] D. Gabai, *Convergence groups are Fuchsian groups*, Ann. of Math. 136 (1992), 447–510.

[H1] W. J. Harvey, *Cyclic groups of automorphisms of a compact Riemann surface*, Quart. J. Math. Oxford Ser. (2) 17 (1966), 86–97.

[H2] W. J. Harvey, *On branch loci in Teichmüller space*, Trans. Amer. Math. Soc. 153 (1971), 387–399.
[H] J. Hempel, 3-*manifolds*, Ann. of Math. Stud. 86, Princeton Univ. Press, 1976.
[J] W. Jaco, *Lectures in Three-Manifold Topology*, CBMS Reg. Conf. Ser. in Math. 43, Amer. Math. Soc., Providence, RI, 1980.
[JK] B. Jahren and S. Kwasik, 3-*dimensional surgery theory*, *UNil-groups and the Borel conjecture*, Topology 42 (2003), 1353–1369.
[JN] M. Jenkins and W. Neumann, *Seifert Manifolds*, notes, Univ. of Maryland, 1981.
[KS1] S. Kwasik and R. Schultz, *Vanishing of Whitehead torsion in dimension four*, Topology 31 (1992), 735–756.
[KS2] —, —, *All* $\mathbb{Z}_q$ *lens spaces have diffeomorphic squares*, ibid. 41 (2002), 321–340.
[L] R. Lee, *Semicharacteristic classes*, ibid. 12 (1973), 183–199.
[Mℓ] W. Metzler, *Diffeomorphismen zwischen Produkten mit dreidimensional Linsenräumen als Faktoren*, Dissertationes Math. 65 (1969).
[M1] J. Milnor, *Two complexes which are homeomorphic but combinatorially distinct*, Ann. of Math. 74 (1961), 575–590.
[M2] —, *Whitehead torsion*, Bull. Amer. Math. Soc. 72 (1963), 358–426.
[Mo] J. Morgan, *A product formula for surgery obstruction*, Mem. Amer. Math. Soc. 201 (1978).
[N] J. Nielsen, *Die Struktur periodischer Transformationen von Flächen*, Denske Vid. Selsk. Mat.-Fys. Medd. 15 (1937), no. 1, 77 pp.
[O] P. Orlik, *Seifert Manifolds*, Lecture Notes in Math. 291, Springer, 1972.
[R] H. Rubinstein, *An algorithm to recognize the* 3-*sphere*, in: Proc. Internat. Congress Math. (Zürich, 1994), Birkhäuser, 1995, 601–611.
[S] G. A. Swarup, *On a theorem of C. B. Thomas*, J. London Math. Soc. (2) 8 (1974), 13–21.
[Sy] P. Symonds, *The cohomology representation of an action of* C_p *on a surface*, Trans. Amer. Math. Soc. 306 (1988), 389–400.
[Th] C. B. Thomas, *Homotopy classification of free actions by finite groups on* S^3, Proc. London Math. Soc. 49 (1980), 284–297.
[TS] W. Threlfall und H. Seifert, *Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensional sphärischen Raumes* (*Schluß*), Math. Ann. 107 (1932), 543–586.
[T] W. Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. 6 (1982), 357–382.
[Tu] V. Turaev, *Towards the topological classification of geometric* 3-*manifolds*, in: Topology and Geometry—Rohlin Seminar, Lecture Notes in Math. 1346, Springer, 1988, 291–323.
[W1] C. T. C. Wall, *Surgery on Compact Manifolds*, Academic Press, 1970.
[W2] —, *On classification of hermitian forms*: *VI*, *Group rings*, Ann. of Math. 103 (1976), 1–80.

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Received 2 December 2003;
in revised form 28 June 2004