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Spaces of ω-limit sets of graph maps

by

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Abstract. Let (X, f) be a dynamical system. In general the set of all ω-limit sets of f is not closed in the hyperspace of closed subsets of X. In this paper we study the case when X is a graph, and show that the family of ω-limit sets of a graph map is closed with respect to the Hausdorff metric.

1. Introduction. A *dynamical system* is a pair (X, f), where X is a compact metric space with a metric d and f is a continuous map from X to itself. For $x \in X$, $\{x, f(x), f^2(x), \ldots\}$ is called the *orbit* of x and denoted by $O(x, f)$. The point x is *periodic* if $f^n(x) = x$ for some $n \in \mathbb{N}$, and the smallest such n is called the *period* of x. If $n = 1$, then x is also called a *fixed point* of f. A system is *transitive* if there exists a dense orbit. Denote the set of all limit points of an orbit $O(x)$ by $\omega(x, f)$ and call it an *ω-limit set* of f. For any $x \in X$, $\omega(x, f)$ is a closed non-empty subset of X and it is strongly invariant (i.e. $f(\omega(x, f)) = \omega(x, f)$). Write $\mathbb{X}(f, \omega) = \{\omega(x, f) : x \in X\}$.

ω-limit sets give fundamental information about asymptotic behavior of a dynamical system. One of the basic tasks is to give their topological characterization. This task is very complicated even in the simplest one-dimensional case of the compact interval ([1]–[4], [6]). Let I be a closed interval in $\mathbb{R}$ and let $f : I \to I$ be a continuous map. Then for any $x \in I$, $\omega(x, f)$ is (i) a periodic orbit, or (ii) an infinite compact nowhere dense set, or (iii) a finite union of connected subintervals which forms a periodic orbit ([3], [6]). Conversely, whenever $A \subseteq I$ has one of the above forms then there is a continuous map $f : I \to I$ such that A is an ω-limit set of f. This result was generalized to graph maps in [7]. Another related problem is, for

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a given f and a closed strongly f-invariant set A, to find a condition deciding whether A is an ω-limit set of f or not. One can find such characterizations of ω-limit sets in [4] and [2].

Let I be a closed interval in $\mathbb{R}$ and let $f : I \to I$ be a continuous map. The map $\omega : I \mapsto \omega(x, f)$ was studied in [5] and shown to be far from continuous. Hence it is somewhat surprising that the image of this map is closed in the Hausdorff metric. This was proved in [4] and the proof is rather long but quite elementary and ingenious. In a similar way, this result was extended to circle maps in [9]. In this paper we show that the family of ω-limit sets of a graph map is closed in the Hausdorff metric. The proof we offer is different from [4], [9] and simpler. Also in the proof we give a characterization of ω-limit sets of a graph map.

2. Preliminaries. In this article, the sets of integers, nonnegative integers, natural numbers, real numbers and complex numbers are denoted by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ respectively.

Let (X, d) be a compact metric space. The *hyperspace* $\mathbb{X}$ is the set of all nonempty closed subsets of X. The Hausdoff metric d_{H} on $\mathbb{X}$ is defined by

$$d_{\mathrm{H}}(V, W) = \max\{d(v, W), d(w, V) : v \in V, w \in W\}, \qquad V, W \in \mathbb{X},$$

where $d(x, Y) = \inf\{d(x, y) : y \in Y\}$ for $x \in X$ and $Y \subseteq X$. It is well known that $(\mathbb{X}, d_{\mathrm{H}})$ is a compact metric space (see [8], for example).

Let (X, f) be a dynamical system. Recall that a subset $A \subseteq X$ is *f-invariant* if $f(A) \subseteq A$, and *strongly f-invariant* if $f(A) = A$. Let $\mathbb{X}_1(f)$ ($\mathbb{X}_2(f)$) be the set of all nonempty (strongly) f-invariant closed subsets. Obviously, $\mathbb{X}(f, \omega) \subseteq \mathbb{X}_2(f) \subseteq \mathbb{X}_1(f)$. It is easy to verify the following proposition.

Proposition 2.1. *$\mathbb{X}_1(f)$ and $\mathbb{X}_2(f)$ are closed subspaces of $(\mathbb{X}, d_{\mathrm{H}})$. In particular, both are compact.*

But generally the space $\mathbb{X}(f, \omega)$ need not be closed in $\mathbb{X}$.

Example 2.2. Let $D = \{re^{i\theta} \in \mathbb{C} : 0 \le r \le 1,\ \theta \in \mathbb{R}\}$ be the unit disc and $f : D \to D$, $re^{i\theta} \mapsto re^{i(\theta + r)}$. Then (D, f) is a dynamical system. It is easy to verify $\mathbb{D}(f, \omega)$ is not closed in $\mathbb{D}$.

In the next section it will be shown that when X is a graph, $\mathbb{X}(f, \omega)$ is closed. Now we recall some definitions concerning graphs. By a *graph* we mean a connected compact one-dimensional polyhedron in $\mathbb{R}^3$. A continuous map from a graph to itself is called a *graph map*. An *arc* is any space which is homeomorphic to the closed interval $[0, 1]$. Then a graph G is a continuum (i.e. a nonempty, compact, connected metric space) which can be written as the union of finitely many arcs, any two of which are either disjoint or intersect only in one or both of their endpoints. Each of these arcs is called an *edge* of G, and its ends are called *vertices*. Since G is a polyhedron in $\mathbb{R}^3$, there are at least three edges in any circle of G. For a given graph G, a *subgraph* of G is a subset of G which is a graph itself. The *valence* of a vertex x is the number of edges that are incident to x, and if the number is n then one writes $\mathrm{val}(x) = n$. A vertex of valence 1 is also called an *end* of G, and a vertex x with $\mathrm{val}(x) \geq 3$ is said to be a *branching point* of G. The set of branching points of G is denoted by $\mathrm{Br}(G)$. A *tree* is a graph without any subset which is homeomorphic to the unit circle. A *star* is either a tree having only one branching point or an arc.

For convenience, we assume that the length of every edge of G is greater than 1. Hence any non-degenerate connected closed subset of G with diameter less than 1 is a star. Let $x, y \in G$. The arc with ends $\{x, y\}$ is denoted by $[x, y]$ or $[y, x]$. Write $(a, b) = [a, b] \setminus \{a, b\}$, and define $[a, b)$ and $(a, b]$ similarly. $[x; y]$ is also used to denote an arc with ends $\{x, y\}$, but in this case the arc is understood to be directed: it starts from x and ends with y.

For a topological space X, the closure of a subset $A \subseteq X$ is denoted by $\overline{A}$. When (X, d) is a metric space and $x \in X$, $Y \subseteq X$ and $\varepsilon > 0$, one writes $B(x, \varepsilon)$ for the ε-ball $\{x' \in X : d(x, x') < \varepsilon\}$, and $B(Y, \varepsilon) = \{x \in X : d(x, Y) < \varepsilon\}$.

3. Spaces of ω-limit sets of graph maps. The following theorem is the main result of this paper.

THEOREM 3.1. *Let G be a graph and let $f : G \to G$ be a continuous map. Then the set of all ω-limit sets of f endowed with the Hausdorff metric is compact.*

Before proving the theorem, one needs some notations and lemmas. Recall that $\mathbb{G}$ is the hyperspace of G and $\mathbb{G}(f, \omega)$ is the set of ω-limit sets of G. Let $v_1, v_2, \ldots$ be an infinite sequence in G. For any $n \in \mathbb{N}$ write

$$V_n = O(v_n, f), \qquad V = \bigcup_{n=1}^{\infty} V_n, \qquad X_n = \omega(v_n, f), \qquad X = \bigcup_{n=1}^{\infty} X_n.$$

Assume that $\{X_n\}_{n=1}^{\infty}$ converges to W in $(\mathbb{G}, d_{\mathrm{H}})$. To prove that $\mathbb{G}(f, \omega)$ is closed, we only need to show that $W \in \mathbb{G}(f, \omega)$.

Let us sketch the idea of the proof. First we reduce the system to the case satisfying Conditions I–III below. The main reason is to exclude the easier case when $(W, f|_W)$ is transitive. Then we study the system under Conditions I–III. Lemma 3.5 gives a condition under which W belongs to $\mathbb{G}(f, \omega)$. The rest is to show that this condition is satisfied.

If there are infinitely many elements of $\{X_n\}_{n=1}^{\infty}$ which are equal, then it is easy to see that $W \in \mathbb{G}(f, \omega)$. So we assume:

CONDITION I. *$X_n \neq X_m$ for $n \neq m$, and $d_{\mathrm{H}}(X_n, W) < 2^{-n-1}$ for all n.*

Observe that if $V_n \cap V_m \neq \emptyset$ then $X_n = X_m$. So in addition one can assume the following condition holds:

Condition II. *$V_n \cap V_m = \emptyset$ for $n \neq m$, and for all n,*

$$d_{\mathrm{H}}(\overline{V}_n, X_n) < 2^{-n-1}, \qquad d_{\mathrm{H}}(\overline{V}_n, W) < 2^{-n}.$$

Lemma 3.2. *Assume that Conditions* I–II *hold. If for any $\varepsilon > 0$ and $N \in \mathbb{N}$, there exist $n \geq N$, $y \in V_n$ and $w \in W$ such that*

$$\sup\{d(f^i(y), f^i(w)) : i \in \mathbb{Z}_+\} \leq \varepsilon,$$

then $(W, f|_W)$ is transitive and hence $W \in \mathbb{G}(f, \omega)$.

Proof. Let $\delta_0 = 1/8$. By the assumption, there are $n_1 \geq 1$, $y_1 \in V_{n_1}$ and $w_1 \in W$ such that $d_{\mathrm{H}}(\overline{V}_{n_1}, X_{n_1}) < \delta_0$, $d_{\mathrm{H}}(X_{n_1}, W) < \delta_0$ and

$$\sup\{d(f^i(y_1), f^i(w_1)) : i \in \mathbb{Z}_+\} \leq \delta_0.$$

Choose $k_1 \in \mathbb{N}$ such that $d_{\mathrm{H}}(O_{k_1}(y_1, f), \overline{O(y_1, f)}) < \delta_0$, where $O_k(x, f) = \{f^j(x) : 0 \leq j \leq k\}$ for $x \in G$ and $k \in \mathbb{N}$. There is some $\delta_1 \in (0, \delta_0/8]$ such that for any $x \in B(w_1, 3\delta_1)$,

$$\sup\{d(f^i(x), f^i(w_1)) : i = 0, 1, \ldots, k_1\} < \delta_0.$$

Then for any $x \in B(w_1, 3\delta_1)$,

$$\begin{aligned} (3.1)\quad d_{\mathrm{H}}(O_{k_1}(x, f), W) &< \delta_0 + d_{\mathrm{H}}(O_{k_1}(w_1, f), W) \\ &\leq 2\delta_0 + d_{\mathrm{H}}(O_{k_1}(y_1, f), W) \\ &< 3\delta_0 + d_{\mathrm{H}}(\overline{O(y_1, f)}, W) \\ &\leq 3\delta_0 + d_{\mathrm{H}}(\overline{O(y_1, f)}, X_{n_1}) + d_{\mathrm{H}}(X_{n_1}, W) \\ &\leq 3\delta_0 + d_{\mathrm{H}}(\overline{V}_{n_1}, X_{n_1}) + d_{\mathrm{H}}(X_{n_1}, W) < 5\delta_0. \end{aligned}$$

By assumption, there are $n_2 > n_1$, $y_2' \in V_{n_2}$ and $w_2' \in W$ such that $d_{\mathrm{H}}(\overline{V}_{n_2}, X_{n_2}) < \delta_1$, $d_{\mathrm{H}}(X_{n_2}, W) < \delta_1$ and

$$\sup\{d(f^i(y_2'), f^i(w_2')) : i \in \mathbb{Z}_+\} \leq \delta_1.$$

Choose $x_2 \in X_{n_2}$ such that $d(w_1, x_2) = d(w_1, X_{n_2}) \leq d_{\mathrm{H}}(W, X_{n_2}) < \delta_1$ and take $j_2 \in \mathbb{Z}_+$ such that $d(f^{j_2}(y_2'), x_2) < \delta_1 - d_{\mathrm{H}}(W, X_{n_2})$. Then $f^{j_2}(y_2') \in B(w_1, \delta_1)$ and $f^{j_2}(w_2') \in B(w_1, 2\delta_1)$. Let $y_2 = f^{j_2}(y_2')$ and $w_2 = f^{j_2}(w_2')$. Choose $k_2 > k_1$ such that $d_{\mathrm{H}}(O_{k_2}(y_2, f), \overline{O(y_2, f)}) < \delta_1$. There is $\delta_2 \in (0, \delta_1/8]$ such that for any $x \in B(w_2, 3\delta_2)$,

$$\sup\{d(f^i(x), f^i(w_2)) : i = 0, 1, \ldots, k_2\} < \delta_1.$$

Then for any $x \in B(w_2, 3\delta_2)$, similarly to (3.1), we get $d_{\mathrm{H}}(O_{k_2}(x, f), W) < 5\delta_1$.

Inductively, we find points $w_1, w_2, \ldots$ in W, positive integers $k_1 < k_2 < \cdots$ and positive numbers $\delta_0 = 1/8 > \delta_1 > \delta_2 > \cdots$ such that for any

$n \in \mathbb{N}$, we have $\delta_n \le \delta_{n-1}/8$, $w_{n+1} \in B(w_n, 2\delta_n)$ and

$$d_{\mathrm{H}}(O_{k_n}(x, f), W) < 5\delta_{n-1}, \qquad \forall x \in B(w_n, 3\delta_n).$$

Hence it is easy to see that $\{w_n\}$ converges to some $w \in W$ and for any $n \in \mathbb{N}$, we have $w \in B(w_n, 3\delta_n)$ and

$$B(O(w, f), 5\delta_{n-1}) \supseteq B(O_{k_n}(w, f), 5\delta_{n-1}) \supseteq W.$$

Thus $O(w, f)$ is dense in W and $f|_W$ is transitive. ∎

Note that the conclusion of Lemma 3.2 holds for any compact space, not only for graph maps. By Lemma 3.2, to show $W \in \mathbb{G}(f, \omega)$ one only needs to consider the case when $(W, f|_W)$ is not transitive. Hence by Lemma 3.2, one can assume:

Condition III. *There exists $\varepsilon_0 \in (0, 1/2]$ such that for any $y \in V$ and $w \in W$,*

$$\sup\{d(f^i(y), f^i(w)) : i \in \mathbb{Z}_+\} > \varepsilon_0.$$

Obviously, if Condition III holds, then $V \cap W = \emptyset$.

Below we always assume that Conditions I–III hold. Fix ε_0 as in Condition III and $\varepsilon \in (0, \varepsilon_0/2]$ such that $f(B(x, \varepsilon)) \subseteq B(f(x), \varepsilon_0)$ for all $x \in G$. Let $\mathbb{Y}(\varepsilon)$ be the set of all non-degenerate connected closed subsets of G contained in $B(W, \varepsilon)$ and with diameter less than ε. By our assumption on G any element on $\mathbb{Y}(\varepsilon)$ is a star.

Definition 3.3. Let $Y, Y' \in \mathbb{Y}(\varepsilon)$. If there is a finite set $\{Y_0, Y_1, \ldots, Y_n\} \subseteq \mathbb{Y}(\varepsilon)$, $n \in \mathbb{N}$, such that $Y = Y_0$, $Y' = Y_n$ and $f(Y_{i-1}) \supseteq Y_i$ for any $i = 1, \ldots, n$, then we write $Y \xrightarrow{(f,\varepsilon)} Y'$.

It is easy to verify the following lemma:

Lemma 3.4.

(1) *“$\xrightarrow{(f,\varepsilon)}$” is a transitive relation on $\mathbb{Y}(\varepsilon)$.*

(2) *If $Y \xrightarrow{(f,\varepsilon)} Y'$, then there exist a connected closed subset Z and $n \in \mathbb{N}$ such that $f^n(Z) = Y'$ and $\bigcup_{i=0}^n f^i(Z) \subseteq B(W, \varepsilon)$.*

The following lemma offers a sufficient condition for a set to belong to $\mathbb{G}(f, \omega)$.

Lemma 3.5. *Let $\varepsilon_0/2 \ge \delta_1 \ge \delta_2 \ge \cdots > 0$ with $\lim_{i\to\infty} \delta_i = 0$ and let $\{Y_i\}_{i=1}^\infty$ be a sequence of non-degenerate connected closed subsets of G. If for any $i \in \mathbb{N}$, $Y_i \in \mathbb{Y}(\delta_i)$, $Y_i \xrightarrow{(f,\delta_i)} Y_{i+1}$ and $W \subseteq B(\bigcup_{j=i}^\infty Y_j, \delta_i)$, then $W \in \mathbb{G}(f, \omega)$.*

Proof. For any $i \in \mathbb{N}$, by Lemma 3.4(2) there exist connected closed subsets Z_i of Y_i and $n_i \in \mathbb{N}$ such that

$$f^{n_i}(Z_i) = Y_{i+1} \quad \text{and} \quad \bigcup_{k=0}^{n_i} f^k(Z_i) \subseteq B(W, \delta_i).$$

Let $m_i = n_1 + \cdots + n_i$ ($m_0 = 0$). Then $\bigcap_{i=1}^{\infty} f^{-m_{i-1}}(Z_i) \neq \emptyset$, so pick $x \in \bigcap_{i=1}^{\infty} f^{-m_{i-1}}(Z_i)$. Since $O(f^{m_i}(x), f) \subseteq B(W, \delta_{i+1})$, we have $\omega(x, f) \subseteq W$.

On the other hand, since $\bigcup_{j=i+1}^{\infty} Y_j \subseteq B(O(f^{m_i}(x), f), \delta_{i+1})$, we have

$$W \subseteq B\Big(\bigcup_{j=i+1}^{\infty} Y_j, \delta_{i+1}\Big) \subseteq B(O(f^{m_i}(x), f), 2\delta_{i+1}).$$

Hence $W \subseteq \omega(x, f)$. Thus $W = \omega(x, f) \in \mathbb{G}(f, \omega)$. ∎

Definition 3.6. Let $Y \in \mathbb{Y}(\varepsilon)$. If $\{i \in \mathbb{N} : Y \cap V_i \neq \emptyset\}$ is infinite, then Y is called a (P, ε)-*star*. Let $A = [w; y] \in \mathbb{Y}(\varepsilon)$ be an arc with $w \in W$ and $y \in V$. If for any $x \in (w, y]$ and $n \in \mathbb{N}$ there is some $i \geq n$ such that $V_i \cap (w, x] \neq \emptyset$, then A is called a (P, ε)-*arc*.

Denote the sets of all (P, ε)-stars and of all (P, ε)-arcs by $\mathbb{Y}(P, \varepsilon)$ and $\mathbb{A}(P, \varepsilon)$ respectively.

By the definition one gets the following lemma readily.

Lemma 3.7.

(1) $\mathbb{A}(P, \varepsilon) \subseteq \mathbb{Y}(P, \varepsilon)$.
(2) *For any $w \in W$, there exists a point $y \in B(w, \varepsilon) \cap V$ such that $[w; y]$ is a (P, ε)-arc.*
(3) *For any $Y \in \mathbb{Y}(P, \varepsilon)$ and $\varepsilon' \in (0, \varepsilon]$, there exists a (P, ε')-arc A with $A \subseteq Y$.*

For $Y \in \mathbb{Y}(P, \varepsilon)$, set

$$\begin{aligned}(3.2)\quad \mathbb{Y}(\varepsilon, Y) = \{Y' \in \mathbb{Y}(\varepsilon) : Y \xrightarrow{(f,\varepsilon)} Y' \text{ and there exists}\\ Y'' \in \mathbb{Y}(P, \varepsilon) \text{ such that } Y' = Y'' \text{ or } Y' \xrightarrow{(f,\varepsilon)} Y''\}.\end{aligned}$$

Lemma 3.8.

(1) *For any $Y \in \mathbb{Y}(P, \varepsilon)$, $\mathbb{Y}(\varepsilon, Y) \neq \emptyset$.*
(2) *If $Y, Y' \in \mathbb{Y}(P, \varepsilon)$ and $Y \xrightarrow{(f,\varepsilon)} Y'$, then $\mathbb{Y}(\varepsilon, Y') \subseteq \mathbb{Y}(\varepsilon, Y)$.*
(3) *If $Y' \in \mathbb{Y}(P, \varepsilon)$ and $Y \in \mathbb{Y}(\varepsilon, Y')$ with* $\operatorname{diam} f(Y) < \varepsilon$, *then $f(Y) \in \mathbb{Y}(\varepsilon, Y')$.*

Proof. (1) and (2) are easy to verify. We now prove (3). Since $\operatorname{diam} f(Y) < \varepsilon$, one has $Y' \xrightarrow{(f,\varepsilon)} Y \xrightarrow{(f,\varepsilon)} f(Y)$. If $f(Y)$ is a (P, ε)-star, then by the definition $f(Y) \in \mathbb{Y}(\varepsilon, Y')$. Now assume that $f(Y)$ is not a (P, ε)-star. By the definition of $\mathbb{Y}(\varepsilon, Y')$, there is some (P,ε)-star Y'' such that $Y' \xrightarrow{(f,\varepsilon)} Y \xrightarrow{(f,\varepsilon)} Y''$. Since $f(Y)$ is not a (P,ε)-star, $f(Y) \neq Y''$. Hence by the definition of $Y \xrightarrow{(f,\varepsilon)} Y''$ it is easy to see that $f(Y) \xrightarrow{(f,\varepsilon)} Y''$. Thus $f(Y) \in \mathbb{Y}(\varepsilon, Y')$. ∎

LEMMA 3.9. *Let $A = [w; y]$ be a (P,ε)-arc and w a periodic point. Then there exist $x \in (w, y)$ and a (P,ε)-star Y such that $[x, y] \xrightarrow{(f,\varepsilon)} Y$.*

Proof. Let m be the period of w and $\mathrm{val}(w) = k$. Take $\delta_{-1} > \delta_0 > \delta_1 > \cdots > \delta_{2k} > 0$ such that $\delta_{-1} < \min\{d(w,y), \varepsilon_0\}$, $(B(w,\delta_{-1}) \setminus \{w\}) \cap \mathrm{Br}(G) = \emptyset$ and for any $i = 0,1,\ldots,2k$ and $j = 1,\ldots,m$,

$$f^j(B(w,\delta_i)) \subseteq B(f^j(w), \delta_{i-1}). \tag{3.3}$$

Let $\delta = \min\{\delta_{2k}, \delta_{i-1} - \delta_i : i = 0,1,\ldots,2k\}/2$. By Condition II and the definition of (P,ε)-arc, there is some $N \in \mathbb{N}$ such that

$$d_{\mathrm{H}}(\overline{V}_N, W) < \delta \quad \text{and} \quad A \cap V_N \cap B(w,\delta) \neq \emptyset. \tag{3.4}$$

Fix $z \in A \cap V_N \cap B(w,\delta)$. By Condition III and (3.3), there are positive integers $n_0 < n_1 < n_2 < \cdots < n_{2k}$ such that for any $i = 0,1,\ldots,2k$, we have $f^{mn_i}(z) \in B(w,\delta_{2k-i-1}) \setminus B(w,\delta_{2k-i})$ and $\{f^{mj}(z) : j = 0,1,\ldots,n_i - 1\} \subseteq B(w,\delta_{2k-i})$. Hence there are integers $0 \le p < q < r \le 2k$ such that $f^{mn_p}(z), f^{mn_q}(z)$ and $f^{mn_r}(z)$ are in the same connected component of $B(w,\delta_{-1}) \setminus \{w\}$. Take $x \in (w,y)$ such that

$$\{f^{mj}(x) : j = 0,1,\ldots,n_r\} \subseteq B(w,\delta).$$

Let $Y = [f^{mn_r}(x), f^{mn_r}(z)]$. Then $[x,y] \supseteq [x,z] \xrightarrow{(f,\varepsilon)} Y$. By (3.4), there is some $w' \in W$ such that $w' \in \overline{B(f^{mn_q}(z),\delta)} \subseteq [w, f^{mn_r}(z)) \setminus B(w,\delta) \subseteq Y$. By Lemma 3.7(2), $\overline{B(f^{mn_p}(z),\delta)}$ is a (P,ε)-star and hence so is Y. ∎

COROLLARY 3.10. *Let $A = [w; y]$ be a (P,ε)-arc. If there is some $n \in \mathbb{N}$ such that $f^n(w)$ is a periodic point, then there exist $x \in (w,y)$ and a (P,ε)-star Y such that $[x,y] \xrightarrow{(f,\varepsilon)} Y$.*

Proof. It is obvious that there exists $z \in (w,y] \cap V$ such that $[f^n(w); f^n(z)]$ is a (P,ε)-arc and $\operatorname{diam} f^i([w,z]) < \varepsilon$ for any $i = 1,\ldots,n$. According to Lemma 3.9, there exist $x' \in (f^n(w), f^n(z))$ and a (P,ε)-star Y such that $[x', f^n(z)] \xrightarrow{(f,\varepsilon)} Y$. Let $x \in f^{-n}(x') \cap (w,z)$. Then

$$[x,y] \supseteq [x,z] \xrightarrow{(f,\varepsilon)} [x', f^n(z)] \xrightarrow{(f,\varepsilon)} Y. \ ∎$$

LEMMA 3.11. *Let $A = [w; y]$ be a (P,ε)-arc. Then there exist $x \in (w,y)$ and a (P,ε)-star Y such that $[x,y] \xrightarrow{(f,\varepsilon)} Y$.*

Proof. According to Corollary 3.10, it suffices to consider the case when $O(w,f)$ is infinite. By the definition of (P,ε)-arc, there are positive integers $k_1 < k_2 < \cdots$ and points $\{x_1, x_2, \ldots\} \subseteq (w,y]$ such that $x_i \in V_{k_i}$ and

$d(x_{i+1}, w) < d(x_i, w)/2$ for all $i \in \mathbb{N}$. By Condition III for every $i \in \mathbb{N}$ there is a unique $n_i \in \mathbb{N}$ such that $\varepsilon/2 \le d(f^{n_i}(x_i), f^{n_i}(w)) < \varepsilon_0$ and

$$d(f^n(x_i), f^n(w)) < \varepsilon/2, \qquad \forall n = 0, 1, \ldots, n_i - 1.$$

Obviously, $\lim_{i\to\infty} n_i = \infty$. Without loss of generality, one can assume $n_1 < n_2 < \cdots$. Let $w_i = f^{n_i}(w)$. Then the w_i are mutually distinct. Passing to a subsequence if necessary one can assume that $\lim_{i\to\infty} w_i = w'$ and for any $i \in \mathbb{N}$,

$$d(w_{i+1}, w') < d(w_i, w')/2 \quad \text{and} \quad w_i \in [w_1, w').$$

Dropping the first finitely many points if necessary one can assume in addition that

$$d(w_1, w') < \varepsilon/4 \quad \text{and} \quad [w_1, w') \cap \mathrm{Br}(G) = \emptyset.$$

Take $\delta > 0$ such that $\delta < d(x_4, w)$ and for any $n = 0, 1, \ldots, n_4$,

$$f^n(B(w,\delta)) \subseteq B(f^n(w), d(w_5, w_4)/2).$$

Then $f^{n_4}(x) \in (w_5, w_3)$ for any $x \in B(w,\delta) \cap (w, y]$. Since $d(f^{n_4}(x_4), f^{n_4}(w)) = d(f^{n_4}(x_4), w_4) > \varepsilon/2$ and $d(w_1, w') < \varepsilon/4$, we have

$$[x, y] \supseteq [x, x_4] \xrightarrow{(f,\varepsilon)} [w', w_5] \quad \text{or} \quad [x, y] \supseteq [x, x_4] \xrightarrow{(f,\varepsilon)} [w_3, w_1].$$

As $w_6 \in W \cap (w', w_5)$ and $w_2 \in W \cap (w_3, w_1)$, Lemma 3.7(2) shows that $[w', w_5]$ and $[w_3, w_1]$ are both (P, ε)-arcs. Thus $[x, y] \xrightarrow{(f,\varepsilon)} Y$ for $Y = [w', w_5]$ or $Y = [w_3, w_1]$. ■

Let $Y \in \mathbb{Y}(P, \varepsilon)$, and write

$$U(P, \varepsilon, Y) = \bigcup\{Y' : Y' \in \mathbb{Y}(\varepsilon, Y)\}. \tag{3.5}$$

LEMMA 3.12. *Let $A = [w; y]$ be a (P, ε)-arc. Then $W \subseteq \overline{U(P, \varepsilon, A)}$.*

Proof. Choose $\varepsilon_1 \in (0, \varepsilon/2]$ such that $f(B(x, \varepsilon_1)) \subseteq B(f(x), \varepsilon/2)$ for all $x \in G$. Fix $v \in V \cap B(w, \varepsilon_1) \cap A$. Then $[w; v]$ and $f([w; v])$ are also (P, ε)-arcs and $f([w; v]) \in \mathbb{Y}(\varepsilon, A)$. Suppose that Lemma 3.12 does not hold. Then there are $w_1 \in W$ and $\delta \in (0, \varepsilon_1]$ such that $B(w_1, \delta) \cap U(P, \varepsilon, A) = \emptyset$.

For a subset Z of G define $N(Z \cap V) = \{i \in \mathbb{N} : V_i \cap Z \ne \emptyset\}$ and $N_1(Z \cap X) = \{i \in \mathbb{N} : X_i \cap X \ne \emptyset\}$. Since $d_{\mathrm{H}}(X_i, W) \to 0$ as $i \to \infty$, $N_1(B(w_1, \delta) \cap X)$ is cofinite. Hence $\mathbb{M} = N([w; v] \cap V) \cap N_1(B(w_1, \delta) \cap X)$ is an infinite subset of $\mathbb{N}$. For any $i \in \mathbb{M}$, choose $x_i \in V_i \cap [w, v]$. Since $f(x_i) \in f([w; v]) \subseteq U(P, \varepsilon, A)$ and $O(x_i, f) \cap B(w_1, \delta) \ne \emptyset$, there exists $y_i \in O(f(x_i), f) \cap U(P, \varepsilon, A)$ such that $f(y_i) \notin U(P, \varepsilon, A)$.

As $y_i \in U(P, \varepsilon, A)$, there is some $Y_i \in \mathbb{Y}(\varepsilon, A)$ such that $y_i \in Y_i$. By Lemma 3.8(3), $\operatorname{diam} Y_i \ge \varepsilon_1$. Let $i_1 < i_2 < \cdots$ in $\mathbb{M}$ be such that $\lim_{j\to\infty} y_{i_j} = y'$, $d(y_{i_1}, y') < \varepsilon_1, [y_{i_1}, y') \cap \mathrm{Br}(G) = \emptyset$, $y_{i_k} \in [y_{i_1}, y')$ and $d(y_{i_{k+1}}, y') < d(y_{i_k}, y')/2$ for all $k \in \mathbb{N}$.

It is obvious that $y' \in W$. If there exists $k \in \mathbb{N}$ such that $y' \in Y_{i_k}$, then $[y_{i_k}, y']$ is a (P, ε_1)-arc contained in Y_{i_k} and hence $f([y_{i_k}, y'])$ is a (P, ε)-star. Thus $f([y_{i_k}, y']) \in \mathbb{Y}(\varepsilon, A)$ and $f(y_{i_k}) \in f([y_{i_k}, y']) \subseteq U(P, \varepsilon, A)$. This contradicts the definition of $\{y_i\}$. So $y' \notin Y_{i_k}$ for any $k \geq 2$. This implies that $[y_{i_1}, y_{i_k}] \subseteq Y_{i_k}$. By Lemma 3.11, there are $x \in (y', y_{i_1})$, a sufficiently large k and a (P, ε)-star Y' such that

$$[y_{i_k}, y_{i_1}] \supseteq [x, y_{i_1}] \xrightarrow{(f,\varepsilon)} Y'.$$

Hence $[y_{i_k}, y_{i_1}] \in \mathbb{Y}(\varepsilon, A)$. By Lemma 3.8(3), $f([y_{i_k}, y_{i_1}]) \in \mathbb{Y}(\varepsilon, A)$ and $f(y_{i_k}) \in f([y_{i_k}, y_{i_1}]) \subseteq U(P, \varepsilon, A)$. This also contradicts the definition of $\{y_i\}$. ∎

Now it is time to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. As discussed at the beginning of this section, one can assume Conditions I–III hold. Choose $\delta_1 \geq \delta_2 \geq \cdots > 0$ such that $\delta_1 < \varepsilon_0/2$, $f(B(x, \delta_1)) \subseteq B(f(x), \varepsilon_0)$ and $\lim_{n\to\infty} \delta_n = 0$. Choose $w_1, w_2, \ldots$ in W such that $\overline{\{w_n, w_{n+1}, \ldots\}} = W$ for any $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, by Lemmas 3.12 and 3.7(3), there are a (P, δ_n)-arc A_n, $Y_n \in \mathbb{Y}(\delta_n)$ and a (P, δ_n)-star Y_n' such that

$$A_n \xrightarrow{(f,\delta_n)} Y_n \xrightarrow{(f,\delta_n)} Y_n' \supseteq A_{n+1} \quad \text{and} \quad d(w_n, Y_n) < \delta_n/2.$$

It is easy to check $Y_1, Y_2, \ldots$ satisfy the condition of Lemma 3.5. Hence W is an ω-limit set. ∎

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