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Centralizers of gap groups

by

Toshio Sumi (Fukuoka)

Dedicated to Professor Masaharu Morimoto on his 60th birthday

Abstract. A finite group G is called a gap group if there exists an $\mathbb{R}G$-module which has no large isotropy groups except at zero and satisfies the gap condition. The gap condition facilitates the process of equivariant surgery. Many groups are gap groups and also many groups are not. In this paper, we clarify the relation between a gap group and the structures of its centralizers. We show that a nonsolvable group which has a normal, odd prime power index proper subgroup is a gap group.

1. Introduction. Let G be a finite group not of prime power order and p a prime. In this paper we regard the trivial group as a p-group. We denote by $\mathcal{S}(G)$ the set of all subgroups of G, by $\mathcal{P}_p(G)$ the set of p-subgroups of G, by $O^p(G)$, called the *Dress subgroup* of type p, the smallest normal subgroup of G whose index is a power of p, possibly 1, and by $\mathcal{L}_p(G)$ the set of subgroups L of G which contain $O^p(G)$. Let $\pi(G)$ be the set of prime divisors of the order of G. Set

$$\mathcal{P}(G) = \bigcup_{p\in\pi(G)} \mathcal{P}_p(G) \quad \text{and} \quad \mathcal{L}(G) = \bigcup_{p\in\pi(G)} \mathcal{L}_p(G).$$

Let V be an $\mathbb{R}G$-module. We always assume that a module is finite-dimensional. We denote by $\mathcal{PH}(G)$ the set of all pairs (P, H) of subgroups of G such that $P<H\leq G$, $P\in\mathcal{P}(G)$, and define a function $d_V\colon \mathcal{PH}(G)\to\mathbb{Z}$ by

$$d_V(P, H) = \dim V^P - 2\dim V^H.$$

An $\mathbb{R}G$-module V is called $\mathcal{L}(G)$*-free* if $\dim V^L = 0$ for any $L \in \mathcal{L}(G)$. The group G is called a *gap group* if there exists an $\mathcal{L}(G)$-free $\mathbb{R}G$-module V such that $d_V(P, H) > 0$ for all $(P, H) \in \mathcal{PH}(G)$. Such an $\mathbb{R}G$-module V is called a *gap* $\mathbb{R}G$*-module*. The inequality arose from equivariant surgery theory [18, 21, 4, 11, 12]. Gap modules play a special role in equivariant surgery,

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and gap groups are the ones for which equivariant surgery techniques are most likely to have direct application. Laitinen and Pawałowski [10] showed a sufficient and necessary condition for a finite perfect group, which is a gap group, to act on a sphere S with just two fixed points, say x and y, such that the tangential representations $T_x(S)$ and $T_y(S)$ are not isomorphic. This problem is related to a question posed by Smith [24] and called the *Smith equivalence problem.* Many researchers attacked related surgery problems [22, 19, 20, 15, 1, 25, 3, 6, 5, 16]. In particular, Pawałowski and Solomon [17] obtained a result on Smith equivalence problem for a gap group G.

If $\mathcal{P}(G) \cap \mathcal{L}(G)$ is not empty, then $\dim V^P = 0$ for any $\mathcal{L}(G)$-free $\mathbb{R}G$-module V and any $P \in \mathcal{P}(G) \cap \mathcal{L}(G)$, and in particular G is not a gap group. For examples of gap groups, see [9, 14, 2, 13, 26, 27]. In this paper we give a necessary and sufficient condition for a group to be a gap group. To state the main theorems, we need some notations.

For an $\mathbb{R}G$- or $\mathbb{C}G$-module U, we denote by $U_{\mathcal{L}(G)}$ the maximal $\mathcal{L}(G)$-free submodule of U, which is isomorphic to

$$U/\sum_{p\in\pi(G)} U^{O^p(G)}.$$

Here, the maximality means that if a submodule W of U contains $U_{\mathcal{L}(G)}$ properly then W is not $\mathcal{L}(G)$-free. We define

$$\begin{aligned}\mathcal{PH}^2(G) = \big\{(P,H) \in \mathcal{PH}(G) \mid [H:P] = 2 = [O^2(G)H : O^2(G)P],\\ O^q(G)P = G \text{ for all odd prime } q\big\}.\end{aligned}$$

The $\mathcal{L}(G)$-free $\mathbb{R}G$-module $\mathbb{R}[G]_{\mathcal{L}(G)}$ has $d_{\mathbb{R}[G]_{\mathcal{L}(G)}}(P,H)$ nonnegative for $(P,H) \in \mathcal{PH}(G)$, and positive if $(P,H) \notin \mathcal{PH}^2(G)$ and $P \notin \mathcal{L}(G)$ [8].

Let K be a normal subgroup of G for which $K \geq O^2(G)$. The *centralizer* $C_G(x)$ of an element x of G is the set of elements of G which commute with x. A 2 power order element is called a 2-*element.*

For an element x of G, we denote by $\psi(x)$ the set of odd primes q such that there exists a subgroup N of G such that $x \in N$ and $O^q(N) \neq N$.

We define $E_2(G,K) \subset G \smallsetminus K$ to be the set of involutions (elements of order 2) x such that $|\psi(x)| > 1$ or $|\pi(C_G(x))| = |\pi(O^2(C_G(x)))| = 2$, and $E_4(G,K)$ to be the subset of 2-elements x of $G \smallsetminus K$ of order ≥ 4 for which $|\psi(x)| > 0$. Set $E(G,K) = E_2(G,K) \cup E_4(G,K)$. Note that $E_2(G,K) = \emptyset$ if $K > O^2(G)$ and $G/O^2(G)$ is cyclic.

We define sets $E_2^o(G,K)$, $E_4^o(G,K)$ and $E^o(G,K)$ as follows. The set $E_4^o(G,K)$ consists of all 2-elements x of $G \smallsetminus K$ of order > 2 such that $C_G(x)$ is not a 2-group. The set $E_2^o(G,K)$ consists of all involutions x of $G \smallsetminus K$ such that $|\pi(C_G(x))| \geq 3$ or $|\pi(O^2(C_G(x)))| \geq 2$. Set $E^o(G,K) = E_2^o(G,K) \cup E_4^o(G,K)$. Note that $E_2^o(G,K)$, $E_4^o(G,K)$ and $E^o(G,K)$ are subsets of $E_2(G,K)$, $E_4(G,K)$ and $E(G,K)$ respectively.

THEOREM 1.1. *Let G be a finite group such that $G/O^2(G)$ is a nontrivial cyclic group. Suppose that G has an index 2 subgroup K which is a gap group. Then the following claims are equivalent:*

(1) *G is a gap group.*
(2) *$E(G,K) \neq \emptyset$.*
(3) *$E^o(G,K) \neq \emptyset$.*
(4) *There exists an $\mathcal{L}(G)$-free $\mathbb{R}G$-module W such that $d_W(P,H) > 0$ for all $(P,H) \in \mathcal{PH}^2(G)$ with $H \not\leq K$.*

The technique of construction of a gap module is elementary. The existence of $\mathcal{L}(G)$-free $\mathbb{R}G$-modules W such that $d_W(P,H) \geq 0$ for (P,H) in $\mathcal{PH}(G)$ plays an important role. If we find such a module W and an $\mathcal{L}(G)$-free $\mathbb{R}G$-module V satisfying $d_V(P,H) > 0$ for any $(P,H) \in \mathcal{PH}(G)$ with $d_W(P,H) = 0$, then

$$V \oplus W^{\oplus \dim V+1}$$

is a gap $\mathbb{R}G$-module.

THEOREM 1.2. *Let G be a finite group such that $G/O^2(G)$ is a nontrivial cyclic group and $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$. Then G is a gap group if and only if*

$$E^o(G,K) \neq \emptyset \quad \text{and} \quad E^o(L, O^2(G)) \neq \emptyset,$$

where K is an index 2 subgroup of G, and L is a subgroup of G for which $L > O^2(G)$ and $[L : O^2(G)] = 2$.

THEOREM 1.3. *Let G be a finite group for which $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$. Then G is not a gap group if and only if there exists a pair (L,K) of subgroups of G such that $L > K \geq O^2(G)$, $[L : K] = 2$, and $E(L,K) = \emptyset$.*

As an application of the transfer homomorphism (Lemma 7.5), we have the following theorem.

THEOREM 1.4. *A nonsolvable group having a normal, odd (> 1) index subgroup is a gap group.*

Note that the condition that K has an odd (> 1) index cannot be omitted: for example, the projective general linear group $\mathrm{PGL}(2,7)$ is not a gap group.

This paper is organized as follows. We discuss the dimension of fixed point sets in Section 2, and inequalities for the order of normalizers and one equality for the order of centralizers in Section 3. In Section 4, for a finite group G not of prime power order and an index 2 subgroup K of G, we show that there exists an $\mathcal{L}(G)$-free $\mathbb{R}G$-module W such that $d_W(P,H) \geq 0$ for any $(P,H) \in \mathcal{PH}(G)$, and $d_W(P,H) > 0$ if $E(G,K) \cap H \neq E(G,K) \cap P$ and $P \notin \mathcal{L}(G)$. We discuss a gap group G for which $G/O^2(G)$ is cyclic by considering separately the cases $[G : O^2(G)] > 2$ and $[G : O^2(G)] = 2$ in Section 5. In Section 6, we study groups G for which $E^o(G,K) \neq \emptyset$. These sections prepare the proofs of Theorems 1.1–1.3, given in Section 7 together with the proof of Theorem 1.4.

2. Dimension of a fixed point set. Let G be a finite group, K an index 2 subgroup, and C a cyclic subgroup of G. The normalizer $N_G(C)$ acts on $(H\backslash G)^C$ on the right. Let P and H be subgroups of G such that $H > P$ and $[H : P] = 2$. Then H acts on the set $P\backslash G/C$ of double cosets by $h \cdot PgC = PhgC$ for $h \in H$.

LEMMA 2.1. *Let $(P, H) \in \mathcal{PH}(G)$ with $[H : P] = 2$. If C is not a subset of the conjugacy class $(P)_G = \bigcup_{g\in G} g^{-1}Pg$, then there is a natural injection from $(H\backslash G)^C$ to $(P\backslash G/C)^H$.*

Proof. If $(P\backslash G/C)^H$ is empty, then C is not a subgroup of $g^{-1}Hg$ for any $g \in G$, and thus $(H\backslash G)^C$ is also empty. Otherwise, we see that

$$\begin{aligned}(P\backslash G/C)^H &= \{PgC \mid g^{-1}Pg \cap C < g^{-1}Hg \cap C\}\\ &\supset \{PgC \mid g^{-1}Hg \geq C\} = \{Hg \mid g^{-1}Hg \geq C\} = (H\backslash G)^C\end{aligned}$$

since $PgC = HgC = HgCg^{-1}g = Hg$. ■

We say that an $\mathcal{L}(G)$-free $\mathbb{C}G$-module W satisfying $\dim_{\mathbb{C}} W^P - \dim_{\mathbb{C}} W^H > 0$ for $(P, H) \in \mathcal{PH}(G)$ is a *gap $\mathbb{C}G$-module*, similarly to a gap $\mathbb{R}G$-module. The complexification of a gap $\mathbb{R}G$-module is a gap $\mathbb{C}G$-module and the realification of a gap $\mathbb{C}G$-module is a gap $\mathbb{R}G$-module. So, we will construct a gap $\mathbb{C}G$-module instead of a gap $\mathbb{R}G$-module.

By Artin's theorem [23, §9.2 Corollary], any $\mathbb{R}G$-module can be written in the real representation ring as a linear combination with rational coefficients of realifications of $\mathbb{C}G$-modules induced from cyclic subgroups of G.

For a $\mathbb{C}G$-module V we let

$$d_{V,\mathbb{C}}(P, H) = \dim_{\mathbb{C}} V^P - 2\dim_{\mathbb{C}} V^H$$

for $(P, H) \in \mathcal{PH}(G)$. Let $\mathcal{PH}^2(G, K)$ be the subset of $\mathcal{PH}^2(G)$ consisting of all (P, H) for which $H \not\leq K$.

LEMMA 2.2 (cf. [13]). *Let C be a subgroup of G and $(P, H) \in \mathcal{PH}^2(G)$. Then*

$$\begin{aligned}d_{(\mathrm{Ind}_C^G W)_{\mathcal{L}(G)},\mathbb{C}}(P, H) = &\sum_{PgC\in(P\backslash G/C)^H} d_{W,\mathbb{C}}(C \cap g^{-1}Pg, C \cap g^{-1}Hg)\\ &- \sum_{O^2(G)PgC\in(O^2(G)P\backslash G/C)^{O^2(G)H}} d_{W,\mathbb{C}}(C \cap g^{-1}O^2(G)Pg, C \cap g^{-1}O^2(G)Hg)\end{aligned}$$

for a $\mathbb{C}C$-module W, and similarly

$$d_{(\mathrm{Ind}_C^G V)_{\mathcal{L}(G)}}(P,H) = \sum_{PgC\in(P\backslash G/C)^H} d_V(C\cap g^{-1}Pg, C\cap g^{-1}Hg) - \sum_{O^2(G)PgC\in(O^2(G)P\backslash G/C)^{O^2(G)H}} d_V(C\cap g^{-1}O^2(G)Pg, C\cap g^{-1}O^2(G)Hg)$$

for an $\mathbb{R}C$-module V.

Lemma 2.2 implies the following proposition.

PROPOSITION 2.3. *For $(P,H)\in\mathcal{PH}^2(G)$, if $O^2(G)C=G$ then*

$$d_{(\mathrm{Ind}_C^G \mathbb{R}[C])_{\mathcal{L}(G)}}(P,H) = |(P\backslash G/C)^H| - 1.$$

We now estimate $d_{(\mathrm{Ind}_C^G \mathbb{C}[C])_{\mathcal{L}(G)},\mathbb{C}}(P,H)$ for an irreducible $\mathbb{C}C$-module ξ over a cyclic subgroup C of G.

LEMMA 2.4. *Let C be a cyclic subgroup of G for which $C\not\le K$, x a generator of C, and ξ_j an irreducible $\mathbb{C}C$-module whose character sends x^k to* $\exp(2jk\pi\sqrt{-1}/|C|)$ $(0\le j<|C|)$. *Then*

$$d_{(\mathrm{Ind}_C^G \xi_j)_{\mathcal{L}(G)},\mathbb{C}}(P,H) = \begin{cases} -|(P\backslash G/C)^H|+1, & j=0,\\ |(P\backslash G/C)^H|-1, & j=|C|/2,\\ 0, & j\ne 0,|C|/2, \end{cases}$$

for any $(P,H)\in\mathcal{PH}^2(G,K)$.

Proof. Note that $O^2(G)C=G$ and $H\smallsetminus P=H\smallsetminus K$. Since $C\cap K=\langle x^2\rangle$, if $g^{-1}(H\smallsetminus P)g\cap C\ne\emptyset$ then $C\le g^{-1}Hg$. By Lemma 2.2,

$$\begin{aligned} d_{(\mathrm{Ind}_C^G \xi_j)_{\mathcal{L}(G)},\mathbb{C}}(P,H) &= \sum_{Hg\in(H\backslash G)^C} d_{\xi_j,\mathbb{C}}(K\cap C,C) - d_{\xi_j,\mathbb{C}}(K\cap C,C)\\ &= -\frac{|(P\backslash G/C)^H|-1}{|C|/2}\sum_{k=1}^{|C|/2}\chi_{\xi_j}(x^{2k-1})\\ &= -\frac{2(|(P\backslash G/C)^H|-1)}{|C|}\exp\Big(\frac{2j\pi\sqrt{-1}}{|C|}\Big)\sum_{k=1}^{|C|/2}\exp\Big(\frac{4j(k-1)\pi\sqrt{-1}}{|C|}\Big)\\ &= \begin{cases} -|(P\backslash G/C)^H|+1, & j=0,\\ |(P\backslash G/C)^H|-1, & j=|C|/2,\\ 0, & j\ne 0,|C|/2. \end{cases} \end{aligned}$$ ∎

3. Cardinality of the fixed point set. In this section, we show one equality for the orders of centralizers and two inequalities for the orders of normalizers.

Let G be a finite group, K an index 2 subgroup of G, and $\mathcal{A}$ the set of conjugacy classes represented by elements of $G \setminus K$. We denote by $(x)_G$, or simply (x), the conjugacy class of x in G. By counting the number of elements of G contained in each conjugacy class, we have

$$\sum_{(g)\in\mathcal{A}} \frac{|G|}{|C_G(g)|} = |G \setminus K| = \frac{|G|}{2},$$

and so

$$\sum_{(g)\in\mathcal{A}} \frac{2}{|C_G(g)|} = 1. \tag{3.1}$$

Let $\{C_j \mid j \in J\}$ be the set of representatives of all conjugacy classes in G of cyclic subgroups C of G for which $C \not\leq K$. Let $J(2)$ be the subset of J containing all $j \in J$ such that C_j is a 2-group. For each $j \in J$, we put

$$s_j = |N_G(C_j)|/|C_j|,$$

where $N_G(C_j)$ is the normalizer of the cyclic group C_j:

$$N_G(C_j) := \{a \in G \mid a^{-1}C_j a = C_j\}.$$

Lemma 3.1. *We have*

$$\sum_{j\in J} \frac{2\varphi(|C_j|)}{|N_G(C_j)|} = 1,$$

where φ is the Euler function; in particular,

$$\sum_{j\in J(2)} s_j^{-1} \leq 1,$$

and equality holds if and only if $J = J(2)$.

Proof. For a cyclic group $C = \langle g\rangle \not\leq K$, the group $N_G(C)/C_G(g)$ acts freely on the set of generators of C consisting of $\varphi(|C|)$ elements. Then there are just $\varphi(|C|)/|N_G(C)/C_G(g)|$ conjugacy classes $(x)_G$ such that $\langle x\rangle$ is conjugate to C. Therefore we have

$$1 = \sum_{(y)\in\mathcal{A}} \frac{2}{|C_G(y)|} = \sum_{j\in J} \frac{2}{|C_G(C_j)|} \cdot \frac{\varphi(|C_j|)}{|N_G(C_j)/C_G(C_j)|} = \sum_{j\in J} \frac{2\varphi(|C_j|)}{|N_G(C_j)|}.$$

If $j \in J(2)$ then $2\varphi(|C_j|) = |C_j|$. Thus,

$$\sum_{j\in J(2)} \frac{|C_j|}{|N_G(C_j)|} \leq 1.$$

Now it is clear that the equality $\sum_{j\in J(2)} s_j^{-1} = 1$ implies $J = J(2)$. ∎

LEMMA 3.2. *Let H be a subgroup of G for which $HK = G$. Then*

$$\sum_{j\in J} s_j^{-1}|(H\backslash G)^{C_j}| \geq 1.$$

Equality holds if and only if $J = J(2)$.

Proof. We consider $N_G(C)$-orbits of $(H\backslash G)^C$. Let

$$(H\backslash G)^C = \coprod_i Hg_iN_G(C).$$

Note that $g_iCg_i^{-1} \leq H$. Since the isotropy subgroup at g_i is $N_G(C) \cap {g_i}^{-1}Hg_i$, we have

$$\begin{aligned}(3.2)\qquad |(H\backslash G)^C| &= \sum_i |Hg_iN_G(C)| = \sum_i \frac{|N_G(C)|}{|N_G(C)\cap {g_i}^{-1}Hg_i|}\\ &= \sum_i \frac{|N_G(C)|}{|N_H(g_iC{g_i}^{-1})|}\\ &= \sum_{HgN_G(C)\in (H\backslash G)^C/N_G(C)} \frac{|N_G(C)|}{|N_H(gCg^{-1})|}.\end{aligned}$$

Let $\{C'_j \mid j \in J_H\}$ be the set of representatives of all conjugacy classes in H of cyclic subgroups C' of H for which $C'K = G$. For a cyclic group C_j ($j \in J$), if C'_i ($i \in J_H$) is conjugate to C_j in G, say $C'_i = gC_jg^{-1}$ for some $g \in G$, then $Hg \in (H\backslash G)^{C_j}$ since $gC_jg^{-1} \leq H$. Combining this with (3.2), we have

$$\begin{aligned}(3.3)\qquad \sum_{i\in J_H} \frac{|C'_i|}{|N_H(C'_i)|} &= \sum_{j\in J}\ \sum_{HgN_G(C_j)\in (H\backslash G)^{C_j}/N_G(C_j)} \frac{|C_j|}{|N_H(gC_jg^{-1})|}\\ &= \sum_{j\in J} s_j^{-1}|(H\backslash G)^{C_j}|.\end{aligned}$$

Let $\{y_i \in H \mid i \in I\}$ be a set of representatives of all conjugacy classes $(y_i)_H$ in H of elements y_i outside K. By applying Lemma 3.2 for $(H, H\cap K)$ instead of (G, K), we have

$$(3.4)\qquad \sum_{j\in J_H} \frac{2\varphi(|C'_j|)}{|N_H(C'_j)|} = 1.$$

Note that $2\varphi(n) \leq n$ for every even integer n, and equality implies that n is a power of 2. Thus by (3.3) and (3.4),

$$\sum_{j\in J} s_j^{-1}|(H\backslash G)^{C_j}| \geq 1,$$

and equality holds if and only if $J = J(2)$. ∎

Next we obtain a result for $E^o(G,K)=\emptyset$.

PROPOSITION 3.3. *Let $G_{\{2\}}$ be a Sylow 2-subgroup of G. Suppose that*

$$E^o(G,K)=\emptyset.$$

If $x,y\in G_{\{2\}}\smallsetminus K$ are conjugate in G, then they are also conjugate in $G_{\{2\}}$.

Proof. First suppose that $C_G(x)$ is a 2-group for any $x\in G\smallsetminus K$. Let $S\subset G_{\{2\}}$ consist of representatives x of elements (x) of $\mathcal{A}$ such that $C_G(x)$ is a subgroup of $G_{\{2\}}$. Then

$$\sum_{x\in S}\frac{2}{|C_{G_{\{2\}}}(x)|}=1$$

by (3.1), since $C_G(x)=C_{G_{\{2\}}}(x)$. By applying again (3.1) for $G_{\{2\}}$ we find that distinct elements x and y of S are not conjugate in $G_{\{2\}}$.

Next, suppose that there exists an $x\in G\smallsetminus K$ such that $C_G(x)$ is not a 2-group. Let $\mathcal{A}_1$ be the set of conjugacy classes (x) of elements $x\in G\smallsetminus K$ of order not divisible by 4, and $\mathcal{A}_2$ the set of conjugacy classes of elements of order divisible by 4. Let $\mathcal{A}_1^0$ be the subset of $\mathcal{A}_1$ consisting of (x) represented by involutions x. We have assumed that $\mathcal{A}_1\neq\mathcal{A}_1^0\neq\emptyset$. Note that $C_G(x)=C_{C_G(y)}(x)$ for every involution y of G and every element x of G with $y\in\langle x\rangle$. Then

$$\sum_{(x)\in\mathcal{A}}\frac{2}{|C_G(x)|}=\sum_{(y)\in\mathcal{A}_1^0}\sum_{\substack{(x)\in\mathcal{A}_1\\ y\in\langle x\rangle}}\frac{2}{|C_{C_G(y)}(x)|}+\sum_{(x)\in\mathcal{A}_2}\frac{2}{|C_G(x)|}. \tag{3.5}$$

Let $y\in G$ with $(y)\in\mathcal{A}_1^0$. If two elements x and x' with $(x),(x')\in\mathcal{A}_1$ and $y\in\langle x\rangle\cap\langle x'\rangle$ are conjugate in G, then they are conjugate in $C_G(y)$. Since $E_2^o(G,K)=\emptyset$, $|x^2|$ is a power of an odd prime, possibly 1, for $(x)\in\mathcal{A}_1$. Noting that $C_{C_G(y)}(x)=C_{C_G(y)}(x^2)$, we have

$$\sum_{\substack{(x)\in\mathcal{A}_1\\ y\in\langle x\rangle}}\frac{2}{|C_{C_G(y)}(x)|}=\sum_{(z)\in\mathcal{B}(y)}\frac{2}{|C_{C_G(y)}(z)|}, \tag{3.6}$$

where $\mathcal{B}(y)$ is the set of conjugacy classes in $C_G(y)$ which are represented by elements of $O^2(C_G(y))$. Since $O^2(C_G(y))$ is a normal, prime power order subgroup of $C_G(y)$, we have

$$\sum_{(z)\in\mathcal{B}(y)}\frac{|C_G(y)|}{|C_{C_G(y)}(z)|}=|O^2(C_G(y))|. \tag{3.7}$$

Consequently, by (3.1) and (3.5)–(3.7) together with $E_4^o(G,K)=\emptyset$, we have

$$\sum_{(y)\in\mathcal{A}_1^0}\frac{2}{|C_G(y)_{\{2\}}|}+\sum_{(x)\in\mathcal{A}_2}\frac{2}{|C_G(x)_{\{2\}}|}=1,$$

where $C_G(z)_{\{2\}}$ is a Sylow 2-subgroup of $C_G(z)$ for $z = x, y$. By comparing this and (3.1) for $G_{\{2\}}$, if two elements of $G_{\{2\}} \smallsetminus K$ are conjugate in G then they are also conjugate in $G_{\{2\}}$. This completes the proof. ∎

4. Properties involving $E(G,K)$ and $E^o(G,K)$. Let G be a finite group not of prime power order and K an index 2 subgroup of G (so G has even order). Recall that $\dim W^P = 0$ if $P \in \mathcal{L}(G)$ for any $\mathcal{L}(G)$-free $\mathbb{R}G$-module W. We denote by $\mathcal{PH}_0(G)$ (resp. $\mathcal{PH}_0(G,K)$) the subset of $\mathcal{PH}(G)$ (resp. $\mathcal{PH}(G,K)$) consisting of all pairs (P,H) for which $P \notin \mathcal{L}(G)$, and by $\mathcal{PH}_0^2(G)$ (resp. $\mathcal{PH}_0^2(G,K)$) the intersection of $\mathcal{PH}_0(G)$ and $\mathcal{PH}^2(G)$ (resp. $\mathcal{PH}^2(G,K)$). We say that an $\mathbb{R}G$-module V is *nonnegative* if $d_V(P,H) \geq 0$ for all $(P,H) \in \mathcal{PH}(G)$. Note that $\mathrm{Ind}_H^G(\mathbb{R}[H]_{\mathcal{L}(H)})$ is a nonnegative $\mathbb{R}G$-module for any $H \leq G$.

Let $\mathcal{B}_1$ be the set of conjugacy classes $(x)_G$ represented by involutions x of $G \smallsetminus O^2(G)$ such that $C_G(x)$ is not a 2-group, and $\mathcal{B}_2$ the set of conjugacy classes represented by 2-elements x of $G \smallsetminus O^2(G)$ of order > 2 such that $C_G(x)$ is not a 2-group. Let $Q(x) = O^2(C_G(x))\langle x\rangle$ for $(x) \in \mathcal{B}_1$ and $Q(x)$ a subgroup $L\langle x\rangle$ of G not of 2 power order for $(x) \in \mathcal{B}_2$, where p is taken as an odd prime dividing $|C_G(x)|$ so that L is a nontrivial Sylow p-subgroup of $C_G(x)$. Put

$$W(G) = \bigoplus_{(x)\in\mathcal{B}_1\cup\mathcal{B}_2} \mathrm{Ind}_{Q(x)}^G(\mathbb{R}[Q(x)]_{\mathcal{L}(Q(x))}) \oplus \mathbb{R}[G]_{\mathcal{L}(G)}. \tag{4.1}$$

This $\mathbb{R}G$-module $W(G)$ is nonnegative.

Proposition 4.1. *$d_{W(G)}(P,H)$ is positive if $(P,H) \in \mathcal{PH}_0(G)$ is such that $(P,H) \notin \mathcal{PH}_0^2(G)$, or $(H \smallsetminus P) \cap E(G,K) \neq \emptyset$, or $P \not\geq O^2(C_G(x))$ for some involution x of $H \smallsetminus P$.*

Proof. For $(P,H) \in \mathcal{PH}_0(G) \smallsetminus \mathcal{PH}_0^2(G)$, $d_{\mathbb{R}[G]_{\mathcal{L}(G)}}(P,H)$ is positive. For $(x) \in \mathcal{B}_2$, $d_{\mathrm{Ind}_{Q(x)}^G(\mathbb{R}[Q(x)]_{\mathcal{L}(Q(x))})}(P,H)$ is positive if $(P,H) \in \mathcal{PH}_0^2(G)$ and $H \cap (x) \neq P \cap (x)$ (cf. [27, Proposition 4.1]). Also we see that for $(x) \in \mathcal{B}_1$, $d_{\mathrm{Ind}_{Q(x)}^G(\mathbb{R}[Q(x)]_{\mathcal{L}(Q(x))})}(P,H)$ is positive if $(P,H) \in \mathcal{PH}_0^2(G)$ and $H \cap (x) \neq P \cap (x)$ (cf. [27, Lemma 4.3]). ∎

5. The cases $[G : O^2(G)] > 2$ and $[G : O^2(G)] = 2$. Let G be a finite group not of prime power order for which $G/O^2(G)$ is nontrivial cyclic and K an index 2 subgroup of G.

First, suppose that $K > O^2(G)$, since $O^2(G) \leq K < G$. Note that $E_2^o(G,K) = \emptyset$. We will show that $E_4^o(G,K) \neq \emptyset$ if and only if there exists an $\mathcal{L}(G)$-free $\mathbb{R}G$-module U such that $d_U(P,H) > 0$ for all $(P,H) \in \mathcal{PH}_0(G,K)$.

Let $W(G)$ be the $\mathcal{L}(G)$-free nonnegative $\mathbb{R}G$-module in (4.1). Let

$$U_0 = \bigoplus_{j\in J(2)} \big((\mathrm{Ind}_{C_j}^G \mathbb{R}[C_j])_{\mathcal{L}(G)}\big)^{\oplus n s_j^{-1}} \quad \text{and} \quad U = U_0 \oplus W(G)^{\oplus m},$$

where $n = \prod_{j\in J(2)} s_j \in \mathbb{Z}$ and $m = \dim U_0 + 1 \in \mathbb{Z}$.

THEOREM 5.1. *If $E_4(G,K) \neq \emptyset$ then*

$$d_U(P,H) > 0 \quad \textit{for } (P,H) \in \mathcal{PH}_0(G,K).$$

Proof. Let $(P,H) \in \mathcal{PH}_0(G,K)$. If $d_{W(G)}(P,H) > 0$ then

$$\begin{aligned} d_U(P,H) &= (\dim U_0^P - \dim U_0^H) + m\, d_{W(G)}(P,H) - \dim U_0^H \\ &\geq m - \dim U_0 > 0. \end{aligned}$$

Suppose that $d_{W(G)}(P,H) = 0$. Then $[H:P] = [O^2(G)H : O^2(G)P] = 2$ and $(H \smallsetminus P) \cap E(G,K) = \emptyset$. The latter implies that $(P\backslash G/C_j)^H = \emptyset$ for $j \in J \smallsetminus J(2)$. Noting that $J(2) \neq J$, we have

$$\begin{aligned} d_U(P,H) &= n \sum_{j\in J(2)} s_j^{-1}\big(|(P\backslash G/C_j)^H| - 1\big) \\ &= n\Big(\sum_{j\in J} s_j^{-1}|(P\backslash G/C_j)^H| - \sum_{j\in J(2)} s_j^{-1}\Big) \\ &\geq n\Big(\sum_{j\in J} s_j^{-1}|(H\backslash G)^{C_j}| - \sum_{j\in J(2)} s_j^{-1}\Big) > 0 \end{aligned}$$

by Lemmas 2.1, 3.1 and 3.2. ∎

On the other hand, we have the following

LEMMA 5.2. *Let V be an $\mathcal{L}(G)$-free $\mathbb{R}G$-module such that $d_V(P,H) \geq 0$ for all $(P,H) \in \mathcal{PH}_0^2(G,K)$. If $E_4^o(G,K)$ is empty then $d_V(C_j, C_j \cap K) = 0$ for all $j \in J$.*

Proof. For an $\mathcal{L}(G)$-free $\mathbb{R}G$-module V' which is the realification of the complexification W of V, we have $d_{V'}(P,H) = 2d_{W,\mathbb{C}}(P,H) = 2d_V(P,H)$. Thus it suffices to show that if $E_4^o(G,K)$ is empty then $d_{W,\mathbb{C}}(C_j, C_j\cap K) = 0$ for all $j \in J$. There exist a rational number n_j, a $\mathbb{C}C_j$-module ξ_j for $j \in J$, a rational number q, and a $\mathbb{C}K$-module η such that

$$W = \sum_{j\in J} n_j \,\mathrm{Ind}_{C_j}^G \xi_j + q\, \mathrm{Ind}_K^G \eta$$

in $\mathbb{Q} \otimes_{\mathbb{Z}} R(G)$. Since V is $\mathcal{L}(G)$-free, we have

$$W = W_{\mathcal{L}(G)} = \sum_{j\in J} n_j (\mathrm{Ind}_{C_j}^G \xi_j)_{\mathcal{L}(G)} + q(\mathrm{Ind}_K^G \eta)_{\mathcal{L}(G)}.$$

For $(P, H) \in \mathcal{PH}_0^2(G, K)$, H is a 2-group and we must have

$$d_{W,\mathbb{C}}(P, H) = \sum_{j \in J} n_j t_j \big(|(P \backslash G/C_j)^H| - 1\big) \geq 0$$

for some integers t_j by Lemma 2.4. Note that $|((C_i \cap K)\backslash G/C_j)^{C_i}| \neq 0$ if and only if $i = j$, and $|((C_j \cap K)\backslash G/C_j)^{C_j}| = s_j$. Since $E_4^o(G, K) = \emptyset$ implies $J = J(2)$, we have

$$\begin{aligned}\sum_{i \in J} s_i^{-1} d_{W,\mathbb{C}}(C_i \cap K, C_i) &= \sum_{j \in J} n_j t_j \sum_{i \in J} s_i^{-1} \big(|((C_i \cap K)\backslash G/C_j)^{C_i}| - 1\big) \\ &= \sum_{j \in J} n_j t_j \Big(1 - \sum_{i \in J} s_i^{-1}\Big) = 0\end{aligned}$$

by Lemma 3.1. Therefore $d_{W,\mathbb{C}}(C_i \cap K, C_i) = 0$ for $i \in J$. ∎

THEOREM 5.3. *If $E_4^o(G, K) = \emptyset$ then there exists no $\mathcal{L}(G)$-free $\mathbb{R}G$-module V such that $d_V(P, H) > 0$ for all $(P, H) \in \mathcal{PH}_0(G, K)$.*

Proof. Assume that, on the contrary, there is an $\mathcal{L}(G)$-free $\mathbb{R}G$-module V such that $d_V(P, H) > 0$ for all $(P, H) \in \mathcal{PH}_0(G, K)$. Then $d_V(C_i, C_i \cap K) = 0$ by Lemma 5.2 and so $C_i \cap K \in \mathcal{L}(G)$ for all $i \in J$. We show that this leads to a contradiction. Note that J is not an empty set. The facts that $C_i \not\leq K$ and $K > O^2(G)$ imply $C_i \cap K \not\leq O^2(G)$. Since $C_i \neq C_i \cap K$, $C_i \cap K \neq O^p(G)$ for all odd primes p. Therefore $C_i \cap K \notin \mathcal{L}(G)$. ∎

PROPOSITION 5.4. *Let L_1 and L_2 be subgroups of G for which $[L_2 : L_1] = 2$ and $L_1 > O^2(G)$. If $E_4(G, K) \neq \emptyset$ then $E_4(L_2, L_1) \neq \emptyset$, and if $E_4^o(G, K) \neq \emptyset$ then $E_4^o(L_2, L_1) \neq \emptyset$.*

Proof. Suppose that there exists an element x of $G \setminus K$ not of prime power order. Then $x^{[G:L_2]}$ is an element of $L_2 \setminus L_1$ not of prime power order. Therefore, $E_4^o(G, K) \neq \emptyset$ implies $E_4^o(L_2, L_1) \neq \emptyset$. Since $\psi(y)$ is a subset of $\psi(z)$ for $z \in \langle y \rangle$, $E_4(G, K) \neq \emptyset$ implies $E_4(L_2, L_1) \neq \emptyset$. ∎

PROPOSITION 5.5. *Suppose that $|G/O^2(G)| = 2^t > 2$. Let G_a for $0 \leq a \leq t$ be subgroups of G for which $G_0 = O^2(G)$, $G_s < G_{s+1}$ and $[G_{s+1} : G_s] = 2$ for $0 \leq s \leq t-1$, and $G_t = G$. Then $E(G, G_0) = \emptyset$ if and only if $E_4(G_2, G_1) = E(G_1, G_0) = \emptyset$, and $E^o(G, G_0) = \emptyset$ if and only if $E_4^o(G_2, G_1) = E^o(G_1, G_0) = \emptyset$.*

Proof. By Proposition 5.4, it is easy to see that $E_4(G, G_0) = \emptyset$ is equivalent to $E_4(G_2, G_1) = \emptyset = E_4(G_1, G_0)$, and $E_4^o(G, G_0) = \emptyset$ is equivalent to $E_4^o(G_2, G_1) = \emptyset = E_4^o(G_1, G_0)$. We clearly see that $E_2(G, G_0) = \emptyset$ (resp. $E_2^o(G, G_0) = \emptyset$) implies $E_2(G_1, G_0) = \emptyset$ (resp. $E_2^o(G_1, G_0) = \emptyset$). Since $E_2(G_1, G_0) = \emptyset$ implies $E_2(G, G_0) = \emptyset$, we have $E(G, G_0) = \emptyset$ if and only if $E_4(G_2, G_1) = E(G_1, G_0) = \emptyset$.

We now show that $E_2^o(G_1, G_0) = \emptyset$ implies $E_2^o(G, G_0) = \emptyset$. Suppose that $E_2^o(G_1, G_0) = \emptyset$. Let h be an involution of $G \smallsetminus G_0$. Then $h \in G_1$. Since $O^2(C_{G_1}(h))$ has odd prime power order, $O^2(C_{G_1}(h))$ is the set of all odd order elements of $C_{G_1}(h)$. Every odd order element of G belongs to G_0. Therefore, the set of all odd order elements of $C_G(h)$ is $O^2(C_{G_1}(h))$ and so $O^2(C_{G_1}(h))$ is a normal subgroup of $C_G(h)$, which means that $E_2^o(G, G_0) = \emptyset$. Therefore, $E_2^o(G_1, G_0) = \emptyset$ implies $E_2^o(G, G_0) = \emptyset$. ∎

Second, suppose that $K = O^2(G)$. The group G is called an *almost gap group* if there exists an $\mathcal{L}(G)$-free $\mathbb{R}G$-module V such that $d_V(P, H) > 0$ for all $(P, H) \in \mathcal{PH}_0(G)$. Such an $\mathbb{R}G$-module V is called an *almost gap* $\mathbb{R}G$*-module*. Clearly if $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$ then an almost gap group G is a gap group.

Let $W(G)$ be the $\mathbb{R}G$-module in (4.1). Let J_1 be the subset of J consisting of $j \in J$ such that a generator of C_j does not belong to $E(G, K)$, and J_2 the subset consisting of $j \in J_1$ such that C_j is a 2-group. Then $J_2 \subseteq J_1 \subseteq J$. Let

$$U_0 = \bigoplus_{j \in J_2} \big((\mathrm{Ind}_{C_j}^G \mathbb{R}[C_j])_{\mathcal{L}(G)}\big)^{\oplus n s_j^{-1}} \quad \text{and} \quad U = U_0 \oplus W(G)^{\oplus m},$$

where $n = \prod_{j \in J_2} s_j \in \mathbb{Z}$ and $m = \dim U_0 + 1 \in \mathbb{Z}$.

THEOREM 5.6. *If $E(G, K) \neq \emptyset$ then U is an almost gap $\mathbb{R}G$-module and G is an almost gap group.*

Proof. Let $(P, H) \in \mathcal{PH}_0(G)$. We will show that $d_U(P, H) > 0$.

If $d_{W(G)}(P, H) > 0$ then

$$\begin{aligned} d_U(P, H) &= (\dim V_0^P - \dim V_0^H) + m\, d_{W(G)}(P, H) - \dim V_0^H \\ &\geq m - \dim V_0 > 0. \end{aligned}$$

Suppose now that $d_{W(G)}(P, H) = 0$. It follows from $(H \smallsetminus P) \cap E(G, K) = \emptyset$ that $(P\backslash G/C_j)^H = \emptyset$ for $j \in J \smallsetminus J_1$. Note that

$$d_U(P, H) = n \sum_{j \in J_2} s_j^{-1} \big(|(P\backslash G/C_j)^H| - 1\big).$$

We consider three cases.

The first case is that H is not a 2-group. Since $[H : P] = 2$, all involutions of H are conjugate. There is a unique $k \in J_2$ such that C_k is subconjugate to H. Note that C_k is an order 2 group. By [28, Lemma 4.4],

$$|(P\backslash G/C_k)^H| = s_k.$$

Therefore,

$$d_U(P, H) = n\Big(1 - \sum_{j \in J_2} s_j^{-1}\Big) > 0$$

by Lemma 3.1.

The second case is that H is a 2-group of order > 2. Since $(P\backslash G/C_j)^H = \emptyset$ for $j \in J \setminus J_2$, we have

$$d_U(P,H) \geq n\Big(\sum_{j\in J} s_j^{-1}|(H\backslash G)^{C_j}| - \sum_{j\in J_2} s_j^{-1}\Big) > 0$$

by Lemmas 2.1, 3.1 and 3.2.

The third case is that H is an order 2 cyclic group. Let h be a generator of H. Then $O^2(C_G(h)) \leq P = \{1\}$ since $d_{W(G)}(P,H) = 0$. Thus, $(P\backslash G/C_j)^H \neq \emptyset$ implies that C_j and H are conjugate. We can apply the argument of the first case and get $d_U(P,H) > 0$. Therefore, U is an almost gap $\mathbb{R}G$-module. ∎

6. More properties involving $E^o(G,K)$. In this section, we assume that G is a finite group not of prime power order and K an index 2 subgroup of G.

LEMMA 6.1. *Let V be an $\mathcal{L}(G)$-free $\mathbb{R}G$-module such that $d_V(P,H) \geq 0$ for all $(P,H) \in \mathcal{PH}_0^2(G,K)$. If $E^o(G,K)$ is empty then $d_V(P_j,H_j) = 0$ for all $j \in J(2)$, where*

$$P_j = O^2(C_G(C_j))(C_j \cap K) \quad \textit{and} \quad H_j = O^2(C_G(C_j))C_j.$$

Proof. Let W be an $\mathcal{L}(G)$-free $\mathbb{C}G$-module such that $d_W(P,H) \geq 0$ for all $(P,H) \in \mathcal{PH}_0^2(G,K)$. Then it suffices to show that if $E^o(G,K)$ is empty then $d_W(P_j,H_j) = 0$ for all $j \in J(2)$. We write W as

$$\sum_{i\in J} n_i \operatorname{Ind}_{C_i}^G \xi_i + q \operatorname{Ind}_K^G \eta$$

in $\mathbb{Q} \otimes_{\mathbb{Z}} R(G)$, where $n_i, q \in \mathbb{Q}$, ξ_i is a $\mathbb{C}C_i$-module for $i \in J$, and η is a $\mathbb{C}K$-module. Since W is $\mathcal{L}(G)$-free, we have

$$W = \sum_{j\in J} n_j (\operatorname{Ind}_{C_j}^G \xi_j)_{\mathcal{L}(G)} + q(\operatorname{Ind}_K^G \eta)_{\mathcal{L}(G)}.$$

Fix a Sylow 2-subgroup $G_{\{2\}}$ of G. For each $j \in J(2)$, we may assume that C_j is a subgroup of $G_{\{2\}}$ and furthermore so is $N_G(C_j)$ if $N_G(C_j)$ is a 2-group without loss of generality. Moreover, we may assume that for $i \in J$ and $j \in J(2)$, if C_j is subconjugate to C_i then $C_j \leq C_i$. For each $j \in J(2)$, we let

$$s'_j = |N_{G_{\{2\}}}(C_j)|/|C_j|.$$

Then $[H_j : P_j] = [O^2(G)H_j : O^2(G)P_j] = 2$. If C_j is a 2-group of order > 2 then $C_G(C_j)$ is also a 2-group, since $E_4^o(G,K) = \emptyset$. If C_j is an order 2 group then $O^2(C_G(C_j)) \in \mathcal{P}(G)$, since $E_2^o(G,K) = \emptyset$. Therefore $(P_j,H_j) \in \mathcal{PH}(G)$ for $j \in J(2)$.

Let $P_jgC_i \in (P_j\backslash G/C_i)^{H_j}$. Then $g^{-1}P_jg \cap C_i$ is an index 2 subgroup of $g^{-1}H_jg \cap C_i$. We show that $C_j \le C_i$ and $(g^{-1}P_jg \cap C_i, g^{-1}H_jg \cap C_i) = (K \cap C_i, C_i)$ by considering two cases.

The first case is where $|C_j|$ is divisible by 4. In this case, $(P_j, H_j) = (K \cap C_j, C_j)$. Moreover, $(P_j\backslash G/C_i)^{H_j}$ is not empty if and only if $i = j$ and

$$(P_j\backslash G/C_j)^{H_j} = P_j\backslash N_G(C_j)/C_j = N_G(C_j)/C_j = N_{G_{\{2\}}}(C_i)/C_i.$$

In particular, we have $g^{-1}H_jg \cap C_j = C_j$, $g^{-1}P_jg \cap C_j = K \cap C_j$ and $|(P_j\backslash G/C_j)^{H_j}| = s'_j$.

The other case is where $|C_j| = 2$. Then

$$(P_j, H_j) = \big(O^2(C_G(C_j)), O^2(C_G(C_j))C_j\big).$$

Since $g^{-1}P_jg$ has odd order, for $i \in J$, if $|(P_j\backslash G/C_i)^{H_j}| \neq \emptyset$ then C_j is subconjugate to C_i. Let h_j be the generator of C_j. For $P_jgC_i \in (P_j\backslash G/C_i)^{H_j}$, there exist $p \in P_j$ and $d \in C_i$ such that $ph_j = gdg^{-1}$. Since $|d|/2$ is odd, $p^{|d|/2}h_j = gh_jg^{-1}$. Since $(p^{|d|/2})^2$ is the identity element and P_j has odd order, $p^{|d|/2}$ itself is the identity element. Thus $h_j = gh_jg^{-1}$, which implies that $g \in C_G(C_j)$. Therefore, $(g^{-1}P_jg \cap C_i, g^{-1}H_jg \cap C_i) = (K \cap C_i, C_i)$ and

$$(P_j\backslash G/C_i)^{H_j} = P_j\backslash N_G(C_j)/C_i.$$

In particular, $|(P_j\backslash G/C_i)^{H_j}| = |N_G(C_j)|/|O^2(N_G(C_j))C_j| = s'_j$.

For each $i \in J(2)$, let $I(i)$ be the set of all $k \in J$ such that $C_i \le C_k$ and define

$$m_i = \sum_{k\in J(i)} n_k d_{\xi_k}(K \cap C_k, C_k).$$

Since J is a disjoint sum of $I(j)$ for $j \in J(2)$, we have

$$d_W(P_i, H_i) = s'_i m_i - \sum_{j\in J(2)} m_j$$

for each $i \in J(2)$. Then

$$\sum_{i\in J(2)} {s'_i}^{-1} d_W(P_i, H_i) = \sum_{i\in J(2)} m_i\Big(1 - \sum_{j\in J(2)} {s'_j}^{-1}\Big) = 0$$

by Proposition 3.3. Therefore $d_W(P_i, H_i) = 0$ for $i \in J(2)$. ∎

THEOREM 6.2. *If G is an almost gap group then either $E^o(G, K) \neq \emptyset$, or $K \in \mathcal{P}(G)$ and G is isomorphic to $K \times D_2$, where D_2 is an order 2 group.*

Proof. Assume that $E^o(G, K) = \emptyset$. If K is an odd prime power order group then $K \times D_2$ is an almost gap group. Thus we may assume that if K is an odd prime power order group then G is not isomorphic to $K \times D_2$. Towards a contradiction, we assume that V is an almost gap $\mathbb{R}G$-module. By Lemma 6.1, $P_i \in \mathcal{L}(G)$ for $i \in J(2)$, where

$$P_i = O^2(C_G(C_i))(C_i \cap K) \quad \text{and} \quad H_i = O^2(C_G(C_i))C_i.$$

We show that this leads to a contradiction. Recall that G is not of prime power order. Suppose that H_i is a 2-group. Then $P_i < H_i \le O^p(G)$ and thus $P_i < O^p(G)$ for any odd prime p. Since P_i is also a 2-group, we have $P_i < O^2(G)$. Hence $P_i \notin \mathcal{L}(G)$, a contradiction. Suppose now that H_i is not a 2-group. Note that C_i is isomorphic to an order 2 group. Since P_i has odd prime power order > 1, we have $P_i < O^p(G)$ for any odd prime p. If $P_i = O^2(G)$ then $P_i = O^2(C_G(C_i)) = K$ and $G = H_i$, which is isomorphic to the direct product group of K with an order 2 cyclic group, a contradiction. ∎

7. Proofs of main theorems. Let G be a finite group such that $G/O^2(G)$ is a nontrivial cyclic group. If $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$ then the subgroup L of G with $L > O^2(G)$ and $[L : O^2(G)] = 2$ is not isomorphic to a direct product group of an odd prime power order group and an order 2 cyclic group, since $O^2(L) = O^2(G)$. We use this fact in the proofs of Theorems 1.1 and 1.2.

In Theorem 1.1, the assumption that K is a gap group implies that $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$. If G is a gap group then $O^2(G)$ is an almost gap group. Thus, we slightly extend Theorem 1.1 as follows.

THEOREM 7.1. *Let G be a finite group such that $G/O^2(G)$ is a nontrivial cyclic group. Suppose that G has an index 2 subgroup K which is an almost gap group not of prime power order. Then the following claims are equivalent:*

(1) *G is an almost gap group.*
(2) *$E(G, K) \ne \emptyset$.*
(3) *$E^o(G, K) \ne \emptyset$.*
(4) *There exists an $\mathcal{L}(G)$-free $\mathbb{R}G$-module W such that $d_W(P, H) > 0$ for all $(P, H) \in \mathcal{PH}_0^2(G)$ with $H \not\le K$.*

Proof. Clearly, (3) implies (2), and (1) implies (4).

We show that (4)⇒(1). Suppose that (4) holds. Let

$$V = W \oplus \big(\mathrm{Ind}_K^G(W_K) \oplus \mathbb{R}[G]_{\mathcal{L}(G)}\big)^{\oplus \dim W+1},$$

where W_K is an almost gap $\mathbb{R}K$-module. For $(P, H) \in \mathcal{PH}_0(G)$, if $(P, H) \notin \mathcal{PH}_0^2(G)$, $(P, H) \in \mathcal{PH}_0^2(G) \setminus \mathcal{PH}_0^2(G, K)$ and $(P, H) \in \mathcal{PH}_0^2(G, K)$, respectively, then $d_{\mathbb{R}[G]_{\mathcal{L}(G)}}(P, H)$, $d_{\mathrm{Ind}_K^G(W_K)}(P, H)$ and $d_W(P, H)$, respectively, are positive. Therefore, $d_V(P, H) > 0$ and V is an almost gap $\mathbb{R}G$-module. Thus (1) and (4) are equivalent.

If $K > O^2(G)$ then (4)⇒(3) follows from Theorem 5.3, and (2)⇒(4) follows from Theorem 5.1. If $K = O^2(G)$, then (2)⇒(1) by Theorem 5.6 and (1)⇒(3) by Theorem 6.2.

Thus, (1)–(4) are equivalent. ∎

Theorem 1.2 is a direct corollary of the following theorem.

Theorem 7.2. *Let G be a finite group not of prime power order such that $G/O^2(G)$ is a nontrivial cyclic group, K an index 2 subgroup of G, and L the subgroup of G for which $L > O^2(G)$ and $[L : O^2(G)] = 2$. The group G is an almost gap group if and only if either $E^o(L, O^2(G)) \neq \emptyset$, or $O^2(G) \in \mathcal{P}(G)$ and L is isomorphic to a direct product group of $O^2(G)$ with an order 2 group, and in addition $E^o_4(G, K) \neq \emptyset$ in the case where $|G/O^2(G)| > 2$.*

Proof. Suppose that G is an almost gap group. Any subgroup H of G with $H \geq O^2(G)$ is also an almost gap group by [13, Proposition 3.1]. Thus we have the "only if" part by Theorems 5.3 and 6.2.

We now show the "if" part. Let $[G : O^2(G)] = 2^t$ and

$$G = G_t > K = G_{t-1} > \cdots > L = G_1 > G_0 = O^2(G)$$

be a sequence of subgroups of G for which $[G_{j+1} : G_j] = 2$ for $j = 0, 1, \ldots, t-1$. By the assumption on L the group G_1 is an almost gap group by Theorem 5.6.

Now assume that $t > 1$. Let V_1 be an almost gap $\mathbb{R}G_1$-module. Since $E^o_4(G, K)$ is not empty, $E^o(G_a, G_{a-1})$ is not empty for $a = 2, \ldots, t$ by Proposition 5.4. By Theorem 5.1, for $a = 2, \ldots, t$, there is an $\mathcal{L}(G_a)$-free $\mathbb{R}G_a$-module V_a such that $d_{V_a}(P, H)$ is positive for $(P, H) \in \mathcal{PH}_0(G_a, G_{a-1})$. Put $W_1 = V_1$ and inductively

$$W_a = V_a \oplus \left(\mathrm{Ind}_{G_{a-1}}^{G_a} W_{a-1} \oplus \mathbb{R}[G_a]_{\mathcal{L}(G_a)}\right)^{\oplus \dim V_a + 1}$$

for $a = 2, \ldots, t$. Then $d_{W_a}(P, H) \geq 0$ for $(P, H) \in \mathcal{PH}(G_a)$. Put

$$V = \bigoplus_{1 \leq a \leq t} \mathrm{Ind}_{G_a}^{G} W_a \oplus \mathbb{R}[G]_{\mathcal{L}(G)}.$$

We show that V is an almost gap $\mathbb{R}G$-module. Let $(P, H) \in \mathcal{PH}_0(G)$. Note that $d_{\mathrm{Ind}_{G_a}^{G} W_a}(P, H) \geq 0$. If $(P, H) \notin \mathcal{PH}_0^2(G)$ then $d_V(P, H) \geq d_{\mathbb{R}[G]_{\mathcal{L}(G)}}(P, H) > 0$. Suppose that $(P, H) \in \mathcal{PH}_0^2(G)$. Since $G/O^2(G)$ is a cyclic group, there exists $1 \leq a \leq t$ such that $[H \cap G_a : P \cap G_a] = 2$. In particular, $P \cap G_a$ is not a Sylow 2-subgroup of G_a. Moreover, $P \cap G_a \neq O^2(G_a)$, since if not, $P = O^2(G) \in \mathcal{L}(G)$. Therefore $P \notin \mathcal{L}(G_a)$ and

$$d_V(P, H) \geq d_{\mathrm{Ind}_{G_a}^{G} W_a}(P, H) \geq d_{W_a}(P \cap G_a, H \cap G_a) > 0.$$

Hence V is an almost gap $\mathbb{R}G$-module. ∎

Theorem 7.3. *Let G be a finite group not of prime power order. The group G is an almost gap group if and only if so is L for any subgroup L with $L > O^2(G)$ and $L/O^2(G)$ cyclic.*

Proof. Recall that $\mathbb{R}[O^2(G)]_{\mathcal{L}(O^2(G))}$ is an almost gap $\mathbb{R}O^2(G)$-module if $O^2(G)$ is not of prime power order. Let $\mathcal{N}$ be the set of subgroups L of G such that $L > O^2(G)$ and $L/O^2(G)$ is a cyclic group, and let $\mathcal{N}_c$ be the minimal subset of $\mathcal{N}$ such that for any $L \in \mathcal{N}$, there is $Q \in \mathcal{N}_c$ satisfying $(Q) \supset (L)$, that is, L is subconjugate to Q in G. Suppose that V_L is an almost gap $\mathbb{R}L$-module for $L \in \mathcal{N}_c$. We show that

$$V = \bigoplus_{(L)\in\mathcal{N}_c} \operatorname{Ind}_L^G V_L \oplus \mathbb{R}[G]_{\mathcal{L}(G)}$$

is an almost gap $\mathbb{R}G$-module. It is easy to see that $\operatorname{Ind}_L^G V_L$ is $\mathcal{L}(G)$-free and $d_{\operatorname{Ind}_L^G V_L}(P,H) \geq 0$ for $(P,H) \in \mathcal{PH}(G)$ (cf. [13, Lemmas 1.2 and 1.7]). Let $(P,H) \in \mathcal{PH}_0(G)$. If $(P,H) \notin \mathcal{PH}_0^2(G)$ then $d_{\mathbb{R}[G]_{\mathcal{L}(G)}}(P,H) > 0$. Suppose that $(P,H) \in \mathcal{PH}_0^2(G)$. There exist $Q \in \mathcal{N}_c$ and $g \in G$ such that $(H \smallsetminus P) \cap g^{-1}Qg \neq \emptyset$. Then $(gPg^{-1} \cap Q, gHg^{-1} \cap Q) \in \mathcal{PH}(Q)$.

We claim that $gPg^{-1} \cap Q \notin \mathcal{L}(Q)$. Indeed, assume otherwise. Then $gPg^{-1} \cap Q = O^2(Q)$ or $gPg^{-1} \cap Q = O^p(Q)$ for an odd prime p. If $gPg^{-1} \cap Q = O^p(Q)$ for an odd prime p, then $O^p(Q)$ is a Sylow 2-subgroup of Q and thus $gHg^{-1} \cap Q = gPg^{-1} \cap Q$, which is a contradiction.

Since $O^2(Q) = O^2(G)$, if $gPg^{-1} \cap Q = O^2(Q)$ then $gPg^{-1} = O^2(G)$. This contradicts $P \notin \mathcal{L}(G)$. Therefore, $d_{V_Q}(gPg^{-1} \cap Q, gHg^{-1} \cap Q) > 0$. Then $\bigoplus_{(L)\in\mathcal{N}_c} \operatorname{Ind}_L^G V_L \oplus \mathbb{R}[G]_{\mathcal{L}(G)}$ is an almost gap $\mathbb{R}G$-module. The converse follows from [13, Proposition 3.1]. ■

Now it is easy to deduce Theorem 1.3 from Theorems 7.2 and 7.3.

Let $\mathcal{PH}^{(2)}(G)$ be the subset of $\mathcal{PH}^2(G)$ consisting of (P,H) such that $O^2(C_G(h))$ is a subgroup of P for any 2-element $h \in H \smallsetminus P$. Note that for $(P,H) \in \mathcal{PH}^{(2)}(G)$, if a 2-element h of $H \smallsetminus P$ has order > 2 then $C_G(h)$ is a 2-group. There exists an $\mathcal{L}(G)$-free nonnegative $\mathbb{R}G$-module W such that $d_W(P,H) > 0$ if $(P,H) \in \mathcal{PH}_0(G) \smallsetminus \mathcal{PH}^{(2)}(G)$ [28, Theorem 2.3]. Put $\mathcal{PH}^{(2)}(G,K) = \mathcal{PH}^2(G,K) \cap \mathcal{PH}^2(G)$.

Proposition 7.4. *Let G be a finite group such that $G/O^2(G)$ is nontrivial cyclic and $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$, and let K be an index 2 subgroup of G. If there exists an $\mathcal{L}(G)$-free nonnegative $\mathbb{R}G$-module W such that $d_W(P,H) > 0$ for some $(P,H) \in \mathcal{PH}^{(2)}(G,K)$, then $E^o(G,K)$ is not empty.*

Proof. Let $(P,H) \in \mathcal{PH}^{(2)}(G,K)$ and let W be an $\mathcal{L}(G)$-free nonnegative $\mathbb{R}G$-module for which $d_W(P,H) > 0$. We show the assertion by considering two cases.

The first case is when $[G : O^2(G)] > 2$. Let $\mathcal{C}$ be the set of all conjugacy classes (C) in G of cyclic 2-groups C for which $C \leq H$ and $C \not\leq P$. Note that H is a 2-group and $P = H \cap K$. Since $[H : P] = 2$, it follows that

$$H \smallsetminus P = \coprod_{(C)\in\mathcal{C}} (C) \cap H \smallsetminus P = \coprod_{(C)\in\mathcal{C}} \coprod_{N_G(C)gH\in N_G(C)\backslash(G/H)^C} g^{-1}Cg \smallsetminus P.$$

By using the character χ_W of W, we see that

$$\begin{aligned} d_W(P,H) &= -\frac{1}{|H|}\sum_{h\in H\smallsetminus P}\chi_W(h) \\ &= -\frac{1}{|H|}\sum_{(C)\in\mathcal{C}}\ \sum_{N_G(C)gH\in N_G(C)\backslash(G/H)^C}\ \sum_{h\in g^{-1}Cg\smallsetminus P}\chi_W(h) \\ &= \frac{1}{|H|}\sum_{(C)\in\mathcal{C}}|g^{-1}Cg|\sum_{N_G(C)gH\in N_G(C)\backslash(G/H)^C} d_W(g^{-1}Cg\cap P, g^{-1}Cg) \\ &= \frac{1}{|H|}\sum_{(C)\in\mathcal{C}}|C|\,|N_G(C)\backslash(G/H)^C|d_W(C\cap P,C) > 0. \end{aligned}$$

Note that $d_W(C\cap K,C)\geq 0$, since $C\cap K = C\cap P$. Therefore, by Lemma 5.2, $E_4^o(G,K)$ is not empty.

The second case is when $[G:O^2(G)]=2$. If H is a 2-group, similarly to the first case, we have $E^o(G,K)\neq\emptyset$. Indeed, suppose that $E^o(G,K)=\emptyset$. Then we may assume that $H\smallsetminus P$ has an involution h such that $|O^2(C_G(h))|\neq 1$ and $P\geq O^2(C_G(h))$. Recall that

$$(P\backslash G/\langle h\rangle)^{H/P} = P\backslash PC_G(h)/\langle h\rangle \cong H\backslash PC_G(h)$$

by [28, Lemma 4.4]. Take $k\in J(2)$ such that $C_k=\langle h\rangle$. In Lemma 6.1 we replace (P_k,H_k) by (P,H) and proceed as in the proof of Lemma 6.1. Then we see that $d_W(P,H)$ must be zero. Therefore $E^o(G,K)$ is not empty.

If K is an almost gap group then G is a gap group by Theorem 7.2. ∎

Now, we show that a nonsolvable group G for which $O^p(G)\neq G$ for some odd prime p is a gap group (Theorem 1.4). We prepare two lemmas.

The following is an application of a transfer homomorphism (cf. [7, Theorem 3.4]).

Lemma 7.5. *Let N be a normal subgroup of a finite group G for which $N\geq O^2(G)$, and $G_{\{2\}}$ a Sylow 2-subgroup of G. Let H be the subgroup of G generated by all elements of the commutator subgroup $[G_{\{2\}},G_{\{2\}}]$ of $G_{\{2\}}$ and by all elements $a^{-1}b$ such that $a,b\in G_{\{2\}}\smallsetminus N$ and a and b are conjugate in G. If involutions of $G_{\{2\}}$ outside N generate $G_{\{2\}}$, then $H=G_{\{2\}}\cap[G,G]$.*

Proof. The group H is a normal subgroup of $G_{\{2\}}$ with $G_{\{2\}}/H$ abelian, since $N\cap G_{\{2\}}$ is normal in $G_{\{2\}}$. Consider the transfer $f\colon G\to G_{\{2\}}/H$. Since P/H is abelian, it follows that $\ker f\geq[G,G]$ and so

$$H\leq G_{\{2\}}\cap[G,G]\leq G_{\{2\}}\cap\ker f. \tag{7.1}$$

We show that f is an epimorphism. Since $[G:N]$ is 2 power and $G_{\{2\}}$ is a Sylow 2-subgroup of G, the set $P\smallsetminus N$ is not empty. Let x be an involution of $G_{\{2\}}$ outside N. Then there exist elements t_j of G, for $1\leq j\leq[G:G_{\{2\}}]$, such that

$$t_j x t_j^{-1} \in G_{\{2\}} \quad \text{and} \quad f(x) = \prod_{j=1}^{[G:P]} (t_j x t_j^{-1})H,$$

since $|x| = 2$ (cf. [7, Theorem 3.3]). Thus we obtain

$$f(x) = \prod_{j=1}^{[G:G_{\{2\}}]} x(x^{-1} t_j x t_j^{-1})H = x^{[G:G_{\{2\}}]}H = xH,$$

since x and $t_j x t_j^{-1}$ are elements of $G_{\{2\}}$ outside N which are conjugate. Since involutions of $G_{\{2\}} \smallsetminus N$ generate $G_{\{2\}}$, we have $f(y) = yH$ for any $y \in G_{\{2\}}$, and so f is an epimorphism. Therefore,

$$[G : \ker f] = [G_{\{2\}} : H]. \tag{7.2}$$

Since $\ker f$ is a normal, 2 power index subgroup of G, we have $G_{\{2\}}(\ker f) = G$ and so $G/\ker f = G_{\{2\}}(\ker f)/\ker f \cong G_{\{2\}}/(G_{\{2\}} \cap \ker f)$. It follows that

$$[G : \ker f] = [G_{\{2\}} : G_{\{2\}} \cap \ker f]. \tag{7.3}$$

By (7.1)–(7.3), we have

$$H = G_{\{2\}} \cap [G, G] = G_{\{2\}} \cap \ker f. \ \blacksquare$$

Lemma 7.6. *Let G be a finite group not of prime power order for which $[G : O^2(G)] = 2$. Suppose that $O^p(G) \neq G$ for some odd prime p, and G is not an almost gap group. Then $O^2(G)$ is an odd order group.*

Proof. Let $G_{\{2\}}$ be a Sylow 2-subgroup of G. Since G is not an almost gap group, there is a unique odd prime p such that $O^p(G) \neq G$ and $E(G, O^2(G)) = \emptyset$. Therefore any 2-element of $G \smallsetminus O^2(G)$ is an involution. By Lemma 7.5, the focal subgroup $G_{\{2\}} \cap [G, G]$ is generated by all elements of $[G_{\{2\}}, G_{\{2\}}]$ and all elements $x^{-1}y$ for which $x, y \in G_{\{2\}} \smallsetminus O^2(G)$ and x is conjugate to y in G. By Proposition 3.3, we find that $G_{\{2\}} \cap [G, G] = [G_{\{2\}}, G_{\{2\}}]$, and it is an index 2 subgroup of $G_{\{2\}}$.

Now, we show that $[K : [K, K]] > 2$ for every 2-subgroup K of order > 2. Recall that every 2-group is solvable. Take an index 2 subgroup K_1 of K and an index 2 subgroup K_2 of K_1.

If there exists an index 2 subgroup H of G such that $K_2 = K_1 \cap H$, then K_2 is a normal subgroup of K and K/K_2 is abelian and thus $[K : [K, K]] > 2$.

Now assume that there does not exist such an index 2 subgroup H. There exists an element x of G such that $K = K_2 \cup xK_2 \cup x^2K_2 \cup x^3K_2$. If $K_2 \cap xK_2x^{-1} = K_2$ then K_2 is an index 4 normal subgroup of G so that G/K_2 is cyclic and thus $[K : [K, K]] > 2$. Suppose that $K_2 \cap xK_2x^{-1} \neq K_2$. Since $K_2 \cup x^2K_2 = K_1$, $H := K_2 \cap xK_2x^{-1}$ is an index 8 normal subgroup of K. For every group P of order 8, we see that $[P : [P, P]] > 2$. Then $[K : [K, K]] > 2$ since $[K : [K, K]] \geq [K/H : [K/H, K/H]] > 2$.

Since $[G_{\{2\}} : G_{\{2\}} \cap [G, G]] = 2$, $G_{\{2\}}$ is an order 2 group, which implies that $O^2(G)$ is an odd order group. ∎

Theorem 1.4 follows from the following theorem.

THEOREM 7.7. *Let G be a finite group not of prime power order such that $G \neq O^2(G)$ and G has an odd index > 1 normal subgroup. If $O^2(G)$ is an even order group, then G is an almost gap group.*

Proof. Let K be an odd index > 1 normal subgroup of G. Since an odd order group is solvable, there exists a prime p such that $O^p(G) \neq G$. Then any 2-element of H outside P is an involution for any pair $(P, H) \in \mathcal{PH}_0^2(G)$. Let L be a subgroup of G for which $L > O^2(G)$ and $[L : O^2(G)] = 2$. Since $L \cap O^p(G) \neq L$, it follows that $O^p(L) \neq L$. By Lemma 7.6, L is an almost gap group. For every pair $(P, H) \in \mathcal{PH}_0^2(G)$, there exists an almost gap subgroup L of G such that $L > O^2(G)$, $[L : O^2(G)] = 2$ and $H \leq L$. Therefore G is an almost gap group. ∎

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Toshio Sumi
Faculty of Arts and Science
Kyushu University
744 Motooka, Nishi-ku
Fukuoka 819-0395, Japan
E-mail: sumi@artsci.kyushu-u.ac.jp

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