Embedding orders into the cardinals with DC_κ

by

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Abstract. Jech proved that every partially ordered set can be embedded into the cardinals of some model of ZF. We extend this result to show that every partially ordered set can be embedded into the cardinals of some model of $\mathsf{ZF} + \mathsf{DC}_{<\kappa}$ for any regular κ. We use this theorem to show that for all κ, the assumption of DC_κ does not entail that there are no decreasing chains of cardinals. We also show how to extend the result to and embed into the cardinals a proper class which is definable over the ground model. We use this extension to give a large-cardinals-free proof of independence of the weak choice principle known as WISC from DC_κ.

1. Introduction. Assuming the axiom of choice, the cardinals trivially form a well-ordered class, but with its failure their order structure can be as complex as desired. An interesting example for this range of possibilities is Jech's theorem that if $(P, \leq)$ is a partial order then there exists a model of ZFA (Zermelo–Fraenkel with Atoms) in which $(P, \leq)$ can be embedded into the cardinals (see [Jec66]). The theorem was complemented by the Jech–Sochor embedding theorem which allowed carrying the consistency result into ZF and removing the need for atoms (see [JS66b, JS66a] and [Jec73, Chapter 6]). This theorem tells us, essentially, that there are no limitations on the order structure of cardinals defined by injections.

Jech's original proof included adding many counterexamples to DC (in the form of Dedekind-finite sets). While it can be modified to allow DC_κ to hold, the Jech–Sochor theorem is not suited to transfer universal statements such as DC_κ. We should point out that Pincus improved upon Jech–Sochor's original work and showed that it is possible to transfer injectively boundable statements (a class of statements which include DC_κ), and more. In this paper we give a direct forcing argument for Jech's proof, and this allows us to preserve DC_κ up to an arbitrary (but fixed) cardinal κ. We then proceed to show that in fact ordered classes (which are definable in

[143]

the ground model) may be embedded into the cardinals while preserving DC_κ.

The authors of [BM90] remark that it is unknown whether or not the assumption that there are no decreasing sequences of cardinals (with respect to the $\leq$ relation) implies that the axiom of choice holds. They write in section four: "The answer [...] is almost certainly negative, but thus far there is no proof" and mention that there has been some disagreement on the topic in the past. We will use the improved embedding theorem (of partial orders into the cardinals) to show that for any κ it is consistent with $\mathsf{ZF}+\mathsf{DC}_\kappa$ that there are decreasing chains of cardinals. While there is no positive result yet, this somehow suggests that the axiom of choice might be equivalent to the assertion "There is no infinite decreasing sequence of cardinals".

Decreasing sequences of cardinals also play a role in answering a question of Feldman & Orhon which appeared in [FOB08]. In the paper the authors prove ([1]) that for any $k \in \omega \setminus \{0,1\}$ the assertion that "Every antichain of cardinals has less than k members" implies the axiom of choice, and the question is about replacing the finite bound by ω. Feldman and Orhon conjectured that "Every antichain of cardinals is finite" does not imply the axiom of choice in ZF. The question is still open, but we will show that $\mathsf{ZF} + \mathsf{DC}_\kappa$ cannot prove that every antichain is finite.

In [Rog90] the author proves that it is consistent relative to the consistency of ZF that for every set of cardinals there exists one incomparable with all of them. We extend this result and show its compatibility with DC_κ. We use this extension to show the consistency of long chains and antichains of cardinals, and to give a large cardinals-free consistency result of the failure of WISC, a recent choice principle related to constructive set theory.

Clarification. After the acceptance and revision of the paper it was pointed out to the author that Takahashi [Tak68] proved some results in the same vein. He shows that a partial order can be embedded into subsets of the continuum, a result which is generalized in Section 3 of this paper. Takahashi also infers the existence of a decreasing chain of cardinals of order ω^*, as we show in Section 5. In this paper we extend both results to a much broader context, and our proofs are written in a modern format using unramified forcing and symmetric models.

2. Basic definitions. Suppose that $\mathfrak{M}$ is a countable transitive model of ZFC, a notion of forcing $\mathbb{P} = (P, \leq) \in \mathfrak{M}$ is a partial order with a maximum denoted by $1_\mathbb{P}$. The elements of P are called conditions, and when $p \leq q$ we say that p extends q, or that p is stronger than q. We say that p and q are

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([1]) The original proof is due to Tarski: see [RR85, Form T3(n), pp. 22–23].

compatible if there is r which extends both of them, otherwise p and q are *incompatible*. If a certain definition of $\mathbb{P}$ does not result in a partial order that has a maximum, we add one artificially. We will also consider only non-trivial notions of forcing, that is, every $p \in P$ has two incompatible extensions.

We define by induction the class of $\mathbb{P}$-names (calculated within $\mathfrak{M}$):

(1) $\mathfrak{M}_0^{\mathbb{P}} = \emptyset$;

(2) $\mathfrak{M}_{\alpha+1}^{\mathbb{P}} = \mathcal{P}(P \times \mathfrak{M}_\alpha^{\mathbb{P}})$;

(3) $\mathfrak{M}_\delta^{\mathbb{P}} = \bigcup_{\alpha < \delta} \mathfrak{M}_\alpha^{\mathbb{P}}$ for a limit ordinal δ.

Finally the class of $\mathbb{P}$-names is

$$\mathfrak{M}^{\mathbb{P}} = \bigcup_{\alpha \in \mathsf{Ord}} \mathfrak{M}_\alpha^{\mathbb{P}},$$

where $\mathcal{P}(x)$ denotes the power set of x, and Ord denotes the class of ordinals. We will use $\dot{x}$ to denote a $\mathbb{P}$-name, and $\check{x}$ to denote a canonical name for $x \in \mathfrak{M}$. If G is a $\mathbb{P}$-generic filter over $\mathfrak{M}$ then $\dot{x}^G$ is the interpretation of $\dot{x}$ by the filter G.

Let $\{\dot{x}_i \mid i \in I\}$ be a class of $\mathbb{P}$-names in $\mathfrak{M}$ (if it is a proper class then we require it to be definable). We denote by $\{\dot{x}_i \mid i \in I\}^\bullet$ the name $\{(1_{\mathbb{P}}, \dot{x}_i) \mid i \in I\}$. We shall also use $(\dot{x}, \dot{y})^\bullet$ to denote the canonical name for the ordered pair, namely $\{\{\dot{x}\}^\bullet, \{\dot{x}, \dot{y}\}^\bullet\}^\bullet$.

Suppose that π is an automorphism of $\mathbb{P}$; we may extend π to an automorphism of $\mathbb{P}$-names by induction,

$$\widetilde{\pi}(\dot{x}) = \{(\pi p, \widetilde{\pi}\dot{y}) \mid (p, \dot{y}) \in \dot{x}\}.$$

From this point, though, we will only use π to denote the automorphism of $\mathbb{P}$ as well the automorphism of the $\mathbb{P}$-names. If $\mathbb{P}$ was defined using parameters from A then a permutation of A can be used to define an automorphism of $\mathbb{P}$. This will be the case in our proofs. It can be shown by induction that if $x \in \mathfrak{M}$ then $\pi\check{x} = \check{x}$ for any $\pi \in \mathrm{Aut}(\mathbb{P})$.

LEMMA (The Symmetry Lemma). *Let $\varphi(u_1, \ldots, u_n)$ be a formula in the language of set theory, and p a condition in $\mathbb{P}$. Let $\dot{x}_1, \ldots, \dot{x}_n$ be $\mathbb{P}$-names, and $\pi \in \mathrm{Aut}(\mathbb{P})$. Then*

$$p \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_n) \Leftrightarrow \pi p \Vdash \varphi(\pi\dot{x}_1, \ldots, \pi\dot{x}_n).$$

The proof is by induction on the complexity of φ, and can be found in [Kun80, Lemma 7.13(c)].

Suppose that $\mathscr{G}$ is a group of permutations of a set A, and $E \subseteq A$; we define the pointwise stabilizer of E as the group $\mathrm{fix}_{\mathscr{G}}(E) = \{\pi \in \mathscr{G} \mid \pi{\restriction}E = \mathrm{id}_E\}$. If $\mathscr{G}$ acts on $\mathbb{P}$-names (through its action on $\mathbb{P}$ in most cases), we define the stabilizer of the name $\dot{x}$ as the group $\mathrm{sym}_{\mathscr{G}}(\dot{x}) = \{\pi \in \mathscr{G} \mid \pi\dot{x} = \dot{x}\}$. We omit $\mathscr{G}$ from these notations if it is clear from context.

If $\mathscr{G}$ is a group, we say that $\mathcal{F} \subseteq \mathcal{P}(\mathscr{G})$ is a *filter of subgroups* if whenever $H \in \mathcal{F}$ and $H \leq K$ then $K \in \mathcal{F}$, and $\mathcal{F}$ is closed under finite intersections. We also require the trivial group is not in $\mathcal{F}$. We say that $\mathcal{F}$ is *normal* if it is closed under conjugation.

If $\mathscr{G}$ is a group of permutations of $\mathbb{P}$ (or acting on it) and $\mathcal{F}$ is a normal filter of subgroups of $\mathscr{G}$, we say that $\dot{x} \in \mathfrak{M}^{\mathbb{P}}$ is an $\mathcal{F}$-*symmetric* name if $\mathrm{sym}(\dot{x}) \in \mathcal{F}$. We define the class of *hereditarily $\mathcal{F}$-symmetric sets* by induction: $\dot{x}$ is hereditarily $\mathcal{F}$-symmetric if and only if $\dot{x}$ is $\mathcal{F}$-symmetric, and for every $(p, \dot{y}) \in \dot{x}$, $\dot{y}$ is hereditarily $\mathcal{F}$-symmetric. We shall denote by $\mathsf{HS}_{\mathcal{F}}$ the class of hereditarily $\mathcal{F}$-symmetric names, and as usual we will omit $\mathcal{F}$ when it is clear from the context.

Let G be a $\mathbb{P}$-generic filter over $\mathfrak{M}$, and denote by $\mathfrak{N}$ the interpretation of the class $\mathsf{HS}_{\mathcal{F}}$ by the filter G, that is, $\mathfrak{N} = (\mathsf{HS}_{\mathcal{F}})^G = \{\dot{x}^G \mid \dot{x} \in \mathsf{HS}_{\mathcal{F}}\}$. Then $\mathfrak{N}$ is called a *symmetric extension* (generated by $\mathcal{F}$) of $\mathfrak{M}$. The proof of the following theorem can be found in [Jec03].

THEOREM. $\mathfrak{N}$ *is a transitive model of* ZF *and* $\mathfrak{M} \subseteq \mathfrak{N} \subseteq \mathfrak{M}[G]$. $\blacksquare$

For a cardinal κ we denote by DC_{κ} the *Principle of Dependent Choice for κ*, which states that for every non-empty set X, if R is a binary relation such that for every ordinal $\alpha < \kappa$ and every $f \colon \alpha \to X$ there is some $y \in X$ such that $f \mathrel{R} y$, then there is $f \colon \kappa \to X$ such that for every $\alpha < \kappa$, $f{\restriction}\alpha \mathrel{R} f(\alpha)$. We shall abbreviate by $\mathsf{DC}_{<\kappa}$ the assertion $(\forall \lambda < \kappa)\mathsf{DC}_{\lambda}$.

The axiom of choice implies that DC_{κ} holds for every κ, and in fact $\forall \kappa . \mathsf{DC}_{\kappa}$ is equivalent to the axiom of choice. One useful consequence of DC_{κ} is that for every set X there is either an injection from X into κ or an injection from κ into X. One can find a thorough treatment of DC_{κ} and related choice principles in [Jec73, Chapter 8].

LEMMA 2.1. *Let $\mathcal{F}$ be a normal filter of subgroups of a group of automorphisms of $\mathbb{P}$, and $\mathfrak{N} = (\mathsf{HS}_{\mathcal{F}})^G$ be the symmetric extension of $\mathfrak{M}$. If $\mathbb{P}$ is κ-closed and $\mathcal{F}$ is a κ-complete filter then $\mathfrak{N} \models \mathsf{DC}_{<\kappa}$.*

Proof. Let $\lambda < \kappa$. We will show that if $f \colon \lambda \to \mathfrak{N}$ is in $\mathfrak{M}[G]$ then $f \in \mathfrak{N}$. From this it follows that $\mathfrak{N} \models \mathsf{DC}_{\lambda}$, because if X and R are elements of $\mathfrak{N}$ as in the assumptions of DC_{λ}, then we can find $f \colon \lambda \to X$ in $\mathfrak{M}[G]$ (as the latter is a model of AC), and by the proof here we will find that $f \in \mathfrak{N}$.

Let $\dot{f}_0$ be a name for f and let p be a condition forcing that $\dot{f}_0$ is a function whose domain is λ and its range is a subset of $\mathfrak{N}$. Because $\mathbb{P}$ is κ-closed, we can extend p to $p_0 \geq p_1 \geq \cdots \geq p_{\alpha} \geq \cdots \geq p_{\lambda}$ such that for all $\alpha < \lambda$, $p_{\alpha} \Vdash \dot{f}_0(\check{\alpha}) = \dot{t}_{\alpha}$, where $\dot{t}_{\alpha} \in \mathsf{HS}$. Then we can define the collection $\{\dot{t}_{\alpha} \mid \alpha < \lambda\}$ in $\mathfrak{M}$, and take $\dot{f} = \{\dot{t}_{\alpha} \mid \alpha < \lambda\}^{\bullet}$. Clearly $\dot{f}^G = f$ whenever $p_{\lambda} \in G$. We need to show that $\dot{f} \in \mathsf{HS}$, but it is enough to show that $\mathrm{sym}(\dot{f}) \in \mathcal{F}$ because all the names appearing in $\dot{f}$ are taken from

HS to begin with. For every $\pi \in \bigcap_{\alpha<\lambda} \mathrm{sym}(\dot{t}_\alpha)$ we have $\pi \dot{f} = \dot{f}$, and by κ-completeness of $\mathcal{F}$ the intersection is in $\mathcal{F}$, and so $\dot{f}$ is in HS as desired. ∎

Remember that if A is a set then $|A|$ is the *cardinal number* of A. While in ZFC cardinal numbers are all ordinals, without the axiom of choice it is not always the case. We define $|A|$ to be the least ordinal bijectible with A if such an ordinal exists; otherwise $|A|$ is the set of those B which are in bijection with A and have minimal rank with respect to this property. If $|A|$ is a finite ordinal we say that A is finite, if it is an infinite ordinal we say that $|A|$ is an *aleph number*; in both cases we may say that $|A|$ is a well-ordered cardinal. If $|A|$ is not a well-ordered cardinal, we say that A is *not well-orderable*.

For sets A, B we define $|A| \leq |B|$ if and only if there is an injection from A into B, and $|A| \leq^* |B|$ if and only if A is empty or there is a surjection from B onto A. Both relations are reflexive and transitive, but only $\leq$ is provably antisymmetric without the axiom of choice. Moreover, $|A| \leq |B|$ implies $|A| \leq^* |B|$. For further analysis of the $\leq^*$ relation see [BM90].

3. Embedding partially ordered sets into cardinals. Let $\mathfrak{M}$ be a countable transitive model of ZFC, and κ a regular cardinal in $\mathfrak{M}$. Let $(\mathcal{Z}, \leq) \in \mathfrak{M}$ be a partially ordered set. We want to embed $(\mathcal{Z}, \leq)$ into the cardinals of some model, but instead we will embed $(\mathcal{P}(\mathcal{Z}), \subseteq)$. We observe that $(\mathcal{Z}, \leq)$ itself embeds into its power set by $z \mapsto \{z' \in \mathcal{Z} \mid z' \leq z\}$, and so it is indeed enough to embed the power set of $\mathcal{Z}$.

We define $\mathbb{P} = (P, \leq)$ to be the following notion of forcing defined within $\mathfrak{M}$: $p \in P$ is a partial function $p \colon (\mathcal{Z} \times \kappa) \times \kappa \to 2$ such that $|\mathrm{dom}\, p| < \kappa$. As usual $p \leq q \Leftrightarrow q \subseteq p$. We note that this forcing is κ-closed and therefore does not collapse cardinals smaller than κ^+. If $\kappa^{<\kappa} = \kappa$, then $\mathbb{P}$ has κ^+-c.c. and no cardinals are collapsed.

If G is $\mathbb{P}$-generic over $\mathfrak{M}$ then $\bigcup G = g$ is a total function from $(\mathcal{Z} \times \kappa) \times \kappa$ to 2 in $\mathfrak{M}[G]$. We define the following generic sets and we give them canonical names:

- For $z \in \mathcal{Z}, \alpha \in \kappa$ we define $r_{z,\alpha} = \{\gamma < \kappa \mid g((z, \alpha), \gamma) = 1\}$, with the canonical name $\dot{r}_{z,\alpha} = \{(p, \check{\gamma}) \mid p((z, \alpha), \gamma) = 1\}$.
- For $z \in \mathcal{Z}$ we define $R_z = \{r_{z,\alpha} \mid \alpha < \kappa\}$ with the canonical name $\dot{R}_z = \{\dot{r}_{z,\alpha} \mid \alpha < \kappa\}^\bullet$.
- For $Q \subseteq \mathcal{Z}$ we define $D_Q = \bigcup_{z \in Q} R_z$. We do not give a canonical name to D_Q, because we allow $Q \notin \mathfrak{M}$.

Let $\mathscr{G}$ the group of all permutations of $\mathcal{Z} \times \kappa$ such that for all (z, α) we have $\pi(z, \alpha) = (z, \beta)$ for some β (note that $\pi(z_1, \alpha_1) = (z_1, \beta)$ and $\pi(z_2, \alpha_2) = (z_2, \beta)$ does not imply $\alpha_1 = \alpha_2$). We define the action of $\mathscr{G}$ on $\mathbb{P}$.

If $\pi \in \mathscr{G}$ we define
$$\pi p(\pi(z, \alpha), \gamma) = p((z, \alpha), \gamma).$$

We extend the action of $\mathscr{G}$ to the class of $\mathbb{P}$-names. We first make the following observation. For all $z \in \mathcal{Z}, \alpha < \kappa$ we have
$$\pi \dot{r}_{z,\alpha} = \{(\pi p, \pi \check{\gamma}) \mid p((z, \alpha), \gamma) = 1\} = \{(p, \check{\gamma}) \mid p(\pi(z, \alpha), \gamma) = 1\} = \dot{r}_{\pi(z,\alpha)},$$
and for any $z \in \mathcal{Z}$ we have $\pi \dot{R}_z = \dot{R}_z$.

Let $I = [\mathcal{Z} \times \kappa]^{<\kappa}$, let $\mathcal{F}$ be the filter generated by $\mathrm{fix}(E)$ for $E \in I$, namely
$$\mathcal{F} = \{H \le \mathscr{G} \mid \exists E \in I : \mathrm{fix}(E) \le H\}.$$
It is left to the reader to verify that $\mathcal{F}$ is indeed a normal filter of subgroups and that $\mathcal{F}$ is κ-closed. Let HS denote the class of hereditarily symmetric $\mathbb{P}$-names, and let $\mathfrak{N}$ be HS^G. If $E \in I$ and $\mathrm{fix}(E) \le \mathrm{sym}(\dot{x})$, we say that E is a *support* of $\dot{x}$.

We know that $\mathfrak{N} \subseteq \mathfrak{M}[G]$ is a model of ZF. It follows from the κ-closure of $\mathbb{P}$ and $\mathcal{F}$ that the conditions for Lemma 2.1 hold, and thus $\mathfrak{N} \models \mathsf{DC}_{<\kappa}$. We will see later that the axiom of choice, indeed DC_κ itself, fails in $\mathfrak{N}$.

PROPOSITION 3.1. *For all $z \in \mathcal{Z}$ and $\alpha < \kappa$, $r_{z,\alpha} \in \mathfrak{N}$ and $R_z \in \mathfrak{N}$.*

Proof. The above observation shows that for every $\pi \in \mathscr{G}$ and $(z, \alpha) \in \mathcal{Z} \times \kappa$, we have
$$\pi \dot{r}_{z,\alpha} = \dot{r}_{\pi(z,\alpha)}, \qquad \pi \dot{R}_z = \dot{R}_z.$$
It follows that $\{(z, \alpha)\}$ is a support for $\dot{r}_{z,\alpha}$ (and clearly every name appearing in $\dot{r}_{z,\alpha}$ is symmetric, being a canonical name of an ordinal). Therefore $\dot{r}_{z,\alpha} \in \mathsf{HS}$. Now all the names appearing in $\dot{R}_z$ are from HS, and having $\emptyset$ as support we have $\dot{R}_z \in \mathsf{HS}$ as well. Therefore the sets $r_{z,\alpha}$, R_z are all in $\mathfrak{N}$. ∎

Two facts which are useful later are:

FACT 3.2. *For every $Q \in \mathfrak{N}$ such that $Q \subseteq \mathcal{Z}$, we have $D_Q \in \mathfrak{N}$. Moreover, the function $Q \mapsto D_Q$ is in $\mathfrak{N}$.*

Proof. We define the name $\dot{F} = \{(\check{z}, \dot{R}_z)^\bullet \mid z \in \mathcal{Z}\}^\bullet$ is in HS, since for every z we have $\mathrm{sym}(\dot{R}_z) = \mathscr{G}$. Let $F = \dot{F}^G$. Then for $z \in \mathcal{Z}$ we have $F(z) = R_z$, and so for $Q \subseteq \mathcal{Z}$ we have $D_Q = \bigcup \{R_z \mid z \in Q\}$. Therefore whenever $Q \in \mathfrak{N}$, $D_Q \in \mathfrak{N}$ as well. ∎

We remark that from a name $\dot{Q} \in \mathsf{HS}$ for a subset of $\mathcal{Z}$ one can give a (relatively) canonical name for D_Q which has the same support as $\dot{Q}$. However by showing that $D_Q \in \mathfrak{N}$ is definable from $Q \in \mathfrak{N}$ we in fact prove that there is such a name.

FACT 3.3. *The following is true in $\mathfrak{N}$. For every $z \in \mathcal{Z}$, R_z can be mapped onto κ, and therefore D_Q can be mapped onto κ for every non-empty Q.*

Proof. The map $r_{z,\alpha} \mapsto \min r_{z,\alpha}$ is well-defined in $\mathfrak{N}$, and by a simple density argument one can see that it is surjective in $\mathfrak{M}[G]$ and therefore in $\mathfrak{N}$ as well. ∎

PROPOSITION 3.4. *In $\mathfrak{N}$, for no $z \in \mathcal{Z}$ can R_z be well-ordered, and therefore the axiom of choice fails.*

Proof. We know that R_z can be mapped onto κ, therefore it suffices to show that there is no injection from κ into R_z. If R_z could have been well-ordered, such a surjection could have been reversed to an injection from κ.

Assume towards a contradiction that $p \Vdash \ulcorner \dot{f} \colon \check{\kappa} \to \dot{R}_z$ is injective $\urcorner$, and $\dot{f} \in \mathsf{HS}$. Let E be a support for $\dot{f}$, and let $q \leq p$ and $\alpha, \tau < \kappa$ be such that $(z, \alpha) \notin E$ and $q \Vdash \dot{f}(\check{\tau}) = \dot{r}_{z,\alpha}$.

Let $\beta \neq \alpha$ be such that $(z, \beta) \notin E$ and $((z, \beta), \gamma) \notin \operatorname{dom} q$ for all γ. We define $\pi \in \mathscr{G}$ by $\pi(z, \alpha) = (z, \beta)$, $\pi(z, \beta) = (z, \alpha)$ and otherwise $\pi(x, y) = (x, y)$. Clearly $\pi \in \operatorname{fix}(E)$, and therefore $\pi \dot{f} = \dot{f}$. By the symmetry lemma we have

$$\pi q \Vdash \dot{f}(\check{\tau}) = \dot{r}_{z,\beta}.$$

If q and πq are compatible then q has an extension which forces that $\dot{f}$ is not a function, a contradiction. Suppose $((t, \varepsilon), \delta) \in \operatorname{dom} q \cap \operatorname{dom} \pi q$. If $t \neq z$ then $\pi(t, \varepsilon) = (t, \varepsilon)$, and by the definition of πq we have

$$\pi q((t, \varepsilon), \delta) = \pi q(\pi(t, \varepsilon), \delta) = q(t, \varepsilon, \delta).$$

Otherwise $t = z$, if $\varepsilon \notin \{\alpha, \beta\}$ then $\pi(t, \varepsilon) = (t, \varepsilon)$, and so $q((t, \varepsilon), \delta) = \pi q((t, \varepsilon), \delta)$. Moreover, if $t = z$ then $\varepsilon \neq \beta$. Recall the choice of β was such that

$$((z, \beta), \delta) \notin \operatorname{dom} q.$$

Finally, if $((z, \alpha), \delta) \in \operatorname{dom} \pi q$ then $(\pi^{-1}(z, \alpha), \delta) = ((z, \beta), \delta) \in \operatorname{dom} q$, and so it is impossible that $t = z$ and $\varepsilon = \alpha$. Therefore q and πq agree on all the points in their common domain, and are compatible, which is our desired contradiction.

Therefore in $\mathfrak{N}$ there is no injection from κ into R_z, and choice fails. ∎

We have in fact shown that κ and R_z have incomparable cardinalities in $\mathfrak{N}$, and therefore DC_κ fails as promised.

THEOREM 3.5. *Let $Q, T \subseteq \mathcal{Z}$ in $\mathfrak{N}$. If $Q \nsubseteq T$ then $\mathfrak{N} \models |D_Q| \nleq |D_T|$ and $|D_Q| \nleq^* |D_T|$.*

Proof. If there were an injection from D_Q into D_T then there would be a surjection from D_T onto D_Q. It is therefore sufficient to argue for the $\nleq^*$ case.

Let $\dot{Q}$ and $\dot{T}$ be names for Q and T respectively, both in HS, and let $\dot{D}_Q$ and $\dot{D}_T$ be names in HS for D_Q and D_T respectively.

Suppose that $p \Vdash \ulcorner \dot{f} \colon \dot{D}_T \to \dot{D}_Q$ is surjective$\urcorner$ for some $\dot{f} \in \mathsf{HS}$. We will prove that $p \Vdash \dot{Q} \subseteq \dot{T}$. Assume towards a contradiction that this is not the case. If p does not decide the statement $\dot{Q} \nsubseteq \dot{T}$ then it has an extension forcing it, and we shall take it instead. So we may assume a stronger assumption towards a contradiction, $p \Vdash \dot{Q} \nsubseteq \dot{T}$. Let E be a support for the names $\dot{f}, \dot{Q}, \dot{T}, \dot{D}_Q, \dot{D}_S$.

Let q be an extension of p such that there are $z, t \in \mathcal{Z}$ and $\alpha, \delta < \kappa$ such that $(z, \alpha) \notin E$, and

$$ q \Vdash \check{z} \in \dot{Q} \setminus \dot{T} \wedge \check{t} \in \dot{T} \wedge \dot{f}(\dot{r}_{t,\delta}) = \dot{r}_{z,\alpha}. $$

This implies that $z \neq t$. Let $\beta \neq \alpha$ be such that $(z, \beta) \notin E$ and there is no $\gamma < \kappa$ for which $((z, \beta), \gamma) \in \operatorname{dom} q$. We define π to be the permutation such that $\pi(z, \alpha) = (z, \beta)$, $\pi(z, \beta) = (z, \alpha)$ and $\pi(x, y) = (x, y)$ otherwise. We have $\pi \in \operatorname{fix}(E)$, and therefore all the names of interest are unchanged by π.

We have $\pi q \Vdash \dot{f}(\dot{r}_{t,\delta}) = \dot{r}_{z,\beta}$. However a simple verification as in the proof of Proposition 3.4 shows that q and πq are compatible, and therefore q has an extension which forces that $\dot{f}$ is not a function, a contradiction. ∎

4. Embedding a proper class. In this section we extend the result by Roguski [Rog90] in which he proves the following theorem:

THEOREM (Roguski). *Let $\mathfrak{M}$ be a countable transitive model of* ZFC, *and* $(I, \preceq)$ *a partially ordered class such that* $I, \preceq$ *are both classes of* $\mathfrak{M}$ *and every initial segment of* $(I, \preceq)$ *belongs to* $\mathfrak{M}$. *Then there is a countable transitive model* $\mathfrak{N}$ *for* ZF, *which is a symmetric extension of* $\mathfrak{M}$ *and a class* $\{S_i \mid i \in I\}$ *in* $\mathfrak{N}$ *such that for all* $i, j \in I$,

$$ i \preceq j \leftrightarrow \mathfrak{N} \models |S_i| \leq |S_j|. $$

From this theorem he draws the consistency of a proper class of pairwise incomparable cardinals. However, it seems that Roguski is proving less than he claims to prove. Roguski embeds a proper class into the cardinals of a model of ZF; however, it is unclear that the class function $i \mapsto S_i$ is definable internally to that model. Roguski's proof shows, instead, that given any set of cardinals, there is one incomparable to all of them. Using Theorem 4.1 we will show that this result can be extended so that $\mathsf{DC}_{<\kappa}$ is preserved for a fixed κ, and that we may replace $|S_i| \leq |S_j|$ by $|S_i| \leq^* |S_j|$.

Let $\mathfrak{M}$ be a countable transitive model of $\mathsf{ZFC} + \mathsf{GCH}$, κ a regular cardinal in $\mathfrak{M}$, and $(I, \preceq)$ a partially ordered class in $\mathfrak{M}$ such that every initial segment of I is a set of $\mathfrak{M}$. Without loss of generality we may assume that there is a class in $\mathfrak{M}$ which well-orders I, for otherwise we can force such a class without adding sets. Therefore we may assume that $I \subseteq \mathsf{Ord}^{\mathfrak{M}}$. In this section we shall prove the following theorem:

THEOREM 4.1. *There exists a class-generic extension $\mathfrak{M}[G]$ with an intermediate model $\mathfrak{N} \subseteq \mathfrak{M}[G]$ such that $\mathfrak{N} \models \mathsf{ZF} + \mathsf{DC}_{<\kappa}$ in which $(I, \preceq)$ can be embedded into the cardinals of $\mathfrak{N}$ with the order $\leq$ or with the order $\leq^*$, such that every initial segment of this embedding is in $\mathfrak{N}$ and the embedding is definable in $\mathfrak{M}[G]$.*

By embedding $(\mathcal{P}(I) \cap \mathfrak{M}, \subseteq)$ into the cardinals of the symmetric extension we will ensure that I has been embedded into it using the same argument as in the previous section. Note that if I is actually a set in $\mathfrak{M}$ then Theorem 3.5 proves the claim, so we may assume that I is a proper class of $\mathfrak{M}$. We aim to mimic the previous proof, therefore for every $i \in I$ we shall add generic subsets to a regular cardinal. In order to preserve $\mathsf{DC}_{<\kappa}$ we require the forcing to be κ-closed, so we will only add subsets to cardinals above κ. We will assume that I is a class of regular cardinals and $\min I \geq \kappa$.

We define the forcing in $\mathfrak{M}$. For every $i \in I$ let $\mathbb{P}_i = (P_i, \leq)$ be the forcing which adds i subsets to i, namely $p \in P_i$ is a partial function from $i \times i$ to 2 such that $|\operatorname{dom} p| < i$, and $p \leq q$ if and only if $q \subseteq p$. Let $\mathbb{P}$ be the Easton support product $\prod_{i \in I} \mathbb{P}_i$. We shall denote by $\mathbb{P}^{\leq i}$ the Easton support product of $\mathbb{P}_j$ for $j \leq i$. This is a product of κ-closed forcings and therefore it is κ-closed, and we also point out that by assuming GCH it does not change cofinalities.

The conditions in $\mathbb{P}$ are functions such that $p(i)$ is a condition in P_i, and for all $i \in I$ we have $|\{j \leq i \mid p(j) \neq 1_{\mathbb{P}_j}\}| < i$. Alternatively we may think about the conditions as functions from $I \times \mathsf{Ord} \times \mathsf{Ord}$ to $\{0,1\}$ such that if (i, α, β) is in $\operatorname{dom} p$ then $\beta, \alpha < i$, and for every $i \in I$ we have $|\{(j, \alpha, \beta) \mid (j, \alpha, \beta) \in \operatorname{dom} p \wedge j \leq i\}| < i$. We will identify $\mathbb{P}^{\leq i}$ with those $p \in \mathbb{P}$ such that $\operatorname{dom} p \subseteq (i+1) \times i \times i$.

If G is a $\mathbb{P}$-generic class over $\mathfrak{M}$ then in $\mathfrak{M}[G]$ it defines i new subsets for every (regular) cardinal in I, and $\mathfrak{M}[G]$ is a model of ZFC. Note that as before $\bigcup G = g$ is a class function $g \colon I \times \mathsf{Ord} \times \mathsf{Ord} \to 2$. We define the following sets from G and give them canonical names:

- For $i \in I$ and $\alpha < i$ the set $r_{i,\alpha} = \{\gamma < i \mid g(i, \alpha, \gamma) = 1\}$ is given the name
$$\dot{r}_{i,\alpha} = \{(p, \check{\gamma}) \mid p(i, \alpha, \gamma) = 1 \wedge p \in \mathbb{P}^{\leq i}\}.$$

- For $i \in I$ we define $R_i = \{r_{i,\alpha} \mid \alpha < i\}$ with the name
$$\dot{R}_i = \{\dot{r}_{i,\alpha} \mid \alpha < i\}^{\bullet}.$$

- For a set $Q \subseteq I$ denote $D_Q = \bigcup_{i \in Q} R_i$. Of course Q might be generic, and as before we do not give a name for D_Q.

We shall now proceed to define the symmetric extension. First we define $\mathscr{G}$ to be a group of automorphisms of $\mathbb{P}$; while this group will be a proper

class, each permutation will only move a set. We say that $\pi \in \mathscr{G}$ if π is a permutation of $I \times \mathsf{Ord} \times \mathsf{Ord}$ such that the following hold:

(1) For all $i \in I$, if $(i, \alpha, \gamma) \in \mathrm{dom}\,\pi$ then $\alpha, \gamma < i$;
(2) whenever $\pi(i, \alpha, \gamma) = (i', \alpha', \gamma')$, we have $i = i'$, $\alpha' < i$, $\gamma = \gamma'$;
(3) $\mathbb{D}_\pi = \{(i, \alpha, \gamma) \mid \pi(i, \alpha, \gamma) \neq (i, \alpha, \gamma)\}$ is a set in $\mathfrak{M}$;
(4) for every $i \in I$, we have $|\{(i, \alpha, \gamma) \mid (i, \alpha, \gamma) \in \mathbb{D}_\pi\}| < i$.

We define the action of $\mathscr{G}$ on $\mathbb{P}$ as before,

$$\pi p(\pi(i, \alpha, \gamma)) = p(i, \alpha, \gamma).$$

For each $i \in I$ we define $\mathscr{G}_i = \{\pi \in \mathscr{G} \mid \mathbb{D}_\pi \subseteq (i+1) \times i \times i\}$. Then $\mathscr{G}_i$ is a group, $i \leq j$ implies $\mathscr{G}_i \leq \mathscr{G}_j$, and $\bigcup_{i \in I} \mathscr{G}_i \simeq \mathscr{G}$. We define $G_{\leq i} = G \cap \mathbb{P}^{\leq i}$. We observe that $\mathfrak{M}^{\mathbb{P}^{\leq i}} \subseteq \mathfrak{M}^{\mathbb{P}^{\leq j}}$ whenever $i \leq j$, and if $\dot{x} \in \mathfrak{M}^{\mathbb{P}^{\leq i}}$ then $\dot{x}^{G_i} = \dot{x}^{G_j}$ as well.

Let $K_i = [(I \cap i^+) \times i \times i]^{<\kappa}$, and let $\mathcal{F}_i$ be the κ-complete filter of subgroups of $\mathscr{G}_i$ generated by $\mathrm{fix}(E)$ for $E \in K_i$,

$$\mathcal{F}_i = \{H \leq \mathscr{G}_i \mid \exists E \in K_i : \mathrm{fix}(E) \leq H\}.$$

Then $\mathcal{F}_i \subseteq \mathcal{F}_j$ for $i \leq j$. For every $i \in I$ let HS_i be $\mathsf{HS}_{\mathcal{F}_i} \subseteq \mathfrak{M}^{\mathbb{P}^{\leq i}}$. Let $\mathcal{F} = \bigcup_{i \in I} \mathcal{F}_i$. Then $\mathsf{HS} = \mathsf{HS}_\mathcal{F} = \bigcup_{i \in I} \mathsf{HS}_i$.

It is standard to define $\mathfrak{M}[G]$ as the union $\bigcup_{i \in I} \mathfrak{M}[G_i]$. For every $i \in I$ we define $\mathfrak{N}_i = (\mathsf{HS}_i)^{G_i} \subseteq \mathfrak{M}[G_i]$ to be a symmetric extension of $\mathfrak{M}$. Then for $i \leq j$ we have $\mathfrak{N}_i \subseteq \mathfrak{N}_j$, and every $\mathfrak{N}_i$ has the same ordinals (and initial ordinals) and satisfies $\mathsf{ZF} + \mathsf{DC}_{<\kappa}$.

Let $\mathfrak{N} = \bigcup_{i \in I} \mathfrak{N}_i$. We first observe that

$$\mathfrak{N} = \bigcup_{i \in I} \mathfrak{N}_i = \bigcup_{i \in I} (\mathsf{HS}_i)^{G_i} = \bigcup_{i \in I} (\mathsf{HS}_i)^G = \left(\bigcup_{i \in I} \mathsf{HS}_i\right)^G = \mathsf{HS}^G.$$

Then $\mathfrak{N}$ is a model of $\mathsf{ZF} + \mathsf{DC}_{<\kappa}$. The model satisfies $\mathsf{DC}_{<\kappa}$ because every $\mathfrak{N}_i$ does. We shall see that $r_{i,\alpha}, R_i$ and D_Q (for $Q \in \mathfrak{N}$) are in $\mathfrak{N}$, and the class $\{D_Q \mid Q \in \mathcal{P}(I) \cap \mathfrak{N}\}$ is a class of $\mathfrak{N}$. Note that the equality above tells us that $x \in \mathfrak{N}$ if and only if there exists $\dot{x} \in \mathsf{HS}$ such that $\dot{x}^G = x$, and therefore there exists $E \in [I \times \mathsf{Ord} \times \mathsf{Ord}]^{<\kappa} \cap \mathfrak{M}$ such that $\mathrm{fix}(E) \leq \mathrm{sym}(\dot{x})$. As before we say that E is a *support* of $\dot{x}$.

PROPOSITION 4.2. *For all $i \in I$ and $\alpha < i$ we have $r_{i,\alpha} \in \mathfrak{N}$ and $R_i \in \mathfrak{N}$. Furthermore $i \mapsto R_i$ is definable in $\mathfrak{M}[G]$ and its initial segments are in $\mathfrak{N}$, and therefore whenever $Q \subseteq I$ is a set in $\mathfrak{N}$ then D_Q is in $\mathfrak{N}$.*

Proof. It is immediate that $\{(i, \alpha, 0)\}$ is a support of $\dot{r}_{i,\alpha}$ and $\dot{r}_{i,\alpha} \in \mathsf{HS}_j$ for any $j > i$. It follows that $\emptyset$ is a support of $\dot{R}_i$, which is also in HS_j. Hence $r_{i,\alpha}$ and R_i are both in $\mathfrak{N}$ for any i and α.

Consider the class name $\dot{F} = \{(\check{i}, \dot{R}_i)^\bullet \mid i \in I\}^\bullet$. For all $j \in I$ the name $\dot{F}_j = \{(\check{i}, \dot{R}_i)^\bullet \mid i \leq j\}^\bullet$ is a symmetric name in HS_j, and $\mathrm{sym}_{\mathscr{G}_j}(\dot{F}_j) = \mathscr{G}_j$.

Therefore $\dot{F} = \bigcup_{i \in I} \dot{F}_i$ is a symmetric class in HS, and its interpretation $F = \dot{F}^G$ is a class of $\mathfrak{M}[G]$ whose initial segments are in $\mathfrak{N}$. It follows that whenever $Q \subseteq I$ is a set in $\mathfrak{N}$ then it appears in some HS_j and therefore

$$D_Q = \bigcup_{i \in Q} F(i) = \{x \mid \exists i \in Q : x \in F(i)\},$$

is in $\mathfrak{N}$ as promised. ∎

We observe that as in Fact 3.3, every R_i can be mapped onto i with the map $r_{i,\alpha} \mapsto \min r_{i,\alpha}$. Obviously there is no R_i that can be mapped onto i^+. Furthermore the proof that $\kappa \not\leq |R_i|$ is the same as the proof of Proposition 3.4.

PROPOSITION 4.3. *For every* $Q, T \subseteq I$, $\mathfrak{N} \models Q \subseteq T \leftrightarrow |D_Q| \leq^* |D_T|$.

Proof. If $Q \subseteq T$ then $D_T \subseteq D_Q$, and the result is trivial. Suppose that $\dot{Q}$ and $\dot{T}$ are names in HS for the sets Q, T respectively. Assume towards a contradiction that $p \Vdash \dot{Q} \not\subseteq \dot{T} \wedge \ulcorner \dot{f} \colon \dot{D}_T \to \dot{D}_Q$ is a surjection$\urcorner$ where $\dot{f}, \dot{D}_Q, \dot{D}_Q \in \mathsf{HS}$ and $\dot{D}_Q, \dot{D}_T$ are names for D_Q and D_T respectively.

The proof of Theorem 3.5 applies here completely by noting that there is some $i \in I$ such that the entire proof is actually carried out in $\mathbb{P}^{\leq i}$. ∎

This concludes the proof of Theorem 4.1, as taking $S_i = \{j \in I \mid j \preceq i\}$ guarantees that $|S_i| \leq^* |S_j| \leftrightarrow |S_i| \leq |S_j| \leftrightarrow i \preceq j$.

5. Extensions of the Theorem. We draw two corollaries from the Theorem and show the independence of two choice principles from DC_κ (for any κ). When a choice principle is not provable by DC_κ, for any κ, it hints that it may be equivalent to the axiom of choice, or that it is "orthogonal" to DC_κ-like principles.

THEOREM 5.1. *For every cardinal μ it is consistent with* $\mathsf{ZF} + \mathsf{DC}_\mu$ *that for every set of cardinals there is one incomparable to all of them, and for every ordinal α there is a decreasing sequence of cardinals of order type α^*.*

Proof. Let $\kappa > \mu$ and consider the model from Section 4 in which we embed the class I of regular cardinals above κ with the discrete order into the cardinals of $\mathfrak{N}$; for better readability we identify the I with its transitive collapse, Ord. We have $\mathfrak{N} \models \mathsf{DC}_{<\kappa}$, and therefore $\mathfrak{N} \models \mathsf{DC}_\mu$. For any $\alpha \in \mathsf{Ord}$ we have R_α as defined in the proof of Theorem 4.1; then $\{|R_\alpha| \mid \alpha < \lambda\}$ is an antichain in both $\leq^*$ and in $\leq$, for every $\lambda \in \mathsf{Ord}$.

Let α be an ordinal, and define the set $D_{\beta^*} = \bigcup\{R_\gamma \mid \beta \leq \gamma < \alpha\}$ for every $\beta < \alpha$. By Proposition 4.3 we conclude that $\{|D_{\beta^*}| \mid \beta < \alpha\}$ is a decreasing chain of cardinals in both $\leq$ and $\leq^*$. ∎

Note that it is impossible to find a decreasing sequence of cardinals of order type Ord^*. Any set whose cardinality is the maximum of such a sequence must have a proper class of different subsets, contrary to the power set axiom.

Finally we will show that the constructive set-theory oriented axiom known as WISC is independent of ZF. This result is due to van den Berg (see [van12]) (^{2}). The proof given by van den Berg assumes some very large cardinals, and we improve it by removing this additional assumption and by showing the compatibility of this failure with DC_μ for arbitrary μ.

The principle WISC (Weakly Initial Set Cover) can be formulated as follows: *For every set X there is a set Y such that whenever Z is a set and $f\colon Z \to X$ is a surjection then there is $q\colon Y \to Z$ such that $f \circ q$ is onto X.* This formulation is due to François Dorais (see [Rob13] for more details).

We will now show that the model from Theorem 5.1 satisfies ¬WISC. Recall that for all α, R_α can be mapped onto κ. We will show that for κ there is no Y as in the requirement of WISC.

THEOREM 5.2. *Let $\mathfrak{N}$ be the model from Theorem 5.1. Then for every set $Y \in \mathfrak{N}$, there is $\alpha \in \mathsf{Ord}$ such that R_α can be mapped onto κ by some function h, but every function $f\colon Y \to R_\alpha$ has range of cardinality $< \kappa$. Therefore there is no such f for which $h \circ f$ is onto κ. Hence $\mathfrak{N} \models \neg\mathsf{WISC}$.*

Proof. We will show that in $\mathfrak{N}$ for every Y there is some α such that any $f\colon Y \to R_\alpha$ must satisfy $|\mathrm{rng}\, f| < \kappa$, and therefore it is impossible that any composition of f with a function from R_α is onto κ.

Let $Y \in \mathfrak{N}$ be any set, and let $\alpha \in \mathsf{Ord}$ be such that for some $\beta < \alpha$ we have $\dot{Y} \in \mathsf{HS}_\beta$. This means that any condition which appears in $\dot{Y}$ appears in $\mathbb{P}^{\leq\beta}$. Suppose that $p \Vdash \dot{f}\colon \dot{Y} \to \dot{R}_\alpha$, and $\dot{f} \in \mathsf{HS}$. If $p \Vdash |\mathrm{rng}\, f| < \kappa$ then we are done, so assume that this is not the case, and $p \Vdash |\mathrm{rng}\, f| \not< \kappa$.

Let $E \in [\mathsf{Ord} \times \mathsf{Ord} \times \mathsf{Ord}]^{<\kappa}$ be a support for $\dot{f}, \dot{Y}$ (recall that $\dot{R}_\alpha$ is supported by any set). Let $q \leq p$ be such that there is $\delta < \alpha$ such that for all $\gamma < \alpha$, $(\alpha, \delta, \gamma) \notin E$, and for some $\dot{y}$ we have $q \Vdash \dot{f}(\dot{y}) = \dot{r}_{\alpha,\delta}$. We can now find $\tau \neq \delta$ such that $(\alpha, \tau, \gamma) \notin E \cup \mathrm{dom}\, q$ for any $\gamma < \alpha$. Let π be the permutation in $\mathscr{G}$ defined as follows: $\pi(\alpha, \delta, \gamma) = (\alpha, \tau, \gamma)$, $\pi(\alpha, \tau, \gamma) = (\alpha, \delta, \gamma)$, and $\pi(x, y, z) = (x, y, z)$ otherwise.

As $\dot{y}$ is a name appearing in $\dot{Y}$, and thus $\dot{y} \in \mathsf{HS}_\beta$, it follows that any condition in $\dot{y}$ appears in $\mathbb{P}^{\leq\beta}$. This means that any permutation in $\mathscr{G}$ which does not move any condition in $\mathbb{P}^{\leq\beta}$ will not move $\dot{y}$ either, in particular this is true for π defined above.

(^{2}) Van den Berg names this axiom the *Axiom of Multiple Choice*, an unfortunate name as it is already the name of a relatively known choice principle.

We have $\pi q \Vdash \dot{f}(\dot{y}) = \dot{r}_{\alpha,\tau}$, and as in the proof of Theorem 3.5, q and πq are compatible, which is a contradiction, and the conclusion follows. ∎

Therefore for every μ, WISC is unprovable from $\mathsf{ZF} + \mathsf{DC}_{\mu}$. This extends the results by Rathjen which establish the independence of a slightly stronger choice principle from ZF by a similar method to van den Berg's (see [Rat06]).

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