FUNDAMENTA
MATHEMATICAE
202 (2009)

Compact spaces that do not map onto finite products

by

Antonio Avilés (Paris)

Abstract. We provide examples of nonseparable compact spaces with the property that any continuous image which is homeomorphic to a finite product of spaces has a maximal prescribed number of nonseparable factors.

1. Introduction. The motivation of this work are several problems coming from [1], [2] and [9], dealing with the possibility of mapping a finite product of spaces onto another product space with more factors.

Let B be the closed euclidean ball of a nonseparable Hilbert space, endowed with its weak topology. It was proven in [2] that B^2 is not homeomorphic to B, and the natural question arises whether B^2 is at least a continuous image of B. We shall prove that the answer is also negative. More generally:

Theorem 1. *Let $n < m$ be natural numbers and suppose $f : B^n \to X_1 \times \cdots \times X_m$ is an onto continuous map. Then there exists $i \leq m$ such that X_i is metrizable.*

We shall also provide an alternative proof of the fact shown in [2] that if B^n is homeomorphic to L^m for some L and some m, then m divides n. Properties of this kind are also proven in [2] for some spaces of probability measures, but we shall show that the methods in our paper do not apply to such spaces.

The second problem deals with linearly ordered spaces. The following result was first obtained by Treybig [13], though there exists a shorter proof by Bula, Dębski and Kulpa [5]:

Theorem 2 (Treybig). *Let L be a linearly ordered compact space, and X_0 and X_1 two infinite compact spaces. If there is a continuous surjection $f : L \to X_0 \times X_1$, then both X_0 and X_1 are metrizable.*

2000 *Mathematics Subject Classification*: 54B10, 54D30, 54F05, 46B50.
Key words and phrases: finite product, Knaster condition, chain conditions, linearly ordered compact, ball of the Hilbert space, euclidean ball.
The author was supported by a Marie Curie Intra-European Fellowship MCEIF-CT2006-038768 (E.U.) and research project MTM2005-08379 (MEC and FEDER).

Mardešić [9] tried to generalize this theorem to higher dimensions, proposing the following conjecture:

Conjecture 3 (Mardešić). *Let $L_1,\dots,L_n$ be linearly ordered compact spaces, let $X_0,\dots,X_n$ be infinite compact spaces and let $f:\prod_1^n L_i\to\prod_0^n X_j$ be an onto continuous map. Then there exist $i,j\le n$, $i\ne j$, such that X_i and X_j are metrizable.*

He proved [9] that the conjecture holds under the assumption that all spaces X_i are separable. Our methods enable us to obtain the following:

Theorem 4. *Let $L_1,\dots,L_n$ be linearly ordered compact spaces, let $X_0,\dots,X_n$ be infinite compact spaces and let $f:\prod_{i=1}^n L_i\to\prod_{j=0}^n X_j$ be an onto continuous map. Then there exist $0\le i,j\le n$, $i\ne j$, such that X_i and X_j are separable.*

Notice that Mardešić's partial answer to the conjecture and our own seem completely unrelated since his hypothesis is stronger than our conclusion. However, in the case when $n=2$ both results can be combined to show that at least one factor must always be metrizable:

Corollary 5. *Let L_1 and L_2 be linearly ordered compact spaces, let X_0, X_1 and X_2 be infinite compact spaces and let $f:L_1\times L_2\to X_0\times X_1\times X_2$ be an onto continuous map. Then there exists $i\in\{0,1,2\}$ such that X_i is metrizable.*

Proof. By Theorem 4, two factors of the last product are separable, say X_0 and X_1. Let Y_3 be an infinite quotient space of X_3 of countable weight. Then there is a quotient $L_1\times L_2\to X_0\times X_1\times Y_3$ where all factors are separable, so by Mardešić's result [9] at least two of them are metrizable, so either X_0 or X_1 is metrizable. ∎

The third problem deals with the spaces $\sigma_n(\Gamma)=\{x\in 2^\Gamma:|\mathrm{supp}(x)|\le n\}$. In our previous work [1] we studied the homeomorphic classification of finite and countable products of these spaces. In this paper, we shall determine when such a product is a continuous image of another one.

We want to express our gratitude to Stevo Todorcevic for calling our attention to the work of Mardešić.

2. Indecomposability properties (I). All along this paper, we work with compact Hausdorff topological spaces. When we talk about compact spaces, the Hausdorff T_2 separation axiom is implicitly assumed.

Definition 6. Let X be a compact space. A *pseudoclopen* of X is a pair $a=(a[0],a[1])$ such that $a[0]$ and $a[1]$ are open subsets of X and $\overline{a[0]}\subset a[1]$.

Notice that every clopen set c can be identified with a pseudoclopen (c,c). Conversely, if K is a totally disconnected compact space, then every pseudoclopen a is interpolated by a clopen set c, meaning $a[0] \subset c \subset a[1]$. The notion of pseudoclopen substitutes the notion of clopen sets in general (not totally disconnected) compact spaces.

DEFINITION 7. An uncountable family $\mathcal{F}$ of sets will be called a *Knaster-disjoint family* if every uncountable subfamily $\mathcal{G} \subset \mathcal{F}$ contains two disjoint elements. An uncountable family $\mathcal{F}$ of pseudoclopens of X will be called *Knaster-disjoint* if $\{a[1] : a \in \mathcal{F}\}$ is a Knaster-disjoint family of sets.

The terminology is motivated by Knaster's well known chain condition: *Every uncountable family of nonempty opens sets contains an uncountable family in which any two elements have nonempty intersection.* Thus, for a completely regular space, the failure of Knaster's condition is equivalent to the existence of an uncountable Knaster-disjoint family of strongly nonempty pseudoclopens (we call a pseudoclopen a *strongly nonempty* if $a[0] \neq \emptyset$).

DEFINITION 8. Let X be a compact space, and n a natural number. We say that X has *property I_n* if for every collection of $n+1$ Knaster-disjoint families $\mathcal{F}_0, \ldots, \mathcal{F}_n$ of pseudoclopens, there exist uncountable subfamilies $\mathcal{G}_i \subset \mathcal{F}_i$ for $i = 0, \ldots, n$ such that for every choice $a_i \in \mathcal{G}_i$ for $i = 0, \ldots, n$ we have $a_0[0] \cap \cdots \cap a_n[0] = \emptyset$.

PROPOSITION 9. *Let X be a compact space with property I_n, and $f : X \to Y$ a continuous onto map. Then Y has property I_n.*

Proof. Let $\mathcal{F}_0, \ldots, \mathcal{F}_n$ be families of Knaster-disjoint pseudoclopens in Y. For every $i \leq n$, let $\mathcal{F}_i' = \{(f^{-1}(a[0]), f^{-1}(a[1])) : a \in \mathcal{F}_i\}$. Then $\mathcal{F}_i'$ is a Knaster-disjoint family of pseudoclopens in X. Because X has property I_n, we can find uncountable subfamilies $\mathcal{G}_i \subset \mathcal{F}_i$ such that whenever $a_i \in G_i$, then $f^{-1}(a_0[0]) \cap \cdots \cap f^{-1}(a_n[0]) = \emptyset$, which implies that $a_0[0] \cap \cdots \cap a_n[0] = \emptyset$ because f is onto. ∎

PROPOSITION 10. *Let $X_0, \ldots, X_n$ be compact spaces such that $X = X_0 \times \cdots \times X_n$ has property I_n. Then there exists $i \leq n$ such that X_i satisfies Knaster's condition.*

Proof. For a set $s \subset X_i$, we write $u_i(s) = \{(x_0, \ldots, x_n) \in X : x_i \in s\}$. Also, for a pseudoclopen a of X_i, $u_i(a) = (u_i(a[0]), u_i(a[1]))$ is a pseudoclopen of X. Now, suppose no X_i satisfies Knaster's condition. Then for every i we can find an uncountable Knaster-disjoint family $\mathcal{F}_i$ of strongly nonempty pseudoclopens in X_i. Associated with them, we have Knaster-disjoint families $\mathcal{F}_i' = \{u_i(a) : a \in \mathcal{F}_i\}$ of pseudoclopens in X. These families contradict property I_n, because $a_0[0] \cap \cdots \cap a_n[0] \neq \emptyset$ whenever $a_i \in \mathcal{F}_i'$. ∎

We now introduce some operations and relations. For two pseudoclopens a and b, we write:

- $b \subset a$ if $b[0] \subset a[0]$ and $b[1] \subset a[1]$,
- $b \prec a$ if $a[0] \subset b[0] \subset b[1] \subset a[1]$ (we say that b is *finer* than a),
- $a \cup b = (a[0] \cup b[0], a[1] \cup b[1])$ (this is a new pseudoclopen).

Definition 11. A family $\mathcal{B}$ of pseudoclopens of X will be called a *basis* if for every pseudoclopen a of X there exist finitely many pseudoclopens $b_1, \dots, b_m$ from $\mathcal{B}$ such that $b_1 \cup \dots \cup b_m \prec a$.

Lemma 12. *If $\mathcal{B}$ is basis for the topology of a compact space X, then the family of all pseudoclopens a such that $a[0] \in \mathcal{B}$ and $a[1] \in \mathcal{B}$ constitutes a basis of pseudoclopens.*

Lemma 13. *If X is a totally disconnected compact space and $\mathcal{B}$ is a basis for the topology of X consisting of clopen sets, then $\mathcal{B}' = \{(c, c) : c \in \mathcal{B}\}$ is a basis of pseudoclopens in X.*

Lemma 14. *In the definition of property I_n we may assume that the families $\mathcal{F}_i$ are subfamilies of some basis of pseudoclopens for the space.*

Proof. Let $\mathcal{F}_0, \dots, \mathcal{F}_n$ be arbitrary Knaster-disjoint families of pseudoclopens of X, and let $\mathcal{B}$ be a basis of pseudoclopens. Because $\mathcal{B}$ is a basis, we can suppose (by passing to finer pseudoclopens) that every $a \in \mathcal{F}_i$ is a finite union of elements of $\mathcal{B}$. Because we are looking for uncountable subfamilies, we can suppose without loss of generality that there exist natural numbers $m_0, \dots, m_n$ such that every element $a \in \mathcal{F}_i$ is a union of exactly m_i elements of $\mathcal{B}$, and we write it in the form $a = a_i^1 \cup \dots \cup a_i^{m_i}$ where each a_i^j is an element of $\mathcal{B}$. For every $i \le n$ and every $j \le m_i$, let $\mathcal{F}_i^j = \{a_i^j : a \in \mathcal{F}_i\}$. For every choice of numbers $j = (j_0, \dots, j_n)$ with $j_i \le m_i$, we can apply our hypothesis to the families $\mathcal{F}_0^{j_0}, \dots, \mathcal{F}_n^{j_n}$, and this implies that there are uncountable subfamilies $\mathcal{G}_i^j \subset \mathcal{F}_i$ such that whenever $a \in \mathcal{G}_i^j$, we have $a_0^{j_0}[0] \cap \dots \cap a_n^{j_n}[0] = \emptyset$. There are only finitely many choices for the tuple $j = (j_0, \dots, j_n)$, call them $j^{(1)}, \dots, j^{(k)}$, so we can successively find the subfamilies $\mathcal{G}_i^j$ satisfying $\mathcal{G}_i^{j^{(r)}} \supset \mathcal{G}_i^{j^{(r+1)}}$. The uncountable families $\mathcal{G}_i^{j^{(k)}} \subset \mathcal{F}_i$ satisfy the desired conclusion. ∎

The following fact is well known; it is the key property behind the fact that, unlike the countable chain condition, Knaster's condition is productive [12]:

Lemma 15. *Let $\{a^0 \times a^1 : a \in \mathcal{F}\}$ be a Knaster-disjoint family of subsets of $X \times Y$ consisting of rectangles. Then one of the two families $\{a^0 : a \in \mathcal{F}\}$ or $\{a^1 : a \in \mathcal{F}\}$ contains an uncountable Knaster-disjoint subfamily.*

PROPOSITION 16. *Let X and Y be compact spaces with property I_n and I_m respectively. Then $X \times Y$ has property I_{n+m}.*

Proof. Let $\mathcal{F}_0, \ldots, \mathcal{F}_{n+m}$ be uncountable Knaster-disjoint families of pseudoclopens of $X \times Y$. By Lemma 14, we can suppose that every $a \in \mathcal{F}_i$ is of the form $a = a^0 \times a^1 = (a^0[0] \times a^1[0], a^0[1] \times a^1[1])$ with a^0 and a^1 pseudoclopens of X and Y. By Lemma 15, we can also suppose that for every $i \leq n+m$ there exists $j(i) \in \{0,1\}$ such that the family $\{a^{j(i)} : a \in \mathcal{F}_i\}$ is Knaster-disjoint. By elementary cardinality reasons, either there exists $S \subset \{0, \ldots, n+m\}$ with $|S| = n+1$ such that $(\forall i \in S)(j(i) = 0)$, or else there exists $T \subset \{0, \ldots, n+m\}$ with $|T| = m+1$ such that $(\forall i \in T)(j(i) = 1)$. In the first case, we finish the proof by appealing to property I_n of X, and in the second case by appealing to property I_m of Y. ■

PROPOSITION 17. *Let X be a compact space and d, n and q natural numbers. Suppose that X^d has property I_n and $(q+1)d > n$. Then X has property I_q.*

Proof. Let $\mathcal{F}_0, \ldots, \mathcal{F}_q$ be Knaster-disjoint families of pseudoclopens of X. For every $i \in \{1, \ldots, d\}$ let $p_i : X^d \to X$ be the projection on the ith coordinate. Let

$$\mathcal{F}_j^i = \{(p_i^{-1}a[0], p_i^{-1}a[1]) : a \in \mathcal{F}_j\}.$$

These are $(q+1)d$ Knaster-disjoint families of pseudoclopens of X^d. Since $(q+1)d > n$, and X^d has property I_n it follows that there are uncountable subfamilies $\mathcal{G}_j^i \subset \mathcal{F}_j^i$ such that

$$\bigcap_{i=1}^{d} \bigcap_{j=0}^{q} c_j^i[0] = \emptyset \quad \text{whenever} \quad c_j^i \in \mathcal{G}_j^i.$$

We claim that there exists $i \in \{1, \ldots, d\}$ such that $b_0[0] \cap \cdots \cap b_q[0] = \emptyset$ whenever $b_0 \in \mathcal{G}_0^i, \ldots, b_q \in \mathcal{G}_q^i$. This claim concludes the proof because then the families $\mathcal{G}_j = \{a \in \mathcal{F}_j : (p_i^{-1}a[0], p_i^{-1}a[1]) \in \mathcal{G}_j^i\}$ are the uncountable subfamilies of $\mathcal{F}_j$ we are looking for. The claim is proved by contradiction. If it were false we could find for every i elements $b_0^i \in \mathcal{G}_0^i, \ldots, b_q^i \in \mathcal{G}_q^i$ such that $b_0^i[0] \cap \cdots \cap b_q^i[0] \neq \emptyset$. Since a clopen from a family $\mathcal{G}_j^i$ depends only on the coordinate i, this implies that $\bigcap_{i=1}^d \bigcap_{j=0}^q b_j^i[0] \neq \emptyset$, which is a contradiction. ■

COROLLARY 18. *Let K be a compact space with property I_n but not I_{n-1}. If K is homeomorphic to X^d for some space X, then d divides n.*

Proof. If d does not divide n, then there exists an integer q such that $q < n/d < q+1$. The previous proposition shows that X has property I_q, so by Proposition 16, $X^d \approx K$ has I_{qd}, and therefore I_{n-1} because $qd < n$, so $qd \leq n-1$. ■

3. The euclidean ball. Let Γ be an uncountable set, and let

$$B = B(\Gamma) = \Big\{x \in [-1,1]^{\Gamma} : \sum_{\gamma\in\Gamma} |x_\gamma| \leq 1\Big\}.$$

We consider this set as a compact space endowed with the pointwise topology. This $B(\Gamma)$ is actually homeomorphic to the ball of the Banach space $\ell_p(\Gamma)$ in the weak topology for $1 < p < \infty$ and also to the dual ball of $c_0(\Gamma)$ in the weak* topology.

Theorem 19. *The space B has property I_1.*

We observe that this result implies Theorem 1. Let $n < m$ and let $f : B^n \to X_1 \times \cdots \times X_n$ be a continuous surjection. By Propositions 9 and 16, $X_1 \times \cdots \times X_n$ has property I_n, and by Proposition 10 there exists $i \leq m$ such that X_i satisfies Knaster's condition, therefore also the countable chain condition (every disjoint family of open sets is countable). But B is an Eberlein compact, so its continuous image X_i is also Eberlein compact [4] and a result of Rosenthal [10] establishes that the countable chain condition implies countable weight for an Eberlein compact.

It is easy to see that $B_+(\Gamma) = B(\Gamma) \cap [0,1]^{\Gamma}$ maps continuously onto $B(\Gamma)$; an onto continuous map $B_+(\Gamma\times 2) \to B(\Gamma)$ is given by $x \mapsto (x_{(\gamma,0)} - x_{(\gamma,1)})_{\gamma\in\Gamma}$. For notational simplicity, we write $\lambda_n = 1/2^{n+1}$. The following totally disconnected compact space maps onto $B_+(\Gamma)$:

$$L = \Big\{x \in \{0,1\}^{\Gamma\times\omega} : \sum_{(\gamma,n)\in\Gamma\times\omega} \lambda_n x_{\gamma,n} \leq 1\Big\}.$$

A surjection $g : L \to B_+(\Gamma)$ is given by $g(x)_\gamma = \sum_{n<\omega} \lambda_n x_{\gamma,n}$. By Proposition 9, Theorem 19 follows from

Theorem 20. *The space L has property I_1.*

Proof. We consider the basis $\mathcal{B}$ of clopen subsets of L consisting of the sets of the form

$$a_U^V = \{x \in L : \forall(\gamma,n) \in U\ x_{\gamma,n} = 1,\ \forall(\gamma,n) \in V\ x_{\gamma,n} = 0\},$$

where U and V are two disjoint finite subsets of $\Gamma\times\omega$. It will be convenient to use the following notations: given $a = a_U^V \in \mathcal{B}$, we write $U(a) = U$ and $V(a) = V$. Also, for a finite set $U \subset \Gamma\times\omega$, we define

$$\sigma(U) = \sum_{(\gamma,n)\in U} \lambda_n = \sum_{n<\omega} \lambda_n |U \cap \Gamma\times\{n\}|.$$

Notice the fundamental property that if $U \cap U' = \emptyset$ then $\sigma(U\cup U') = \sigma(U) + \sigma(U')$. We need to know when two elements of $\mathcal{B}$ are disjoint:

(DC) Let $a, b \in \mathcal{B}$. Then $a \cap b = \emptyset$ if and only if one of the following three conditions holds: either $U(a) \cap V(b) \neq \emptyset$ or $V(a) \cap U(b) \neq \emptyset$ or $\sigma(U(a)\cup U(b)) > 1$.

Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be two Knaster-disjoint families of clopen sets from $\mathcal{B}$. By Lemmas 13 and 14, it is enough to check property I_1 on such two families of clopen sets.

By the familiar Δ-system lemma (cf. [6, Theorem 9.18]), we can assume that the families $\{U(a) : a \in \mathcal{F}_0\}$, $\{U(a) : a \in \mathcal{F}_1\}$, $\{V(a) : a \in \mathcal{F}_0\}$ and $\{V(a) : a \in \mathcal{F}_1\}$ are Δ-systems with roots R_0, R_1, S_0 and S_1 respectively. We write $U(a) = R_i \cup U_i(a)$ and $V(a) = S_i \cup V_i(a)$ for $a \in \mathcal{F}_i$, separating the root and the disjoint part of the Δ-system in such a way that $\{U_i(a) : a \in \mathcal{F}_i\}$ and $\{V_i(a) : a \in \mathcal{F}_i\}$ are disjoint families of finite sets for $i = 0, 1$. By a further refinement we can also suppose that the whole family

$$(\star) \qquad \{U_0(a), U_1(b), V_0(a), V_1(b) : a \in \mathcal{F}_0, b \in \mathcal{F}_1\} \cup \{R_0 \cup R_1 \cup S_0 \cup S_1\}$$

is a disjoint family.

The number $\sigma(U)$ is always a rational number, so we can also suppose that $\sigma(U_0(a)) = q_0$ and $\sigma(U_1(b)) = q_1$ are rational numbers independent of $a \in \mathcal{F}_0$ and $b \in \mathcal{F}_1$.

Claim. $\sigma(R_i) + 2q_i > 1$ *for* $i = 0, 1$.

Proof of the claim. Since $\mathcal{F}_i$ is a Knaster-disjoint family, we can pick two different elements $a, b \in \mathcal{F}_i$ such that $a \cap b = \emptyset$. Thus, one of the three alternatives of the disjointness criterion (DC) must hold. But the first two alternatives are impossible. For example, the disjointness of the family $(\star)$ above implies that $U(a) \cap V(b) = R_i \cap S_i \subset U(a) \cap V(a) = \emptyset$. Therefore, the third alternative holds:

$$1 < \sigma(U(a) \cup U(b)) = \sigma(R_i) + \sigma(U_i(a)) + \sigma(U_i(b)) = \sigma(R_i) + 2q_i.$$

We finish the proof by showing that, after all these refinements, $a \cap b = \emptyset$ whenever $a \in \mathcal{F}_0$ and $b \in \mathcal{F}_1$. Using again the disjointness criterion (DC) we prove that $\sigma(U(a) \cup U(b)) > 1$. We have

$$\sigma(U(a) \cup U(b)) = \sigma(R_0 \cup R_1) + \sigma(U_0(a)) + \sigma(U_1(b)) = \sigma(R_0 \cup R_1) + q_0 + q_1.$$

If say $q_i = \min(q_0, q_1)$, then by the Claim,

$$\sigma(U(a) \cup U(b)) = \sigma(R_0 \cup R_1) + q_0 + q_1 \geq \sigma(R_i) + 2q_i > 1. \quad \square$$

Corollary 21 (Avilés, Kalenda). *Let X be a compact space and m, n natural numbers such that B^n is homeomorphic to X^m. Then m divides n.*

Proof. Apply the preceding theorem and Corollary 18. ■

4. Remarks about spaces $P(K)$. Given a compact space K, we denote by $P(K)$ the space of Radon probability measures on K endowed with the weak* topology. Results analogous to Corollary 21 are proven in [2] for certain spaces of probability measures (like $P(\sigma_n(\Gamma))$ and $P(\sigma_1(\Gamma)^n)$), so it is a natural question whether such spaces have property I_1. We show in this section that property I_n on $P(K)$ imposes very restrictive conditions on K.

PROPOSITION 22. *Let X be a compact space which contains n open subsets whose closures are pairwise disjoint and fail the countable chain condition. Then $P(X)$ maps continuously onto B^n, and in particular $P(X)$ does not have property I_{n-1}.*

Proof. Let $V_1,\dots,V_n$ be open subsets of X with $\overline{V}_i\cap\overline{V}_j=\emptyset$ for $i\neq j$, and for every i, let $\mathcal{U}_i$ be an uncountable disjoint family of nonempty open subsets of V_i. For every $u\in\bigcup_{i=1}^n\mathcal{U}_i$ let $h_u: X\to[0,1]$ be a continuous function such that $h_u(X\setminus u)=0$ and $\max\{h_u(x): x\in u\}=1$. For every $i\leq n$ let also $g_i: X\to[0,1]$ be a continuous function such that $g_i(x)=0$ if $x\in V_i$ and $g_i(x)=1$ if $x\in V_j$ for some $j\neq i$. Let

$$\chi_n(t)=\begin{cases} n & \text{if } t\geq 1-1/n,\\ 1/(1-t) & \text{if } t<1-1/n.\end{cases}$$

For a Radon measure μ on X and a continuous function $\phi: X\to\mathbb{R}$, we put $\mu(\phi)=\int\phi\,d\mu$. We define $f: P(X)\to[0,1]^{\mathcal{U}_1}\times\dots\times[0,1]^{\mathcal{U}_n}$ as follows:

$$f(\mu)_u=\chi_n(\mu(g_i))\cdot\mu(h_u),\qquad u\in\mathcal{U}_i.$$

For a set Γ, recall that $B_+(\Gamma)=\{x\in[0,1]^\Gamma:\sum_{\gamma\in\Gamma}x_\gamma\leq 1\}$. Then $B(\Gamma)$ is a continuous image of $B_+(\Gamma)$, and we shall show that $f(P(X))=B_+(\mathcal{U}_1)\times\dots\times B_+(\mathcal{U}_n)$.

First, we check that $f(P(X))\subset B_+(\mathcal{U}_1)\times\dots\times B_+(\mathcal{U}_n)$. For fixed μ in $P(X)$ and $i\leq n$,

$$\sum_{u\in\mathcal{U}_i}f(\mu)_u=\chi_n(\mu(g_i))\sum_{u\in\mathcal{U}_i}\mu(h_u)\leq\frac{1}{1-\mu(g_i)}\sum_{u\in\mathcal{U}_i}\mu(h_u)\leq 1$$

because g_i has disjoint support from all h_u's, so $\mu(g_i)+\sum_{u\in\mathcal{U}_i}\mu(h_u)\leq 1$.

Conversely, we now prove that any $x\in B_+(\mathcal{U}_1)\times\dots\times B_+(\mathcal{U}_n)$ belongs to $f(P(X))$. For every $i\leq n$ let $\xi_i\in\overline{V}_i\setminus\bigcup\{u: u\in\mathcal{U}_i\}$, and for every $u\in\mathcal{U}_i$ let $\zeta_u\in u$ be such that $h_u(\zeta_u)=1$. We define $\mu\in P(X)$ to be a discrete probability measure on X such that $\mu\{\zeta_u\}=x_u/n$ for every u, and $\mu\{\xi_n\}=(1-\sum_{u\in\mathcal{U}_n}x_u)/n$. We have $\mu(h_u)=x_u/n$ for $u\in\mathcal{U}_n$, $\mu(g_i)=1-1/n$, and $f(\mu)=x$. ∎

Recall that $\sigma_1(\Gamma)$ is the one-point compactification of a discrete set Γ. The space $P(\sigma_1(\Gamma))$ is homeomorphic to $B_+(\Gamma)$, a continuous image of $B(\Gamma)$, so it has property I_1 by Theorem 19. Also, if K_n is a discrete union of n disjoint copies of $\sigma_1(\Gamma)$, then $P(K_n)$ has property I_n, because it is a continuous image of $P(\sigma_1(\Gamma))^n\times B_+(\{1,\dots,n\})$. We may ask if there is some sufficient condition on L so that $P(L)$ has property I_n.

5. Indecomposability properties (II). A family $\mathcal{F}$ of pseudoclopens will be called *disjoint* if $a[1] \cap b[1] = \emptyset$ for any $a, b \in \mathcal{F}$, $a \neq b$.

DEFINITION 23. Let X be a compact space, and n a natural number. We say that X has *property* I_n^* if for every collection of n Knaster-disjoint families $\mathcal{F}_1, \ldots, \mathcal{F}_n$ of pseudoclopens, and every infinite disjoint family $\mathcal{F}_0$ of pseudoclopens, there exist uncountable subfamilies $\mathcal{G}_i \subset \mathcal{F}_i$ for $i = 1, \ldots, n$, and an infinite subfamily $\mathcal{G}_0 \subset \mathcal{F}_0$ such that for every choice $a_i \in \mathcal{G}_i$ for $i = 0, \ldots, n$ we have $a_0[0] \cap \cdots \cap a_n[0] = \emptyset$.

PROPOSITION 24. *Let X be a compact space with property I_n^*, and $f : X \to Y$ a continuous onto map. Then Y has property I_n^*.*

Proof. Analogous to Proposition 9. ∎

PROPOSITION 25. *Let $X_0, \ldots, X_n$ be infinite compact spaces such that $X = X_0 \times \cdots \times X_n$ has property I_n^*. Then there exist $i, j \leq n$, $i \neq j$, such that X_i and X_j satisfy Knaster's condition.*

Proof. This is equivalent to saying that for every $i \leq n$ there exists $j \neq i$ such that X_j satisfies Knaster's condition. We prove this statement for $i = 0$. By contradiction, if this were false, then we could find an uncountable Knaster-disjoint family $\mathcal{F}_j$ of strongly nonempty pseudoclopens in X_j, for every $j = 1, \ldots, n$. Let also $\mathcal{F}_0$ be an infinite disjoint family of pseudoclopens of X_0. Just as in Proposition 10, these families can be lifted to families of pseudoclopens of X that violate property I_n^*. ∎

DEFINITION 26. A family $\mathcal{B}_0$ of pseudoclopens of X will be called a *strong basis* if for every pseudoclopen a of X there exists $b \in \mathcal{B}_0$ such that $b \prec a$.

Notice that if X is a totally disconnected compact space, then the family $\mathcal{B}_0$ of pseudoclopen sets of the form (c, c), c clopen, constitutes a strong basis.

LEMMA 27. *Let $\mathcal{B}$ and $\mathcal{B}_0$ be a basis and a strong basis of pseudoclopens of X respectively. Then the following condition is sufficient for X to have property I_n^*: For every collection of n Knaster-disjoint families $\mathcal{F}_1, \ldots, \mathcal{F}_n$ of pseudoclopens from $\mathcal{B}$ and every infinite disjoint family $\mathcal{F}_0$ of pseudoclopens from $\mathcal{B}_0$, there exist uncountable subfamilies $\mathcal{G}_i \subset \mathcal{F}_i$ for $i = 1, \ldots, n$ and an infinite subfamily $\mathcal{G}_0 \subset \mathcal{F}_0$ such that for every choice $a_i \in G_i$ for $i = 0, \ldots, n$ we have $a_0[0] \cap \cdots \cap a_n[0] = \emptyset$.*

Proof. Analogous to Lemma 14. Just note that we need $\mathcal{B}_0$ to be a strong basis and not just a basis, because when dealing with infinite instead of uncountable families, it is not possible to fix the length of finite unions by passing to a further subfamily. ∎

6. Linearly ordered spaces

Lemma 28. *Let $\mathcal{F}$ be a Knaster-disjoint family of sets. Then there exist at most countably many elements $a \in \mathcal{F}$ such that $\{b \in \mathcal{F} : a \cap b = \emptyset\}$ is countable.*

Proof. Suppose by contradiction that there are uncountably many such elements. Then it is possible to construct by induction a transfinite sequence $\{a_\alpha : \alpha < \omega_1\} \subset \mathcal{F}$ of such elements such that $a_\alpha \cap a_\beta \neq \emptyset$ for all $\alpha < \beta < \omega_1$. This contradicts the assumption that $\mathcal{F}$ is Knaster-disjoint. ∎

Theorem 29. *Every linearly ordered compact space L has property I_1.*

Proof. Every compact linearly ordered space L is the continuous image of a compact linearly ordered totally disconnected space (one can consider the lexicographical product $L \times \{0,1\}$. Therefore, we can suppose that L is totally disconnected. By Lemma 14, we have to show that for any Knaster-disjoint families $\mathcal{F}_0$ and $\mathcal{F}_1$ of clopen intervals of L, there are further uncountable subfamilies for which all crossed intersections are empty. By Lemma 28, we can suppose that each element of $\mathcal{F}_i$ is disjoint from uncountably many elements of $\mathcal{F}_i$, $i = 0,1$. Notice that Knaster-disjoint families are point-countable, that is, every element of L belongs to at most countably many intervals from $\mathcal{F}_i$. Suppose that some interval $a \in \mathcal{F}_0$ intersects uncountably many intervals from $\mathcal{F}_1$. Except those that contain one of the two extremes of a, which are at most countably many, the others are actually contained in a. In this case it is enough to take $\mathcal{G}_0 = \{b \in \mathcal{F}_0 : b \cap a = \emptyset\}$ and $\mathcal{G}_1 = \{b \in \mathcal{F}_1 : b \subset a\}$. The remaining case is that every element of $\mathcal{F}_0$ intersects at most countably many elements of $\mathcal{F}_1$ and vice versa. In this case we can produce by induction two ω_1-sequences $(a_\alpha)_{\alpha<\omega_1} \subset \mathcal{F}_0$ and $(b_\alpha)_{\alpha<\omega_1} \subset \mathcal{F}_1$ with $a_\alpha \cap b_\beta = \emptyset$ for all $\alpha, \beta < \omega_1$. ∎

Theorem 30. *If $L_1, \ldots, L_n$ are linearly ordered compact spaces, then $K = L_1 \times \cdots \times L_n$ has property I_n^*.*

Proof. Again, we assume that the spaces L_j are totally disconnected. Using Lemma 27, let $\mathcal{F}_0$ be a countably infinite disjoint family of clopens of K and $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be Knaster-disjoint families of clopen boxes of K (by a *box* we mean a product of clopen intervals),

$$\mathcal{F}_i = \{a^\alpha[i] = a_1^\alpha[i] \times \cdots \times a_n^\alpha[i] : \alpha < \omega_1\}, \quad i > 0.$$

By Lemma 15 we can suppose that for every i there exists $j(i)$ such that $\{a_{j(i)}^\alpha[i] : \alpha < \omega_1\}$ is Knaster-disjoint. We can actually assume that the map $i \mapsto j(i)$ is a bijection, since if there existed $i \neq i'$ with $j(i) = j(i') = j$ we would be done by applying the fact that L_j has property I_1. After relabelling we suppose that each family $\mathcal{H}_i = \{a_i^\alpha[i] : \alpha < \omega_1\}$ is a Knaster-disjoint family of clopen intervals of L_i. We consider two cases:

Case 1: There exists $c \in \mathcal{F}_0$ and a box $b_1 \times \cdots \times b_n \subset c$ such that $A_i = \{\alpha < \omega_1 : a_i^\alpha[i] \subset b_i\}$ is uncountable for all $i = 1, \ldots, n$. In this case, we can take $\mathcal{G}_0 = \mathcal{F}_0 \setminus \{c\}$ and $\mathcal{F}_i = \{a^\alpha[i] : \alpha \in A_i\}$.

Case 2: The previous case does not hold, so for every $c \in \mathcal{F}_0$ and every box $b = b_1 \times \cdots \times b_n \subset c$ there exists $i = i(b) \in \{1, \ldots, n\}$ such that the set $A_{i(b)}(b) = \{\alpha < \omega_1 : a_i^\alpha[i] \subset b_i\}$ is countable. Actually, the set $A_i'(b) = \{\alpha < \omega_1 : a_i^\alpha[i] \cap b_i \neq \emptyset\}$ is also countable, because the family $\mathcal{H}_i = \{a_i^\alpha[i] : \alpha < \omega_1\}$ is point-countable, so only countably many intervals from it can hit one of the two extremes of b_i. Every clopen $c \in \mathcal{F}_0$ is a finite union of boxes, so there is a countable family $\mathcal{B}$ of boxes contained in elements of $\mathcal{F}_0$ such that every element of $\mathcal{F}_0$ is a finite union of elements of $\mathcal{B}$. The subfamilies $\mathcal{G}_0 = \mathcal{F}_0$ and $\mathcal{G}_i = \{a^\alpha[i] : \alpha < \omega_1, \alpha \notin \bigcup_{b\in\mathcal{B}} A'_{i(b)}(b)\}$ have the desired properties. ∎

In order to complete the proof of Theorem 4, having Theorem 30, Proposition 24 and Proposition 25, it only remains to pass from Knaster's condition to separability. It is a classical result of Knaster [8] that a linearly ordered space which satisfies Knaster's condition is separable. We need just a little more:

Proposition 31. *Let $L_1, \ldots, L_n$ be linearly ordered compact spaces and $f : L_1 \times \cdots \times L_n \to X$ an onto continuous map. If X satisfies Knaster's condition, then X is separable.*

Proof. Let $Y \subset L_1 \times \cdots \times L_n$ be a closed subspace such that $f : Y \to X$ is an irreducible onto map. Recall that a continuous function is *irreducible* if $f(Y') \neq X$ whenever $Y' \neq Y$ is a proper closed subset of Y. A standard argument using Zorn's lemma yields the existence of such a Y. Knaster's condition is preserved by irreducible preimages, so Y satisfies Knaster's condition (to see this, associate to an uncountable family $\mathcal{F}$ of nonempty open sets of Y the family $\{X \setminus f(X \setminus U) : U \in \mathcal{F}\}$ of nonempty open sets of X). Now let $p_i : L_1 \times \cdots \times L_n \to L_i$ be the natural projection, and $K_i = p_i(Y)$. Then $p_i(Y)$ is a linearly ordered compact space with Knaster's property, so by Knaster's result [8] it is separable. On the other hand, $Y \subset p_1(Y) \times \cdots \times p_n(Y)$, so $f(p_1(Y) \times \cdots \times p_n(Y)) = X$, therefore X is also separable. ∎

7. Spaces of finite sets. For a natural number n and an uncountable set Γ, let $\sigma_n(\Gamma)$ denote the family of subsets of Γ of cardinality less than or equal to n. This is a closed subset of 2^Γ, so we view $\sigma_n(\Gamma)$ as a compact topological space. A basis for its topology consists of the sets of the form

$$\Phi_A^B = \{C \in \sigma_n(\Gamma) : A \subset C \subset \Gamma \setminus B\},$$

where A and B are finite subsets of Γ.

The topological classification of the spaces which are finite or countable products of spaces $\sigma_n(\Gamma)$ is studied in [1]. In the case of finite products, which we are now interested in, if $\sigma_1(\Gamma)^{e_1} \times \cdots \times \sigma_n(\Gamma)^{e_n}$ is homeomorphic to $\sigma_1(\Gamma)^{f_1} \times \cdots \times \sigma_n(\Gamma)^{f_n}$, where n and each e_i, f_i are natural numbers, then $e_i = f_i$ for every i.

In this section, we will determine when a finite product of spaces $\sigma_n(\Gamma)$ can be mapped continuously onto another such product. An obvious sufficient condition for the existence of a continuous onto map between such finite products is the following:

LEMMA 32. *Let $(n_1, \ldots, n_r)$ and $(m_1, \ldots, m_s)$ be two finite sets of natural numbers. Suppose that there exist sets $S_i \subset \{1, \ldots, r\}$ for $i = 1, \ldots, s$ which are pairwise disjoint and such that $m_i \leq \sum_{j \in S_i} n_j$ for every $i \in \{1, \ldots, s\}$. Then the space $\sigma_{n_1}(\Gamma) \times \cdots \times \sigma_{n_r}(\Gamma)$ maps continuously onto $\sigma_{m_1}(\Gamma) \times \cdots \times \sigma_{m_s}(\Gamma)$.*

Proof. First, $\sigma_n(\Gamma)$ maps continuously onto $\sigma_m(\Gamma)$ if $m \leq n$. Namely, fix $\gamma_0 \in \Gamma$ and then define $f : \sigma_n(\Gamma) \to \sigma_{n-1}(\Gamma \setminus \{\gamma_0\})$ by $f(x) = x \setminus \{\gamma_0\}$ if $\gamma_0 \in x$, and $f(x) = \emptyset$ if $\gamma_0 \notin x$. Secondly, for a finite set S of natural numbers whose sum is $\Sigma(S)$, we have the union map

$$u : \prod_{n \in S} \sigma_n(\Gamma) \to \sigma_{\Sigma(S)}(\Gamma), \qquad u(x) = \bigcup_{n \in S} x_n.$$

These two remarks show that $\prod_{n \in S} \sigma_n(\Gamma)$ maps onto $\sigma_m(\Gamma)$ whenever $m \leq \Sigma(S)$. The proof of the lemma is obtained by applying this fact to $S = S_i$ for every i, and considering product maps. ∎

In Theorem 34 below we shall prove that the sufficient condition of the previous lemma is actually necessary. Indeed, we shall obtain stronger indecomposability properties. An *m-point family of sets* is a family $\mathcal{F}$ such that every subfamily of cardinality $m+1$ has empty intersection.

DEFINITION 33. Let $m_* = (m_1, \ldots, m_s)$ be a finite sequence of natural numbers. We say that a compact space has *property $I[m_*]$* if for any uncountable families $\mathcal{F}_1, \ldots, \mathcal{F}_s$ of clopen sets such that $\mathcal{F}_i$ is an m_i-point family for every i, there exist uncountable subfamilies $\mathcal{G}_i \subset \mathcal{F}_i$ such that $\bigcup_{i=1}^{s} \mathcal{G}_i$ is a $((\sum_{i=1}^{s} m_i) - 1)$-point family.

THEOREM 34. *Let $(n_1, \ldots, n_r)$ and $(m_1, \ldots, m_s)$ be two finite sets of natural numbers. The following are equivalent*:

(1) *$K = \sigma_{n_1}(\Gamma) \times \cdots \times \sigma_{n_r}(\Gamma)$ does not have property $(m_1, \ldots, m_s)$.*
(2) *$\sigma_{n_1}(\Gamma) \times \cdots \times \sigma_{n_r}(\Gamma)$ maps continuously onto $\sigma_{m_1}(\Gamma) \times \cdots \times \sigma_{m_s}(\Gamma)$.*
(3) *There exist disjoint sets $S_i \subset \{1, \ldots, r\}$ for $i = 1, \ldots, s$ such that $m_i \leq \sum_{j \in S_i} n_j$ for every $i \in \{1, \ldots, s\}$.*

Proof. (3)⇒(2) is Lemma 32, (2)⇒(1) is clear: Let $g:\sigma_{n_1}(\Gamma)\times\cdots\times\sigma_{n_r}(\Gamma)\to\sigma_{m_1}(\Gamma)\times\cdots\times\sigma_{m_s}(\Gamma)$ be onto, and consider the families

$$\mathcal{F}_i=\{\{x:\gamma\in g(x)_i\}:\gamma\in\Gamma\},\qquad i=1,\dots,s.$$

These families witness the failure of $I[m_*]$. It remains to prove (1)⇒(3). As K does not have property $(m_1,\dots,m_s)$, there exist families $\mathcal{F}_1,\dots,\mathcal{F}_s$ in K such that $\mathcal{F}_i$ is an m_i-point family, but for any uncountable subfamilies $\mathcal{G}_i\subset\mathcal{F}_i$, the union $\bigcup_{i=1}^s\mathcal{G}_i$ is not a $(\sum_{i=1}^s m_i-1)$-point family.

For a fixed i, we can suppose that each clopen $x\in\mathcal{F}_i$ is of the form

$$x=\bigcup_{p=1}^{k(i)}\Phi^{b(x,p,1)}_{a(x,p,1)}\times\cdots\times\Phi^{b(x,p,r)}_{a(x,p,r)},\tag{$\star$}$$

where $a(x,p,j)$ and $b(x,p,j)$ are finite subsets of Γ, and $\Phi^{b(x,p,j)}_{a(x,p,j)}$ is a basic clopen subset of $\sigma_{n_j}(\Gamma)$. Moreover, we can suppose that for fixed i,p,j, $\{a(x,p,j):x\in\mathcal{F}_i\}$ and $\{b(x,p,j):x\in\mathcal{F}_i\}$ form Δ-systems of constant cardinality with roots $A(i,p,j)$ and $B(i,p,j)$. We write $a(x,p,j)=A(i,p,j)\cup\alpha(x,p,j)$ and $b(x,p,j)=B(i,p,j)\cup\beta(x,p,j)$ separating the root and the disjoint part of the Δ-system. We also write $|\alpha|(i,p,j)=|\alpha(x,p,j)|$, $x\in\mathcal{F}_i$. By passing to further uncountable subfamilies we can also assume that

$$(\alpha(x,p,j)\cup\beta(x,p,j))\cap(a(x',p',j')\cup b(x',p',j'))=\emptyset\tag{$\star\star$}$$

whenever $x\neq x'$.

Because each $\mathcal{F}_i$ is an m_i-point family but the family $\bigcup\mathcal{F}_i$ is not a $(\sum_{i=1}^s m_i-1)$-point family, there must exist $x_1^i,\dots,x_{m_i}^i\in\mathcal{F}_i$ such that $\bigcap_{i=1}^r\bigcap_{q=1}^{m_i}x_q^i\neq\emptyset$.

Each element $x\in S_i$ is of the form ($\star$) so there also exist $p_i^1,\dots,p_i^{m_i}$ for every i such that

$$\bigcap_{i=1}^r\bigcap_{q=1}^{m_i}\Phi^{b(x_i^q,p_i^q,1)}_{a(x_i^q,p_i^q,1)}\times\cdots\times\Phi^{b(x_i^q,p_i^q,r)}_{a(x_i^q,p_i^q,r)}\neq\emptyset.$$

We define the sets S_i appearing in statement (3) of the theorem in the following way:

$$S_i=\Big\{j\in\{1,\dots,r\}:\exists\overline{q}:|\alpha|(i,p_i^{\overline{q}},j)+\Big|\bigcup_{q=1}^{m_i}a(x_i^q,p_i^q,j)\Big|>n_j\Big\}.$$

CLAIM 1. $S_i\cap S_{i'}=\emptyset$.

Proof. Suppose $j\in S_i\cap S_{i'}$. Consider witnesses $\overline{q}$ and $\overline{q}'$ for $j\in S_i$ and $j\in S_{i'}$ respectively. Assume that $|\alpha|(i,p_i^{\overline{q}},j)\le|\alpha|(i',p_{i'}^{\overline{q}'},j)$. We know that

$$\Big(\bigcap_{q=1}^{m_i}\Phi^{b(x_i^q,p_i^q,j)}_{a(x_i^q,p_i^q,j)}\Big)\cap\Big(\bigcap_{q=1}^{m_{i'}}\Phi^{b(x_{i'}^q,p_{i'}^q,j)}_{a(x_{i'}^q,p_{i'}^q,j)}\Big)\neq\emptyset.$$

In particular,

$$\Big|a(x_{i'}^{\overline{q}'},p_{i'}^{\overline{q}'},j)\cup\bigcup_{q=1}^{m_i}a(x_i^q,p_i^q,j)\Big|\le n_j,$$

but

$$\begin{aligned}n_j<|\alpha|(i,p_i^{\overline{q}},j)+\Big|\bigcup_{q=1}^{m_i}a(x_i^q,p_i^q,j)\Big|&\le|\alpha|(i',p_{i'}^{\overline{q}'},j)+\Big|\bigcup_{q=1}^{m_i}a(x_i^q,p_i^q,j)\Big|\\ &\le\Big|a(x_{i'}^{\overline{q}'},p_{i'}^{\overline{q}'},j)\cup\bigcup_{q=1}^{m_i}a(x_i^q,p_i^q,j)\Big|,\end{aligned}$$

a contradiction.

Claim 2. *Fix $i\in\{1,\dots,s\}$. For every $\overline{q}\in\{1,\dots,m_i\}$ there exists $j\in\{1,\dots,r\}$ such that $|\bigcup_{q=1}^{m_i}a(x,p_i^q,j)|+|\alpha|(p_i^{\overline{q}},i,j)>n_j$.*

Proof. Given $x\in\mathcal{F}_i\setminus\{x_i^q:q=1,\dots,m_i\}$, because $\mathcal{F}_i$ is an m_i-point family,

$$\varPhi_{a(x,p_i^{\overline{q}},1)}^{b(x,p_i^{\overline{q}},1)}\times\cdots\times\varPhi_{a(x,p_i^{\overline{q}},r)}^{b(x,p_i^{\overline{q}},r)}\cap\bigcap_{q=1}^{m_i}\varPhi_{a(x_i^q,p_i^q,1)}^{b(x_i^q,p_i^q,1)}\times\cdots\times\varPhi_{a(x_i^q,p_i^q,r)}^{b(x_i^q,p_i^q,r)}=\emptyset,$$

thus there exists j such that

$$\varPhi_{a(x,p_i^{\overline{q}},j)}^{b(x,p_i^{\overline{q}},j)}\cap\bigcap_{q=1}^{m_i}\varPhi_{a(x_i^q,p_i^q,j)}^{b(x_i^q,p_i^q,j)}=\emptyset.$$

Since we know that $\bigcap_{q=1}^{m_i}\varPhi_{a(x_i^q,p_i^q,j)}^{b(x_i^q,p_i^q,j)}\ne\emptyset$, there are only three possibilities that such an intersection of basic clopen sets of the form $\varPhi_F^G$ is empty in $\sigma_{n_j}(\varGamma)$: either

(1) $|a(x,p_i^{\overline{q}},j)\cup\bigcup_{q=1}^{m_i}a(x_i^q,p_i^q,j)|>n_j$, or
(2) $a(x,p_i^{\overline{q}},j)\cap\bigcup_{q=1}^{m_i}b(x_i^q,p_i^q,j)\ne\emptyset$, or
(3) $b(x,p_i^{\overline{q}},j)\cap\bigcup_{q=1}^{m_i}a(x_i^q,p_i^q,j)\ne\emptyset$.

The first option leads immediately to the desired conclusion. The second and third options cannot occur. The reason is that we assumed the $\varDelta$-systems to satisfy the disjointness property $(\star\star)$. Thus, for instance if (2) happened we would have $A(i,p_i^{\overline{q}},j)\cap B(i,p_i^{\hat{q}},j)\ne\emptyset$ for some $\hat{q}$, which implies that $\varPhi_{a(x_i^{\overline{q}},p_i^{\overline{q}},j)}^{b(x_i^{\overline{q}},p_i^{\overline{q}},j)}\cap\varPhi_{a(x_i^{\hat{q}},p_i^{\hat{q}},j)}^{b(x_i^{\hat{q}},p_i^{\hat{q}},j)}=\emptyset$, which is a contradiciton.

Claim 3. $\sum_{j\in S_i}n_j\ge m_i$.

Proof. Consider the function $J:\{1,\dots,m_i\}\to\{1,\dots,r\}$ which associates to every $\overline{q}$ an element $j=J(\overline{q})$ as in Claim 2. It is enough to notice that $|J^{-1}(j)| \leq n_j$ for every j. Namely,

$$\bigcap_{q \in J^{-1}(j)} \Phi_{a(x_i^q, p_i^q, j)}^{b(x_i^q, p_i^q, j)} \neq \emptyset,$$

so $|\bigcup_{\overline{q} \in J^{-1}(j)} a(x_i^q, p_i^q, j)| \leq n_j$. But on the other hand, $|\alpha|(i, p_i^{\overline{q}}, j) > 0$ for every $\overline{q} \in J^{-1}(j)$, so $|J^{-1}(j)| \leq n_j$. ∎

8. Remarks

Remark 35. Theorem 1 is in a sense best possible, because B^n maps continuously onto $B^n \times [0,1]^\omega$. This is a consequence of a result of Kalenda [7] that $B_+(\Gamma) \approx P(\sigma_1(\Gamma))$ is homeomorphic to $B_+(\Gamma) \times [0,1]$ together with the facts that B_+ and B are continuous images of each other, and $[0,1]^\omega$ is a continuous image of $[0,1]$.

Remark 36. Mardešić's conjecture is also best possible. For example, any nonmetrizable nonscattered compact space maps continuously onto $[0,1]^\omega$. Hence we can easily get continuous maps $L_1 \times \cdots \times L_n \to L_1 \times \cdots \cdots \times L_{n-1} \times [0,1]^\omega$.

Remark 37. The reader might wonder why we deal with Knaster-disjoint families instead of simply disjoint families. The reason is that Knaster-disjointness behaves better with respect to products. For instance, if we define similar properties to I_n or I_n^* with disjoint families instead of Knaster-disjoint, then the proofs of Proposition 16 and Theorem 30 do not work any more, unless we assume that certain colorings of the uncountable have uncountable monochromatic sets: In the first case this can be overcome if the compact spaces X and Y have property (Q) of Bell [3], and in the second case one may need to assume that Suslin lines do not exist.

Remark 38. Given an uncountable regular cardinal $\aleph$, we can define indecomposability properties $I_n(\aleph)$, $I_n^*(\aleph)$ or $I[m_*](\aleph)$ in a similar way but substituting "for any uncountable families... there exist uncountable subfamilies" by "for any families of cardinality $\aleph$... there exist subfamilies of cardinality $\aleph$". All the results in this note can be rewritten in this more general way, and in particular the assertions of Theorems 19, 29, 30 and 34 hold for these properties relative to $\aleph$.

Remark 39. After Rudin's result [11] that every monotonically normal compact space is the continuous image of a linearly ordered space, in Theorem 4 the assumption that the spaces L_i are linearly ordered can be substituted by the assumption that the spaces L_i are monotonically normal.

References

[1] A. Avilés, *Countable products of spaces of finite sets*, Fund. Math. 186 (2005), 147–159.

[2] A. Avilés and O. F. K. Kalenda, *Fiber orders and compact spaces of uncountable weight*, preprint.

[3] M. Bell, *A Ramsey theorem for polyadic spaces*, Fund. Math. 150 (1996), 189–195.

[4] Y. Benyamini, M. E. Rudin and M. Wage, *Continuous images of weakly compact subsets of Banach spaces*, Pacific J. Math. 70 (1977), 309–324.

[5] W. Bula, W. Dębski and W. Kulpa, *A short proof that a compact ordered space cannot be mapped onto a nonmetric product*, Proc. Amer. Math. Soc. 82 (1981), 312–313.

[6] T. Jech, *Set Theory. The third millennium edition, revised and expanded*. Springer Monogr. Math., Springer, Berlin.

[7] O. F. K. Kalenda, *On products with the unit interval*, Topology Appl. 155 (2008), 1098–1104.

[8] B. Knaster, *Sur une propriété caractéristique de l'ensemble des nombres réels*, Mat. Sb. 16 (1945), 284–288.

[9] S. Mardešić, *Mapping products of ordered compacta onto products of more factors*, Glasnik Mat. Ser. III 5 (25) (1970) 163–170.

[10] H. Rosenthal, *On injective Banach spaces and the spaces $L^\infty(\mu)$ for finite measures μ*, Acta Math. 124 (1970), 205–248.

[11] M. E. Rudin, *Nikiel's conjecture*, Topology Appl. 116 (2001), 305–331.

[12] E. Szpilrajn (E. Marczewski), *Remarque sur les produits cartésiens d'espaces topologiques*, C. R. (Dokl.) Acad. Sci. URSS 31 (1941), 525–527.

[13] L. B. Treybig, *Concerning continuous images of compact ordered spaces*, Proc. Amer. Math. Soc. 15 (1964), 866–871.

Université Paris 7
UMR 7056
2, Place Jussieu, Case 7012
75251 Paris, France
E-mail: aviles@logique.jussieu.fr, avileslo@um.es

Received 14 May 2008