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Whitney arcs and 1-critical arcs

by

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Abstract. A simple arc $\gamma \subset \mathbb{R}^n$ is called a Whitney arc if there exists a non-constant real function f on γ such that $\lim_{y\to x,\, y\in\gamma} |f(y)-f(x)|/|y-x| = 0$ for every $x \in \gamma$; γ is 1-critical if there exists an $f \in C^1(\mathbb{R}^n)$ such that $f'(x) = 0$ for every $x \in \gamma$ and f is not constant on γ. We show that the two notions are equivalent if γ is a quasiarc, but for general simple arcs the Whitney property is weaker. Our example also gives an arc γ in $\mathbb{R}^2$ each of whose subarcs is a monotone Whitney arc, but which is not a strictly monotone Whitney arc. This answers completely a problem of G. Petruska which was solved for $n \geq 3$ by the first author in 1999.

1. Introduction. A famous example of Whitney [10] shows that there exist a simple arc $\gamma \subset \mathbb{R}^2$ and a C^1 function f on $\mathbb{R}^2$ such that each point of γ is critical for f, and f is not constant on γ. A slightly weaker example was independently constructed by Choquet in [1]. Namely, he constructed a simple arc $\gamma \subset \mathbb{R}^2$ which is *Whitney* by the following terminology introduced in [8] and used in [4].

Definition 1.1. We say that a simple arc $\gamma \subset \mathbb{R}^n$ is a *Whitney arc* if there exists a non-constant real function f on γ such that

$$(1)\qquad \lim_{y\to x,\, y\in\gamma} \frac{|f(y)-f(x)|}{|y-x|} = 0 \quad \text{for each } x \in \gamma.$$

It seems that the difference between Whitney arcs thus defined and arcs considered by Whitney is not sufficiently emphasized in the literature (see e.g. remarks in [9, p. 399] on Choquet's results). The aim of the present article is to study this difference. First we recall the terminology of [9] which corresponds precisely to the example of Whitney.

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Definition 1.2. We say that a simple arc $\gamma \subset \mathbb{R}^n$ is a 1-*critical arc* if there exists a C^1 function on $\mathbb{R}^n$ which is not constant on γ and $f'(x) = 0$ for each $x \in \gamma$.

Of course, each 1-critical arc is Whitney but the opposite implication does not hold. If the convergence in (1) were uniform in $x \in \gamma$ then Whitney's extension theorem would imply that f can be extended to $\mathbb{R}^n$ as a C^1 function with derivative 0 at the points of γ; however, without assuming uniform convergence this is not the case. In Section 3 we will construct a Whitney arc γ in $\mathbb{R}^2$ (slightly modifying the original construction of Whitney) which is not 1-critical.

No full characterization of 1-critical arcs or Whitney arcs is known (even in $\mathbb{R}^2$). However, there are interesting necessary or sufficient conditions. It is not difficult to prove (see [1] and Lemma 4.1 below) that no Whitney arc has σ-finite 1-dimensional Hausdorff measure. Choquet also proved that no graph of a continuous $f : [a, b] \to \mathbb{R}$ is Whitney. This result easily implies [5] that if $\gamma \subset \mathbb{R}^n$ has a parametrization whose $n-1$ coordinates have finite variation, then γ is not a Whitney arc. Interesting necessary [8, Theorem 3] and sufficient [8, Theorem 2] conditions for $\gamma \subset \mathbb{R}^n$ to be Whitney were proved by Laczkovich and Petruska.

Norton [9] proved that each simple arc γ in $\mathbb{R}^n$ which is a quasiarc and has Hausdorff dimension greater than 1 is 1-critical, and noted that such arcs "are in the plentiful supply (e.g. as Julia sets for certain rational maps in the plane)". (Note that all arcs constructed in [1], [4], [8] and [10] are quasiarcs.) We prove (Theorem 2.2) that if a Whitney arc in $\mathbb{R}^n$ is a quasiarc, then it is 1-critical. That is, for quasiarcs the two notions are equivalent.

A modification of the construction of Whitney (see Section 3) is used as a basic building block in an iterative construction in Section 4, which gives an example of a Whitney arc which is not 1-critical and also has other interesting properties. To describe them, recall that a real function f defined (at least) on a simple arc $\gamma \subset \mathbb{R}^n$ is said to be *monotone* (resp. *strictly monotone*) along γ if $f \circ \varphi$ is monotone (resp. strictly monotone) for each homeomorphic parametrization φ of γ. Following [4], we say that a simple arc $\gamma \subset \mathbb{R}^n$ is a *monotone* (resp. *strictly monotone*) *Whitney arc* if there exists a non-constant f on γ that is monotone (resp. strictly monotone) along γ and satisfies (1).

Petruska raised the question whether there exists a simple arc γ for which every subarc is Whitney, but for which there is no parametrization φ of γ satisfying

$$\lim_{t \to t_0} \frac{|t - t_0|}{|\varphi(t) - \varphi(t_0)|} = 0, \qquad t_0 \in [0, 1]$$

(which is clearly equivalent to γ not being a strictly monotone Whitney arc).

This question was answered affirmatively in [4] for $n \geq 3$, and it remained open in $\mathbb{R}^2$ (see Problem 4 in [4]). Our example gives an affirmative answer also for $n = 2$. We construct an arc $\gamma \subset \mathbb{R}^2$ such that each of its subarcs is a monotone Whitney arc but any Lipschitz function satisfying (1) on any subarc γ' of γ is constant on γ'. From the last property it will easily follow that each function satisfying (1) on γ is locally constant on a relatively open dense subset of γ (and so γ is not a strictly monotone Whitney arc).

For the sake of completeness we remark that Theorem 2.2 implies that every Whitney quasiarc is a monotone Whitney arc. However, if γ is not a quasiarc then this is no longer true: Kolář ([6]) recently constructed a 1-critical arc in $\mathbb{R}^2$ which is not a monotone Whitney arc (and since each 1-critical arc is a Whitney arc, this solves Problem 2 in [4]).

2. Whitney quasiarcs are 1-critical. We denote by λ the Lebesgue measure on $\mathbb{R}$. In the following we will use the well-known notion of a quasiarc.

Definition 2.1. We say that a simple arc $\gamma \subset \mathbb{R}^n$ is a *quasiarc* if there exists $K > 0$ such that, for any distinct $x, y \in \gamma$, the subarc of γ "between x and y" (in the natural sense) is contained in some ball of radius $K|x - y|$.

Theorem 2.2. *Let $\gamma \subset \mathbb{R}^n$ be a Whitney arc which is a quasiarc. Then there exists a C^1 function f on $\mathbb{R}^n$ that is non-constant monotone along γ, and $f'(x) = 0$ for every $x \in \gamma$. In particular, γ is 1-critical.*

Proof. Let $\varphi : [0, 1] \to \mathbb{R}^n$ be a continuous injective parametrization of γ. Choose a non-constant $f : \gamma \to \mathbb{R}$ such that (1) holds. We can suppose that $g := f \circ \varphi$ is not non-increasing (otherwise we take $-f$ instead of f). So we can choose $0 \leq a < b \leq 1$ such that $g(a) < g(b)$. For each $y \in [g(a), g(b)]$ put $\omega(y) = \min\{x \in [a, b] : g(x) = y\}$. Since g is continuous, ω is clearly (strictly) increasing. Using Lusin's theorem and then the Cantor–Bendixson theorem we can choose a set $T^* \subset [g(a), g(b)]$ such that $\lambda(T^*) > 0$ and $\omega|_{T^*}$ is continuous. Put $T := \omega(T^*)$. Then $g_0 := g|_T$ is an increasing homeomorphism between T and T^*, and $g_0 = f_0 \circ \varphi|_T$ where $f_0 := f|_{\varphi(T)}$ is a homeomorphism between $\varphi(T)$ and T^*.

Let, for $x \in \gamma$,

$$\eta_k(x) := \sup\left\{ \frac{|f(y) - f(x)|}{|y - x|} : y \in \gamma,\, 0 < |y - x| < 1/k \right\}.$$

Then $\lim_{k\to\infty} \eta_k(x) \to 0$ for every $x \in \gamma$. It is easy to prove that $p_k := \eta_k \circ f_0^{-1}$ is a Borel function on T^*. Since $p_k \to 0$ at every point of T^*, applying Egorov's theorem (see [3, 2.3.7]) we can find a closed $H^* \subset T^*$ with $\lambda(H^*) > 0$ such that $p_k \to 0$ uniformly on H^*. That is, the limit in (1) is uniform on $f_0^{-1}(H^*)$.

Set $H := g_0^{-1}(H^*)$. We can define a (strictly) increasing continuous function $\widetilde{g}$ on $[0,1]$ which extends $g_0|_H$ and is linear on each component of $[0,1] \setminus H$. Put $q(t) := \lambda((-\infty, t] \cap H^*)$ and $F := q \circ \widetilde{g} \circ \varphi^{-1}$. Then F is a non-constant function monotone along γ. We will prove that

$$(2) \qquad \lim_{y \to x,\, x \in \gamma} \frac{|F(y) - F(x)|}{|y - x|} = 0 \quad \text{uniformly with respect to } x \in \gamma.$$

To this end consider an arbitrary $\varepsilon > 0$. Let $K \geq 1$ witness the fact that γ is a quasiarc. Note that $F = q \circ f$ on $\varphi(H)$ and q is Lipschitz with constant 1, therefore $|F(y) - F(x)| \leq |f(y) - f(x)|$ for each $x, y \in \varphi(H)$. Using also the fact that the limit (1) is uniform with respect to $x \in \varphi(H) = f_0^{-1}(H^*)$, we can find $\delta > 0$ such that

$$(3) \qquad \frac{|F(x) - F(y)|}{|x - y|} < \frac{\varepsilon}{2K} \quad \text{whenever } x, y \in \varphi(H) \text{ and } 0 < |x - y| < \delta.$$

Let $x, y \in \gamma$ be arbitrary points with $0 < |x - y| < \delta(4K)^{-1}$ and $F(x) \neq F(y)$. We can suppose that $x = \varphi(t_x)$ and $y = \varphi(t_y)$ with $t_x < t_y$.

Since F is constant on the intervals contiguous to $\varphi(H)$ and $F(x) \neq F(y)$, we see that H has at least two points in $[t_x, t_y]$. Define

$$s_x := \min(H \cap [t_x, t_y]) \quad \text{and} \quad s_y = \max(H \cap [t_x, t_y]).$$

Clearly $t_x \leq s_x < s_y \leq t_y$ and F is constant on $\varphi([t_x, s_x])$ and $\varphi([s_y, t_y])$. The definition of K gives

$$|\varphi(s_x) - \varphi(s_y)| \leq 2K|\varphi(t_x) - \varphi(t_y)| \leq 2K \frac{\delta}{4K} < \delta$$

and thus (3) gives

$$\frac{|F(x) - F(y)|}{|x - y|} = \frac{|F(\varphi(t_x)) - F(\varphi(t_y))|}{|\varphi(t_x) - \varphi(t_y)|} \leq \frac{|F(\varphi(s_x)) - F(\varphi(s_y))|}{\frac{1}{2K}|\varphi(s_x) - \varphi(s_y)|} < 2K \frac{\varepsilon}{2K} = \varepsilon,$$

which proves (2).

Whitney's extension theorem (see e.g. [2, p. 245]) and (2) immediately imply that there exists an extension $\widetilde{F}$ of F such that $\widetilde{F} \in C^1(\mathbb{R}^n)$ and $(\widetilde{F})'(x) = 0$ for each $x \in \gamma$. Since F is a non-constant monotone function along γ, we have proved Theorem 2.2. ∎

3. A modified Whitney's example: a Whitney arc which is not 1-critical. In this section we slightly modify the original construction of Whitney to obtain a class of Whitney arcs (called here MW-arcs for short) and prove some of their properties that are used in this section to give a simple construction of a Whitney arc which is not 1-critical, and are also used in Section 4 for constructing our main example.

3.1. For the convenience of the reader we first repeat (almost word for word) the construction of Whitney from [10].

Let $Q = Q_\emptyset := [0,1]^2$. Let Q_0, Q_1, Q_2, Q_3 be closed squares of side $1/3$ lying inside to Q in clockwise order, each at distance $1/12$ from the boundary of Q as in Figure 1. Let q and q' be the centres of the sides of Q along Q_0, Q_1, and along Q_3, Q_0. Let q_i and q'_i be the centres of two adjacent edges of Q_i ($i = 0,1,2,3$), as in Figure 1. Let A_i ($i = 0,1,2,3,4$) be the line segments as in Figure 1.

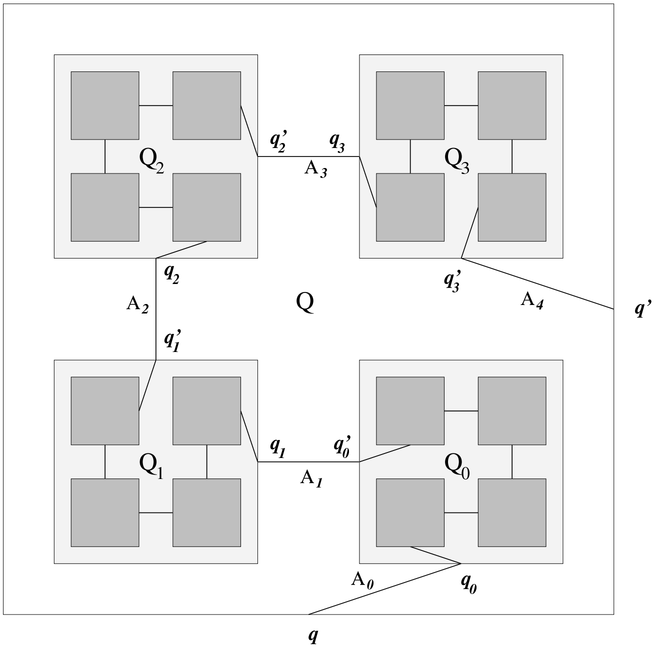

Fig. 1. Construction

Suppose we have constructed squares $Q_{i_1\ldots i_t}$, points $q_{i_1\ldots i_t}$, $q'_{i_1\ldots i_t}$, and line segments $A_{i_1\ldots i_t,j}$ (each $i_k = 0,1,2,3$; $j = 0,1,2,3,4$) for $t < s$. By taking a square $Q_{i_1\ldots i_{s-2}}$, shrinking it to a third of its size, and turning it around and upside down if necessary, we may place it in $Q_{i_1\ldots i_{s-1}}$ so that $q_{i_1\ldots i_{s-2}}$ and $q'_{i_1\ldots i_{s-2}}$ go into $q_{i_1\ldots i_{s-1}}$ and $q'_{i_1\ldots i_{s-1}}$, and thus construct four new squares $Q_{i_1\ldots i_s}$ ($i_s = 0,1,2,3$) as images of $Q_{i_1\ldots i_{s-2}i_s}$, furthermore points $q_{i_1\ldots i_s}$, $q'_{i_1\ldots i_s}$ and segments $A_{i_1\ldots i_{s-1}j}$ for $j = 0,1,2,3,4$ as images of $q_{i_1\ldots i_{s-2}i_s}$, $q'_{i_1\ldots i_{s-2}i_s}$ and $A_{i_1\ldots i_{s-2}j}$, respectively. We denote the point $Q \cap Q_{i_1} \cap Q_{i_1 i_2} \cap \cdots$ by $Q_{i_1 i_2 \ldots}$.

It is not difficult to see that the line segments $A_{i_1\ldots i_s}$ together with the points $Q_{i_1 i_2\ldots}$ form a simple arc A (a canonical parametrization is described in [10]).

Now define F on A as follows:

$$F(x) := \begin{cases} \dfrac{i_1}{4} + \dfrac{i_2}{4^2} + \cdots + \dfrac{i_s}{4^s}, & x \in A_{i_1 i_2 \ldots i_s}, \\ \dfrac{i_1}{4} + \dfrac{i_2}{4^2} + \cdots, & x = Q_{i_1 i_2 \ldots}. \end{cases}$$

Whitney proved that F is a restriction of a C^1 function F^* defined on the plane such that each point of A is critical for F^*.

3.2. Now we will make some modifications which lead to a class of Whitney (but not 1-critical) arcs.

In the following, the symbol $\underline{i}$ will always denote a sequence $\underline{i} = i_1 \ldots i_k$ where $i_n \in \{0,1,2,3\}$ (for $k = 0$ we set $\underline{i} = \emptyset$); we define $|\underline{i}| := k$. For each $\underline{i} = i_1 \ldots i_k$ and $j \in \{0,1,2,3,4\}$ we choose an arbitrary simple arc $\gamma_{\underline{i},j}$ lying (except the endpoints) in $\operatorname{int} Q_{\underline{i}} \setminus (Q_{\underline{i}0} \cup Q_{\underline{i}1} \cup Q_{\underline{i}2} \cup Q_{\underline{i}3})$ that connects the same points as $A_{\underline{i},j}$, such that the arcs $\gamma_{\underline{i},j}$ are pairwise disjoint and

$$\operatorname{dist}(\gamma_{\underline{i},j}, \gamma_{\underline{i},j+1}) < 1/5^k, \quad j = 0,1,2,3. \tag{4}$$

It is easy to show that the arcs $\gamma_{\underline{i},j}$ together with the points $Q_{i_1 i_2 \ldots}$ form a simple arc γ. We will choose points $a_{\underline{i},j}, b_{\underline{i},j} \in \gamma_{\underline{i},j}$ such that

$$\operatorname{dist}(a_{\underline{i},j}, b_{\underline{i},j+1}) < 1/5^k, \quad j = 0,1,2,3. \tag{5}$$

We will call any arc constructed in this way an *MW-arc* (that is, an arc obtained by the modified Whitney construction).

3.3. We show that each MW-arc γ is a monotone Whitney arc. To this end consider the function f on γ which agrees with F at the points $Q_{i_1 i_2 \ldots}$ and is constant on each $\gamma_{\underline{i},j}$ with the same value as F has on $A_{\underline{i},j}$. Clearly f is monotone along γ. We will show that (1) holds. It is immediate that (1) holds at the points of the arcs $\gamma_{\underline{i},j}$, since f is constant on these arcs and each such arc has, in the space γ, a neighbourhood formed by three (or two) arcs $\gamma_{\underline{i},j}$. Now let $x = Q_{i_1 i_2 \ldots}$ and let y be an arbitrary point of γ different from x. Consider the largest k with $y \in Q_{i_1 \ldots i_k} = Q_{\underline{i}}$. Then we can see that $|f(x) - f(y)| \le 1/4^k$, while $|x - y| \ge \operatorname{dist}(Q_{\underline{i}i_{k+1}i_{k+2}}, \partial Q_{\underline{i}i_{k+1}}) = 1/(12 \cdot 3^{k+1})$. This shows

$$\lim_{y \to x,\, y \in \gamma} \frac{|f(y) - f(x)|}{|y - x|} \le \lim_{k \to \infty} \frac{12 \cdot 3^{k+1}}{4^k} = 0.$$

3.4. Now we will show that if f is a Lipschitz function on an MW-arc γ, then

$$\lambda(f(\gamma)) \le \sum_{\underline{i},\, 0 \le j \le 4} \lambda(f(\gamma_{\underline{i},j})). \tag{6}$$

Let f be Lipschitz with constant K. For each $k \in \mathbb{N}$, let

$$I_k := \bigcup_{|\underline{i}|\le k,\, 0\le j\le 4} f(\gamma_{\underline{i},j}) \cup \bigcup_{|\underline{i}|=k,\, 0\le j\le 3} [f(a_{\underline{i},j}), f(b_{\underline{i},j+1})].$$

It is easy to see that I_k is a closed interval, since it is clearly connected and closed; let $I_k =: [u,v]$. Now observe that $f(\gamma) \subset [u - K\sqrt{2}/3^{k+1}, v + K\sqrt{2}/3^{k+1}]$. This follows by the Lipschitz property of f, the definition of I_k and the obvious fact that $\operatorname{dist}(c, \bigcup_{|\underline{i}|\le k,\, 0\le j\le 4} \gamma_{\underline{i},j}) \le \sqrt{2}/3^{k+1}$ for every $c \in \gamma$.

Clearly

$$\lambda\Big(I_k \setminus \bigcup_{|\underline{i}|\le k,\, 0\le j\le 4} f(\gamma_{\underline{i},j})\Big) \le \sum_{|\underline{i}|=k,\, 0\le j\le 3} |f(b_{\underline{i},j+1}) - f(a_{\underline{i},j})| \le \frac{K4^{k+1}}{5^k}.$$

Therefore

$$\lambda(f(\gamma)) \le \sum_{\underline{i},\, 0\le j\le 4} \lambda(f(\gamma_{\underline{i},j})) + \frac{K4^{k+1}}{5^k} + 2K\sqrt{2}/3^{k+1},$$

which easily implies (6).

Similarly to (6), we find that for each $\underline{i}^* = i_1^* \ldots i_s^*$,

$$\lambda(f(\gamma \cap Q_{\underline{i}^*})) \le \sum_{\underline{i},\, 0\le j\le 4} \lambda(f(\gamma_{\underline{i},j} \cap Q_{\underline{i}^*})). \tag{7}$$

3.5. Now we can prove the following result:

THEOREM 3.1. *There exists a Whitney arc $\gamma \subset \mathbb{R}^2$ which is not* 1-*critical. Moreover*, *there exists no non-constant Lipschitz function f on γ which satisfies* (1).

Proof. We choose γ as an arbitrary MW-arc for which all the arcs $\gamma_{\underline{i},j}$ are polygons. Thus γ is a (monotone) Whitney arc.

Now suppose that f is a Lipschitz function on γ which satisfies (1) on γ. Then, since a polygon is not a Whitney arc, $\lambda(f(\gamma_{\underline{i},j})) = 0$ for each arc $\gamma_{\underline{i},j}$ and hence (6) implies that $\lambda(f(\gamma)) = 0$ and thus f is constant on γ. Since each C^1 function on $\mathbb{R}^2$ is Lipschitz on γ, we have proved that the arc γ is not 1-critical. ■

4. The main example. We will need the following result (see [1, p. 49]).

LEMMA 4.1. *Suppose that $A \subset \mathbb{R}^n$ has σ-finite one-dimensional Hausdorff measure and f is a real function on A such that*

$$\lim_{y\to x,\, y\in A} \frac{|f(y) - f(x)|}{|y-x|} = 0 \quad \textit{for each } x \in A.$$

Then $\lambda(f(A)) = 0$.

Using the generalized Whitney construction from Section 3 we will now prove the following main result of the present article.

THEOREM 4.2. *There exists a simple arc $\gamma \subset \mathbb{R}^2$ such that*:

(i) *Each subarc of γ is a monotone Whitney arc.*
(ii) *There is no non-constant Lipschitz function f on any subarc γ^* of γ such that f satisfies* (1) *on γ^*.*
(iii) *Each function satisfying* (1) *on γ is locally constant on a relatively open dense subset of γ. In particular*, *γ is not a strictly monotone arc.*

Proof. First note that (iii) is an easy consequence of (ii). Indeed, suppose that (ii) holds, f satisfies (1) on γ, and γ^* is an arbitrary subarc of γ. For each $n \in \mathbb{N}$, let Z_n denote the set of all $x \in \gamma^*$ such that $|f(y)-f(x)| \le |y-x|$ whenever $y \in \gamma$ and $|y-x| \le 1/n$. Since each Z_n is closed and $\gamma^* = \bigcup Z_n$, the Baire category theorem implies that there exists $n \in \mathbb{N}$ and a subarc γ^{**} of γ^* with $\operatorname{diam} \gamma^{**} < 1/n$ and $\gamma^{**} \subset Z_n$. Then f is Lipschitz on γ^{**} and thus constant on γ^{**} by (ii), and (iii) follows.

Now we fix an arbitrary MW-arc $\widetilde{\gamma}$ for which all the arcs $\widetilde{\gamma}_{\underline{i},j}$ are polygons and we will construct γ by an iterative procedure, as follows.

STEP 1. Let $\gamma^1 := \widetilde{\gamma}$. We choose a countable set $\mathcal{Q}^1$ of disjoint closed squares such that each square in $\mathcal{Q}^1$ is inside $Q_{\underline{i}} \setminus (Q_{\underline{i}0} \cup Q_{\underline{i}1} \cup Q_{\underline{i}2} \cup Q_{\underline{i}3})$ for some $\underline{i}$, it meets precisely one arc $\widetilde{\gamma}_{\underline{i},j}$, and its intersection with $\widetilde{\gamma}_{\underline{i},j}$ is a line segment that connects the centres of two adjacent edges of the square. We also require that $\bigcup \mathcal{Q}^1$ covers a dense subset of $\bigcup_{\underline{i},j} \widetilde{\gamma}_{\underline{i},j}$,

(8) $\qquad$ no point $a_{\underline{i},j}$ or $b_{\underline{i},j}$ (cf. (5)) is contained in $\bigcup \mathcal{Q}^1$ and

$$r := \sum_{Q^* \in \mathcal{Q}^1} \text{edge length of } Q^* < 1. \tag{9}$$

Step 1 concludes with the arc $\gamma^1 = \widetilde{\gamma}$ and the set of squares $\mathcal{Q}^1$. For any $m \ge 1$, the mth step will conclude with a simple arc γ^m and a set of disjoint squares $\mathcal{Q}^m$ such that γ^m intersects each square $Q^* \in \mathcal{Q}^m$ in a line segment that connects the centres of two adjacent edges of Q^*. Observe that, using (8), we easily deduce that

(10) $\quad$ any simple arc $\eta \subset \bigcup \mathcal{Q}^1 \cup \gamma^1$ such that $\eta \setminus \bigcup \mathcal{Q}^1 = \gamma^1 \setminus \bigcup \mathcal{Q}^1$ is an MW-arc.

STEP m. Suppose that γ^{m-1} and $\mathcal{Q}^{m-1}$ have been defined. We will repeat the same construction as in Step 1 inside each of the squares of $\mathcal{Q}^{m-1}$:

For each $Q^* \in \mathcal{Q}^{m-1}$ choose a similarity ψ_{Q^*} of the plane that maps the unit square $Q = [0,1]^2$ onto Q^*, such that the segment between q and q' is mapped onto the segment $Q^* \cap \gamma^{m-1}$. Let

$$\gamma^m = \Big(\gamma^{m-1} \setminus \bigcup \mathcal{Q}^{m-1}\Big) \cup \bigcup_{Q^* \in \mathcal{Q}^{m-1}} \psi_{Q^*}(\gamma^1),$$

$$\mathcal{Q}^m = \{\psi_{Q^*}(\widetilde{Q}) : Q^* \in \mathcal{Q}^{m-1},\ \widetilde{Q} \in \mathcal{Q}^1\}.$$

It is easy to see by induction on m that

$$r^m = \sum_{Q^* \in \mathcal{Q}^m} \text{edge length of } Q^*. \tag{11}$$

Let $\gamma := \bigcap_{m=1}^{\infty}(\gamma^m \cup \bigcup \mathcal{Q}^m)$. It is geometrically obvious and not difficult to prove that γ is a simple arc. For a precise proof we have at least two possibilities. The more straightforward one is to define inductively "natural" parametrizations of γ^m and to check that the limit of these parametrizations is an injective parametrization of γ. The other possibility is to apply [7, Theorem 3, Section V, §47] which gives a sufficient condition for a set to be a simple arc, which is rather easy to verify for our set γ. (We choose $C_n := \gamma^n \cup \bigcup \mathcal{Q}^n$; for the definition of A_n and B_n we use the natural order on γ^n.)

Using (10), we find that γ is an MW-arc. Also, for each $Q^* \in \bigcup_{m=1}^{\infty} \mathcal{Q}^m$, we infer by (10) that

$$\psi_{Q^*}^{-1}(\gamma \cap Q^*) \quad \text{is an MW-arc} \tag{12}$$

and therefore $\gamma \cap Q^*$ is a monotone Whitney arc. Therefore each subarc of γ is a monotone Whitney arc.

For each $Q^* \in \bigcup_{m=1}^{\infty} \mathcal{Q}^m$, let $\widetilde{\gamma}_{\underline{i},j,Q^*} := \psi_{Q^*}(\widetilde{\gamma}_{\underline{i},j})$ and $\gamma_{\underline{i},j,Q^*}$ be the subarc of γ with the same endpoints as $\widetilde{\gamma}_{\underline{i},j,Q^*}$.

To prove (ii), first suppose that $f : \gamma \to \mathbb{R}$ is a Lipschitz function defined on the whole arc γ that satisfies (1). Let K denote the Lipschitz constant of f.

Consider an arbitrary $Q \in \mathcal{Q}^k$ and an arbitrary arc $\gamma_{\underline{i},j,Q}$. Since

$$\gamma_{\underline{i},j,Q} = (\gamma_{\underline{i},j,Q} \cap \widetilde{\gamma}_{\underline{i},j,Q}) \cup \bigcup_{Q^* \in \mathcal{Q}^{k+1},\, Q^* \cap \gamma_{\underline{i},j,Q} \neq \emptyset} (Q^* \cap \gamma)$$

and $\gamma_{\underline{i},j,Q} \cap \widetilde{\gamma}_{\underline{i},j,Q}$ is rectifiable, Lemma 4.1 implies $\lambda(f(\gamma_{\underline{i},j,Q} \cap \widetilde{\gamma}_{\underline{i},j,Q})) = 0$ and therefore

$$\lambda(f(\gamma_{\underline{i},j,Q})) \le \sum_{Q^* \in \mathcal{Q}^{k+1},\, Q^* \cap \gamma_{\underline{i},j,Q} \neq \emptyset} \lambda(f(Q^* \cap \gamma)). \tag{13}$$

By (12) and (6) we obtain

$$\lambda(f(Q \cap \gamma)) \le \sum_{\underline{i},j} \lambda(f(\gamma_{\underline{i},j,Q})).$$

Using also (13) we obtain

$$\lambda(f(Q\cap\gamma))\le \sum_{Q^*\subset Q,\, Q^*\in\mathcal{Q}^{k+1}} \lambda(f(Q^*\cap\gamma)).$$

Using this inequality and (11), we conclude by induction that, for any $m\in\mathbb{N}$,

$$\lambda(f(\gamma))\le \sum_{Q^*\in\mathcal{Q}^m}\lambda(f(Q^*\cap\gamma))\le K\sum_{Q^*\in\mathcal{Q}^m}\operatorname{diam} Q^*\le K\sqrt{2}\, r^m$$

and therefore $\lambda(f(\gamma))=0$.

Now let $f:\gamma^*\to\mathbb{R}$ be a Lipschitz function defined on a subarc γ^* of γ that satisfies (1) on γ^*. If γ^* is of the form $\gamma^*=\gamma\cap T$, where T is a square of the form $T=\psi_{Q^*}(Q_{\underline{i}})$ (where $\underline{i}$ is a finite sequence (possibly empty), $Q^*\in\bigcup_{m=0}^{\infty}\mathcal{Q}^m$, $\mathcal{Q}^0:=\{Q_\emptyset=[0,1]^2\}$ and $\psi_{Q_\emptyset}$ is the identity), then we deduce that f is constant using (7) and the same argument as above for $\gamma^*=\gamma$. A general γ^* can be written as a union of countably many subarcs of the above form and a σ-rectifiable set. Indeed, consider any point $x\in\gamma^*$ which is not an endpoint of γ^*. If $x\in\bigcup\mathcal{Q}^m$ for every m then $\{x\}=\bigcap_{m=1}^{\infty}Q^m$ for some $Q^m\in\mathcal{Q}^m$, and if m is large enough then $x\in Q^m\cap\gamma=\psi_{Q^m}(Q_\emptyset)\cap\gamma\subset\gamma^*$. If x is not of this form, then there is a largest m so that $x\in Q^*$ for some $Q^*\in\bigcup\mathcal{Q}^m$. Then $x\in\psi_{Q^*}(\widetilde{\gamma})$, therefore either $x=\psi_{Q^*}(Q_{i_1i_2\ldots})$ for an infinite sequence $i_1i_2\ldots$ (in which case $x\in\psi_{Q^*}(Q_{i_1i_2\ldots i_s})\cap\gamma\subset\gamma^*$ if s is sufficiently large), or $x\in\psi_{Q^*}(\widetilde{\gamma}_{\underline{i},j})$ for some $\underline{i},j$ (and $\psi_{Q^*}(\widetilde{\gamma}_{\underline{i},j})$ is a polygon, therefore it is rectifiable).

Thus, using also Lemma 4.1, we obtain $\lambda(f(\gamma^*))=0$ and so f is constant on γ^*. ∎

5. Notes on k-critical arcs. The following definition is used in [9].

DEFINITION 5.1. We say that a simple arc $\gamma\subset\mathbb{R}^n$ is *k-critical* if there exists a C^k function on $\mathbb{R}^n$ which is not constant on γ and $f'(x)=0$ for each $x\in\gamma$.

The following related notion was implicitly used in [10].

DEFINITION 5.2. We say that a simple arc $\gamma\subset\mathbb{R}^n$ is *k^*-critical* if there exists a C^k function on $\mathbb{R}^n$ which is not constant on γ and $f^{(j)}(x)=0$ for each $x\in\gamma$ and $1\le j\le k$.

Note that the Morse–Sard theorem implies that there is no k-critical arc in $\mathbb{R}^n$ for $k\ge n$. On the other hand, Whitney [10, p. 517] has showed how the (above) planar construction can be generalized to obtain an $(n-1)^*$-critical arc in $\mathbb{R}^n$.

The following notion is implicitly used in [1].

Definition 5.3. We say that a simple arc $\gamma \subset \mathbb{R}^n$ is k_*-*critical* if there exists a non-constant real function f on γ such that

$$\lim_{y \to x,\, y \in \gamma} \frac{|f(y) - f(x)|}{|y - x|^k} = 0 \quad \text{for each } x \in \gamma. \tag{14}$$

Note that Choquet [1] observed that no k_*-critical arc has σ-finite k-dimensional Hausdorff measure (cf. [5], where also some sufficient and some necessary conditions are presented).

Proceeding as in the proof of Theorem 2.2, we easily obtain a generalization.

Theorem 5.4. *Let $\gamma \subset \mathbb{R}^n$ be a k_*-critical arc which is a quasiarc. Then there exists a C^k function f on $\mathbb{R}^n$ such that f is a non-constant monotone function along γ, and $f'(x) = \cdots = f^{(k)}(x) = 0$ for every $x \in \gamma$. In particular*, *γ is k^*-critical and thus also k-critical.*

Modifying the above mentioned Whitney construction of an $(n-1)^*$-critical arc in $\mathbb{R}^n$ in the same way as in the proof of Theorem 3.1, we obtain the following result.

Theorem 5.5. *There exists an $(n-1)_*$-critical arc γ in $\mathbb{R}^n$ which is not* 1*-critical. Moreover*, *there exists no non-constant Lipschitz function f on γ which satisfies* (1).

We do not know whether each k-critical arc is k^*-critical.

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