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The Lindelöf property and σ-fragmentability

by

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Abstract. In the previous paper, we, together with J. Orihuela, showed that a compact subset X of the product space $[-1,1]^D$ is fragmented by the uniform metric if and only if X is Lindelöf with respect to the topology $\gamma(D)$ of uniform convergence on countable subsets of D. In the present paper we generalize the previous result to the case where X is K-analytic. Stated more precisely, a K-analytic subspace X of $[-1,1]^D$ is σ-fragmented by the uniform metric if and only if $(X,\gamma(D))$ is Lindelöf, and if this is the case then $(X,\gamma(D))^{\mathbb{N}}$ is also Lindelöf. We give several applications of this theorem in areas of topology and Banach spaces. We also show by examples that the main theorem cannot be extended to the cases where X is Čech-analytic and Lindelöf or countably K-determined.

1. Introduction. In the paper [6], we, together with J. Orihuela, have investigated conditions for a compact subset K of the product $[-1,1]^D$ to be fragmented by the uniform metric. We discovered, among other results, that for K to be fragmented by the uniform metric, it is necessary and sufficient that K is Lindelöf with respect to the topology $\gamma(D)$ of uniform convergence on countable subsets of D, and if this is the case, then $(K,\gamma(D))^{\mathbb{N}}$ is Lindelöf. Although this topological result provided us with a number of applications in topology and Banach spaces, we have been keenly aware of the limitation on K to be compact.

In this paper we present a generalization of the result stated above to the class of K-analytic spaces. More specifically, a K-analytic subset X in the product $[-1,1]^D$ is σ-fragmented by the uniform metric if and only if $(X,\gamma(D))$ is Lindelöf, and if this is the case then $(X,\gamma(D))^{\mathbb{N}}$ is also Lindelöf. The proof of this main theorem, which is far more involved than that for the compact case, is given in the next section, where the mathematical terms used above are defined.

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The subsequent sections are devoted to the applications of the main theorem and examples. In Section 3, we show that the main theorem gives an easy proof of the theorem proved by Gul'ko [11] and Orihuela [23]: If K is a Corson compact space, then $(C(K), \gamma(K))$ is Lindelöf. In this case, [23] proves that $(C(K), \gamma(K))^n$ is Lindelöf for each $n \in \mathbb{N}$, but we can do a bit better: $(C(K), \gamma(K))^{\mathbb{N}}$ is Lindelöf. The section concludes with an example showing that the converse of the last statement is not true.

In Section 4, we apply the main theorem to investigate K-analytic Tikhonov spaces X. Specifically we give a number of conditions on X or $C(X)$, each equivalent to X being σ-scattered. We also give in this section examples to show that our main theorem cannot be further generalized to the cases where X is Čech-analytic and Lindelöf or countably K-determined.

In Section 5, we consider a class of Banach spaces wider than that of representable Banach spaces introduced by Godefroy and Talagrand [10]. The new class includes all dual Banach spaces and the class of spaces considered in [6, Section 5]. Our main theorem is applied once again to prove that a Banach space in this class is weakly Lindelöf if, and only if, its dual unit ball endowed with the weak* topology is countably tight and has the property that its separable subsets are metrizable.

2. σ-fragmentability and the Lindelöf property for $\gamma(D)$. We recall some topological terms. Let (T, τ) be a topological space and let δ be a metric on T. Let S be a subset of T. Then we say that (S, τ) (or simply, S) is *fragmented by δ down to ε* for some $\varepsilon > 0$ if, whenever A is a non-empty subset of S, there is a τ-open set U in T such that $A \cap U \neq \emptyset$ and δ-diam$(A \cap U) < \varepsilon$. The subspace (S, τ) (or simply, set S) is *fragmented by δ* if it is fragmented by δ down to each $\varepsilon > 0$. The space (T, τ) is *σ-fragmented by δ* if, for each $\varepsilon > 0$, T can be written as $T = \bigcup_{n=1}^{\infty} T_n$, where each T_n is fragmented by δ down to ε. If the metric δ is that of a norm $\| \ \|$, then instead of "fragmented by the metric of the norm", we say *norm-fragmented* or *$\| \ \|$-fragmented.*

A topological space (T, τ) is said to be *K-analytic* if there is an upper-semicontinuous set-valued map $F : \mathbb{N}^{\mathbb{N}} \to 2^T$ such that $F(\sigma)$ is compact for each $\sigma \in \mathbb{N}^{\mathbb{N}}$ and $F(\mathbb{N}^{\mathbb{N}}) := \bigcup\{F(\sigma) : \sigma \in \mathbb{N}^{\mathbb{N}}\} = T$. Here the set-valued map F is called *upper-semicontinuous* if for each $\sigma \in \mathbb{N}^{\mathbb{N}}$ and for an open subset U of T such that $F(\sigma) \subset U$ there exists a neighborhood V of σ with $F(V) \subset U$. Our basic reference for K-analytic spaces is [26]. A subset S of T is said to be K-analytic if S with the relative topology, i.e. (S, τ), is K-analytic. We use repeatedly the fact that each K-analytic Hausdorff space is Lindelöf (see [26, Theorem 2.7.1]).

Let (M, ϱ) be a metric space with the metric ϱ bounded, and let D be an index set. We consider various topologies on the product space M^D in addition to the product (= pointwise) topology τ_p. If S is a subset of D, we define the pseudo-metric d_S on M^D by

$$d_S(x,y) = \sup\{\varrho(x(p), y(p)) : p \in S\}$$

for all $x, y \in M^D$. Note that d_D is the *uniform metric* on M^D and we denote it by d. Throughout this paper, we let $\mathcal{C}$ denote the family of all countable subsets of D. Finally we let $\gamma(D)$ denote the topology on M^D of uniform convergence on members of $\mathcal{C}$. This is the topology of the uniformity generated by the family $\{d_A : A \in \mathcal{C}\}$ of pseudo-metrics.

Using the notation above our main theorem is the following.

THEOREM 2.1. *Let X be a K-analytic subspace of M^D, where (M, ϱ) is a metric space with ϱ bounded. Then the following statements are equivalent.*

(a) *The space (X, τ_p) is σ-fragmented by d.*
(b) *For each compact subset K of (X, τ_p), (K, τ_p) is fragmented by d.*
(c) *For each $A \in \mathcal{C}$, the pseudo-metric space (X, d_A) is separable.*
(d) *$(X, \gamma(D))$ is Lindelöf.*

Proof. (Easy parts.) (a)$\Leftrightarrow$(b). This follows from [12, Theorem 4.1]. (A simpler proof in [19].)

(c)$\Rightarrow$(b). (c) implies that, for each compact $K \subset X$, (K, d_A) is separable whenever $A \in \mathcal{C}$. Then (K, τ_p) is fragmented by d by e.g. [6, Theorem 2.1].

(d)$\Rightarrow$(c). This is clear because if $A \in \mathcal{C}$, then the topology of d_A is weaker than $\gamma(D)$.

In order to prove (a)$\Rightarrow$(c), we need the following simple lemma.

LEMMA 1. *Let (T, τ) be metrizable and separable* (*or more generally, hereditarily Lindelöf*) *and let δ be a metric on T. Then (T, τ) is σ-fragmented by δ if and only if (T, δ) is separable.*

Proof. If (T, δ) is not separable, then there exist an $\varepsilon > 0$ and an uncountable subset S of T such that $\delta(t, t') \geq \varepsilon$ whenever t, t' are distinct elements of S. If (T, τ) is σ-fragmented by δ, then T can be written as $T = \bigcup\{T_n : n \in \mathbb{N}\}$, where, for each n, T_n is fragmented by δ down to $\varepsilon/2$. Choose n so that $T_n \cap S$ is uncountable. Since (T, τ) is hereditarily Lindelöf, there is an uncountable subset B of $T_n \cap S$ without a τ-isolated point. Because of the property of T_n, there is a τ-open subset U of T such that $U \cap B \neq \emptyset$ and δ-diam$(U \cap B) \leq \varepsilon/2$. Since B is without a τ-isolated point, $U \cap B$ contains two distinct points t, t'. Recalling that $B \subset S$, we obtain $\varepsilon \leq \delta(t, t') \leq \delta$-diam$(U \cap B) \leq \varepsilon/2$, a contradiction.

Conversely if (T, δ) is separable, then for each ε, Y is a countable union of subsets of δ-diameter $< \varepsilon$. So with any topology, T is σ-fragmented by δ. ∎

Proof of (a)$\Rightarrow$(c) *of Theorem 2.1.* Let $A \in \mathcal{C}$ and let $r : M^D \to M^A$ be the restriction map. Then r is continuous with respect to the product topologies as well as with respect to d_D and d_A, and these metrics are lower-semicontinuous in respective product topologies. Since (X, τ_p) is K-analytic and σ-fragmented by d_D, by [12, Theorem 5.1], $(r(X), \tau_p)$ is σ-fragmented by d_A. Moreover, being the continuous image of a Lindelöf space, $(r(X), \tau_p)$ is Lindelöf. Since A is countable, (M^A, τ_p) is metrizable and therefore $(r(X), \tau_p)$ is metrizable and separable. Hence by Lemma 1, $(r(X), d_A)$ is separable. It follows that (X, d_A) is separable.

This completes the proof of the equivalence of (a), (b) and (c) and they are implied by (d). It remains to prove that (c)⇒(d). We do this in the next two subsections 2.1 and 2.2.

2.1. *Preliminary remarks.* We use the following convention: If $\sigma = n_1, n_2, \ldots \in \mathbb{N}^{\mathbb{N}}$ and if $k \in \mathbb{N}$, then $\sigma|k = n_1, n_2, \ldots, n_k$. Let $\mathcal{A}$ be a family of subsets of a set T. Then a Suslin($\mathcal{A}$)-*set* is a subset S of T that can be represented as

$$S = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{k=1}^{\infty} S(\sigma|k),$$

where $S(\sigma|k) \in \mathcal{A}$ for each $\sigma \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$. The family of all Suslin($\mathcal{A}$)-sets is denoted by Suslin($\mathcal{A}$). The family Suslin($\mathcal{A}$) is closed under countable intersections and countable unions ([26, Corollary 2.3.3]). If $\mathcal{A}$ consists of K-analytic subsets of a Hausdorff space (T, τ), then each Suslin($\mathcal{A}$)-set is again K-analytic ([26, Theorem 2.5.4]). The intersection of a K-analytic subset of T and a closed subset of T is K-analytic ([26, Theorem 2.5.3]).

We also recall some facts concerning Baire sets. Let (T, τ) be a Tikhonov space. A subset Z of T is called a *zero-set* (in T) if $Z = f^{-1}(0)$ for some continuous function $f : T \to \mathbb{R}$. Let $\mathcal{Z}$ (or $\mathcal{Z}(T)$) denote the family of all zero-sets in T. Then $\mathcal{Z}$ is closed under finite unions and countable intersections. If $f : T \to \mathbb{R}$ is continuous, then $f^{-1}(F) \in \mathcal{Z}$ for each closed subset F of $\mathbb{R}$. The σ-algebra generated by $\mathcal{Z}$ is denoted by Baire(T) and the elements of Baire(T) are called the *Baire sets* in T. If $Z \in \mathcal{Z}$, then $T \setminus Z$ is a countable union of members of $\mathcal{Z}$. Hence $T \setminus Z \in$ Suslin($\mathcal{Z}$). Since the family

$$\{S \subset T : S, T \setminus S \in \mathrm{Suslin}(\mathcal{Z})\}$$

is a σ-algebra, it follows that Baire(T) $\subset$ Suslin($\mathcal{Z}$).

Using the earlier notation, let X be a K-analytic subset of M^D. Then each zero-set in X, being closed, is K-analytic and therefore each member of Suslin($\mathcal{Z}$) is K-analytic. It follows that each Baire set in X is K-analytic, hence Lindelöf relative to τ_p.

Our proof of (c)⇒(d) is by contradiction. So suppose henceforward that (c) holds and (d) fails for a fixed K-analytic subset X of (M^D, τ_p), and we agree upon the following notation. All topological terms (such as K-analytic, Baire(X), Lindelöf, etc.) are relative to τ_p unless otherwise specified.

Notation. Given $x \in X$, $S \subset D$ and $\varepsilon > 0$ we write

$$U(x,S,\varepsilon) := \{y \in X : d_S(y,x) < \varepsilon\},$$
$$V(x,S,\varepsilon) := \{y \in X : d_S(y,x) \le \varepsilon\}.$$

Let $\mathcal{U} = \{U_j : j \in J\}$ be a $\gamma(D)$-open cover of X without a countable subcover. We may assume that each U_j is of the form

$$U_j = U(x_j, A_j, \varepsilon_j) = \{y \in X : d_{A_j}(y, x_j) < \varepsilon_j\},$$

where $x_j \in X$, $A_j \in \mathcal{C}$, $\varepsilon_j > 0$ for each $j \in J$. For each $A \in \mathcal{C}$, let

$$U(A) = \bigcup\{U_j : j \in J,\ A_j \subset A\}.$$

Clearly $U(A) \subset U(A')$ whenever $A \subset A'$. Since $\mathcal{U}$ covers X, $X = \bigcup\{U(A) : A \in \mathcal{C}\}$.

Lemma 2. *Under the notation above, the following statements hold.*

(i) $U(x,A,\varepsilon) \in \mathrm{Baire}(X)$ *whenever* $x \in X$, $A \in \mathcal{C}$, $\varepsilon > 0$.

(ii) $U(A) \in \mathrm{Baire}(X)$ *for each* $A \in \mathcal{C}$. *In particular* $U(A)$ *is* K*-analytic and Lindelöf for each* $A \in \mathcal{C}$.

(iii) *A subset* S *of* X *is covered by a countable subfamily of* $\mathcal{U}$ *if and only if* $S \subset U(A)$ *for some* $A \in \mathcal{C}$.

Proof. (i) Since $U(x,A,\varepsilon) = \bigcup\{V(x,A,\varepsilon - 1/n) : n \in \mathbb{N}\}$, it is sufficient to show $V(x,A,\varepsilon) = \bigcap\{V(x,\{a\},\varepsilon) : a \in A\} \in \mathrm{Baire}(X)$. Since $y \mapsto \varrho(x(a), y(a))$ is continuous on (X,τ_p), $V(x,\{a\},\varepsilon) \in \mathcal{Z}(X)$, and because A is countable, $V(x,A,\varepsilon) \in \mathrm{Baire}(X)$.

(ii) The set $U_j = U(x_j, A_j, \varepsilon_j)$ is d_A-open whenever $A_j \subset A$. Since, by (c), (X, d_A) is hereditarily Lindelöf, $U(A)$ is a countable union of sets U_j with $A_j \subset A$. Therefore, by (i), $U(A) \in \mathrm{Baire}(X)$.

(iii) If $S \subset \bigcup\{U_j : j \in J_0\}$ for some countable subset J_0 of J, then $S \subset U(A)$ where $A = \bigcup\{A_j : j \in J_0\}$. Conversely if $S \subset U(A)$, then from the proof of (ii), we see that S is covered by a countable subfamily of $\mathcal{U}$. ■

2.2. *Proof of* (c)⇒(d). All the assumptions and notation of subsection 2.1 are retained in this section. Let $\mathcal{Y}$ be the family of all K-analytic subsets Y of (X,τ_p) such that there is no countable subfamily of $\mathcal{U}$ that covers Y, i.e. for no $A \in \mathcal{C}$, $Y \subset U(A)$. By our assumption, $X \in \mathcal{Y}$. If $Y \in \mathcal{Y}$ with $Y \subset Z \subset X$ and if Z is K-analytic then $Z \in \mathcal{Y}$.

We distinguish two cases:

A. For each $Y \in \mathcal{Y}$ and each $\varepsilon > 0$, there is a $Z \in \mathcal{Y}$ such that $Z \subset Y$ and d-$\mathrm{diam}\, Z \le \varepsilon$.

B. For some $Z \in \mathcal{Y}$ and some $\varepsilon > 0$, d-$\mathrm{diam}\, Y > \varepsilon$ whenever $Y \in \mathcal{Y}$ and $Y \subset Z$. (The negation of Case **A**.)

We show that each case leads to a contradiction.

Lemma 3. *Case* **A** *leads to a contradiction.*

Proof. Assume Case **A**. Let $G : \mathbb{N}^{\mathbb{N}} \to 2^X$ be a compact-set-valued upper-semicontinuous map such that $G(\mathbb{N}^{\mathbb{N}}) = X$. Recall that if $\sigma = n_1, \ldots \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$, then we let $\sigma|k = n_1, \ldots, n_k$, and if $n_1, \ldots, n_k$ is a finite sequence in $\mathbb{N}$, we let $[n_1, \ldots, n_k] = \{\sigma \in \mathbb{N}^{\mathbb{N}} : \sigma|k = n_1, \ldots, n_k\}$. We note that $[n_1, \ldots, n_k]$ is a clopen subset of $\mathbb{N}^{\mathbb{N}}$ and that the family of the sets of this form constitutes a base for the topology of $\mathbb{N}^{\mathbb{N}}$. For convenience we set $[\emptyset] = \mathbb{N}^{\mathbb{N}}$. Note that each set of the form $G([n_1, \ldots, n_k])$ is K-analytic.

By induction, we construct a decreasing sequence $F_0 \supset F_1 \supset F_2 \supset \ldots$ of closed subsets of (X, τ_p) and a sequence $\sigma = n_1, \ldots \in \mathbb{N}^{\mathbb{N}}$ such that

(i) $F_k \cap G([\sigma|k]) \in \mathcal{Y}$ for each $k \ge 0$.
(ii) d-diam $F_k \le 1/k$ for each $k \ge 1$.

Construction. To start the induction, let $F_0 = X$. Clearly (i) holds and (ii) does not apply. Inductively assume that $F_0, F_1, \ldots, F_k$ and $n_1, \ldots, n_k$ have been constructed. Since

$$F_k \cap G([n_1, \ldots, n_k]) = \bigcup_{i=1}^{\infty} F_k \cap G([n_1, \ldots, n_k, i]) \in \mathcal{Y},$$

there is an $i \in \mathbb{N}$ with

$$F_k \cap G([n_1, \ldots, n_k, i]) \in \mathcal{Y}.$$

By the assumption of Case **A**, there is a $Z \in \mathcal{Y}$ such that

$$Z \subset F_k \cap G([n_1, \ldots, n_k, i]), \qquad d\text{-diam}\, Z \le 1/(k+1).$$

Let $F_{k+1} = \overline{Z} \subset F_k$ and $n_{k+1} = i$, where the closure is taken in (X, τ_p). Then d-diam $F_{k+1} \le 1/(k+1)$. Clearly $F_{k+1} \cap G([n_1, \ldots, n_{k+1}])$ is K-analytic and contains Z which is a member of $\mathcal{Y}$. Hence (i) holds if k is replaced by $k+1$. This completes the construction.

By [26, Lemma 3.1.1], $(\bigcap\{F_k : k \in \mathbb{N}\}) \cap G(\sigma) \ne \emptyset$. Hence by (ii), the set $\bigcap\{F_k : k \in \mathbb{N}\}$ is a singleton $\{a\}$. Now there is an $A \in \mathcal{C}$ such that $a \in U(A)$. Since $U(A)$ is d_A-open, there is a $\delta > 0$ such that $U(a, A, \delta) \subset U(A)$. By (ii), there is a $k \in \mathbb{N}$ such that d-diam $F_k < \delta$. Then $F_k \subset U(a, A, \delta) \subset U(A)$. This contradicts (i), and completes the proof. ■

Before we take up Case **B**, we prove the following lemma. Recall that all topological terms are relative to τ_p unless otherwise mentioned.

Lemma 4. *Let $Y \in \mathcal{Y}$. Then there is a subset Q of Y such that*

(i) *$Q \in \mathcal{Y}$.*
(ii) *If V is an open subset of (Y, τ_p) with $V \cap Q \ne \emptyset$, then for no $A \in \mathcal{C}$, $V \subset U(A)$.*

Proof. Let
$$Q = Y \setminus \bigcup\{\mathrm{int}_Y(U(A)\cap Y) : A \in \mathcal{C}\},$$
where int_Y indicates the τ_p-interior relative to Y. Then clearly Q is closed in (Y,τ_p). Hence Q is K-analytic. We show that $Q \in \mathcal{Y}$ by contradiction. Suppose then that $Q \subset U(A_0)$ for some $A_0 \in \mathcal{C}$. Then
$$\bigcup\{\mathrm{int}_Y(U(A)\cap Y) : A \in \mathcal{C}\} = Y \setminus Q \supset Y \setminus U(A_0) = Y \cap (X \setminus U(A_0)).$$
By Lemma 2, $U(A_0) \in \mathrm{Baire}(X)$ and hence $X \setminus U(A_0) \in \mathrm{Baire}(X)$. Consequently, the set $X \setminus U(A_0)$ is K-analytic. It follows that the intersection $Y \cap (X \setminus U(A_0))$ is K-analytic and hence Lindelöf. Therefore, there is a sequence $\{A_n : n \in \mathbb{N}\}$ in $\mathcal{C}$ such that
$$Y \setminus U(A_0) \subset \bigcup\{\mathrm{int}_Y(U(A_n)\cap Y) : n \in \mathbb{N}\} \subset \bigcup\{U(A_n)\cap Y : n \in \mathbb{N}\}.$$
Let $B = \bigcup\{A_n : n \in \{0\}\cup\mathbb{N}\} \in \mathcal{C}$. Then $Y \subset U(B)$, contradicting $Y \in \mathcal{Y}$. This proves (i).

Next, suppose that V is an open subset of (Y,τ_p) such that $Q \cap V \neq \emptyset$. If the conclusion of (ii) is not true, then $V \subset U(A_0)$ for some $A_0 \in \mathcal{C}$. This implies that $V \subset \mathrm{int}_Y(U(A_0)\cap Y) \subset Y \setminus Q$ and $Q \cap V = \emptyset$, contradicting the assumption. ∎

LEMMA 5. *Case* **B** *leads to a contradiction.*

Proof. Let $Z \in \mathcal{Y}$ and $\varepsilon > 0$ be fixed so that d-diam $Y > \varepsilon$ whenever $Y \in \mathcal{Y}$ and $Y \subset Z$. We let $H : \mathbb{N}^{\mathbb{N}} \to 2^Z$ be a compact-set-valued upper-semicontinuous map such that $H(\mathbb{N}^{\mathbb{N}}) = Z$. All the following construction takes place in (Z,τ_p).

Let $2^{(\mathbb{N})}$ be the set all finite sequences s of 0's and 1's, and in this case let $|s|$ denote the length of s. $\mathbb{N}^{(\mathbb{N})}$ is similarly defined. For each $s \in 2^{(\mathbb{N})}$, we construct a closed subset $F(s)$ of (Z,τ_p), an $\ell(s) \in \mathbb{N}^{(\mathbb{N})}$ and $p(s) \in D$ satisfying the following conditions.

(i) $F(\emptyset) = Z$.
(ii) For each $s \in 2^{(\mathbb{N})}$, $F(s,0)\cup F(s,1) \subset F(s)$, $|\ell(s)| = |s|$ and $\ell(s,0)$, $\ell(s,1)$ extend $\ell(s)$.
(iii) For each $s \in 2^{(\mathbb{N})}$, $\varrho(x(p(s)), y(p(s))) > \varepsilon$ whenever $x \in F(s,0)$, $y \in F(s,1)$.
(iv) For each $s \in 2^{(\mathbb{N})}$, $Y(s) := F(s) \cap H([\ell(s)]) \in \mathcal{Y}$.

CONSTRUCTION. The construction is by induction on $|s|$. When $|s| = 0$, we let $F(\emptyset) = Z$ and $\ell(\emptyset) = \emptyset$. Inductively assume that $F(s), \ell(s)$ have been constructed for all s with $|s| \le n$ and $p(s)$ for $|s| \le n-1$ so that (ii), (iii) hold for $|s| \le n-1$ and (iv) for $|s| \le n$.

Let $s \in 2^{(\mathbb{N})}$ with $|s| = n$. Then by (iv), $Y(s) \in \mathcal{Y}$. Hence by Lemma 4, there exists a subset Q of $Y(s)$ such that $Q \in \mathcal{Y}$, and whenever V is an open subset of $(Y(s),\tau_p)$ with $Q\cap V\neq\emptyset$, then $V\subset U(A)$ for no $A\in\mathcal{C}$. By hypothesis d-diam $Q>\varepsilon$. So there are $x_0,x_1\in Q$ and a $p\in D$ such that $\varrho(x_0(p),x_1(p))>\varepsilon+\delta$ for some $\delta>0$. Let

$$W_0=\{x\in Y(s):\varrho(x_0(p),x(p))<\delta/2\},$$
$$W_1=\{x\in Y(s):\varrho(x_1(p),x(p))<\delta/2\}.$$

Note first that if $x\in\overline{W}_0$ and $y\in\overline{W}_1$, where closures are taken in (Z,τ_p), then $\varrho(x(p),y(p))>\varepsilon$. Next W_0 is open in $(Y(s),\tau_p)$ and $W_0\cap Q\neq\emptyset$. Hence, for no $A\in\mathcal{C}$, $W_0\subset U(A)$. Similarly for W_1. It follows that $\overline{W}_0\cap Y(s)$ is K-analytic and $\overline{W}_0\cap Y(s)\subset U(A)$ for no $A\in\mathcal{C}$. Hence $\overline{W}_0\cap Y(s)\in\mathcal{Y}$. Now

$$\overline{W}_0\cap Y(s)=\overline{W}_0\cap F(s)\cap H([\ell(s)])=\bigcup_{k\in\mathbb{N}}\overline{W}_0\cap F(s)\cap H([\ell(s),k]).$$

Hence for some k, $\overline{W}_0\cap F(s)\cap H([\ell(s),k]\in\mathcal{Y}$. Let $F(s,0)=\overline{W}_0\cap F(s)$, $\ell(s,0)=\ell(s),k$ and $p(s)=p$. Then (iv) holds for $(s,0)$. Similarly we let $F(s,1)=\overline{W_1}\cap F(s)$ and choose $\ell(s,1)$ to make (iv) hold for $(s,1)$. Clearly conditions (ii) and (iii) are satisfied as well. This completes the construction.

Now for each $\sigma\in 2^{\mathbb{N}}$, let $\ell(\sigma)$ be the unique element in $\mathbb{N}^{\mathbb{N}}$ such that $\ell(\sigma)|n=\ell(\sigma|n)$ for each $n\in\mathbb{N}$. Then by (iv), $F(\sigma|n)\cap H([\ell(\sigma)|n])\neq\emptyset$ for each $n\in\mathbb{N}$. Therefore, by [26, Lemma 3.1.1],

$$K(\sigma):=\Big(\bigcap\{F(\sigma|n):n\in\mathbb{N}\}\Big)\cap H(\ell(\sigma))$$

is a non-empty compact subset of Z. For each $\sigma\in 2^{\mathbb{N}}$, choose a point $x(\sigma)$ in $K(\sigma)$, and let B be the countable set $\{p(s):s\in 2^{(\mathbb{N})}\}\subset D$. Then from (ii) and (iii) it follows that $d_B(x(\sigma),x(\sigma'))>\varepsilon$ whenever σ and σ' are distinct elements of the uncountable space $2^{\mathbb{N}}$. This contradicts (c), which proves Lemma 5. ∎

The proof of Theorem 2.1 now follows from Lemmas 3 and 5.

The proof of the next corollary is almost the same as the one for Corollary 2.2 in our previous paper [6]. For the convenience of the reader, a part of the proof is repeated here.

Corollary 2.2. *Let X,M,D be as in Theorem* 2.1. *If X satisfies one* (*hence all*) *of the four conditions of Theorem* 2.1, *then $(X,\gamma(D))^{\mathbb{N}}$ is Lindelöf.*

Proof. We may assume that the metric ϱ of the space M is bounded by 1. Let $\varphi:(M^D)^{\mathbb{N}}\to(M^{\mathbb{N}})^D$ be the map defined by $\varphi(\xi)(p)(j)=\xi(j)(p)$ for all $\xi\in(M^D)^{\mathbb{N}},p\in D,j\in\mathbb{N}$. Clearly φ is a homeomorphism when the product topology is used throughout. Now the space $M^{\mathbb{N}}$ is metrizable, and we use the metric $\varrho_\infty(m,m'):=\sum_{j\in\mathbb{N}}2^{-j}\varrho(m(j),m'(j))$ for $m,m'\in M^{\mathbb{N}}$.

Let d_∞ be the metric on $(M^{\mathbb{N}})^D$ given by

$$d_\infty(x,x') = \sup\{\varrho_\infty(x(p),x'(p)) : p\in D\} \quad \text{for } x,x'\in (M^{\mathbb{N}})^D.$$

Now, by [26, Theorem 2.5.5], $X^{\mathbb{N}}$ is K-analytic, hence so is $\varphi(X^{\mathbb{N}})$. We show that each compact subset of $\varphi(X^{\mathbb{N}})$ is fragmented by d_∞. For this it is sufficient to prove that each set of the form $\varphi(K)$ is fragmented by d_∞, where $K=\prod\{K_j : j\in\mathbb{N}\}$ with each K_j compact in X. Let $\varepsilon>0$, let C be a non-empty subset of K and let $\pi_j : K\to K_j$ be the jth projection. Then, since each K_j is fragmented by d according to (b) of Theorem 2.1, we can construct inductively a decreasing sequence $V_1\supset V_2\supset\ldots$ of non-empty relatively open subsets of C such that d-diam $\pi_j(V_j)<\varepsilon/2$ for each $j\in\mathbb{N}$. Choose $k\in\mathbb{N}$ so that $2^{-k}<\varepsilon/2$, and let $\xi,\xi'\in V_k$. Then for each $p\in D$,

$$\begin{aligned}\varrho_\infty(\varphi(\xi)(p),\varphi(\xi')(p)) &\le \sum_{j\le k} 2^{-j}\varrho(\xi(j)(p),\xi'(j)(p)) + \sum_{j\ge k+1} 2^{-j}\\ &< \sum_{j\le k} 2^{-j} d(\pi_j(\xi),\pi_j(\xi')) + \varepsilon/2 \le \varepsilon/2+\varepsilon/2=\varepsilon.\end{aligned}$$

Thus $\varphi(V_k)$ is a non-empty relatively open subset of $\varphi(C)$ with d_∞-diameter not greater than ε. It follows that $\varphi(K)$ is fragmented by d_∞.

Hence by Theorem 2.1, $\varphi(X^{\mathbb{N}})$ is $\gamma(D)$-Lindelöf. So we finish the proof by noting that φ maps $(M^D,\gamma(D))^{\mathbb{N}}$ homeomorphically onto $((M^{\mathbb{N}})^D,\gamma(D))$. This fact is shown at the end of the proof of [6, Corollary 2.2]. ∎

Remark 2.3. In Theorem 2.1 and Corollary 2.2 we have restricted ourselves to metric spaces (M,ϱ) with ϱ bounded, because if ϱ is unbounded, then ϱ can always be replaced by $\varrho' := \varrho\wedge 1 = \min\{\varrho,1\}$ without changing the uniformity. However in applications, there are cases when this replacement of ϱ by $\varrho\wedge 1$ is not necessary. More specifically, suppose ϱ is unbounded, but $X\subset M^D$ is so situated that

$$d(x,y) = \sup\{\varrho(x(p),y(p)) : p\in D\} < \infty$$

for each $(x,y)\in X\times X$. In this case the uniformities and the topologies of d, d_A and $\gamma(D)$ are unaffected by whether ϱ or ϱ' is used in our definitions. Hence Theorem 2.1 and Corollary 2.2 continue to hold for the original unbounded metric ϱ.

Remark 2.4. In Theorem 2.1, the equivalence of (a) and (b) is valid under a less restrictive assumption than that of K-analyticity. In the unpublished "Note of 8 December 1980", D. H. Fremlin defined the weaker notion of *Čech-analyticity*. We shall not repeat the definition here but refer instead to [12, Section 8]. According to [12, Theorem 4.1], statements (a) and (b) are equivalent when X is assumed to be Čech-analytic. This may lead one to conjecture that Theorem 2.1 is true when X is only assumed to be Čech-analytic and Lindelöf. A counter-example to this conjecture is discussed in Section 4.

3. Corson compact spaces. Let $I=[-1,1]$ and let Γ be an arbitrary index set. For an $x\in I^\Gamma$, let us write $\operatorname{supp}(x)=\{\gamma\in\Gamma : x(\gamma)\neq 0\}$. We define two special subsets of I^Γ as follows:

$$\mathcal{F}(\Gamma)=\{x\in[-1,1]^\Gamma : \operatorname{supp}(x) \text{ is finite}\},$$
$$\Sigma(\Gamma)=\{x\in[-1,1]^\Gamma : \operatorname{supp}(x) \text{ is countable}\}.$$

Unless otherwise stated, the topology of I^Γ and its subsets is the product (= pointwise) topology τ_p.

Recall that a compact Hausdorff space K is said to be *Corson compact* if K is homeomorphic to a τ_p-compact subset of $\Sigma(\Gamma)$. From the definition, it follows that if A is a countable subset of a Corson compact space K, then the closure of A is compact and metrizable. A topological space T is said to be *countably tight* if, whenever S is a subset of T and $t\in\overline{S}$, then for some countable subset A of S, $t\in\overline{A}$. One can show easily that the space $(\Sigma(\Gamma),\tau_p)$ defined above is countably tight (see [13, Lemma 1.6]). Hence the Corson compact space K is countably tight.

As the first application of our main theorem, we show that for any Corson compact space K the space $(C(K),\gamma(K))$ is Lindelöf, where $\gamma(K)$ stands for the topology in $C(K)$ of uniform convergence on countable subsets of K. This result, which implies that $C_p(K):=(C(K),\tau_p(K))$ is Lindelöf, was first proved by Gul'ko [11] by a direct method based on the abundance of retracts in K. Orihuela [23] gave a different proof based on Banach space techniques. That $C_p(K)$ is Lindelöf also follows from the result of Alster and Pol [2] obtained independently by yet a different method.

We need the following simple lemma first. If S is a subset of a linear space, the convex hull and the absolute convex hull of S are denoted respectively by $\operatorname{co}(S)$ and $\operatorname{aco}(S)$. The linear span of the set S is denoted by $\operatorname{span} S$.

Lemma 6. *Let Γ be an index set and let H be a norm bounded subset of $\ell^\infty(\Gamma)\subset\mathbb{R}^\Gamma$. If*

$$\overline{\operatorname{aco}(H)}^{\tau_p}=\overline{\operatorname{aco}(H)}^{\|\ \|}, \tag{1}$$

then $X:=\overline{\operatorname{span} H}^{\|\ \|}$ is K-analytic with respect to the pointwise topology τ_p of $\mathbb{R}^\Gamma$. In particular, if H is a norm bounded τ_p-compact subset of $\ell^\infty(\Gamma)$ that is norm-fragmented, then $\overline{\operatorname{span} H}^{\|\ \|}$ is K-analytic relative to τ_p.

Proof. Let $W=\overline{\operatorname{aco}(H)}^{\tau_p}$. Then W is τ_p-compact and the equation (1) implies the equality $X=\overline{\operatorname{span} H}^{\|\ \|}=\overline{\bigcup_n nW}^{\|\ \|}$. We define the set-valued map $\varphi:\mathbb{N}^\mathbb{N}\to 2^{\ell^\infty(\Gamma)}$ as follows. For $\alpha=(a_k)\in\mathbb{N}^\mathbb{N}$, let

$$\varphi(\alpha)=\bigcap_{k=1}^{\infty}\Big(a_kW+\frac{1}{k}B\Big), \tag{2}$$

where B denotes the closed unit ball of $\ell^\infty(\Gamma)$ which is τ_p-compact. Each $\varphi(\alpha)$ is a non-empty τ_p-compact set contained in X. Now, we prove that the set-valued map φ is upper-semicontinuous relative to τ_p. Let U be a τ_p-open set such that $\varphi(\alpha) \subset U$. Then by the definition (2) for $\varphi(\alpha)$, we see that there is an $m \in \mathbb{N}$ such that

$$\bigcap_{k=1}^{m} \left(a_k W + \frac{1}{k} B\right) \subset U.$$

Then $\varphi([\alpha|m]) \subset U$, where $[\alpha|m]$ stands for the open neighborhood of α defined by $[\alpha|m] = \{\beta \in \mathbb{N}^{\mathbb{N}} : \beta|m = \alpha|m\}$. This proves that $X = \varphi(\mathbb{N}^{\mathbb{N}})$ is K-analytic with respect to τ_p.

Suppose that H satisfies the assumptions of the second part of the lemma. If we regard $\ell^\infty(\Gamma)$ as the dual of the Banach space $\ell^1(\Gamma)$, then, on norm bounded subsets, τ_p is identical with the weak* topology. Now as easily seen (cf. proof of [18, Theorem 2.5]) $\widehat{H} := \{th : t \in [-1, 1],\, h \in H\}$ is again norm bounded, τ_p-compact and norm-fragmented. Since $\operatorname{aco}(H) = \operatorname{co}(\widehat{H})$, we see that H satisfies equation (1) by applying [18, Theorem 2.3] to $\widehat{H}$. This completes the proof. ∎

The following proposition is the basis for all the results in this section. If K is a compact Hausdorff space and if S is a subset of K, then we let $\gamma(S)$ denote the topology for $C(K)$ of uniform convergence on countable subsets of S.

Proposition 3.1. . *Let K be a compact subset of I^Γ such that $K \cap \mathcal{F}(\Gamma)$ is dense in K. Then $(C(K), \gamma(K \cap \Sigma(\Gamma)))^{\mathbb{N}}$ is Lindelöf.*

Proof. Let $D = K \cap \mathcal{F}(\Gamma)$. Then by hypothesis, D is dense in K. For each $\gamma \in \Gamma$, let $\pi_\gamma : K \to I$ be the γ-th projection, i.e. $\pi_\gamma(x) = x(\gamma)$ for $x \in K$, and let $G = \{\pi_\gamma : \gamma \in \Gamma\} \cup \{1\}$. Then G is a subset of the unit ball of $C(K)$ separating points of K. As in [6, Example B and C], we enlarge G as follows. First for each $n \in \mathbb{N}$, let

$$G^n = \{g_1 \ldots g_n : g_i \in G,\ i = 1, \ldots, n\} \subset B_{C(K)}.$$

Then for each $x \in D$, $\{g \in G^n : g(x) \neq 0\}$ is finite, and hence $G^n \setminus U$ is finite for each $\tau_p(D)$-neighborhood U of 0 in $C(K)$. Let $H = \bigcup\{(1/n)G^n : n \in \mathbb{N}\} \cup \{0\}$. Then again $H \setminus U$ is finite for each $\tau_p(D)$-neighborhood U of 0. It follows that H is $\tau_p(D)$-compact and each non-zero element of H is isolated. Hence H is also a norm-fragmented subset of $C(K)$. Furthermore by the Stone–Weierstrass theorem $C(K) = \overline{\operatorname{span} H}^{\|\ \|}$. Since

$$C(K) \subset \ell^\infty(D) \subset \mathbb{R}^D$$

and the norm of $C(K)$ is that of $\ell^\infty(D)$, we conclude from Lemma 6 that $C(K)$ is K-analytic relative to $\tau_p(D)$. We claim that $X := C(K)$ satisfies condition (c) of Theorem 2.1. In fact, for any countable subset A of D, the closure $\overline{A} \subset K$ is metrizable, since it is homeomorphic to a subset of I^S, where we have written $S = \bigcup\{\mathrm{supp}(a) : a \in A\}$. Hence, the Banach space $(C(\overline{A}), \|\ \|_\infty)$ is separable, and from this we can conclude that $C(K)$ is separable with respect to the pseudo-metric d_A. Consequently, by Corollary 2.2 and Remark 2.3, $(C(K), \gamma(D))^{\mathbb{N}}$ is Lindelöf. Note that $D \subset K \cap \Sigma(\Gamma) \subset K$. Hence, D is dense in $K \cap \Sigma(\Gamma)$ and $K \cap \Sigma(\Gamma)$ is dense in K. Since $\Sigma(\Gamma)$ is countably tight, each element of $K \cap \Sigma(\Gamma)$ is in the closure of a countable subset of D. It follows that on $C(K)$ the topologies $\gamma(D)$ and $\gamma(K \cap \Sigma(\Gamma))$ agree, and hence $(C(K), \gamma(K \cap \Sigma(\Gamma)))^{\mathbb{N}}$ is Lindelöf. ∎

A compact Hausdorff space K is said to be *Valdivia compact* if K can be so embedded in the space (I^Γ, τ_p) that $K \cap \Sigma(\Gamma)$ is dense in K. The spaces which satisfy the hypothesis of the previous theorem are Valdivia compact. Obviously Corson compact spaces are Valdivia compact. The next theorem, stated in the context of Banach spaces, is due to Orihuela [23].

Theorem 3.2. *Let K be a Valdivia compact subset of I^Γ with $K \cap \Sigma(\Gamma)$ dense in K. Then $(C(K), \gamma(K \cap \Sigma(\Gamma)))^{\mathbb{N}}$ is Lindelöf.*

Proof. Let the map

$$\varphi : [0,1]^\Gamma \times K \to I^\Gamma$$

be defined by $\varphi((t_\gamma)_{\gamma\in\Gamma}, (x_\gamma)_{\gamma\in\Gamma}) = (t_\gamma x_\gamma)_{\gamma\in\Gamma}$. Then φ is continuous. Therefore $\widehat{K} := \varphi([0,1]^\Gamma \times K)$ is a compact subset of (I^Γ, τ_p) containing K and $\widehat{K} \cap \mathcal{F}(\Gamma)$ is dense in $\widehat{K}$. Hence by Proposition 3.1, $(C(\widehat{K}), \gamma(\widehat{K} \cap \Sigma(\Gamma))^{\mathbb{N}}$ is Lindelöf. Since the restriction map $C(\widehat{K}) \to C(K)$ is surjective and $\gamma(\widehat{K} \cap \Sigma(\Gamma))$-$\gamma(K \cap \Sigma(\Gamma))$-continuous, the conclusion of the corollary follows. ∎

Remark. As the proof shows, the conclusion of the theorem above is true for any compact subset K of I^Γ. The assumption of K being Valdivia compact makes the space $(C(K), \gamma(K \cap \Sigma(\Gamma)))$ Hausdorff.

The next corollary is an immediate consequence of the previous theorem.

Corollary 3.3. *If K is a Corson compact space, then $(C(K), \gamma(K))^{\mathbb{N}}$ is Lindelöf. In particular $C_p(K)^{\mathbb{N}}$ is Lindelöf.*

Example. The converse of the preceding corollary is false. To show this we use the space X used by R. Pol in [24]. The compact Hausdorff space X is defined as follows. Let $\Omega = [0, \omega_1)$, i.e. the set of all countable ordinals, let Λ be the set of all limit ordinals in Ω and let $\Gamma = \Omega \setminus \Lambda$. For each $\lambda \in \Lambda$, choose an increasing sequence $s_\lambda : \mathbb{N} \to \Gamma$ that converges to λ and let $S_\lambda = \{\lambda\} \cup s_\lambda(\mathbb{N})$. The topology on Ω is defined as follows: each point in Γ is open and, for each $\lambda \in \Lambda$, the family $\{S_\lambda \setminus F : F \subset \Gamma,\ F \text{ is finite}\}$

is a base of open neighborhoods of λ. Thus the space Ω is locally compact and Hausdorff; let $X = \Omega \cup \{\omega_1\}$ be its one-point compactification. The space X is scattered and X is not Eberlein compact [24]. Consequently, X is not Corson compact (cf. [1]). However, $(C(X), \gamma(X))^{\mathbb{N}}$ is Lindelöf, showing that the converse of the preceding corollary is false. The proof that $(C(X), \gamma(X))^{\mathbb{N}}$ is Lindelöf consists of a result from [8, Section 4] as well as modifications of ones in [24]. Below, we give a general remark and an outline of the proof. We gratefully acknowledge the helpful exchanges of e-mail concerning this example with Professor R. Pol.

1. Let K be a compact Hausdorff space and let (M, ϱ) be a metric space, where ϱ is not necessarily bounded. We let $C(K, M)$ denote the space of all continuous maps $K \to M$. Since $C(K, M) \subset M^K$, the various topologies defined at the beginning of Section 2 can be localized to $C(K, M)$, and Remark 2.3 applies to $C(K, M)$. Whereas [24] is concerned with the pointwise topology, we are interested in $\gamma(K)$ for $C(K, M)$, which is, of course, stronger. Throughout this Example only, we denote $(C(K, M), \gamma(K))$ by $C_\gamma(K, M)$.

The following general remark is helpful when modifying the proofs in [24] for $C_\gamma(K, M)$. For a subset A of M and a subset B of M, we let $W(A, B) = \{f \in C(K, M) : f(A) \subset B\}$. Then one can see easily that *the family of the sets of the form $W(L, U)$, where L is a compact separable subset of K and U a non-empty open subset of M, form a subbase for the topology $\gamma(K)$. Hence $\gamma(K)$ depends only on the topology of (M, ϱ).*

2. [24, Lemma 1] can be modified as follows: *Let S be a compact zero-dimensional space. Then the space $C_\gamma(S, \mathbb{R})^{\mathbb{N}}$ is Lindelöf if and only if the product $C_\gamma(S, D)^{\mathbb{N}}$ is Lindelöf. Moreover, given a point $p \in S$, the space $C_\gamma(S, D)$ can be replaced in this equivalence by the space $G_p = \{f \in C(S, D) : f(p) = 0\}$.* Here D denotes the two-point space $\{0, 1\}$, the discrete group of order two. The proof follows the one for the original lemma. The exponential law involving $\gamma(S)$ has already been alluded to at the end of the proof of Corollary 2.2. Keeping in mind that $\gamma(S)$ is stronger than the pointwise topology, one can follow the proof in [24] to conclude that $C_\gamma(S, P)$ is Lindelöf. Hence, the proof is complete if it is shown that $C_\gamma(S, \mathbb{R})^{\mathbb{N}} \equiv C_\gamma(S, \mathbb{R}^{\mathbb{N}})$ is a continuous image of $C_\gamma(S, P)$. As in [24], choose a continuous, open and onto map $u : P \to \mathbb{R}$. Since P and $P^{\mathbb{N}}$ are homeomorphic, u gives rise to a continuous, open and onto map $u^* : P \to \mathbb{R}^{\mathbb{N}}$, which induces the continuous map $F^* : C_\gamma(S, P) \to C_\gamma(S, \mathbb{R}^{\mathbb{N}})$ by $F^*(f) = u^* \circ f$. That F^* is onto can be seen exactly as in [24]. The second part of the assertion follows from $C_\gamma(S, D) \equiv (G_p, \gamma(S)) \times D$.

3. Now, let X be the space defined above. As in [24], let $G = \{f \in C(X, D) : f(\omega_1) = 0\}$, and write $G_\gamma = (G, \gamma(X))$. Then for each $f \in G$, the sets of the form $\{f \in G : f|A = g|A\}$, with $A \subset X$ countable, constitute a $\gamma(X)$-neighborhood base of f. This means that on G, $\gamma(X)$ coincides with the topology generated by the G_δ-subsets of (G, τ_p) (cf. Section 4). We must prove that $G_\gamma^{\mathbb{N}}$ is Lindelöf. For this we apply [24, Lemma 3] to G_γ, which obviously is an Abelian topological group, with a suitably chosen E. Here we follow [8]. For each $\lambda \in \Lambda$, S_λ is a compact and open subset of X. Hence the characteristic function f_λ of S_λ is in G. Define

$$E = \{f_\lambda : \lambda \in \Lambda\} \cup \{\chi_F : F \subset \Gamma,\ F \text{ is finite}\} \subset G.$$

Then a special case of the result in [8, Section 4] shows that $(E^{\mathbb{N}}, \gamma(X))$ is Lindelöf. Note that each element of G is the characteristic function of a compact open subset of Ω, i.e. a set of the form $F \triangle \bigcup\{S_\lambda : \lambda \in L\}$, where F and L are finite subsets of Γ and Λ respectively. It follows that each element of G is the finite sum of elements in E, and therefore the set E satisfies the conditions of [24, Lemma 3].

4. K-analytic spaces without compact perfect subsets. Let (X, τ) be a Tikhonov (completely regular and T_1) space, and let $C(X, I)$ be the space of all continuous functions $f : X \to I = [0, 1]$. Then the map $\Phi : X \to I^{C(X,I)}$, given by $\Phi(x)(f) = f(x)$ for $x \in X$, $f \in C(X, I)$, embeds X topologically in $(I^{C(X,I)}, \tau_p)$ (see e.g. [15]). Here τ_p denotes the product (= pointwise) topology as before. Thus X may be regarded as a subspace of I^D with $D = C(X, I)$, and this makes it possible to apply our main theorem and its corollary to the space X when it is K-analytic. In the next paragraphs, we interpret the topological properties mentioned in Theorem 2.1 for our situation here.

The uniform metric d on X is given by

$$d(x, x') = \sup\{|f(x) - f(x')| : f \in C(X, I)\}$$

for $x, x' \in X$. Hence if $x \neq x'$, then $d(x, x') = 1$, i.e. d is the *discrete metric.*

Given a topological space (Z, τ), the *G_δ-topology associated to* τ is the topology τ_δ on Z whose basis is the family of all G_δ-sets in Z, i.e. the family of sets of the form $\bigcap\{U_n : U_n \in \tau,\ n \in \mathbb{N}\}$. When no confusion is likely, we simply write Z for the topological space (Z, τ) and refer to τ_δ as its G_δ-topology. The proof of the next lemma is omitted, since it is a verbatim repetition of the short one given for [6, Lemma 2].

Lemma 7. *Let X be a Tikhonov space. Then the G_δ-topology for X is identical with $\gamma(C(X, I))$ on X.*

Let d denote the discrete metric as above for the space (X, τ). For $S \subset X$, (S, τ) is fragmented by d down to ε, $0 < \varepsilon < 1$, if and only if each non-empty subset of S contains an isolated point, i.e. S is *scattered.* Therefore (X, τ) is σ-fragmented by d if and only if X is *σ-scattered*, that is, X is a countable union of scattered subsets. One can easily check that each compact subset of (X,τ) is fragmented by d if and only if there is no compact perfect subset of X (i.e. a compact subset of X without an isolated point). In the context of the present section, Theorem 2.1 and its Corollary 2.2 can now be translated as:

THEOREM 4.1. *Let (X,τ) be a K-analytic Tikhonov space. Then the following statements are equivalent.*

(a) *X is σ-scattered.*
(b) *X does not contain a compact perfect subset.*
(c) *(X,τ_δ) is Lindelöf.*
(d) *$(X,\tau_\delta)^{\mathbb{N}}$ is Lindelöf.*

To the list of conditions of the theorem above, we wish to add several more. For this we need some more definitions. A Hausdorff topological space Z is said to be *Fréchet–Urysohn* if, whenever, $S\subset Z$ and $z\in\overline{S}$, z is the limit of a sequence in S. We use the following simple fact: for Z to be Fréchet–Urysohn, it is sufficient that Z be countably tight and each separable subset of Z be metrizable. A subset S of Z is said to be *sequentially closed* if the limit of each sequence in S is in S. The topological space is said to be *sequential* if each sequentially closed subset is closed. The topological space Z is called a *k-space* if a subset S of Z is closed provided $S\cap C$ is closed for each compact subset C of Z. The space Z is called a *k_R-space* if a real-valued function f on Z is continuous whenever its restriction $f|C$ is continuous for each compact subset C of Z. For a Tikhonov space X, $B_1(X)$ denotes the space all functions f on X which are the pointwise limits of sequences in $C(X)$.

Finally we recall two facts. The first one is due to Arkhangel'skiĭ [3, Theorem II.1.1]: *If Z is a topological space such that Z^n is Lindelöf for each $n\in\mathbb{N}$, then $(C(Z),\tau_p)$ is countably tight.* The second one is the following simple lemma quoted from [6].

LEMMA 8. *Let Z be a Lindelöf space, and let $H\subset C(Z)$ be equicontinuous. Then (H,τ_p) is metrizable.*

COROLLARY 4.2. *Let (X,τ) be a K-analytic Tikhonov space. Then each of the statements of the theorem above is equivalent to each of the following.*

(i) *For any countable set $A\subset C(X)$, $\overline{A}^{\tau_p}$ (closure in $\mathbb{R}^X$) is τ_p-metrizable.*
(ii) *$(B_1(X),\tau_p)$ is Fréchet–Urysohn.*
(iii) *$(C(X),\tau_p)$ is Fréchet–Urysohn.*
(iv) *$(C(X),\tau_p)$ is sequential.*
(v) *$(C(X),\tau_p)$ is a k-space.*
(vi) *$(C(X),\tau_p)$ is a k_R-space.*

Proof. We first remark that if A is a countable subset of $C(X)$ then it is τ_δ-equicontinuous. Hence $\overline{A}^{\tau_p}$ (closure in $\mathbb{R}^X$) is again τ_δ-equicontinuous and is a subset of $C(X,\tau_\delta)$.

(c)$\Rightarrow$(i). This is clear from the remark above and Lemma 8.

(d)$\Rightarrow$(ii). By (d) and the Arkhangel'skiĭ theorem above, $(C(X,\tau_\delta),\tau_p)$ is countably tight. By the remark at the beginning of the proof, $B_1(X)\subset C(X,\tau_\delta)$ and so $(B_1(X),\tau_p)$ is countably tight. Hence to show (ii), it is sufficient to prove that $\overline{A}^{\tau_p}$ is τ_p-metrizable for each countable subset A of $B_1(X)$. In fact, if A is a countable subset of $B_1(X)$, then, for some countable $C\subset C(X)$, $A\subset \overline{C}^{\tau_p}$ and $\overline{C}^{\tau_p}$ is τ_p-metrizable by (i) (which is a consequence of (c) and hence of (d)). Hence $\overline{A}^{\tau_p}$ is τ_p-metrizable as desired.

(i)$\Rightarrow$(iii). Since (X,τ) is K-analytic, the product space $(X,\tau)^{\mathbb{N}}$ is also K-analytic (see [26, Theorem 2.5.5]). It follows that $(X,\tau)^n$ is Lindelöf for each $n\in\mathbb{N}$. Hence by the Arkhangel'skiĭ theorem, $(C(X),\tau_p)$ is countably tight, and by (i), if A is a countable subset of $C(X)$, $\overline{A}^{\tau_p}\cap C(X)$ is τ_p-metrizable. This shows (iii).

The implications (ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(vi) are obvious.

To complete the proof we show that (vi)$\Rightarrow$(b) by contradiction. So we assume (vi) and that there is a compact perfect subset K of X, and try to reach a contradiction. Let $\pi:(C(X),\tau_p)\to(C(K),\tau_p)$ be the restriction map $f\mapsto f|K$. Then f is continuous and open. By applying the Tietze extension theorem to βX, one can see that π is onto. Hence π is a quotient map, and this fact together with (vi) implies that $(C(K),\tau_p)$ is a k_R-space. Since $\mathbb{R}$ is homeomorphic to the interval $(-1,1)$, $(C(K,(-1,1)),\tau_p)$ is also a k_R-space. We show that this is not the case.

Since K is compact and perfect, there is a continuous onto map $\varphi: K\to[0,1]$ (cf. [26, Proposition 5.4.1]). Let λ denote the Lebesgue measure on $[0,1]$. Since the map φ induces the map of all Radon probability measures on K onto that of $[0,1]$, there exists a Radon probability measure μ on K such that $\lambda(B)=\mu(\varphi^{-1}(B))$ for each Borel subset B of $[0,1]$. In particular $\mu(\{x\})=0$ for each $x\in K$. We show that the function $\Psi:(C(K,(-1,1)),\tau_p)\to\mathbb{R}$ given by

$$\Psi(f)=\int_K f\,d\mu,$$

for each $f\in C(K,(-1,1))$, is continuous when restricted to compact subsets of $(C(K,(-1,1)),\tau_p)$ but it is not τ_p-continuous on the whole of $C(K,(-1,1))$.

Let H be a τ_p-compact subset of $C(K,(-1,1))$ and let $C\subset H$ be arbitrary. If $f\in\overline{C}^{\tau_p}$, then there is a sequence $(f_n)_n$ in C that converges to f pointwise (cf. [16, Theorem 2.8.20]). Then by the Dominated Convergence

Theorem we have $\Psi(f) = \lim_n \Psi(f_n)$. It follows that $\Psi(\overline{C}^{\tau_p}) \subset \overline{\Psi(C)}^{\tau_p}$. This shows that $\Psi|H$ is τ_p-continuous.

On the other hand, suppose that Ψ is continuous on $C(K,(-1,1))$ at, say, 0. Then there is a finite subset F of K and an $\varepsilon > 0$ such that $\Psi(f) < 1/4$ whenever $f \in W := \{g \in C(K,(-1,1)) : |g(x)| < \varepsilon \text{ for each } x \in F\}$. Since $\mu(F) = 0$, there is an open subset U of K with $F \subset U$ and $\mu(U) < 1/2$. Let $L = K \setminus U$. Then by Urysohn's lemma [15, Lemma 4, p. 115], there is a continuous function $h : K \to [0, 1/2]$ such that $h|F \equiv 0$ and $h|L \equiv 1/2$. Then $h \in W$ but $\Psi(h) > 1/4$ since $\mu(L) > 1/2$. This contradiction proves that Ψ is not τ_p-continuous. ■

Examples. We give examples to show that Theorem 4.1 cannot be generalized to the case where (X,τ) is a Čech-analytic Lindelöf space or a countably K-determined space.

A. *Čech-analytic Lindelöf space, assuming CH.* The following example has been communicated to us by Professor V. Tkachuk in response to a related question. A similar construction has been used by him in [21]. We gratefully acknowledge his permission for us to use the example here. Let L be a Lusin set in $\mathbb{R}$, i.e. a subset L of $\mathbb{R}$ of cardinality continuum such that, whenever N is a nowhere dense subset of $\mathbb{R}$, $L \cap N$ is countable. Recall that N is nowhere dense in $\mathbb{R}$ if the interior of its closure in $\mathbb{R}$ is empty. Such a set L can be constructed assuming the Continuum Hypothesis (CH) (see e.g. [17, Theorem 2.1]). Note that if a subset A of L is nowhere dense relative to L, then it is nowhere dense in $\mathbb{R}$ and so it is countable. Let λ be the usual topology of $\mathbb{R}$ relativized to L and let D be a countable dense subset of (L,λ). Now the family $\mathcal{B}$ of subsets of L given by

$$\mathcal{B} = \{\{x\} : x \in L \setminus D\} \cup \{U \in \lambda : U \cap D \neq \emptyset\}$$

is a base for a unique topology τ for L. Clearly the space (L,τ) is Hausdorff and regular. The following are additional properties.

(1) (L,τ) is Lindelöf. For suppose $\mathcal{U}$ is a covering of L by a subfamily of $\mathcal{B}$. Then $\mathcal{V} := \{V \in \mathcal{U} : V \cap D \neq \emptyset\}$ has a countable subcover $\mathcal{W}$ of D. Let $W = \bigcup \mathcal{W}$. Then W is open dense in (L,λ) and so $L \setminus W$ is nowhere dense in (L,λ) and hence countable. It follows that $\mathcal{U}$ has a countable subcover of L.

(2) Since (L,τ) is regular and Lindelöf, it is normal by Tikhonov's Lemma (see [15, Lemma 3.1]). Thus (L,τ) is a Tikhonov space.

(3) The space (L,τ) is σ-discrete. In fact, $L \setminus D$ is discrete and D is countable.

(4) The space (L,τ) is Čech-analytic. A discrete space, being Čech-complete, is Čech-analytic. Since the family of Čech-analytic sets is closed under the Suslin operation, (L,τ) is Čech-analytic by (3).

(5) The G_δ-topology τ_δ is the discrete topology. Hence the space (L, τ_δ) is not Lindelöf.

In Theorem 4.1, assume only that (X, τ) is a Čech-analytic Lindelöf space. Then the equivalence of (a) and (b) still holds because of [12, Theorem 4.1]. However, the example above shows that (a)⇒(c) fails. In fact, (2) and (4) show that (L, τ) is Tikhonov, Lindelöf and Čech-analytic, and (3) shows that (a) holds. However, (5) shows that (c) fails. Consequently, as pointed out in Remark 2.4, Theorem 2.1 is not valid when X is assumed to be Čech-analytic and Lindelöf in lieu of "K-analytic".

B. *Countably K-determined spaces.* There is another kind of generalization of K-analyticity. A topological space (T, τ) is said to be *countably K-determined* if there is an upper-semicontinuous set-valued map $F : M \to 2^T$ for some separable metric space M such that $F(M) = T$ and $F(m)$ is compact for each $m \in M$. Obviously any separable metric space is countably K-determined. Let $B \subset \mathbb{R}$ be a *Bernstein set*, i.e. an uncountable set B such that each compact subset of B is countable (see [28, Corollary 1.5.14]). Then (B, τ) is countably K-determined, where τ is the relativization of the usual topology for $\mathbb{R}$. The space (B, τ) clearly satisfies the condition (b) of Theorem 4.1. Since each scattered subset of (B, τ) is countable, (B, τ) is not σ-scattered. Also the G_δ-topology of (B, τ) is discrete, and hence (B, τ_δ) is not Lindelöf. Thus conditions (a) and (c) fail in (B, τ).

Remarks. We add a few comments on known results vis-à-vis our Theorem 4.1 and Corollary 4.2.

(I) The equivalence of (iii), (iv) and (v) is valid for an arbitrary Tikhonov space X (see [3, Section II.3]). Our results show that if we impose the condition that X is K-analytic, then any one of (iii), (iv) or (v) implies that X is σ-scattered.

(II) The equivalence of (b) and (c) is one of the main results of Blasco in [4]. We acknowledge that some of the techniques in the proof of our Theorem 2.1 were inspired by studying his paper.

(III) A topological space is called a *P-space* if its G_δ-topology agrees with the original one. Noble has shown in [20] that the product of a countable family of Lindelöf P-spaces is again Lindelöf. Hence the equivalence of (c) and (d) holds for an arbitrary topological space (X, τ).

(IV) A topological space T is said to be *angelic* if, whenever C is a relatively countably compact subset of T, its closure $\overline{C}$ is compact and each element of $\overline{C}$ is a limit of a sequence in C. It is known that $(C(X), \tau_p)$ is angelic whenever X is K-analytic (see [22]). Condition (iii) of Corollary 4.2 shows that there is a big difference between angelicity and Fréchet–Urysohn property of $C(X)$.

(V) In [14] Kąkol and López-Pellicer state the equivalence (iii), (vi) and condition (a′) below:

(a′) *The space X is scattered*,

in case X is Čech-complete and Lindelöf (see [14, Theorem 2]). Since, in this case, X is hereditarily Baire, (a) and (a′) are actually equivalent (see e.g. [12, Corollary 3.1.2]). Now, one can show that a Čech-complete Lindelöf space is a $K_{\sigma\delta}$ subset of its compactification, hence K-analytic. So, Theorem 2 in [14] is also a consequence of our Theorem 4.1 and Corollary 4.2.

5. Applications to Banach spaces. In this section, we abstract some of the arguments in the previous sections in the setting of Banach spaces. Let X be a Banach space and X^* its dual Banach space. The unit ball $\{x \in X : \|x\| \leq 1\}$ is denoted by B_X. Thus the unit ball of X^* is B_{X^*}. If S is a subset of X^*, then $\sigma(X, S)$ denotes the weakest topology for X that makes each member of S continuous, or equivalently, the topology of pointwise convergence on S. Dually, if S is a subset of X, then $\sigma(X^*, S)$ is the topology for X^* of pointwise convergence on S. In particular $\sigma(X, X^*)$ and $\sigma(X^*, X)$ are the weak (w) and weak* (w*) topologies respectively. Of course, $\sigma(X, S)$ is always a locally convex topology and it is Hausdorff if and only if $X^* = \overline{\operatorname{span} S}^{\mathrm{w}^*}$ and similarly for $\sigma(X^*, S)$. When $S \subset X^*$, $\gamma(X, S)$ (or simply $\gamma(S)$) is the topology for X of uniform convergence on countable subsets of S. The Banach space is said to have *property* (C) (after Corson) if each collection of (norm) closed convex subsets of X with empty intersection contains a countable subcollection with empty intersection. A subset B of X^* is said to be *norming* if the function p of X given by $p(x) = \sup\{|x^*(x)| : x^* \in B\}$ is a norm equivalent to $\| \ \|$. This is the case if and only if B is norm bounded and $kB_{X^*} \subset \overline{\operatorname{aco} B}^{\mathrm{w}^*}$ for some $k > 0$. Without loss of generality we assume that $k = 1$ so that $\|x\| \leq p(x)$ for $x \in X$.

In [10], Godefroy and Talagrand called a Banach space X *representable* if there is a countable norming subset B of X^* such that $(X, \sigma(X, B))$ is analytic. In this section, we consider a wider class of Banach spaces X such that $(X, \sigma(X, B))$ is K-analytic for some norming subset B of X^*.

We need the following lemma that improves Proposition 4.1 in [6].

Lemma 9. *Let X be a Banach space and $B \subset X^*$ a norming subset. If X has property (C), then $\gamma(B)$ is stronger than the weak topology of X.*

Proof. We simply have to show that for each $x^* \in B_{X^*}$ and $\varepsilon > 0$ the weak open neighborhood of the origin

$$V = \{x \in X : |x^*(x)| < \varepsilon\} = \{x \in X : x^*(x) < \varepsilon\} \cap \{x \in X : x^*(x) > -\varepsilon\}$$

is a $\gamma(B)$-neighborhood of the origin, or equivalently that the weak open semi-spaces

$$U_0 = \{x \in X : x^*(x) < \varepsilon\} \quad \text{and} \quad U_1 = \{x \in X : x^*(x) > -\varepsilon\}$$

are $\gamma(B)$-neighborhoods of the origin. For each $b^* \in B$, let

$$D_{b^*} = \{x \in X : |b^*(x)| \leq \varepsilon/2\}.$$

Clearly for $i \in \{0,1\}$,

$$\bigcap\{D_{b^*} : b^* \in B\} \subset \{x \in X : \|x\| \leq \varepsilon/2\} \subset U_i,$$

or equivalently, $(X \setminus U_i) \cap \bigcap\{D_{b^*} : b^* \in B\} = \emptyset$.

Now fix $i \in \{0,1\}$. Since X has property (C) and each entry in the intersection above is convex closed, there is a countable subset A of B such that

$$(X \setminus U_i) \cap \bigcap\{D_{b^*} : b^* \in A\} = \emptyset,$$

or equivalently

$$\bigcap\{D_{b^*} : b^* \in A\} \subset U_i,$$

which means that

$$\{x \in X : \sup_{b^* \in A} |b^*(x)| \leq \varepsilon/2\} \subset U_i.$$

This shows that U_i is a $\gamma(B)$-neighborhood of the origin for each $i \in \{0,1\}$. ∎

Proposition 5.1. *Let X be a Banach space such that, for some norming subset B of X^*, $(X, \sigma(X,B))$ is K-analytic. Then the following statements are equivalent.*

(i) *X has property (C) and $(X, \sigma(X,B))$ is σ-fragmented by the norm.*
(ii) *(X, w) is Lindelöf.*
(iii) *(B_{X^*}, w^*) is countably tight and its w^*-separable subsets are metrizable.*

Proof. (i)⇒(ii). $(X, \gamma(B))$ is Lindelöf after Theorem 2.1. By Lemma 9, $\gamma(B)$ is stronger than the weak topology, and consequently (X, w) is Lindelöf.

(ii)⇒(i). Property (C) is obvious. We use [7, Theorem B] to deduce that each $\sigma(X,B)$-compact subset of X is norm fragmented. Now, by Theorem 2.1, $(X, \sigma(X,B))$ is σ-fragmented.

(i)&(ii)⇒(iii). If we assume (i)&(ii) then we know that $\gamma(B)$ is stronger than the weak topology by Lemma 9 and that $(X, \gamma(B))^{\mathbb{N}}$ is Lindelöf by Corollary 2.2. Consequently, being a continuous image of the space $(X, \gamma(B))^{\mathbb{N}}$, $(X, \mathrm{w})^{\mathbb{N}}$ is Lindelöf. Hence by Arkhangel'skiĭ's theorem (see Section 4), $(X^*, \mathrm{w}^*) \subset (C(X, \mathrm{w}), \tau_p)$ is countably tight.

We prove that each w^*-separable subset M of B_{X^*} is w^*-metrizable. Since $B_{X^*} \subset \overline{\operatorname{aco} B}^{\mathrm{w}^*}$ and (X^*, w^*) is countably tight, $M \subset \overline{\operatorname{aco} A}^{\mathrm{w}^*}$ for some countable subset A of B. Then $\operatorname{aco} A$ is $\gamma(B)$-equicontinuous on the Lindelöf space $(X, \gamma(B))$. Consequently, $\overline{\operatorname{aco} A}^{\mathrm{w}^*}$ is $\gamma(B)$-equicontinuous, and hence w^*-metrizable by Lemma 8.

(iii)⇒(i). By [25, Theorem 3.4], if (B_{X^*}, w^*) is countably tight then X has property (C). To prove that $(X, \sigma(X, B))$ is σ-fragmented by the norm, by Theorem 2.1, it is enough to show that (X, d_A) is separable for each countable subset A of B, where d_A is a pseudo-metric on X given by

$$d_A(x, y) = \sup\{|x^*(x) - x^*(y)| : x^* \in A\} = \sup\{|x^*(x) - x^*(y)| : x^* \in \overline{A}^{\mathrm{w}^*}\}.$$

But this is obvious since $X|\overline{A}^{\mathrm{w}^*} \subset C(\overline{A}^{\mathrm{w}^*}, \mathrm{w}^*)$ and $C(\overline{A}^{\mathrm{w}^*}, \mathrm{w}^*)$ is norm separable on account of $(\overline{A}^{\mathrm{w}^*}, \mathrm{w}^*)$ being compact and metrizable by (iii). ■

It should be noted that σ-fragmentability cannot be dropped in statement (i) of Proposition 5.1. An example follows. Take $X = JT^*$, the dual of the James tree space JT, and B the unit ball of JT. In this case $\sigma(JT^*, B) = \mathrm{w}^*$ and we have:

(i) $(X, \sigma(X, B))$ is σ-compact;
(ii) X has property (C) according to Example 5.8 of [9];
(iii) $(X, \sigma(X, B))$ is not σ-fragmented by the norm (i.e. JT^* does not have the RNP) and (X, w) is not Lindelöf.

There are plenty of Banach spaces X for which there is a norming set $B \subset B_{X^*}$ such that the space $(X, \sigma(X, B))$ is K-analytic. The following are some of the examples.

• *Weakly K-analytic Banach spaces.* If X is a Banach space that is K-analytic for its weak topology then for any norming set $B \subset X^*$ the space $(X, \sigma(X, B))$ is K-analytic too because it is a continuous image of (X, w). We refer to the paper by Talagrand [27] for an account concerning weakly K-analytic Banach spaces.

• *Dual Banach spaces.* If $X = Y^*$ is a dual Banach space, and we write $B := B_Y$, then $B \subset B_{X^*}$ is norming and $X = \bigcup_n nB_{Y^*}$ is σ-compact with respect to $\sigma(X, B)$.

• *Representable Banach spaces.* As mentioned earlier the class of Banach spaces that satisfies condition of Proposition 5.1 includes the class of representable Banach spaces introduced in [10]. In this paper Godefroy and Talagrand proved that if X is representable, then (X, w) is Lindelöf if and only if X is separable (see the proof of (1)⇔(3) of [10, Théorème 7]). Proposition 5.1 gives an alternative proof of this fact.

Corollary 5.2 (Godefroy and Talagrand [10]). *Let X be a representable Banach space. Then X is weakly Lindelöf if, and only if, X is separable.*

Proof. We prove the non-obvious direction. Assume that (X, w) is Lindelöf. According to the definition of representability, there is a countable norming subset B of X^* such that $(X, \sigma(X, B))$ is analytic. Hence by Proposition 5.1, $(X, \sigma(X, B))$ is σ-fragmented by the norm. Since B is countable $(X, \sigma(X, B))$ is metrizable and separable. So by Lemma 1, X is separable for the norm topology. ∎

• *Banach spaces generated by RN-compact subsets.* Let X be a Banach space, B a norming subset of B_{X^*} and $H \subset X$ a $\sigma(X, B)$-compact set which is fragmented by the norm. If $X = \overline{\mathrm{span}(H)}^{\|\ \|}$ then X is said to be generated by the RN-compact set H (see [6]). In this case $(X, \sigma(X, B))$ is K-analytic by Lemma 6.

Open problems. In Proposition 5.1, can one replace statement (iii) with the stronger one, viz. (B_{X^*}, w^*) *is Corson compact*?

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