

On finite groups acting on a connected sum of 3-manifolds $S^2 \times S^1$

by

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Abstract. Let H_g denote the closed 3-manifold obtained as the connected sum of g copies of $S^2 \times S^1$, with free fundamental group of rank g . We prove that, for a finite group G acting on H_g which induces a faithful action on the fundamental group, there is an upper bound for the order of G which is quadratic in g , but there does not exist a linear bound in g . This implies then a Jordan-type bound for arbitrary finite group actions on H_g which is quadratic in g . For the proofs we develop a calculus for finite group actions on H_g , by codifying such actions by handle-orbifolds and finite graphs of finite groups.

1. Introduction. All finite group actions in the present paper will be faithful, smooth and orientation-preserving, and all manifolds and orbifolds will be orientable. For the case of surfaces, the famous Hurwitz bound states that the order of a finite group acting on a closed surface of genus $g > 1$ is bounded above by $84(g - 1)$. In a similar spirit, the order of a finite group acting on a 3-dimensional handlebody V_g of genus $g > 1$ is bounded by $12(g - 1)$ ([Z1], [MMZ, Theorem 7.2]).

We consider finite groups G acting on the connected sum $H_g = \sharp_g(S^2 \times S^1)$ of g copies of $S^2 \times S^1$; we will call H_g a *closed handle* or just a *handle* of genus g in the following. Similar as for a handlebody of genus g , the fundamental group of a closed handle H_g is the free group F_g of rank g . For a handlebody V_g of genus $g > 1$, a finite group G acting on V_g acts faithfully on its fundamental group, meaning that it induces an injection $G \rightarrow \text{Out } \pi_1(V_g)$ into the outer automorphism group $\text{Out } \pi_1(V_g)$ (see Proposition 2), and hence induces a subgroup of $\text{Out } F_g$ of order at most $12(g - 1)$. However, the maximum order of a general finite subgroup of $\text{Out } F_g$ is $2^g g!$ for $g > 2$ and 12 for $g = 2$ ([WZ]) (based on the result in [Z2] that each finite subgroup

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of $\text{Out } F_g$ can be induced by an action of the group on a finite graph, and then also on some handlebody of sufficiently high dimension).

Since H_g admits S^1 -actions (see [R]), it admits finite cyclic group actions of arbitrarily large order acting trivially on the fundamental group (i.e., inducing the trivial homomorphism to $\text{Out } F_g$), so in contrast to the situation for handlebodies of genus $g > 1$ there is no upper bound for the orders of finite groups G acting on a closed handle H_g . However, let G_0 denote the normal subgroup of all elements of G acting trivially on $\pi_1(H_g)$; then G_0 is cyclic for $g > 1$ (Proposition 2), the quotient H_g/G_0 is again a closed handle of the same genus g , and the factor group G/G_0 acts faithfully on the fundamental group of the quotient $H_g/G_0 \cong H_g$. Hence one is led to consider actions of finite groups G on H_g which act faithfully on the fundamental group; in Section 4, we prove the following:

THEOREM 1. *Let G be a finite group acting on a closed handle H_g of genus g such that the induced action on the fundamental group is faithful. Then, for $g \geq 15$, there is the quadratic bound*

$$|G| \leq 24g(g-1).$$

If G is cyclic then, for $g \geq 3$,

$$|G| \leq 4(g-1)^2.$$

There does not exist a linear bound in g for the order of G .

By the above, Theorem 1 implies the following Jordan-type bound for arbitrary finite groups acting on a closed handle (i.e., not necessarily faithful on the fundamental group):

COROLLARY. *Let G be a finite group acting on a closed handle H_g of genus $g > 1$. Then G has a cyclic normal subgroup C (the subgroup acting trivially on the fundamental group) such that, for $g \geq 15$, the order of G/C is bounded above by $24g(g-1)$.*

We note that by the classical Jordan bound each finite subgroup G of a linear group $\text{GL}(n, \mathbb{C})$ has a normal abelian subgroup A such that the order of G/A is bounded by a constant depending only on n ; by [C], for $n \geq 71$, the optimal bound here is $(n+1)!$, realized by the symmetric group S_{n+1} occurring as a subgroup of $\text{GL}(n, \mathbb{C})$.

After the various linear bounds for finite group actions on surfaces and handlebodies, Theorem 1 seems to present the first instance of a quadratic bound in such a situation. There remains the problem to determine the optimal quadratic bound (the optimal coefficient for g^2), both for cyclic and arbitrary finite groups; in Section 3 we will construct some explicit examples of finite cyclic and finite group actions on closed handles which seem to come close to these optimal bounds.

In order to prove Theorem 1, in Section 2 we shall develop a calculus for finite group actions on closed handles H_g (see Theorem 2), in analogy with the theory of finite group actions on handlebodies V_g [MMZ] (see also [MZ] for applications to group actions of large order on handlebodies and [Z3] for the case of closed 3-manifolds). This uses the language of handle-orbifolds and of finite graphs of finite groups which codify the quotient orbifolds H_g/G .

We note that the maximum order of a finite cyclic group acting on a closed surface of genus $g > 1$ is $4g + 2$, for a handlebody of genus $g > 1$ it is $2g + 2$ if g is even, and $2g - 2$ if g is odd [MMZ]. By contrast, for general finite cyclic subgroups of $\text{Out } F_g$, it is proved in [LN] and [B] that the maximum order behaves approximatively like the Landau estimate $\exp(\sqrt{g \log g})$ for the maximum orders of elements of the symmetric group of degree g . The maximum orders of the finite cyclic subgroups of $\text{Aut}(F_g)$ and $\text{Out}(F_g)$, for $g \leq 300$, can be found in [LN, Table 1].

We close the Introduction with the question of what happens for the higher-dimensional analogues $\sharp_g(S^d \times S^1)$ of a closed handle H_g for $d > 2$ (all with free fundamental group of rank g). In analogy with Theorem 1, is there a polynomial bound in g for the order of G , and of what degree? (See also [Z2] for a discussion of finite group actions on higher-dimensional analogues of handlebodies.)

2. Handle orbifolds and associated finite graphs of finite groups.

Let G be a finite group acting faithfully and orientation-preservingly on a handle $H_g = \sharp_g(S^2 \times S^1)$ of genus g . Denoting by E the group generated by all lifts of elements of G to the universal covering of H_g and by F_g the normal subgroup of covering transformations, we have a group extension $1 \rightarrow F_g \rightarrow E \rightarrow G \rightarrow 1$ which belongs to the abstract kernel $G \rightarrow \text{Out } \pi_1(H_g) \cong \text{Out } F_g$ induced by the action of G on $\pi_1(H_g)$. Using the equivariant sphere theorem (see [MSY] for an approach by minimal surface techniques, and [Du] and [JR] for purely topological-combinatorial proofs), we will associate to the action of G a handle-orbifold \mathcal{H} and a finite graph of finite groups (Γ, \mathcal{G}) whose fundamental group $\pi_1(\Gamma, \mathcal{G})$ is isomorphic to the extension E .

By the equivariant sphere theorem there exists an embedded, homotopically nontrivial 2-sphere S^2 in $H = H_g$ such that $x(S^2) = S^2$ or $x(S^2) \cap S^2 = \emptyset$ for all $x \in G$. We cut H along the system of disjoint 2-spheres $G(S^2)$, by removing the interiors of G -equivariant regular neighbourhoods $S^2 \times [-1, 1]$ of each of these 2-spheres, and call each of these regular neighbourhoods $S^2 \times [-1, 1]$ a 1-handle. The result is a collection of 3-manifolds with 2-sphere boundaries, with an induced action of G . We close each of the 2-sphere boundaries by a 3-ball and extend the action of G by taking the cone over the center of each of these 3-balls, so G permutes these

3-balls and their centers. The result is a finite collection of closed handles of lower genus on which G acts. Applying inductively the procedure of cutting along 2-spheres, we finally end up with a finite collection of 3-spheres which we call *0-handles* (these are just the closed handles of genus 0). Note that the construction gives a finite graph $\tilde{\Gamma}$ on which G acts, whose vertices correspond to the 0-handles and the edges to the 1-handles.

On each 3-sphere (0-handle) there are finitely many points which are the centers of the attached 3-balls (their boundaries are the 2-spheres along which the 1-handles are attached). For each of these 3-spheres we consider its stabilizer G_v in G . By the recent geometrization of finite group actions on 3-manifolds following Thurston and Perelman, we can assume that the action of a stabilizer G_v on the corresponding 3-sphere is standard, i.e. orthogonal; we call such a quotient S^3/G_v a *0-handle orbifold* (we note that the geometrization is not really essential for the construction and its applications, it just says that each 0-handle orbifold is standard). Similarly, considering the stabilizers G_e in G of the 1-handles $S^2 \times [-1, 1]$, we can assume that each stabilizer G_e preserves the product structure of $S^2 \times [-1, 1]$. If some element of a stabilizer G_e acts as a reflection on $[-1, 1]$, we split the 1-handle into two 1-handles by introducing a new 0-handle obtained from a small regular neighbourhood $S^2 \times [-\epsilon, \epsilon]$ of $S^2 \times \{0\}$ by closing up with two 3-balls. Hence we can assume that each stabilizer G_e of a 1-handle $S^2 \times [-1, 1]$ does not interchange its two boundary 2-spheres; equivalently, G acts without inversions on the graph $\tilde{\Gamma}$. We call such a quotient $(S^2 \times [-1, 1])/G_e \cong (S^2/G_e) \times [-1, 1]$ a *1-handle orbifold*.

As a result, the quotient orbifold $\mathcal{H} = H/G$ is obtained from a finite collection of 0-handle orbifolds S^3/G_v by removing the interiors of disjoint 3-ball neighbourhoods of finitely many points and attaching 1-handle orbifolds along the resulting 2-sphere boundaries (respecting singular sets and branching orders as well as orientations); we call such a structure a *closed handle-orbifold* or just a *handle-orbifold*. Summarizing, we have:

PROPOSITION 1. *The quotients of closed handles H_g by finite group actions have the structure of closed handle-orbifolds.*

We note also the following easy but crucial observation:

OBSERVATION. If around a point of a 0-handle S^3 of H a 1-handle is attached, then the stabilizer G_e of the 1-handle is exactly the subgroup of the stabilizer G_v of the 0-handle which fixes the considered point (since otherwise, some larger subgroup of G_v would stabilize the 1-handle).

To each handle-orbifold $\mathcal{H} = H/G$ a graph of groups (Γ, \mathcal{G}) is associated in a natural way. The underlying graph Γ is just the quotient $\tilde{\Gamma}/G$. The vertices (resp. edges) of the graph Γ correspond to 0-handle orbifolds

S^3/G_v (resp. 1-handle orbifolds $(S^2/G_e) \times [-1, 1]$), and to each vertex (resp. edge) the corresponding stabilizer G_v (resp. G_e) is associated (choosing an isomorphic lift of a maximal tree of Γ to $\tilde{\Gamma}$, and then also lifts of the remaining edges). In particular, the vertex groups G_v of (Γ, \mathcal{G}) are isomorphic to finite subgroups of the orthogonal group $\text{SO}(4)$, and the edge groups G_e to finite subgroups of $\text{SO}(3)$. We can also assume that the graph of groups has no trivial edges, i.e. has no edges with two different vertices such that the edge group coincides with one of the two vertex groups (by collapsing the edge, i.e. amalgamating the two 0-handles into a single 0-handle). We say that such a handle-orbifold \mathcal{H} and the associated graph of groups are *in normal form*.

We have associated to each handle-orbifold $\mathcal{H} = H/G$ a graph of groups (Γ, \mathcal{G}) in normal form. By the orbifold version of Van Kampen's theorem (see [HD]), the orbifold fundamental group $\pi_1(\mathcal{H})$ is isomorphic to the fundamental group $\pi_1(\Gamma, \mathcal{G})$ of the graph of groups (Γ, \mathcal{G}) (which is the iterated free product with amalgamation and HNN-extension of the vertex groups over the edge groups, starting with a maximal tree in Γ ; see [Se], [ScW] or [Z4] for the standard theory of graphs of groups, their fundamental groups and the connection with groups acting on trees and graphs). We also have a canonical surjection $\phi : \pi_1(\mathcal{H}) \cong \pi_1(\Gamma, \mathcal{G}) \rightarrow G$, injective on vertex and edge groups, whose kernel is isomorphic to the fundamental group $\pi_1(H) \cong F_g$ of the handle H , and the group extension

$$1 \rightarrow \pi_1(H) \cong F_g \rightarrow \pi_1(\mathcal{H}) \cong \pi_1(\Gamma, \mathcal{G}) \rightarrow G \rightarrow 1$$

is equivalent to the group extension $1 \rightarrow F_g \rightarrow E \rightarrow G \rightarrow 1$. In particular, H is the orbifold covering of \mathcal{H} associated to the kernel of the surjection ϕ .

Conversely, suppose we have a finite graph of finite groups (Γ, \mathcal{G}) associated to a handle-orbifold \mathcal{H} and a surjection $\phi : \pi_1(\Gamma, \mathcal{G}) \rightarrow G$ onto a finite group G which is injective on the vertex groups. Then the orbifold covering of \mathcal{H} associated to the kernel of ϕ is a closed handle H_g of some genus g on which G acts as the group of covering transformations. The genus g can be computed as follows. Denoting by

$$\chi(\Gamma, \mathcal{G}) = \sum \frac{1}{|G_v|} - \sum \frac{1}{|G_e|}$$

the Euler characteristic of the graph of groups (Γ, \mathcal{G}) (the sum is taken over all vertex groups G_v resp. edge groups G_e of (Γ, \mathcal{G})), we have

$$g - 1 = -\chi(\Gamma, \mathcal{G})|G|$$

(see [ScW], [Z4]).

Finally, we note that the induced action of G on the fundamental group of H is effective (faithful) if and only if the corresponding group extension $1 \rightarrow F_g \rightarrow E \rightarrow G \rightarrow 1$ is effective (i.e., by considering F_g as a subgroup

of E , the homomorphism $G \rightarrow \text{Out } F_g$ induced by conjugation of F_g by preimages in E of elements in G is injective). It is easy to see that this is the case if and only if the extension group $E \cong \pi_1(\Gamma, \mathcal{G})$ has no nontrivial finite normal subgroups: If a preimage $e \in E$ of an element $x \in G$ induces by conjugation an inner automorphism of F_g , then another preimage of x in E induces the trivial or identity automorphism of F_g ; since a power of e lies in F_g and the center of F_g is trivial, e must have finite order, and clearly the subgroup of elements of E acting by conjugation trivially on F_g is a finite normal subgroup of E .

Summarizing, we have:

THEOREM 2. *A finite group G acts on a closed handle H_g of genus g if and only if there is a finite graph of finite groups (Γ, \mathcal{G}) in normal form associated to a handle-orbifold \mathcal{H} , and a surjection $\phi : \pi_1(\Gamma, \mathcal{G}) \rightarrow G$ which is injective on the vertex groups such that*

$$g = -\chi(\Gamma, \mathcal{G})|G| + 1.$$

The induced action of G on the fundamental group of H_g is faithful if and only if $\pi_1(\Gamma, \mathcal{G})$ has no nontrivial finite normal subgroups.

See Section 3 for some significant examples. As noted in the Introduction, the following also holds:

PROPOSITION 2.

- (i) *Let G be a finite group acting on closed handle H_g of genus $g > 1$. Then the normal subgroup G_0 of all elements of G inducing a trivial action on the fundamental group is cyclic, and the quotient H_g/G_0 is again homeomorphic to a closed handle of genus g .*
- (ii) *Let G be a finite group acting faithfully on a handlebody V_g . If $g > 1$ then the induced action of G on the fundamental group is faithful.*

Proof. (i) Consider the action of G on a graph $\tilde{\Gamma}$ associated to a handle-decomposition of H_g as before; in particular, $\tilde{\Gamma}$ has no vertices of degree 1. Since $g > 1$, by [Z5, Lemma 1] (considering homology and the Hopf trace formula) the action of G_0 on $\tilde{\Gamma}$ has to be trivial (or by a direct combinatorial argument). Hence G_0 maps each 1-handle $S^2 \times [-1, 1]$ and each 0-handle S^3 to itself, and is isomorphic to a subgroup of $\text{SO}(3)$. Moreover, G_0 has to be a cyclic group since a noncyclic group G_0 would have only two global fixed points in each 0-handle S^3 around which a 1-handle can be attached, so the graph $\tilde{\Gamma}$ would be a circular graph and $g = 1$ (a cyclic group instead has a circle of fixed points in each 0-handle along which arbitrarily many 1-handles can be attached).

(ii) This follows, as in the proof of (i), from the analogous theory of finite group actions on handlebodies, replacing the equivariant sphere theorem

by the equivariant Dehn lemma/loop theorem. Denoting by B^n the closed n -ball, now the stabilizers of the 1-handles $B^2 \times [-1, 1]$ are finite subgroups of $\text{SO}(2)$, the stabilizers of the 0-handles B^3 are finite subgroups of $\text{SO}(3)$, and each 1-handle is attached along its two boundary components to the boundary of one or two 0-handles B^3 ; since $g > 1$, G_0 has to be trivial now (alternatively, one may apply [Z1, Korollar 1.3]).

3. Examples. We construct first an infinite series of finite cyclic group actions on closed handles, faithful on the fundamental group. For large g , this realizes the maximum order for cyclic group actions which we know at present.

For an odd positive integer a which is divisible by 3, consider $G \cong \mathbb{Z}_n$ where $n = a(a + 1)$. Let (Γ, \mathcal{G}) be the graph of groups which consists of two edges with edge groups \mathbb{Z}_a and \mathbb{Z}_{a+1} , and three vertices with vertex groups $\mathbb{Z}_{2a}, \mathbb{Z}_{3(a+1)}$ (both of valence 1) and $\mathbb{Z}_{a(a+1)}$ (the middle vertex of valence 2); its fundamental group is the free product with amalgamation

$$\pi_1(\Gamma, \mathcal{G}) \cong \mathbb{Z}_{2a} *_{\mathbb{Z}_a} \mathbb{Z}_{a(a+1)} *_{\mathbb{Z}_{a+1}} \mathbb{Z}_{3(a+1)};$$

note that $\pi_1(\Gamma, \mathcal{G})$ has no nontrivial finite normal subgroups (cf. Theorem 2).

Choose an orthogonal action of $\mathbb{Z}_{a(a+1)}$ on S^3 such that the subgroups \mathbb{Z}_a and \mathbb{Z}_{a+1} have two disjoint circles of fixed points, and associate a handle-orbifold \mathcal{H} to (Γ, \mathcal{G}) , with 0-handles $S^3/\mathbb{Z}_{a(a+1)}, S^3/\mathbb{Z}_{2a}$ and $S^3/\mathbb{Z}_{3(a+1)}$ such that $\pi_1 \mathcal{H} \cong \pi_1(\Gamma, \mathcal{G})$ (cf. the Observation after Proposition 1).

There is an obvious surjection $\phi : \pi_1(\Gamma, \mathcal{G}) \rightarrow \mathbb{Z}_n$, injective on vertex groups, which defines an action of \mathbb{Z}_n on a closed handle H_g (the orbifold covering of the handle-orbifold \mathcal{H} corresponding to the kernel of ϕ). Now

$$-\chi = -\chi(\Gamma, \mathcal{G}) = \frac{1}{a} + \frac{1}{a+1} - \frac{1}{2a} - \frac{1}{a(a+1)} - \frac{1}{3(a+1)} = \frac{7a-3}{6a(a+1)},$$

$$g-1 = -\chi n = \frac{7a-3}{6}, \quad a = \frac{6g-3}{7},$$

and finally

$$n = a(a+1) = \frac{6g-3}{7} \frac{6g+4}{7}$$

which is quadratic in g . In particular, there cannot exist a linear bound in g for the order of G , proving the final assertion of Theorem 1.

Next we discuss an infinite series of actions of noncyclic groups on closed handles, faithful on the fundamental group; again for large values of g , this realizes the maximum order for arbitrary finite group actions which we know at present.

For arbitrary finite groups G , the vertex groups of a graph of groups (Γ, \mathcal{G}) as in Theorem 2 are finite subgroups of the orthogonal group $\text{SO}(4)$, and the edge groups are finite subgroups of $\text{SO}(3)$. The orthogonal group

$SO(4)$ is isomorphic to the central product $S^3 \times_{\mathbb{Z}_2} S^3$ of two copies of the unit quaternions S^3 (i.e., with identified centers \mathbb{Z}_2); for a fixed pair of unit quaternions $(q_1, q_2) \in S^3 \times S^3$, the orthogonal action on S^3 is given by $x \mapsto q_1^{-1}xq_2$. There is a 2-fold covering $S^3 \rightarrow SO(3)$ whose kernel is the central subgroup $\mathbb{Z}_2 = \{\pm 1\}$ of S^3 , and the finite subgroups of S^3 are exactly the binary polyhedral groups which are the preimages of the polyhedral groups in $SO(3)$; we denote by \mathbb{D}_{4a}^* the binary dihedral group of order $4a$ which is the preimage of the dihedral group \mathbb{D}_{2a} of order $2a$, with $\mathbb{D}_{4a}^*/\mathbb{Z}_2 \cong \mathbb{D}_{2a}$.

For an integer $a \geq 2$, let (Γ, \mathcal{G}) be the graph of groups consisting of a single edge and two vertices, with vertex groups $\mathbb{D}_{4a}^* \times_{\mathbb{Z}_2} \mathbb{D}_{4a}^* \subset S^3 \times_{\mathbb{Z}_2} S^3$ (of order $8a^2$) and $\mathbb{D}_{2a} \times \mathbb{Z}_2$ (of order $4a$), where \mathbb{D}_{2a} denotes the diagonal subgroup of $\mathbb{D}_{4a}^* \times_{\mathbb{Z}_2} \mathbb{D}_{4a}^*$ (which is the maximal subgroup fixing the points $\pm 1 \in S^3$) and \mathbb{Z}_2 is the central subgroup of S^3 ; the edge group is the dihedral group \mathbb{D}_{2a} . Its fundamental group is the free product with amalgamation

$$\pi_1(\Gamma, \mathcal{G}) \cong (\mathbb{D}_{4a}^* \times_{\mathbb{Z}_2} \mathbb{D}_{4a}^*) *_{\mathbb{D}_{2a}} (\mathbb{D}_{2a} \times \mathbb{Z}_2),$$

and there is an obvious surjection, injective on vertex groups,

$$\phi : \pi_1(\Gamma, \mathcal{G}) \rightarrow \mathbb{D}_{4a}^* \times_{\mathbb{Z}_2} \mathbb{D}_{4a}^*.$$

Any finite group G onto which $\pi_1(\Gamma, \mathcal{G})$ surjects, injectively on vertex groups, has some order $8xa^2$. Now

$$\begin{aligned} -\chi &= -\chi(\Gamma, \mathcal{G}) = \frac{1}{2a} - \frac{1}{8a^2} - \frac{1}{4a} = \frac{2a-1}{8a^2}, \\ g-1 &= -\chi 8xa^2 = x(2a-1), \quad g = x(2a-1) + 1, \\ \frac{|G|}{g^2} &= \frac{8xa^2}{x^2(2a-1)^2 + 1 + 2x(2a-1)} \\ &\leq \frac{8a^2}{(2a-1)^2 + 1 + 2(2a-1)} = \frac{8a^2}{4a^2} = 2. \end{aligned}$$

It follows that $|G| \leq 2g^2$, and

$$|G| = 2g^2$$

if and only if $x = 1$ and $G \cong \mathbb{D}_{4a}^* \times_{\mathbb{Z}_2} \mathbb{D}_{4a}^*$.

4. Proof of Theorem 1. Suppose that the finite group G of order n acts on a closed handle $H = H_g$ of genus $g > 1$, faithfully on the fundamental group. By Theorem 2, there is a finite graph of finite groups (Γ, \mathcal{G}) in normal form associated to a handle-orbifold \mathcal{H} whose fundamental group $\pi_1(\mathcal{H}) \cong \pi_1(\Gamma, \mathcal{G})$ has no nontrivial finite normal subgroups, and there is a surjection $\phi : \pi_1(\Gamma, \mathcal{G}) \rightarrow G$, injective on vertex groups, such that the action of G on H is given by the orbifold covering of \mathcal{H} associated to the kernel of ϕ .

The edge groups of (Γ, \mathcal{G}) are finite subgroups of $\text{SO}(3)$, that is, cyclic, dihedral, tetrahedral of order 12, octahedral of order 24 or dodecahedral of order 60. Let $\chi = \chi(\Gamma, \mathcal{G})$ denote the Euler characteristic of (Γ, \mathcal{G}) ; note that $-\chi > 0$ since $g > 1$, and that for any graph of groups (Γ, \mathcal{G}) in normal form associated to a handle-orbifold \mathcal{H} one has $-\chi \geq 0$ unless Γ consists of a single vertex v , i.e. \mathcal{H} consists just of a single 0-handle orbifold S^3/G_v .

Let e be any edge of Γ and denote by a the order of its edge group; we will show that $n/a \leq 6(g-1)$.

Suppose first that e is a closed edge (i.e., an edge that is a closed loop). If e is the only edge of (Γ, \mathcal{G}) , then

$$-\chi \geq \frac{1}{a} - \frac{1}{2a} = \frac{1}{2a}, \quad g-1 = -\chi n \geq \frac{n}{2a}, \quad \frac{n}{a} \leq 2(g-1).$$

If e is closed and not the only edge, then

$$-\chi \geq \frac{1}{a}, \quad g-1 = -\chi n \geq \frac{n}{a}, \quad \frac{n}{a} \leq g-1.$$

Suppose that e is not closed. If e is the only edge of (Γ, \mathcal{G}) , then both vertices of e are isolated and

$$-\chi \geq \frac{1}{a} - \frac{1}{2a} - \frac{1}{3a} = \frac{1}{6a}, \quad g-1 = -\chi n \geq \frac{n}{6a}, \quad \frac{n}{a} \leq 6(g-1).$$

If e is not closed, not the only edge, and has exactly one isolated vertex, then

$$-\chi \geq \frac{1}{a} - \frac{1}{2a} = \frac{1}{2a}, \quad g-1 = -\chi n \geq \frac{n}{2a}, \quad \frac{n}{a} \leq 2(g-1).$$

Finally, if e is not closed, not the only edge, and has no isolated vertex, then

$$-\chi \geq \frac{1}{a}, \quad g-1 = -\chi n \geq \frac{n}{a}, \quad \frac{n}{a} \leq g-1.$$

Concluding, in all cases we have

$$\frac{n}{a} \leq 6(g-1).$$

In particular, if (Γ, \mathcal{G}) has an edge whose edge group has order $a \leq 60$, then $n \leq 6a(g-1) \leq 360(g-1)$, which is linear in g ; in particular, the quadratic bound of Theorem 1 holds if $g \geq 15$. So for the proof of the first part of the theorem we can assume that all edge groups of (Γ, \mathcal{G}) are cyclic or dihedral.

Again, let e be any edge of Γ ; its edge group is either cyclic of order $a = b$ or dihedral of order $a = 2b$. In each case this gives a cyclic subgroup \mathbb{Z}_b of G which has a global fixed point in H_g . Let (Γ', \mathcal{G}') be a graph of groups in normal form associated to the action of \mathbb{Z}_b on H_g (applying again Theorem 2), and $\chi' = \chi(\Gamma', \mathcal{G}')$. Since \mathbb{Z}_b has a global fixed point in H_g , the graph of groups (Γ', \mathcal{G}') must have a vertex v with vertex group \mathbb{Z}_b such that, for the corresponding 0-handle orbifold S^3/\mathbb{Z}_b , the action of the vertex group \mathbb{Z}_b on the 0-handle S^3 has a global fixed point. Since no cyclic

subgroup of prime order can have two circles of fixed points, any nontrivial subgroup of \mathbb{Z}_b has exactly the same circle of fixed points on the 0-handle S^3 . Hence all edges of (Γ', \mathcal{G}') which have v as a vertex have either trivial edge group or edge group \mathbb{Z}_b (by the Observation). We will show that $b \leq 2g$.

If all edges of (Γ', \mathcal{G}') with vertex v are closed, then one must have trivial edge group (since $\pi_1(\Gamma', \mathcal{G}')$ has no nontrivial finite normal subgroups), hence

$$-\chi' \geq 1 - \frac{1}{b} = \frac{b-1}{b}, \quad g-1 = -\chi'b \geq b-1, \quad b \leq g.$$

Otherwise there is a nonclosed edge e' in (Γ', \mathcal{G}') with vertex v , and the edge group of e' must be trivial (since (Γ', \mathcal{G}') is in normal form, i.e. without trivial edges).

If no vertex of e' is isolated then

$$-\chi' \geq 1, \quad g-1 = -\chi'b \geq b, \quad b \leq g-1.$$

If exactly one vertex of e' is isolated then

$$-\chi' \geq 1 - \frac{1}{2} = \frac{1}{2}, \quad g-1 = -\chi'b \geq \frac{b}{2}, \quad b \leq 2(g-1).$$

If both vertices of e' are isolated then

$$-\chi' \geq 1 - \frac{1}{2} - \frac{1}{b} = \frac{b-2}{2b}, \quad g-1 = -\chi'b \geq \frac{b-2}{2}, \quad b \leq 2g.$$

In conclusion, in all cases we have

$$b \leq 2g.$$

Combining this with the inequality

$$\frac{n}{a} \leq 6(g-1)$$

from above, we obtain (since $a = b$ or $a = 2b$)

$$n = |G| = \frac{n}{a} \cdot a \leq 6(g-1) \cdot 4g = 24g(g-1).$$

This proves the first part of Theorem 1.

Now suppose that G is a finite cyclic group of order n ; then all vertex and edge groups of (Γ, \mathcal{G}) are cyclic groups whose orders divide n . Consider an edge e of (Γ, \mathcal{G}) with edge group $G_e \cong \mathbb{Z}_a$ whose order a realizes the minimum order over all edge groups. If e is the only edge of (Γ, \mathcal{G}) (with one or two distinct vertices), then $a = 1$ (otherwise \mathbb{Z}_a would be a nontrivial finite normal subgroup of $\pi_1(\Gamma, \mathcal{G})$), hence

$$-\chi \geq 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}, \quad g-1 = -\chi n \geq \frac{n}{6}, \quad n \leq 6(g-1);$$

in particular, the quadratic bound of Theorem 1 in the cyclic case holds if $g \geq 3$.

Suppose then that e is not the only edge of Γ ; consider another edge with edge group \mathbb{Z}_b which has a common vertex v with e . The vertex v corresponds to a 0-handle S^3/G_v of \mathcal{H} . The action of the finite cyclic group G_v on S^3 is orthogonal, so the union of the fixed point sets of nontrivial elements of G_v consists of at most two disjoint circles S^1 ; also, a nontrivial subgroup of G_v of prime order cannot fix two circles (if we do not assume that the action of G_v is orthogonal, this follows from Smith fixed point theory). Since by the Observation each edge group is a maximal subgroup of a vertex group G_v fixing the point around which the corresponding 1-handle is attached, this implies that either $a = b$ or $(a, b) = 1$ (the greatest common divisor).

If $a = b$, since the edge group $G_e \cong \mathbb{Z}_a$ is not normal in $\pi_1(\Gamma, \mathcal{G})$, there must occur a situation of two edges with a common vertex, one with edge group \mathbb{Z}_a and the other with some edge group \mathbb{Z}_b such that $(a, b) = 1$. This implies $ab \leq n$ and $a \leq \sqrt{n}$ (since $a \leq b$). If the edge with edge group \mathbb{Z}_a has an isolated vertex (i.e., of degree or valency 1), then

$$-\chi \geq \frac{1}{a} - \frac{1}{2a} = \frac{1}{2a} \geq \frac{1}{2\sqrt{n}}, \quad g-1 = -\chi n \geq \frac{n}{2\sqrt{n}} = \frac{\sqrt{n}}{2}, \quad n \leq 4(g-1)^2.$$

If the edge has no isolated vertex, one obtains stronger inequalities

$$-\chi \geq \frac{1}{a} \geq \frac{1}{\sqrt{n}}, \quad g-1 \geq \sqrt{n}, \quad n \leq (g-1)^2.$$

This proves Theorem 1 also in the cyclic case.

Since the final assertion of Theorem 1 about the nonexistence of a linear bound follows from the examples constructed in Section 3, this completes the proof of Theorem 1.

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