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Tangency properties of sets with finite geometric curvature energies

by

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Abstract. We investigate tangential regularity properties of sets of fractal dimension, whose inverse thickness or integral Menger curvature energies are bounded. For the most prominent of these energies, the integral Menger curvature

$$\mathcal{M}_p^\alpha(X) := \int_X\int_X\int_X \kappa^p(x,y,z)\, d\mathcal{H}_X^\alpha(x)\, d\mathcal{H}_X^\alpha(y)\, d\mathcal{H}_X^\alpha(z),$$

where $\kappa(x,y,z)$ is the inverse circumradius of the triangle defined by x, y and z, we find that $\mathcal{M}_p^\alpha(X) < \infty$ for $p \geq 3\alpha$ implies the existence of a *weak approximate α-tangent* at every point of the set, if some mild density properties hold. This includes the scale invariant case $p = 3$ for $\mathcal{M}_p^1$, for which, to the best of our knowledge, no regularity properties have been established before. Furthermore we prove that for $\alpha = 1$ these exponents are sharp, i.e., if p lies below the threshold value of scale invariance, then there exists a set containing points with no weak approximate 1-tangent, but such that the energy is still finite. Moreover we demonstrate that weak approximate tangents are the most we can expect. For the other curvature energies analogous results are shown.

1. Introduction. In [13] J.-C. Léger proved a remarkable theorem ([1]) which states that one-dimensional Borel sets in $\mathbb{R}^n$ with finite integral Menger curvature $\mathcal{M}_2^1$ are 1-rectifiable. Here, the integral Menger ([2]) curvature of a set $X \subset \mathbb{R}^n$ is the triple integral over the squared inverse circumradius ([3]),

$$\mathcal{M}_p^\alpha(X) := \int_X\int_X\int_X [r(x,y,z)]^{-p}\, d\mathcal{H}_X^\alpha(x)\, d\mathcal{H}_X^\alpha(y)\, d\mathcal{H}_X^\alpha(z),$$

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([1]) Léger refers to an unpublished work of G. David that inspired his work.

([2]) Named after Karl Menger, because in [20] Menger introduced the limit of the inverse circumradius, when the three points in the argument converge to a single point, as a pointwise curvature.

([3]) For other applications of the circumradius see [23].

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for $p=2$ and $\alpha=1$, where $\mathcal{H}^\alpha_X$ is the Hausdorff measure on X. The circumradius $r(x,y,z)$ is the radius of the unique circle on which the vertices of the non-degenerate triangle $\{x,y,z\}$ lie; in the case of a degenerate triangle it is set to be infinite. These results for $\mathcal{M}^1_2$ were later extended to metric spaces in [10], and in [17] to sets of fractional dimension, where C^1-α-rectifiability of measurable sets with positive and finite $\mathcal{H}^\alpha$ measure was shown if $\mathcal{M}^\alpha_{2\alpha}$ is finite and $\alpha \in (0,1/2]$ under the additional assumption that these sets are α-Ahlfors regular (4). As a consequence Léger's theorem also ensures that an $\mathcal{H}^1$ measurable set $E \subset \mathbb{R}^n$ with $\mathcal{M}^1_2(E) < \infty$ has approximate 1-tangents at $\mathcal{H}^1$ a.e. point. By an *approximate α-tangent* at a point x we mean a direction $s \in \mathbb{S}^{n-1}$ such that

$$\lim_{r\downarrow 0} \frac{\mathcal{H}^\alpha([X \setminus C_{s,\varepsilon}(x)] \cap \overline{B}_r(x))}{(2r)^\alpha} = 0 \quad \text{for all } \varepsilon > 0,$$

where $C_{s,\varepsilon}(x)$ is the double cone with opening angle ε in direction s about x (cf. [19, p. 203]; for different tangential regularity properties cf. also [18]). One might think of it as a kind of geometric or measure-theoretic counterpart to differentiability. Roughly speaking it means that the set is locally well approximated by the approximate tangent. For example a regular, differentiable curve has approximate 1-tangents at all points and these tangents coincide with the usual tangent, but the arc length parametrisation of the set $S := \{(x,0) \mid x \in [0,1]\} \cup \{(x,x^2) \mid x \in [0,1]\}$ has no tangent at $(0,0)$, despite the set having an approximate 1-tangent at this point (see Remark 4.5).

Now one could ask if the condition $\mathcal{M}^1_2(X) < \infty$ also guarantees that the set has approximate 1-tangents at *all* points, or, if this is not the case, what influence, if any, the exponent p of the energy $\mathcal{M}^1_p$ has on these matters. This question and related topics are the subject of this paper.

Complementary to this research, where highly irregular sets are permitted, was the investigation of rectifiable curves, which have a classic tangent $\mathcal{H}^1$ a.e. to begin with, of finite $\mathcal{M}^1_p$ energy. It turns out (see [27]) that for $p > 3$ this guarantees that the curve is simple and that the arc length parametrisation is of class $C^{1,1-3/p}$, which can be interpreted as a geometric Morrey–Sobolev imbedding. In [1] it was shown that the space of curves with finite $\mathcal{M}^1_p$ for $p > 3$ is that of Sobolev–Slobodetskiĭ embeddings of class $W^{2-2/p,p}$. The same program has also been conducted for a different kind of energy, the so-called tangent point energy in [33, 2].

We would like to point out the important role of integral Menger curvature for $p=2$ in the solution of the Painlevé problem, i.e. to find geometric

(4) It was also shown that these results are sharp, i.e. wrong for $s \in (1/2,1)$, but that there is no hope of maintaining these results for $s \in (0,1)$ if one drops the α-Ahlfors regularity.

characterisations of removable sets for bounded analytic functions; see [21, 5] for a detailed presentation and references.

Besides integral Menger curvature there are other interesting curvature energies that have been investigated in the same vein. In [8] Gonzalez and Maddocks proposed their notion of *thickness*

$$\Delta[X] := \inf_{\substack{x,y,z\in X\\ x\neq y\neq z\neq x}} r(x,y,z)$$

of a knot X, which is the infimum of the circumradius $r(x,y,z)$ over all triangles $\{x,y,z\}$ on the curve, and also suggested investigating different integral curvature energies

$$\mathcal{U}_p^\alpha(X) := \int_X \Big[\inf_{\substack{y,z\in X\\ x\neq y\neq z\neq x}} r(x,y,z)\Big]^{-p} d\mathcal{H}_X^\alpha(x),$$

$$\mathcal{I}_p^\alpha(X) := \int_X \int_X \Big[\inf_{\substack{z\in X\\ x\neq y\neq z\neq x}} r(x,y,z)\Big]^{-p} d\mathcal{H}_X^\alpha(x)\, d\mathcal{H}_X^\alpha(y),$$

and $\mathcal{M}_p^\alpha$, where the inverse circumradius is integrated to the power p and the infimisations are successively replaced by integrations. That arc length parametrisations of curves with finite inverse thickness are actually of class $C^{1,1}$, and the existence of ideal knots, which are minimizers of the inverse thickness in a knot class under the restriction of fixed length, were shown in [9, 4, 7]; for further research in this direction see also [24, 25]. In the series of works [30, 26, 27] the integral curvature energies $\mathcal{U}_p^1$, $\mathcal{I}_p^1$ and $\mathcal{M}_p^1$ have been investigated for closed rectifiable curves, to find that the arc length parametrisations of curves with finite energy for $p \in [1,\infty)$, $p \in (2,\infty)$ and $p \in (3,\infty)$, respectively, are simple and actually belong to the class $C^{1,\beta_{\mathcal{F}}(p)}$, where $\beta_{\mathcal{U}}(p) = 1 - 1/p$, $\beta_{\mathcal{I}}(p) = 1 - 2/p$ and $\beta_{\mathcal{M}}(p) = 1 - 3/p$. In [1] it was shown that the space of curves with finite $\mathcal{I}_p^1$ for $p > 2$ and $\mathcal{M}_p^1$ for $p > 3$ is that of Sobolev–Slobodetskiĭ embeddings of class $W^{2-1/p,p}$ and $W^{2-2/p,p}$, respectively. Similar energies for surfaces and higher-dimensional sets have been examined in [28, 29, 31, 14, 15, 11, 12, 32, 3].

As mentioned at the very beginning, the purpose of this paper is to investigate which pointwise tangential properties can be expected of sets in Euclidean space with finite energy. To be more precise we will investigate if a set X possesses an approximate α-tangent or at least a weak approximate α-tangent at every point x. A *weak approximate α-tangent* is a mapping $s : (0,\rho) \to \mathbb{S}^{n-1}$ such that

$$\lim_{r\downarrow 0} \frac{\mathcal{H}^\alpha([X \setminus C_{s(r),\varepsilon}(x)] \cap \overline{B}_r(x))}{(2r)^\alpha} = 0 \quad \text{for all } \varepsilon > 0.$$

For the T-shaped set $E := ([-1,1]\times\{0\}) \cup (\{0\}\times[0,1])$ it is shown that $\mathcal{M}_2^1(E) < \infty$ does *not* suffice to infer that the set has weak approximate 1-

tangents at all points with positive lower density (see Lemma 6.1). So it seems that these properties might depend on the exponent p and the parameter α of the integral curvature energies $\mathcal{U}_p^\alpha$, $\mathcal{I}_p^\alpha$ and $\mathcal{M}_p^\alpha$. Thus our aim is to find conditions on p and α that ensure the existence of α-tangents at all points with positive lower density. We shall solve this question completely, to be honest with one minor additional technical requirement in the case of $\mathcal{M}_p^\alpha$, namely $\Theta^{*\alpha}(\mathcal{H}^\alpha, X, x) < \infty$ (for the notation, see Definition 3.1), which, despite our best efforts, we have not been able to remove. We have gathered the findings from different sections of the present paper in Theorem 1.1 below. Note that compared to [17] we do not require the set to be measurable and α-Ahlfors regular and have more detailed information on *which* points do possess tangents, but we pay for that by a more restrictive requirement on the parameter p. We also remark that in [16, 1.5 Corollary, p. 13] it is shown that for $\alpha > 1$ and an $\mathcal{H}^\alpha$ measurable set $X \subset \mathbb{R}^n$ with $0 < \mathcal{H}^\alpha(X) < \infty$ we always have $\mathcal{M}_{2\alpha}^\alpha(X) = \infty$, which somewhat restricts the extent of the next theorem for $\alpha > 1$. On the other hand, however, there are a lot more sets allowed in the theorem that still could have finite $\mathcal{M}_{2\alpha}^\alpha$.

Theorem 1.1 (Main result). *Let $X \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, $\alpha \in (0,\infty)$.*

- *Let $1/\Delta[X] < \infty$ and $\mathcal{H}^1(X) < \infty$. Then X has an approximate 1-tangent at x.*
- *Let $p \in [\alpha,\infty)$ and $\mathcal{U}_p^\alpha(X) < \infty$. Then X has an approximate α-tangent at x.*
- *Let $p \in [2\alpha,\infty)$, $\mathcal{I}_p^\alpha(X) < \infty$ and $\Theta_*^\alpha(\mathcal{H}^\alpha, X, x) > 0$. Then X has a weak approximate α-tangent at x.*
- *Let $p \in [3\alpha,\infty)$, $\mathcal{M}_p^\alpha(X) < \infty$ and*

$$0 < \Theta_*^\alpha(\mathcal{H}^\alpha, X, x) \le \Theta^{*\alpha}(\mathcal{H}^\alpha, X, x) < \infty.$$

  Then X has a weak approximate α-tangent at x.

To the best of our knowledge these are the first results regarding regularity that incorporate the critical cases $p = 2$ for $\mathcal{I}_p^1$ and $p = 3$ for $\mathcal{M}_p^1$. Moreover, we show that the exponents are sharp for $\alpha = 1$, that is, there is a set, namely the T-shaped set E above, that contains a point without weak approximate 1-tangent and has finite energy if p is below the respective threshold value.

Proposition 1.2 (Exponents are sharp for $\alpha = 1$). *For $E := ([-1,1] \times \{0\}) \cup (\{0\} \times [0,1])$ we have*

- *$\mathcal{U}_p^1(E) < \infty$ for $p \in (0,1)$,*
- *$\mathcal{I}_p^1(E) < \infty$ for $p \in (0,2)$,*
- *$\mathcal{M}_p^1(E) < \infty$ for $p \in (0,3)$.*

Furthermore we demonstrate that there is a set F that has a point without an approximate 1-tangent and finite $\mathcal{I}_p^1$ and $\mathcal{M}_p^1$ for all $p \in (0,\infty)$. Hence there is no hope of obtaining the analog to the main result for $\mathcal{U}_p^1$ for these two energies.

Proposition 1.3 (Weak approximate 1-tangents are optimal for $\alpha = 1$). *There is a set $F \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$ such that F has no approximate 1-tangent at x and*

- $\mathcal{I}_p^1(F) < \infty$ *for* $p \in (0,\infty)$,
- $\mathcal{M}_p^1(F) < \infty$ *for* $p \in (0,\infty)$.

Finally we mention that, using the techniques of this paper, we can show that the $\mathcal{M}_p^1$ energy of all simple polygons is finite if and only if $p \in (0,3)$; see [22]. Similar statements hold for $\mathcal{U}_p^1$ and $\mathcal{I}_p^1$ for p below the scale invariant threshold value.

The paper is organised as follows: Section 2 introduces integral curvature energies for arbitrary metric spaces, as this is not more complicated than doing so for arbitrary sets in $\mathbb{R}^n$ and even provides a simpler notation. Then, in Section 3, we give lower bounds for the Hausdorff measure of annuli under certain conditions on the Hausdorff density. We also introduce a new and slightly wider notion of Hausdorff density for set-valued mappings. In Section 4 we give some examples and simple properties of the different notions of tangents. Finally we prove the main Theorem 1.1 in Section 5. Then Section 6 shows Proposition 1.2 by estimating the energies $\mathcal{U}_p^\alpha$, $\mathcal{I}_p^\alpha$ and $\mathcal{M}_p^\alpha$ of the set E. Section 7 contains the proof of Proposition 1.3. To improve readability we have deferred several technical issues to the appendix.

2. Curvature energies and notation. In a metric space (X,d) we denote by $B_r(x)$ the *open ball* of radius r about $x \in X$ and by $\overline{B}_r(x) := \{y \in X \mid d(x,y) \le r\}$ the *closed ball* of radius r about $x \in X$. For a set X with outer measure $\mathcal{V}$ we write $\mathcal{C}(\mathcal{V})$ for the $\mathcal{V}$ *measurable sets* of X, i.e. those sets E which are measurable in the sense of Carathéodory:

$$\mathcal{V}(M) = \mathcal{V}(M \cap E) + \mathcal{V}(M \setminus E) \quad \text{for all } M \subset X.$$

Let (X,τ) be a topological space (in this paper the topology is always induced by a metric). Then $\mathcal{B}(X)$ denotes the *Borel sets* of (X,τ). For two measurable spaces $(X,\mathcal{A})$ and $(Y,\mathcal{B})$ we say that a function $f : (X,\mathcal{A}) \to (Y,\mathcal{B})$ is $\mathcal{A}$-$\mathcal{B}$ *measurable* if $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$. By $\mathcal{H}^\alpha$ we denote the α*-dimensional Hausdorff measure* on a metric space (X,d) and by $\mathcal{L}^n$ the n*-dimensional Lebesgue measure* on $\mathbb{R}^n$. The *extended real numbers* are indicated by $\overline{\mathbb{R}}$. Furthermore we write $\operatorname{Pot}(X)$ for the *power set* of a set X. For the notation in case of $X \subset \mathbb{R}^n$, where it might not be clear if $\mathcal{H}^\alpha$ is the Hausdorff measure on $\mathbb{R}^n$ or X, see the comment after Lemma 2.8.

The thickness of a set was introduced by O. Gonzalez and J. Maddocks in [8], where they also suggested investigating the integral curvature energies $\mathcal{U}_p^1, \mathcal{I}_p^1$ and $\mathcal{M}_p^1$, which will be defined subsequently.

Definition 2.1 (Circumradius, intermediate and global radius of curvature, thickness). Let (X,d) be a metric space. We define the *circumradius* of three distinct points $x,y,z \in X$ as the circumradius of the triangle defined by the (unique up to Euclidean motions) isometric embedding of these points in the Euclidean plane, i.e.

$$r : \{(x,y,z) \in X^3 \mid d(x,y), d(y,z), d(z,x) > 0\} =: D \to \overline{\mathbb{R}},$$
$$(x,y,z) \mapsto \frac{abc}{\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}},$$

where $a := d(x,y)$, $b := d(y,z)$, $c := d(z,x)$ and $\alpha/0 = \infty$ for any $\alpha > 0$. We also write $X_0 := X^3 \setminus D$. Now we define the mappings $\rho : X^2 \setminus \operatorname{diag}(X) \to \overline{\mathbb{R}}$ and $\rho_G : X \to \overline{\mathbb{R}}$ by

$$\rho(x,y) := \inf_{\substack{w\in X\\ y\neq w\neq x}} r(x,y,w) \quad \text{and} \quad \rho_G(x) := \inf_{\substack{v,w\in X\\ x\neq v\neq w\neq x}} r(x,v,w),$$

which are often called *intermediate* and *global radius of curvature*, respectively. Here $\operatorname{diag}(X) := \{(x,x) \mid x \in X\}$ denotes the *diagonal* of X. The *thickness* is then defined to be

$$\Delta[X] := \inf_{\substack{u,v,w\in X\\ u\neq v\neq w\neq u}} r(u,v,w).$$

Remark 2.2 (Different formulas for the circumradius). We note that in $\mathbb{R}^n$ there are various formulas for the circumradius, for example one has the following representations for $x,y,z \in \mathbb{R}^n$ mutually distinct [21, (14) and (15), p. 29]

$$r(x,y,z) = \frac{\|x-y\|}{2|\sin(\measuredangle(x,z,y))|} = \frac{\|x-z\|\,\|y-z\|}{2\operatorname{dist}(z, L_{x,y})},$$

where $L_{x,y} := x + \mathbb{R}(x-y)$ is the straight line connecting x and y.

Lemma 2.3 (Reciprocal radii of curvature are l.s.c. and measurable). *Let (X,d) be a metric space. Then the functions*

$$\kappa_G : X \to \overline{\mathbb{R}}, \qquad x \mapsto 1/\rho_G(x),$$
$$\kappa_i : X^2 \to \overline{\mathbb{R}}, \qquad (x,y) \mapsto \begin{cases} 1/\rho(x,y), & (x,y) \in X^2 \setminus \operatorname{diag}(X), \\ 0, & \textit{else}, \end{cases}$$
$$\kappa : X^3 \to \overline{\mathbb{R}}, \qquad (x,y,z) \mapsto \begin{cases} 1/r(x,y,z), & (x,y,z) \in X^3 \setminus X_0, \\ 0, & \textit{else}, \end{cases}$$

with the convention $1/0 = \infty$ *and* $1/\infty = 0$, *are lower semicontinuous and* $\mathcal{B}(X)$-$\mathcal{B}(\overline{\mathbb{R}})$, $\mathcal{B}(X^2)$-$\mathcal{B}(\overline{\mathbb{R}})$ *and* $\mathcal{B}(X^3)$-$\mathcal{B}(\overline{\mathbb{R}})$ *measurable, respectively.*

Proof. It is easy to see that r is continuous on $X^3\backslash X_0$ and hence ρ and ρ_G are upper semicontinuous on $X^2 \setminus \operatorname{diag}(X)$ and X, respectively. This holds, because infima of upper semicontinuous functions are upper semicontinuous. Now it is clear that the reciprocals of these functions are lower semicontinuous and, considering that the excluded sets $\operatorname{diag}(X)$ and X_0 are closed, and the functions are non-negative on the whole space and 0 on these sets, we see that they are lower semicontinuous on the entire space. Therefore we directly obtain Borel measurability. ∎

Definition 2.4 (A menagerie of integral curvature energies). Let (X, d) be a metric space and $\alpha, p \in (0, \infty)$. We are now able to define the following two-parameter families of integral curvature energies:

$$\begin{aligned}
\mathcal{U}_p^\alpha(X) &:= \int_X \kappa_G^p(x)\, d\mathcal{H}^\alpha(x),\\
\mathcal{I}_p^\alpha(X) &:= \int_X \int_X \kappa_i^p(x,y)\, d\mathcal{H}^\alpha(x)\, d\mathcal{H}^\alpha(y),\\
\mathcal{M}_p^\alpha(X) &:= \int_X \int_X \int_X \kappa^p(x,y,z)\, d\mathcal{H}^\alpha(x)\, d\mathcal{H}^\alpha(y)\, d\mathcal{H}^\alpha(z).
\end{aligned}$$

The last of these energies, $\mathcal{M}_p^\alpha$, is often called the *α-dimensional (integral) p-Menger curvature.*

Remark 2.5 (Subtle differences in possible definitions of energies). We remark that in the Euclidean case the measure in the integrals is the Hausdorff measure on the set X (with respect to the subspace metric, i.e. the restriction of the metric of $\mathbb{R}^n$ to the set X), in contrast to the Hausdorff measure on $\mathbb{R}^n$. As we shall see shortly, this enables us to include non-measurable sets, in contrast to the other approach, where the energy might not exist on non-measurable sets, which can easily be seen by the example of a Vitali type set on the unit circle. We suspect that the gain of permitted sets when comparing [10] for $\mathbb{R}^n$ to [13], where only Borel sets were permitted, might be related to this matter.

Lemma 2.6 (Integral curvature energies are well-defined). *Let* (X, d) *be a metric space. Then for all* $\alpha, p \in (0, \infty)$ *the curvature energies* $\mathcal{U}_p^\alpha(X)$, $\mathcal{I}_p^\alpha(X)$ *and* $\mathcal{M}_p^\alpha(X)$ *are well defined.*

Proof. Repeatedly use Lemma 2.3 and Fatou's Lemma. ∎

Lemma 2.7 (Inequalities between integral curvature energies). *Let* (X, d) *be a metric space with* $\mathcal{H}^\alpha(X) < \infty$ *and* $\alpha, p \in (0, \infty)$. *Then*

$$\mathcal{M}_p^\alpha(X) \le \mathcal{H}^\alpha(X)\mathcal{I}_p^\alpha(X) \le \mathcal{H}^\alpha(X)^2\mathcal{U}_p^\alpha(X) \le \frac{\mathcal{H}^\alpha(X)^3}{\Delta[X]^p}.$$

For $0 < p < q < \infty$ *we have*

$$\begin{aligned}\mathcal{U}_p^\alpha(X) &\le \mathcal{H}^\alpha(X)^{(1-p/q)}\mathcal{U}_q^\alpha(X)^{p/q},\\ \mathcal{I}_p^\alpha(X) &\le \mathcal{H}^\alpha(X)^{2(1-p/q)}\mathcal{I}_q^\alpha(X)^{p/q},\\ \mathcal{M}_p^\alpha(X) &\le \mathcal{H}^\alpha(X)^{3(1-p/q)}\mathcal{M}_q^\alpha(X)^{p/q}.\end{aligned}$$

Proof. The first part is a direct consequence of the definition of the integrands and the second part is easily proved by successively using the Hölder inequality for $a = q/p > 1$ and $b = q/(q-p)$ from the inner to the outer integral. ∎

Later on we often use the contrapositive of the following lemma to show that a set has infinite curvature energy.

LEMMA 2.8 ($\mathcal{F}(B_r) \to 0$ if $\mathcal{F}(X) < \infty$). *Let* (X,d) *be a metric space with* $\mathcal{H}^\alpha(X) < \infty$, $\alpha, p \in (0,\infty)$, $\mathcal{F} \in \{\mathcal{U}_p^\alpha, \mathcal{I}_p^\alpha, \mathcal{M}_p^\alpha\}$. *If* $\mathcal{F}(X) < \infty$ *then for all* $x \in X$,

$$\lim_{r\downarrow 0} \mathcal{F}(B_r(x)) = 0.$$

Proof. Let $x_0 \in X$ and assume that there is a decreasing sequence $(r_n)_{n\in\mathbb{N}}$ with $r_n > 0$ and $\lim_{n\to\infty} r_n = 0$ such that $\mathcal{F}(B_{r_n}(x_0)) \ge c > 0$ for all $n \in \mathbb{N}$. We first note that as $B_r(x_0) \in \mathcal{C}(\mathcal{H}^\alpha)$ and measures are continuous on decreasing sets E_j if E_1 has finite measure [6, Theorem 1.1, (b), p. 2], we have

$$\lim_{n\to\infty} \mathcal{H}^\alpha(B_{r_n}(x_0)) = \mathcal{H}^\alpha\Big(\lim_{n\to\infty} B_{r_n}(x_0)\Big) = \mathcal{H}^\alpha(\{x_0\}) = 0.$$

Let

$$f \in \Big\{x \mapsto \kappa_G^p(x), y \mapsto \int_X \kappa_i^p(x,y)\, d\mathcal{H}^\alpha(x), \\ z \mapsto \int_X\int_X \kappa^p(x,y,z)\, d\mathcal{H}^\alpha(x)\, d\mathcal{H}^\alpha(y)\Big\}$$

be the integrand corresponding to $\mathcal{F}$. It can be easily seen that f is measurable by showing that it is lower semicontinuous, using Lemma 2.3 and Fatou's Lemma. Furthermore,

$$\int_{B_{r_n}(x_0)} f\, d\mathcal{H}^\alpha \ge \mathcal{F}(B_{r_n}(x_0)) \ge c > 0.$$

To conclude the proof we remark that, in order for $\mathcal{F}(X) = \int_X f\, d\mathcal{H}^\alpha$ to be finite, we must have $\int_{B_{r_n}(x_0)} f\, d\mathcal{H}^\alpha \to 0$, by the Monotone Convergence Theorem. Thus we obtain the desired contradiction. ∎

When dealing with subsets of $\mathbb{R}^n$, which is always the case from now on except for Definition 3.1 and Lemma 3.2, we denote the α-dimensional Hausdorff measure on $\mathbb{R}^n$ by $\mathcal{H}^\alpha$, and the α-dimensional Hausdorff measure on $X \subset \mathbb{R}^n$, induced by the subspace metric, by $\mathcal{H}^\alpha_X$. Additionally $B_r(x)$ and $\overline{B}_r(x)$ denote the open and closed balls in $\mathbb{R}^n$. Note that for $A \subset X \subset \mathbb{R}^n$ we have $\mathcal{H}^\alpha(A) = \mathcal{H}^\alpha_X(A)$, but for example X is trivially $\mathcal{H}^\alpha_X$ measurable, while X might not be $\mathcal{H}^\alpha$ measurable.

DEFINITION 2.9 (Double cone in direction s with opening angle ε). Let $x \in \mathbb{R}^n$, $s \in \mathbb{S}^{n-1}$ and $\varepsilon > 0$. By $C_{s,\varepsilon}(x)$ we denote the open double cone centred at x in direction s, i.e.

$$C_{s,\varepsilon}(x) := \{y \in \mathbb{R}^n \setminus \{x\} \mid \min\{\measuredangle(y,x,x-s), \measuredangle(y,x,x+s)\} < \varepsilon\}.$$

Just after the first version of this paper had been written up, Martin Meurer (who also did a higher dimensional version of this) and the author proved the following lemma, which permits including sets with infinite measure in our subsequent theorems.

LEMMA 2.10 (Finite energy implies finite measure on all balls). *Let $\alpha \in [1,\infty)$, $p \in (0,\infty)$, $\mathcal{F} \in \{\mathcal{U}^\alpha_p, \mathcal{I}^\alpha_p, \mathcal{M}^\alpha_p\}$ and $X \subset \mathbb{R}^n$ be a set with $\mathcal{F}(X) < \infty$. Then for all $x \in \mathbb{R}^n$ and $R > 0$ we have $\mathcal{H}^\alpha(X \cap \overline{B}_R(x)) < \infty$.*

Proof. Assume this is not the case.

STEP 1. We show that there is an $x_0 \in \overline{B}_R(x)$ with

$$\mathcal{H}^\alpha(X \cap B_r(x_0)) = \infty \quad \text{for all } r > 0. \tag{2.1}$$

According to our assumption there exist $x \in \mathbb{R}^n$ and $R > 0$ such that $\mathcal{H}^\alpha(X \cap B_R(x)) = \infty$. By a covering argument, for any $n \in \mathbb{N}$ there is an $x_n \in B_R(x)$ such that $\mathcal{H}^\alpha(X \cap B_{1/n}(x_n)) = \infty$. As $\overline{B}_R(x)$ is compact, there is a subsequence such that $x_{n_k} \to x_0 \in \overline{B}_R(x)$. Then $\mathcal{H}^\alpha(X \cap B_r(x_0)) = \infty$ for all $r > 0$, because

$$\sup_{y \in B_{1/n_k}(x_{n_k})} d(x_0,y) \le d(x_0,x_{n_k}) + 1/n_k \to 0.$$

STEP 2. For $\rho > 0$ we can find $r = r(\rho)$ and $A := B_\rho(x_0) \setminus B_r(x_0)$ such that $\mathcal{H}^\alpha(X \cap A) \ge 3\rho$, because $B_\rho(x_0) \setminus B_r(x_0) \in \mathcal{C}(\mathcal{H}^\alpha_X)$ and by the continuity of measures on increasing sets [6, Theorem 1.1(a), p. 2] we have

$$\begin{aligned} \infty = \mathcal{H}^\alpha(X \cap B_\rho(x_0)) &= \mathcal{H}^\alpha_X(B_\rho(x_0)) = \mathcal{H}^\alpha_X(B_\rho(x_0) \setminus \{x_0\}) \\ &= \mathcal{H}^\alpha_X\Big(\bigcup_{n\in\mathbb{N}} B_\rho(x_0) \setminus B_{1/n}(x_0)\Big) = \lim_{n\to\infty} \mathcal{H}^\alpha_X(B_\rho(x_0) \setminus B_{1/n}(x_0)). \end{aligned}$$

Then there exists a direction $s \in \mathbb{S}^{n-1}$ and an $\varepsilon > 0$ such that

$$\mathcal{H}^\alpha(X \cap A \cap C_{s,\varepsilon}(x_0)) > 0 \quad \text{and} \quad \mathcal{H}^\alpha([X \cap A] \setminus C_{s,2\varepsilon}(x_0)) > 0, \tag{2.2}$$

because, by a covering and compactness argument similar to that of Step 1, there is a direction s such that $\mathcal{H}^\alpha(X \cap A \cap C_{s,\varepsilon}(x_0)) > 0$ for all $\varepsilon > 0$. If we assume that $\mathcal{H}^\alpha([X \cap A] \setminus C_{s,2\varepsilon}(x_0)) = 0$ for all $\varepsilon > 0$, we obtain a contradiction for $N_n := [X \cap A] \setminus C_{s,1/n}(x_0)$, because

$$\mathcal{H}^\alpha([X \cap A] \setminus L) = \mathcal{H}^\alpha\Big(\bigcup_{n\in\mathbb{N}} N_n\Big) \le \sum_{n\in\mathbb{N}} \mathcal{H}^\alpha(N_n) = 0, \tag{2.3}$$

which follows from

$$\begin{aligned} 3\rho \le \mathcal{H}^\alpha(X \cap A) &= \mathcal{H}^\alpha([X \cap A] \setminus L) + \mathcal{H}^\alpha(X \cap A \cap L) \\ &= \mathcal{H}^\alpha(X \cap A \cap L) \le 2\rho, \end{aligned}$$

where $L = x_0 + [-\rho, \rho]s$. For the last inequality we needed $\alpha \in [1, \infty)$.

Step 3. Denote by $C := X \cap A \cap C_{s,\varepsilon}(x_0)$ and $C' := [X \cap A] \setminus C_{s,2\varepsilon}(x_0)$ the sets from (2.2). By Lemma A.1 (see Appendix) we have $\operatorname{dist}(L_{x,y}, x_0) \ge \sin(\varepsilon)r/2$ for $x \in C$ and $y \in C'$, so that for all $z \in B_{\sin(\varepsilon)r/4}(x_0)$ we have

$$\operatorname{dist}(L_{x,y}, z) \ge \operatorname{dist}(L_{x,y}, x_0) - d(z, x_0) \ge \sin(\varepsilon)r/4,$$

and hence

$$\begin{aligned} \mathcal{M}_p^\alpha(X) &\ge \int_C \int_{C'} \int_{X\cap B_{\sin(\varepsilon)r/4}(x_0)} \frac{[\sin(\varepsilon)r/4]^p}{r^{2p}}\, d\mathcal{H}_X^\alpha(z)\, d\mathcal{H}_X^\alpha(y)\, d\mathcal{H}_X^\alpha(x) \\ &\ge \mathcal{H}^\alpha(C)\mathcal{H}^\alpha(C')\mathcal{H}^\alpha(X \cap B_{\sin(\varepsilon)r/4}(x_0)) \frac{[\sin(\varepsilon)r/4]^p}{r^{2p}} \overset{(2.1)}{=} \infty. \end{aligned}$$

For the other energies the argument is similar. ∎

Corollary 2.11 (Finite energy implies that $\mathcal{H}_X^\alpha$ is a Radon measure). *Let $\alpha \in [1, \infty)$, $p \in (0, \infty)$, $\mathcal{F} \in \{\mathcal{U}_p^\alpha, \mathcal{I}_p^\alpha, \mathcal{M}_p^\alpha\}$ and $X \subset \mathbb{R}^n$ be a set with $\mathcal{F}(X) < \infty$. Then $\mathcal{H}_X^\alpha$ is a Radon measure.*

Proof. This is a direct consequence of Lemma 2.10. ∎

For $\alpha \in (0, 1)$ it can happen that $\mathcal{H}^\alpha(X) = \infty$, but $\mathcal{F}(X) < \infty$, even $\mathcal{F}(X) = 0$ is possible, as can be seen by the example of the bounded set $X = [0, 1] \times \{0\}$. Therefore we have to find an appropriate version of Lemma 2.10 that takes this into account, but still enables us later on to draw the same conclusions regarding the tangency properties we want to investigate.

Lemma 2.12 (Consequences of finite energy for $\alpha \in (0, 1)$). *Let $\alpha \in (0, 1)$, $p \in (0, \infty)$, $\mathcal{F} \in \{\mathcal{U}_p^\alpha, \mathcal{I}_p^\alpha, \mathcal{M}_p^\alpha\}$ and $X \subset \mathbb{R}^n$ be a set with $\mathcal{F}(X) < \infty$. For all $x_0 \in X$, either*

- *there is $r > 0$ such that $\mathcal{H}^\alpha(X \cap \overline{B}_r(x_0)) < \infty$, or*
- *there is a direction $s \in \mathbb{S}^{n-1}$ such that $\mathcal{H}^\alpha(X \setminus [x_0 + \mathbb{R}s]) = 0$.*

Proof. We can argue as in the proof of Lemma 2.10, because now (the negation of) the additional second item together with (2.3) yields the contradiction needed to prove the second part of (2.2). ■

3. Hausdorff density and lower estimates of annuli. In this section we recall the definition of Hausdorff density, introduce a slightly wider notion for set-valued mappings and prove some properties of these densities. More importantly, we estimate the Hausdorff measure of annuli from below under the assumption that the densities fulfill certain conditions.

DEFINITION 3.1 (Hausdorff density for set-valued mappings). Let (X,d) be a metric space, $x \in X$, $\alpha \in (0,\infty)$ and $A : (0,\rho) \to \operatorname{Pot}(X)$. Then

$$\Theta_*^\alpha(\mathcal{H}^\alpha, A(r), x) := \liminf_{r\downarrow 0} \frac{\mathcal{H}^\alpha(A(r) \cap \overline{B}_r(x))}{(2r)^\alpha},$$
$$\Theta^{*\alpha}(\mathcal{H}^\alpha, A(r), x) := \limsup_{r\downarrow 0} \frac{\mathcal{H}^\alpha(A(r) \cap \overline{B}_r(x))}{(2r)^\alpha}$$

are called the *lower* and *upper α-dimensional Hausdorff density of A at x*. If the upper and lower densities coincide we call their common value the *Hausdorff density* and denote it by $\Theta^\alpha(\mathcal{H}^\alpha, A(r), x)$. If $A(r) \equiv A$ is constant we will usually identify the mapping with the constant and neglect the argument. Note that $\Theta^\alpha(\mathcal{H}^\alpha, A(r), x)$ does not depend on r, but we use this notation to emphasize that set-valued mappings are allowed.

LEMMA 3.2 (Simultaneous estimate of annuli). *Let (X,d) be a metric space, $\alpha \in (0,\infty)$, $A, B : (0,\rho) \to \operatorname{Pot}(X)$, $x \in X$ with*

$$\Theta_*^\alpha(\mathcal{H}^\alpha, A(r), x) > 0, \quad \Theta^{*\alpha}(\mathcal{H}^\alpha, B(r), x) > 0 \quad \textit{and} \quad \Theta^{*\alpha}(\mathcal{H}^\alpha, X, x) < \infty.$$

Then there exists a $q_0 \in (0,1)$, a sequence $(r_n)_{n\in\mathbb{N}}$ with $r_n > 0$ and $\lim_{n\to\infty} r_n = 0$ and a constant $c > 0$ such that

$$cr_n^\alpha \le \min\{\mathcal{H}^\alpha(A(r_n) \cap \overline{B}_{r_n}(x) \setminus B_{q_0 r_n}(x)), \mathcal{H}^\alpha(B(r_n) \cap \overline{B}_{r_n}(x) \setminus B_{q_0 r_n}(x))\}.$$

Proof. STEP 1. By the hypotheses $\Theta^{*\alpha}(\mathcal{H}^\alpha, B(r), x) =: \delta_0 > 0$ and $\Theta^{*\alpha}(\mathcal{H}^\alpha, X, x) =: \theta/4^\alpha < \infty$, there are $r_n > 0$ with $r_n \to 0$ such that

$$\delta_0 r_n^\alpha \le \mathcal{H}^\alpha(B(r_n) \cap \overline{B}_{r_n}(x))$$

and

$$\mathcal{H}^\alpha(B(r_n) \cap \overline{B}_{qr_n}(x)) \le \mathcal{H}^\alpha(\overline{B}_{qr_n}(x)) \le \theta q^\alpha r_n^\alpha \quad \text{for all } q \in (0,1),$$

as the upper density is positive. Together this means that

$$\begin{aligned}&\mathcal{H}^\alpha(B(r_n) \cap [\overline{B}_{r_n}(x) \setminus B_{qr_n}(x)]) \\ &\quad \ge \mathcal{H}^\alpha(B(r_n) \cap \overline{B}_{r_n}(x)) - \mathcal{H}^\alpha(B(r_n) \cap \overline{B}_{qr_n}(x)) \ge (\delta_0 - \theta q^\alpha) r_n^\alpha \ge \delta_0 r_n^\alpha/2\end{aligned}$$

if we choose $q^\alpha \le \delta_0/(2\theta) < 1$.

Step 2. As $0 < \delta_1 := \Theta^\alpha_*(\mathcal{H}^\alpha, A(r), x)$ we know that for n large enough,

$$\delta_1 r_n^\alpha \le \mathcal{H}^\alpha(A(r_n) \cap \overline{B}_{r_n}(x)).$$

Now we can use the argument from Step 1 to obtain

$$\mathcal{H}^\alpha(A(r_n) \cap [\overline{B}_{r_n}(x) \setminus B_{qr_n}(x)]) \ge (\delta_1 - \theta q^\alpha) r_n^\alpha \ge \delta_1 r_n^\alpha / 2$$

if we choose $q^\alpha \le \delta_1/(2\theta) < 1$.

Step 3. Combining the results from the previous steps we obtain the proposition for $q_0 = [\min\{\delta_1, \delta_2\}/(2\theta)]^{1/\alpha} \in (0,1)$ and $c = \min\{\delta_1, \delta_2\}/2$. ∎

Note that in case $X \subset \mathbb{R}^n$ we do not require $x \in X$ in Lemma 3.2. We remind the reader that the angle $\measuredangle(s, 0, s')$ is a metric, denoted by $d_{\mathbb{S}^{n-1}}(s, s')$, on the sphere $\mathbb{S}^{n-1}$, so that $(\mathbb{S}^{n-1}, d_{\mathbb{S}^{n-1}})$ is a complete metric space.

Lemma 3.3 (Uniform estimate of cones if $\Theta^\alpha_*(\mathcal{H}^\alpha, X, x) > 0$). *Suppose $X \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $\Theta^\alpha_*(\mathcal{H}^\alpha, X, x) > 0$. Then there is a $\rho > 0$ and a mapping $s : (0, \rho) \to \mathbb{S}^{n-1}$ such that for all $\varepsilon > 0$ there is $c(\varepsilon) > 0$ with*

$$\mathcal{H}^\alpha(X \cap \overline{B}_r(x) \cap C_{s(r),\varepsilon}(x)) \ge c(\varepsilon) r^\alpha \quad \text{for all } r \in (0, \rho).$$

Proof. Step 1. Fix $x \in \mathbb{R}^n$. Let $0 < \varphi < \psi$, $s \in \mathbb{S}^{n-1}$ and define

$$M(s, \varphi, \psi) := \min\Big\{|I| \;\Big|\; C_{s,\psi}(x) \subset \bigcup_{i \in I} C_{s_i,\varphi}(x),\ s_i \in \mathbb{S}^{n-1},\ d_{\mathbb{S}^{n-1}}(s, s_i) < \psi\Big\},$$

where $|I|$ is the number of elements in I. As $x + \mathbb{S}^{n-1}$ is compact in $\mathbb{R}^n$ we can select from $\{C_{s',\varphi}(x) \mid s' \in \mathbb{S}^{n-1}, d_{\mathbb{S}^{n-1}}(s, s') < \psi\}$ a finite subcover of $\overline{C}_{s,\psi}(x)$, and consequently $M(s, \varphi, \psi)$ is finite. It is clear that $M(s, \varphi, \psi) = M(\tilde{s}, \varphi, \psi)$ for all $s, \tilde{s} \in \mathbb{S}^{n-1}$, since we can transform s to $\tilde{s}$ by a rotation. Therefore we write $M(\varphi, \psi) := M(s, \varphi, \psi)$.

Step 2. We define $s_0(r) := e_1 = (1, 0, \dots, 0)$ and $\varepsilon_0 := 2\pi 2^{-0} = 2\pi$. As the lower density is positive, there are $\rho > 0$ and $c > 0$ such that

$$\mathcal{H}^\alpha(X \cap \overline{B}_r(X)) = \mathcal{H}^\alpha(X \cap \overline{B}_r(X) \cap C_{s_0(r),\varepsilon_0}(x)) \ge c r^\alpha \quad \text{for all } r \in (0, \rho).$$

Now we set $\varepsilon_{k+1} = 2\pi 2^{-(k+1)}$ and find, with the help of Step 1, a direction $s_{k+1}(r) \in \mathbb{S}^{n-1}$ with $d_{\mathbb{S}^{n-1}}(s_k(r), s_{k+1}(r)) < \varepsilon_k$ such that

$$\begin{aligned}\mathcal{H}^\alpha(X \cap \overline{B}_r(X) \cap C_{s_{k+1}(r),\varepsilon_{k+1}}(x)) &\ge \frac{\mathcal{H}^\alpha(X \cap \overline{B}_r(X) \cap C_{s_k(r),\varepsilon_k}(x))}{M(\varepsilon_{k+1}, \varepsilon_k)} \\ &\ge \dots \ge \frac{c}{\prod_{i=0}^k M(\varepsilon_{i+1}, \varepsilon_i)} r^\alpha \quad \text{for all } r \in (0, \rho).\end{aligned}$$

Since the sphere is a complete metric space, we know that for all $r \in (0, \rho)$

there are $s(r) \in \mathbb{S}^{n-1}$ such that $s_k(r) \to s(r)$ with

$$\begin{aligned} d_{\mathbb{S}^{n-1}}(s_k(r), s(r)) &\le \sum_{i=k}^{\infty} \varepsilon_i = \sum_{i=k}^{\infty} 2\pi 2^{-i} \\ &= 2\pi \left[\frac{1}{1-1/2} - \frac{1-1/2^{-k}}{1-1/2} \right] = 2\pi 2^{-(k-1)} = \varepsilon_{k-1}. \end{aligned}$$

Step 3. Let $\varepsilon > 0$. Then, as $\varepsilon_k \to 0$, there is a k such that $\varepsilon > \varepsilon_{k-1} + \varepsilon_k$. Because $d_{\mathbb{S}^{n-1}}(s, s') + \varphi \le \psi$ implies $C_{s',\varphi}(x) \subset C_{s,\psi}(x)$ and we already know $d_{\mathbb{S}^{n-1}}(s_k(r), s(r)) \le \varepsilon_{k-1}$ by Step 2, we have $C_{s_k(r),\varepsilon_k}(x) \subset C_{s(r),\varepsilon}(x)$ and hence

$$\begin{aligned} \mathcal{H}^\alpha(X \cap \overline{B}_r(x) \cap C_{s(r),\varepsilon}(x)) &\ge \mathcal{H}^\alpha(X \cap \overline{B}_r(x) \cap C_{s_k(r),\varepsilon_k}(x)) \\ &\ge \frac{c}{\prod_{i=0}^{k-1} M(\varepsilon_{i+1}, \varepsilon_i)} r^\alpha = c(\varepsilon) r^\alpha \quad \text{for all } r \in (0, \rho). \ \blacksquare \end{aligned}$$

4. Approximate tangents, counterexamples. We now fix our notation regarding the tangency properties we wish to investigate. Also we give some remarks and examples in this context. In this section we finally leave the setting of metric spaces and are only concerned with subsets of $\mathbb{R}^n$.

Definition 4.1 (Weakly α-linearly approximable). Let $\alpha \in (0, \infty)$. We say that a set $X \subset \mathbb{R}^n$ is *weakly α-linearly approximable* at a point $x \in \mathbb{R}^n$ if there is a $\rho > 0$ and a mapping $s : (0, \rho) \to \mathbb{S}^{n-1}$ such that for every $\varepsilon > 0$ and every $\delta > 0$, there is a $\rho(\varepsilon, \delta) \in (0, \rho)$ with

$$\mathcal{H}^\alpha([X \cap \overline{B}_r(x)] \setminus C_{s(r),\varepsilon}(x)) \le \delta r^\alpha \quad \text{for all } r \in (0, \rho(\varepsilon, \delta)).$$

Definition 4.2 (Weak and strong approximate α-tangents). Let $X \subset \mathbb{R}^n$ be a set and $x \in \mathbb{R}^n$, $\alpha \in (0, \infty)$. We say that X has a (*strong*) *approximate α-tangent* at x if there is a direction $s \in \mathbb{S}^{n-1}$ such that

$$\Theta^\alpha(\mathcal{H}^\alpha, X \setminus C_{s,\varepsilon}(x), x) = 0 \quad \text{for all } \varepsilon > 0,$$

and we say that X has a *weak approximate α-tangent* at x if there is a $\rho > 0$ and a mapping $s : (0, \rho) \to \mathbb{S}^{n-1}$ such that

$$\Theta^\alpha(\mathcal{H}^\alpha, X \setminus C_{s(r),\varepsilon}(x), x) = 0 \quad \text{for all } \varepsilon > 0.$$

We will also sometimes call the direction s and the mapping $s : (0, \rho) \to \mathbb{S}^{n-1}$ a (strong) approximate α-tangent and weak approximate α-tangent, respectively.

Lemma 4.3 (Weak α-linear approximability and weak approximate α-tangents). *Let $X \subset \mathbb{R}^n$ be a set and $x \in \mathbb{R}^n$, $\alpha \in (0, \infty)$. Then the following are equivalent:*

- *X is weakly α-linearly approximable at x,*
- *X has a weak approximate α-tangent at x.*

Proof. One direction is an immediate consequence of the definitions, and the other can easily be seen by taking a closer look at what it means to have zero density. ∎

Remark 4.4 (Differences to standard terminology). We should warn the reader that our definitions of 1-linear approximability and approximate 1-tangents differ from the standard use in the literature [19, 15.7 & 15.10 Definition, p. 206 and 15.17 Definition, p. 212] in that we do not impose additional density requirements, like $\Theta^{*1}(\mathcal{H}^1, X, x) > 0$ in the case of approximate 1-tangents. This is simply due to the fact that in the following sections we obtain simpler formulations of our results, because some distinction of cases can be omitted, as we cannot expect a set with finite curvature energy to have positive upper density at any point.

Remark 4.5 (Difference between approximate 1-tangents and tangents). What it means for a set to have an approximate 1-tangent at a point is, in some respects, quite different to having an actual tangent at this point. To illustrate this, consider

$$S := \{(x,0) \mid x \in [0,1]\} \cup \{(x,x^2) \mid x \in [0,1]\}.$$

As $x \mapsto x^2$ is convex there is $r(\varepsilon)$ such that $S \cap B_{r(\varepsilon)}(0) \subset C_{e_1,\varepsilon}(0)$ and hence S has an approximate 1-tangent at $(0,0)$, but an arc length parametrisation γ of S does not possess a derivative, and hence a tangent, at $\gamma^{-1}((0,0))$.

Example 4.6 (A set with weak approximate but no approximate 1-tangents). Set $a_n := 2^{-n^n n^3}$, $A_n := [a_n/2, a_n]$ and

$$F := \Big[\bigcup_{n\in\mathbb{N}} A_{2n} \times \{0\}\Big] \cup \Big[\bigcup_{n\in\mathbb{N}} \{0\} \times A_{2n-1}\Big].$$

For $\varepsilon > 0$ we have

$$\begin{aligned} \mathcal{H}^1(F \cap C_{e_1,\varepsilon}(0) \cap \overline{B}_{a_{2n}}(0)) &\geq \mathcal{H}^1([a_{2n}/2, a_{2n}]) = a_{2n}/2, \\ \mathcal{H}^1(F \cap C_{e_2,\varepsilon}(0) \cap \overline{B}_{a_{2n+1}}(0)) &\geq \mathcal{H}^1([a_{2n+1}/2, a_{2n+1}]) = a_{2n+1}/2. \end{aligned} \tag{4.1}$$

Now (4.1) tells us that no approximate 1-tangent exists, because for every $s \in \mathbb{S}^{n-1}$ there are $\varepsilon_s > 0$ and $i_s \in \{1,2\}$ such that $C_{e_{i_s},\varepsilon_s}(0) \cap C_{s,\varepsilon_s}(0) = \emptyset$ and hence by (4.1) there are $r_n = r_n(s) > 0$ with $r_n \to 0$ and

$$\Theta^{*1}(\mathcal{H}^1, F \setminus C_{s,\varepsilon_s}(0), 0) \geq \lim_{n\to\infty} \frac{\mathcal{H}^1([F \cap C_{e_{i_s},\varepsilon_s}(0)] \cap \overline{B}_{r_n}(0))}{2r_n} \geq \frac{1}{4}.$$

On the other hand,

$$\begin{aligned} \mathcal{H}^1([F \cap \overline{B}_r(0)] \setminus C_{e_1,\varepsilon}(0)) &\leq \mathcal{H}^1([0, a_{2n+1}]) = 2^{-(2n+1)^{2n+1}(2n+1)^3} \\ &\leq 2^{-2n} 2^{-(2n)^{2n}(2n)^3 - 1} = 2^{-2n} \frac{a_{2n}}{2} \leq 2^{-2n} r \end{aligned}$$

for all $r \in [a_{2n}/2, a_{2n-1}/2]$ and

$$\mathcal{H}^1([F \cap \overline{B}_r(0)] \setminus C_{e_2,\varepsilon}(0)) \le \mathcal{H}^1([0, a_{2(n+1)}]) = 2^{-(2[n+1])^{2[n+1]}(2[n+1])^3}$$
$$\le 2^{-(2n+1)} 2^{-(2n+1)^{2n+1}(2n+1)^3-1} = 2^{-(2n+1)} \frac{a_{2n+1}}{2} \le 2^{-(2n+1)} r$$

for all $r \in (a_{2n+1}/2, a_{2n}/2)$. We have thus verified the definition of F having a weak approximate 1-tangent for

$$s : (0, 1/2) \to \mathbb{S}^1, \qquad r \mapsto \begin{cases} e_1, & r \in \bigcup_{n\in\mathbb{N}} [a_{2n}/2, a_{2n-1}/2], \\ e_2, & r \in \bigcup_{n\in\mathbb{N}} (a_{2n+1}/2, a_{2n}/2). \end{cases}$$

Lemma 4.7 (Density estimates for sets with no weak approximate tangent). *Let $X \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, $\alpha \in (0,\infty)$ and $\Theta^\alpha_*(\mathcal{H}^\alpha, X, x) > 0$. If X has no weak approximate α-tangent at x, then there is $\rho > 0$, a mapping $s : (0,\rho) \to \mathbb{S}^{n-1}$, and $\varepsilon_0 > 0$ such that*

$$\Theta^\alpha_*(\mathcal{H}^\alpha, X \cap C_{s(r),\varepsilon_0/2}(x), x) > 0, \qquad \Theta^{*\alpha}(\mathcal{H}^\alpha, X \setminus C_{s(r),\varepsilon_0}(x), x) > 0. \tag{4.2}$$

Proof. By Lemma 3.3 we find a mapping s such that the left inequality of (4.2) holds for every $\varepsilon_0 > 0$. Now Lemma 4.3 and Definition 4.1 give an $\varepsilon_0 > 0$ such that the right inequality of (4.2) holds for this s. ∎

The next lemma, used in the omitted proof of the first two items of Theorem 1.1, is the counterpart to Lemma 4.7.

Lemma 4.8 (Density estimates for sets with no approximate tangent). *Let $X \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, $\alpha \in (0,\infty)$ and $\Theta^{*\alpha}(\mathcal{H}^\alpha, X, x) > 0$. If X has no approximate α-tangent at x, then there are $s \in \mathbb{S}^{n-1}$ and $\varepsilon_0 > 0$ such that*

$$\Theta^{*\alpha}(\mathcal{H}^\alpha, X \cap C_{s,\varepsilon_0/2}(x), x) > 0 \quad \text{and} \quad \Theta^{*\alpha}(\mathcal{H}^\alpha, X \setminus C_{s,\varepsilon_0}(x), x) > 0.$$

Proof. Assuming that there exists no approximate α-tangent at $x \in X$ we know that for all directions $s \in \mathbb{S}^{n-1}$ there is an $\varepsilon_s > 0$ such that $\Theta^{*\alpha}(\mathcal{H}^\alpha, X \setminus C_{s,\varepsilon_s}(x), x) > 0$. As $\mathbb{S}^{n-1}$ is compact and $\{C_{s,\varepsilon_s/2}(x)\}_{s\in\mathbb{S}^{n-1}}$ is an open cover of $x + \mathbb{S}^{n-1}$ there exists a finite subcover $\{C_{s_i,\varepsilon_{s_i}/2}(x)\}_{i=1}^N$. Clearly this subcover also covers the whole $\mathbb{R}^n \setminus \{x\}$. Since $\Theta^{*\alpha}(\mathcal{H}^\alpha, X, x) = \Theta^{*\alpha}(\mathcal{H}^\alpha, X \setminus \{x\}, x) > 0$, there must be $j \in \{1, \dots, N\}$ with $\Theta^{*\alpha}(\mathcal{H}^\alpha, X \cap C_{s_j,\varepsilon_j/2}(x), x) > 0$. ∎

5. Finite integral curvature energy implies (weak) approximate tangents for p above the threshold value. In this section we prove our main result, Theorem 1.1. As the proofs for the different energies are very similar, we will only present the one for the most difficult energy $\mathcal{M}^\alpha_p$. That is, we show that for $p \in [3\alpha, \infty)$ a set with $\mathcal{M}^\alpha_p$ finite has a weak approximate α-tangent at all points where the lower density is positive and the upper density is finite.

LEMMA 5.1 (Necessary conditions for finite Menger curvature). *Let $X \subset \mathbb{R}^n$, $z_0 \in \mathbb{R}^n$, $\alpha \in (0,\infty)$, $\mathcal{H}^\alpha(X) < \infty$, $\Theta_*^\alpha(\mathcal{H}^\alpha, X, z_0) > 0$. Let $\varepsilon > 0$, $c > 0$, $q_0 \in (0,1)$ and let two sequences of sets $A_n, B_n \subset X$ as well as a sequence $(r_n)_{n\in\mathbb{N}}$ with $r_n > 0$ and $r_n \to 0$ be given, with the following properties:*

- *for all $n \in \mathbb{N}$ and all $x \in A_n \setminus \{z_0\}$ and $y \in B_n \setminus \{z_0\}$ we have $\pi - \varepsilon \geq \measuredangle(x, z_0, y) \geq \varepsilon$,*
- *for all $n \in \mathbb{N}$ we have*

$$\min\{\mathcal{H}^\alpha(A_n \cap [\overline{B}_{r_n}(z_0) \setminus B_{q_0 r_n}(z_0)]), \mathcal{H}^\alpha(B_n \cap [\overline{B}_{r_n}(z_0) \setminus B_{q_0 r_n}(z_0)])\} \geq c r_n^\alpha.$$

Then $\mathcal{M}_p^\alpha(X) = \infty$ for all $p \geq 3\alpha$.

Proof. Let $p \geq 3\alpha$ and suppose for contradiction that $\mathcal{M}_p^\alpha(X) < \infty$. We set

$$\tilde{A}_n := A_n \cap [\overline{B}_{r_n}(z_0) \setminus B_{q_0 r_n}(z_0)] \quad \text{and} \quad \tilde{B}_n := B_n \cap [\overline{B}_{r_n}(z_0) \setminus B_{q_0 r_n}(z_0)].$$

From Lemma A.1 we know that for all $x \in \tilde{A}_n \setminus \{z_0\}$ and $y \in \tilde{B}_n \setminus \{z_0\}$ we have $\operatorname{dist}(L_{x,y}, z_0) \geq \sin(\varepsilon) q_0 r_n/2$ and therefore for all $z \in B_{\sin(\varepsilon)q_0 r_n/4}(z_0)$,

$$\operatorname{dist}(L_{x,y}, z) \geq \operatorname{dist}(L_{x,y}, z_0) - d(z_0, z) \geq \frac{\sin(\varepsilon)}{4} q_0 r_n.$$

There exists a constant $c_1 > 0$ such that

$$\mathcal{H}^\alpha(X \cap \overline{B}_{\sin(\varepsilon)q_0 r_n/4}(z_0)) \geq c_1 (\sin(\varepsilon) q_0 r_n/4)^\alpha$$

for all $n \in \mathbb{N}$. Then

$$\begin{aligned}
&\mathcal{M}_p^\alpha(X \cap B_{2r_n}(z_0)) \\
&\quad\geq \int\limits_{X\cap \overline{B}_{\sin(\varepsilon)q_0 r_n/4}(z_0)} \int\limits_{\tilde{A}_n} \int\limits_{\tilde{B}_n} \left(\frac{2\operatorname{dist}(L_{x,y}, z)}{\|x-z\|\,\|y-z\|} \right)^p d\mathcal{H}_X^\alpha(x)\, d\mathcal{H}_X^\alpha(y)\, d\mathcal{H}_X^\alpha(z) \\
&\quad\geq \int\limits_{X\cap \overline{B}_{\sin(\varepsilon)q_0 r_n/4}(z_0)} \int\limits_{\tilde{A}_n} \int\limits_{\tilde{B}_n} \left(\frac{2\frac{\sin(\varepsilon)}{4} q_0 r_n}{4 r_n^2} \right)^p d\mathcal{H}_X^\alpha(x)\, d\mathcal{H}_X^\alpha(y)\, d\mathcal{H}_X^\alpha(z) \\
&\quad\geq \left(\frac{\sin(\varepsilon) q_0}{8} \right)^p \mathcal{H}^\alpha(X \cap \overline{B}_{\sin(\varepsilon)q_0 r_n/4}(z_0)) \mathcal{H}^\alpha(\tilde{A}_n) \mathcal{H}^\alpha(\tilde{B}_n) \left(\frac{1}{r_n} \right)^p \\
&\quad\geq \left(\frac{\sin(\varepsilon) q_0}{8} \right)^p c_1 \left(\frac{\sin(\varepsilon) q_0 r_n}{4} \right)^\alpha c^2 r_n^{2\alpha} \left(\frac{1}{r_n} \right)^p \\
&\quad\geq \left(\frac{\sin(\varepsilon) q_0}{8} \right)^{p+\alpha} 2^\alpha c_1 c^2 r_n^{3\alpha - p} \geq c' > 0
\end{aligned}$$

for all $n \in \mathbb{N}$. Hence Lemma 2.8 tells us that $\mathcal{M}_p^\alpha(X) = \infty$ (note that for this we needed $\mathcal{H}^\alpha(B_{2r_n}(x) \cap X) < \infty$). This is absurd as we assumed $\mathcal{M}_p^\alpha(X) < \infty$. ■

PROPOSITION 5.2 (Finite $\mathcal{M}_p^\alpha$, $p \geq 3\alpha$, implies weak approximate tangents). *Let $X \subset \mathbb{R}^n$ be a set, and let $\alpha \in (0,\infty)$ and $x \in \mathbb{R}^n$ with*

$$0 < \Theta_*^\alpha(\mathcal{H}^\alpha, X, x) \leq \Theta^{*\alpha}(\mathcal{H}^\alpha, X, x) < \infty.$$

If $p \in [3\alpha, \infty)$ and $\mathcal{M}_p^\alpha(X) < \infty$ then X has a weak approximate α-tangent at x.

Proof. Assume that this is not the case. By Lemmas 2.10 and 2.12 we can assume that $\mathcal{H}^\alpha(X \cap \overline{B}_r(x)) < \infty$ for all small radii. Then by Lemma 4.7 there is a mapping $s : (0,\rho) \to \mathbb{S}^{n-1}$ with $\rho > 0$ and $\varepsilon_0 > 0$ such that

$$\Theta_*^\alpha(\mathcal{H}^\alpha, X \cap C_{s(r),\varepsilon_0/2}(x), x) > 0, \qquad \Theta^{*\alpha}(\mathcal{H}^\alpha, X \setminus C_{s(r),\varepsilon_0}(x), x) > 0.$$

This means that the hypotheses of Lemma 3.2 hold for

$$A(r) := X \cap C_{s(r),\varepsilon_0/2}(x) \quad \text{and} \quad B(r) := X \setminus C_{s(r),\varepsilon_0}(x),$$

so that there exists a $q_0 \in (0,1)$, a sequence $(r_n)_{n\in\mathbb{N}}$ with $r_n > 0$ and $\lim_{n\to\infty} r_n = 0$ and a constant $c > 0$ such that

$$\min\{\mathcal{H}^\alpha(A(r_n) \cap \overline{B}_{r_n}(x) \setminus B_{q_0 r_n}(x)), \mathcal{H}^\alpha(B(r_n) \cap \overline{B}_{r_n}(x) \setminus B_{q_0 r_n}(x))\} \geq c r_n^\alpha.$$

Hence the hypotheses of Lemma 5.1 are fulfilled for $\varepsilon := \varepsilon_0/2$ (note that $\mathcal{H}^\alpha(X \cap \overline{B}_r(x)) < \infty$ for small radii), and we have proven the proposition. ■

6. Finite integral curvature energy does not imply (weak) approximate tangents for $\alpha = 1$ and p below threshold value. In this section we prove Proposition 1.2 by estimating the energies $\mathcal{U}_p^1$, $\mathcal{I}_p^1$ and $\mathcal{M}_p^1$ of the T-shaped set E, defined by

$$E := ([-1,1] \times \{0\}) \cup (\{0\} \times [0,1]) \subset \mathbb{R}^2, \tag{6.1}$$

for p below the scale invariant threshold value. Clearly E does not have a weak approximate 1-tangent at $(0,0)$. Further we set $E_1 := [-1,0] \times \{0\}$, $E_2 := \{0\} \times [0,1]$ and $E_3 := [0,1] \times \{0\}$.

PROPOSITION 6.1 (The set E has finite $\mathcal{U}_p^1$ for $p \in (0,1)$). *For $p \in (0,1)$ we have*

$$\mathcal{U}_p^1(E) \leq \frac{6}{1-p}.$$

Proof. For all $x \in E \setminus \{0\}$ and $y, z \in B_{\|x\|}(x) \cap E$, $y \neq z$, we have $\kappa(x,y,z) = 0$, so that to have $\kappa(x,y,z) > 0$ we need $\|x-y\| \geq \|x\|$ or $\|x-z\| \geq \|x\|$, which both result in $r(x,y,z) \geq \|x\|/2$ and consequently

$$\sup_{\substack{y,z\in E\setminus\{x\} \\ y\neq z}} \kappa(x,y,z) \leq \frac{2}{\|x\|},$$

so that for $p \in (0,1)$,

$$\mathcal{U}_p^1(E) = \int\limits_{E\setminus\{0\}} \underbrace{\Big(\sup_{\substack{y,z\in E\setminus\{x\}\\ y\neq z}} \kappa(x,y,z)\Big)^p}_{\leq 2/\|x\|} d\mathcal{H}_E^1(x)$$

$$\leq 3\int\limits_{E_2} \frac{2}{\|x\|^p}\, d\mathcal{H}_E^1(x) = 6\int\limits_0^1 \frac{1}{s^p}\, d\mathcal{L}^1(s) = \frac{6}{1-p} < \infty. \blacksquare$$

PROPOSITION 6.2 (The set E has finite $\mathcal{I}_p^1$ for $p \in (1,2)$). *Let E be the set from* (6.1). *For $p \in (1,2)$ we have*

$$\mathcal{I}_p^1(E) \leq \frac{9\cdot 2^{3p/2+1}(2^{1-p}-1)}{(1-p)(2-p)}$$

and consequently $\mathcal{I}_p^1(E) < \infty$ for $p \in (0,2)$.

Proof. Let $x, y \in E\setminus\{0\}$, $x \neq y$. We are interested in the maximal value of $\kappa(x,y,z)$ for $z \in E\setminus\{x,y\}$. As κ is invariant under isometries we can restrict ourselves to the cases $x, y \in E_1$; $x \in E_1$, $y \in E_3$; and $x \in E_1$, $y \in E_2$. In each of these cases we want to estimate $\kappa(x,y,z)$ independently of z. We denote the non-zero components of x, y, z by ξ, η, ζ respectively.

CASE 1. If $x, y \in E_1$, $xy \neq 0$ we can clearly assume $z \in E_2\setminus\{0\}$ and hence

$$\kappa(x,y,z) = \frac{2\zeta}{\sqrt{\xi^2+\zeta^2}\sqrt{\eta^2+\zeta^2}} = \frac{2}{\sqrt{\zeta^2+\xi^2+\eta^2+\xi^2\eta^2/\zeta^2}}.$$

By taking the first and second derivatives of $f(u) = \alpha u + \beta/u$, $\alpha, \beta > 0$, we easily see that $\min_{u>0} f(u) = f(\sqrt{\beta/\alpha})$, so that for all $\zeta > 0$ we have

$$\zeta^2 + \frac{\xi^2\eta^2}{\zeta^2} \geq \xi\eta + \frac{\xi^2\eta^2}{\xi\eta} = 2\xi\eta$$

and therefore

$$\kappa(x,y,z) \leq \frac{2}{\sqrt{\xi^2+\eta^2+2\xi\eta}} = \frac{2}{|\xi|+|\eta|}.$$

CASE 2. If $x \in E_1$, $y \in E_3$, $xy \neq 0$ we need $z \in E_2$ in order to have $\kappa(x,y,z) > 0$, but then $\kappa(x,y,z) = \kappa(x,-y,z)$, so that we can assume that $y \in E_1$. This was already done in Case 1.

CASE 3. If $x \in E_1$, $y \in E_2$, $xy \neq 0$ we note that $\kappa(x,y,z) = \kappa(x,y,-z)$ for $z \in E_3$, so that we may assume $z \in E_1$. Then

$$\kappa(x,y,z) = \frac{2\eta}{\sqrt{\xi^2+\eta^2}\sqrt{\zeta^2+\eta^2}} \leq \frac{2\eta}{\sqrt{\xi^2+\eta^2}\sqrt{\eta^2}} = \frac{2}{\sqrt{\xi^2+\eta^2}} \leq \frac{2\sqrt{2}}{|\xi|+\eta}.$$

In all cases we have

$$\kappa(x,y,z) \le \frac{2\sqrt{2}}{|\xi|+|\eta|} \quad \text{for all } z \in E \setminus \{x,y\},$$

which for $p \in (1,2)$ gives us

$$\begin{aligned}
\mathcal{I}_p^1(E) &\le 9 \cdot 2^{3p/2} \int_0^1\int_0^1 \Big(\frac{1}{s+t}\Big)^p \, d\mathcal{L}^1(s)\, d\mathcal{L}^1(t) \\
&= \frac{9 \cdot 2^{3p/2}}{1-p} \int_0^1 [(1+t)^{1-p} - t^{1-p}]\, d\mathcal{L}^1(t) \\
&= \frac{9 \cdot 2^{3p/2}}{(1-p)(2-p)} [(1+t)^{2-p} - t^{2-p}]_0^1 \\
&= \frac{9 \cdot 2^{3p/2}}{(1-p)(2-p)} \big[[2^{2-p}-1] - [1-0]\big] = \frac{9 \cdot 2^{3p/2+1}(2^{1-p}-1)}{(1-p)(2-p)}.
\end{aligned}$$

Now the rest of the proposition follows from Lemma 2.7. ∎

Proposition 6.3 (The set E has finite $\mathcal{M}_p^1$ for $p \in [2,3)$). *Let E be the set from* (6.1). *For $p \in [2,3)$ we have*

$$\mathcal{M}_p^1(E) \le \frac{72\pi}{(3-p)^2},$$

and consequently $\mathcal{M}_p^1(E) < \infty$ for $p \in (0,3)$.

Proof. Step 1. We set

$$\mathcal{F}_p(A,B,C) := \int_C\int_B\int_A \kappa^p(x,y,z)\, d\mathcal{H}_A^1(x)\, d\mathcal{H}_B^1(y)\, d\mathcal{H}_C^1(z).$$

Since the integrand κ^p vanishes on certain sets, we have

$$\sum_{\substack{i,j,k\in\{1,2,3\}\\ \#\{i,j,k\}=1}} \mathcal{F}_p(E_i,E_j,E_k) + \sum_{i,j,k\in\{1,3\}} \mathcal{F}_p(E_i,E_j,E_k) = 0,$$

furthermore

$$\begin{aligned}
\mathcal{M}_p^1(E_1 \cup E_2) &= \sum_{\substack{i,j,k\in\{1,2\}\\ \#\{i,j,k\}=2}} \mathcal{F}_p(E_i,E_j,E_k) = \sum_{\substack{i,j,k\in\{2,3\}\\ \#\{i,j,k\}=2}} \mathcal{F}_p(E_i,E_j,E_k) \\
&= \mathcal{M}_p^1(E_2 \cup E_3),
\end{aligned}$$

as the energy is invariant under isometries. We obtain

$$\mathcal{M}_p^1(E_1 \cup E_2) = 3(\mathcal{F}_p(E_1,E_1,E_2) + \mathcal{F}_p(E_1,E_2,E_2)) = 6\mathcal{F}_p(E_1,E_1,E_2),$$

and the same for $\mathcal{M}_p^1(E_2 \cup E_3)$, where the last equality is, again, due to the invariance of the integrand under isometries. By considering the integrand

κ^p in the form

$$\kappa^p(x,y,z) = \left(\frac{2\,\mathrm{dist}(x, L_{zy})}{d(x,y)d(x,z)}\right)^p$$

for $x \in E_2, y \in E_1$ and $z \in E_3$ we note that $\kappa^p(x,y,z) = \kappa^p(x,y,-z)$; by mapping E_3 onto E_1 via $z \mapsto -z$ we find $\mathcal{F}_p(E_3, E_1, E_2) = \mathcal{F}_p(E_1, E_1, E_2)$, so that

$$\sum_{\substack{i,j,k\in\{1,2,3\}\\ \#\{i,j,k\}=3}} \mathcal{F}_p(E_i, E_j, E_k) = 6\mathcal{F}_p(E_1, E_1, E_2).$$

All in all we obtain

$$\begin{aligned}
&\mathcal{M}_p^1(E)\\
&= \Bigg(\sum_{\substack{i,j,k\in\{1,2,3\}\\ \#\{i,j,k\}=1}} + \sum_{\substack{i,j,k\in\{1,3\}\\ \#\{i,j,k\}=2}} + \sum_{\substack{i,j,k\in\{1,2\}\\ \#\{i,j,k\}=2}} + \sum_{\substack{i,j,k\in\{2,3\}\\ \#\{i,j,k\}=2}} + \sum_{\substack{i,j,k\in\{1,2,3\}\\ \#\{i,j,k\}=3}}\Bigg)\mathcal{F}_p(E_i, E_j, E_k)\\
&= 18\mathcal{F}_p(E_1, E_1, E_2) = 18\mathcal{F}_p(E_2, E_1, E_1).
\end{aligned}$$

STEP 2. Let us first choose parametrisations

$$\gamma_1 : [0,1] \to \mathbb{R}^2,\ t \mapsto (-t, 0) \quad \text{and} \quad \gamma_2 : [0,1] \to \mathbb{R}^2,\ t \mapsto (0, t)$$

of E_1 and E_2, respectively. This gives

$$\begin{aligned}
\mathcal{F}_p(E_2, E_1, E_1) &= \int_0^1\int_0^1\int_0^1 \left(\frac{2x}{\sqrt{x^2+y^2}\sqrt{x^2+z^2}}\right)^p d\mathcal{L}^1(x)\,d\mathcal{L}^1(y)\,d\mathcal{L}^1(z)\\
&\overset{\text{Lemma A.2}}{\leq} \int_0^1\int_0^1 2^p\frac{\pi}{2^p}(zy)^{-(p-1)/2}\,d\mathcal{L}^1(y)\,d\mathcal{L}^1(z)\\
&= \pi\int_0^1 z^{(1-p)/2}\left[\frac{2}{3-p}y^{(3-p)/2}\right]_0^1 d\mathcal{L}^1(z) = \pi\left[\frac{2}{3-p}z^{(3-p)/2}\right]_0^1\frac{2}{3-p}\\
&= \frac{4\pi}{(3-p)^2}.
\end{aligned}$$

Notice that the range $p \geq 2$ was necessary to apply Lemma A.2. Now the rest of the proposition follows from Lemma 2.7. ∎

7. Weak approximate tangents are optimal for $\alpha = 1$. The weak approximate 1-tangents in the results for $\mathcal{I}_p^1$ and $\mathcal{M}_p^1$ are optimal in the following sense:

PROPOSITION 7.1 (A set with no approximate tangent and finite $\mathcal{I}_p^1$ for all $p \in (0,\infty)$). *Set* $a_n := 2^{-n^n n^3}$, $A_n := [a_n/2, a_n]$ *and*

$$F := \Big[\bigcup_{n\in\mathbb{N}} \underbrace{A_{2n} \times \{0\}}_{=:B_{2n}} \Big] \cup \Big[\bigcup_{n\in\mathbb{N}} \underbrace{\{0\} \times A_{2n-1}}_{=:B_{2n-1}} \Big].$$

Then F does not have an approximate 1-tangent at 0 and

- $\mathcal{I}_p^1(F) < \infty$ *for all* $p \in (0,\infty)$,
- $\mathcal{M}_p^1(F) < \infty$ *for all* $p \in (0,\infty)$.

Proof. In Example 4.6 it was already shown that F does not have an approximate 1-tangent at 0, so it remains to prove the finiteness of the energies.

STEP 1. For $l \neq k$ we denote

$$\mu := \min\{k,l\} \quad \text{and} \quad M := \max\{k,l\}.$$

Then

$$\begin{aligned}\operatorname{dist}(B_k, B_l) &\geq \operatorname{dist}(A_k, A_l) = 2^{-(\mu^\mu \mu^3+1)} - 2^{-M^M M^3} \\ &= 2^{-(\mu^\mu \mu^3+1)}(1 - 2^{(\mu^\mu\mu^3+1)-M^M M^3}) \geq 2^{-(\mu^\mu\mu^3+2)} = a_\mu/4.\end{aligned}$$

Let $y \in B_k$, $z \in B_l$ with $k \neq l$. Then

$$\kappa_i(y,z) \leq \frac{2}{\operatorname{dist}(B_k,B_l)} \leq \frac{8}{a_\mu} = \frac{8}{a_{\min\{k,l\}}} = \frac{8}{\max\{a_k,a_l\}}.$$

STEP 2. Let $q > 1$. We now compute some inequalities for the indices. Let $k, m \in \mathbb{N}$, $k < m$, i.e. $m = k + i$ for some $i \in \mathbb{N}$. Then

$$m^3 = (k+i)^3 = k^3 + 3k^2 i + 3ki^2 + i^3,$$

so that

$$-m^3 + k^3 = -(3k^2 i + 3ki^2 + i^3) \leq -3(k+i) = -3m. \tag{7.1}$$

As $qk^k \leq m^m$ for $1 < q \leq k < m$ we have

$$\begin{aligned}-m^m m^3 + qk^k k^3 &\leq -qk^k m^3 + qk^k k^3 = qk^k(-m^3+k^3) \\ &\overset{(7.1)}{\leq} qk^k(-3m) \leq -3m.\end{aligned}$$

Consequently, for all $1 < q \leq k < m$,

$$\frac{a_m}{a_k^q} = \frac{2^{-m^m m^3}}{2^{-qk^k k^3}} = 2^{-m^m m^3 + qk^k k^3} \leq 2^{-3m}. \tag{7.2}$$

STEP 3. As $\mathcal{H}^1(B_n) = a_n/2$ we have, for $p \geq 3$, and $q = p-1 > 1$,

$$\begin{aligned}
\sum_{\substack{k,m\in\mathbb{N}\\ k\neq m}} \int_{B_k}\int_{B_m} \kappa_i^p(y,z)\, d\mathcal{H}^1_F(y)\, d\mathcal{H}^1_F(z) &\leq \sum_{\substack{k,m\in\mathbb{N}\\ k\neq m}} \left[\frac{8}{\max\{a_k,a_m\}}\right]^p \frac{a_k a_m}{4}\\
&\leq \frac{2\cdot 8^p}{4} \sum_{\substack{k,m\in\mathbb{N}\\ 1\leq k<m}} \frac{a_k a_m}{\max\{a_k,a_m\}^p}\\
&\leq 4\cdot 8^{p-1} \sum_{\substack{k,m\in\mathbb{N}\\ 1\leq k\leq q\\ k<m}} \frac{a_k a_m}{\max\{a_k,a_m\}^p} + 4\cdot 8^{p-1} \sum_{\substack{k,m\in\mathbb{N}\\ q\leq k<m}} \frac{a_m}{a_k^{p-1}}\\
&\leq 4\cdot 8^{p-1} \sum_{\substack{k,m\in\mathbb{N}\\ 1\leq k\leq q\\ k<m}} \frac{a_k a_m}{a_{\lceil q\rceil}^p} + 4\cdot 8^{p-1} \sum_{\substack{k,m\in\mathbb{N}\\ q\leq k<m}} \frac{a_m}{a_k^q}\\
&\overset{(7.2)}{\leq} \frac{4\cdot 8^{p-1}}{a_{\lceil q\rceil}^p} \sum_{k,m\in\mathbb{N}} 2^{-k}2^{-m} + 4\cdot 8^{p-1} \sum_{\substack{k,m\in\mathbb{N}\\ q\leq k<m}} 2^{-3m}\\
&\leq \frac{4\cdot 8^{p-1}}{a_{\lceil q\rceil}^p} + 4\cdot 8^{p-1} \sum_{\substack{k,m\in\mathbb{N}\\ q\leq k<m}} 2^{-k}2^{-m} \leq \frac{4\cdot 8^{p-1}}{a_{\lceil q\rceil}^p} + 4\cdot 8^{p-1} \sum_{k,m\in\mathbb{N}} 2^{-k}2^{-m}\\
&= 4\cdot 8^{p-1}\left(\frac{1}{a_{\lceil q\rceil}^p} + 1\right).
\end{aligned}$$

STEP 4. Let $y,z \in B_n$. Then $\kappa(x,y,z) > 0$ if and only if $x \in B_k$ for $(k-n) \bmod 2 = 1$. To simplify matters we may without loss of generality assume that k is even and n is odd. We now have (cf. Remark 2.2)

$$\kappa(x,y,z) = \frac{2\xi}{\sqrt{\xi^2+\eta^2}\sqrt{\xi^2+\zeta^2}},$$

where we denote the non-zero entries of x, y and z by ξ, η and ζ, respectively. If we set $f(\xi) := \kappa(x,y,z)/2$ for fixed y and z, we have

$$\begin{aligned}
f'(\xi) &= \frac{1}{\sqrt{\xi^2+\eta^2}\sqrt{\xi^2+\zeta^2}} - \frac{\xi^2}{\sqrt{\xi^2+\eta^2}^3\sqrt{\xi^2+\zeta^2}} - \frac{\xi^2}{\sqrt{\xi^2+\eta^2}\sqrt{\xi^2+\zeta^2}^3}\\
&= \frac{(\xi^2+\eta^2)\zeta^2 - \xi^2(\xi^2+\zeta^2)}{\sqrt{\xi^2+\eta^2}^3\sqrt{\xi^2+\zeta^2}^3} = \frac{\eta^2\zeta^2 - \xi^4}{\sqrt{\xi^2+\eta^2}^3\sqrt{\xi^2+\zeta^2}^3},
\end{aligned}$$

which is 0 if and only if $\xi = \sqrt{\eta\zeta}$, because $\xi,\eta,\zeta > 0$. That f attains its maximum at $\xi = \sqrt{\eta\zeta}$ is clear from $f' \geq 0$ on $[0,\sqrt{\eta\zeta}]$ and $f' \leq 0$ on $[\sqrt{\eta\zeta},\infty)$. Since $\sqrt{\eta\zeta} \in A_n$ we have $(\sqrt{\eta\zeta},0) \notin F$, as n is odd, so that $\kappa_i(y,z) = \sup_{x\in F}\kappa(x,y,z)$ is attained for $x = (\xi,0)$, $\xi \in \{a_{n+1}, a_{n-1}/2\}$.

We have

$$f(a_{n+1}) = \frac{a_{n+1}}{\sqrt{a_{n+1}^2+\eta^2}\sqrt{a_{n+1}^2+\zeta^2}} \le \frac{a_{n+1}}{a_{n+1}^2 + a_n^2/4} \le 4\frac{a_{n+1}}{a_n^2}$$

and

$$\begin{aligned} f(a_{n-1}/2) &= \frac{a_{n-1}/2}{\sqrt{a_{n-1}^2/4+\eta^2}\sqrt{a_{n-1}^2/4+\zeta^2}} \\ &\le \frac{a_{n-1}/2}{a_{n-1}^2/4 + a_n^2/4} \le 2\frac{a_{n-1}}{a_{n-1}^2} \le \frac{4}{a_{n-1}}. \end{aligned}$$

As $2n^n n^3 \le (n+1)(n+1)^n(n+1)^3 = (n+1)^{n+1}(n+1)^3$ and $a_{n-1} \le 1$ we have $a_{n+1}a_{n-1} \le a_n^2$ and hence for $n \ge 2$,

$$\kappa_i(y,z) = 2\max\{f(a_{n+1}), f(a_{n-1}/2)\} \le 2\max\left\{\frac{4a_{n+1}}{a_n^2}, \frac{4}{a_{n-1}}\right\} = \frac{8}{a_{n-1}}.$$

Consequently, for $p \ge 3$ we have

$$\begin{aligned} &\sum_{n=1}^{\infty}\int_{B_n}\int_{B_n} \kappa_i^p(y,z)\,d\mathcal{H}^1_F(y)\,d\mathcal{H}^1_F(z) \\ &\qquad \le \frac{2^p}{\operatorname{dist}(B_1, \mathbb{R}\times\{0\})^p}\left(\frac{1}{4}\right)^2 + \sum_{n=2}^{\infty}\frac{8^p}{a_{n-1}^p}\mathcal{H}^1(B_n)\mathcal{H}^1(B_n) \\ &\qquad \le \frac{2^p}{(1/4)^p}\left(\frac{1}{4}\right)^2 + \sum_{n=2}^{\infty}\frac{8^p}{a_{n-1}^p}\frac{a_n^2}{4} \le \frac{8^p}{16} + 8^p\sum_{n=2}^{\infty}\frac{a_n}{a_{n-1}^p} \\ &\qquad \le \frac{8^p}{16} + 8^p\sum_{n=2}^{\lceil p\rceil+1}\frac{a_n}{a_{n-1}^p} + 8^p\sum_{n=\lceil p\rceil+1}^{\infty}\frac{a_n}{a_{n-1}^p} \\ &\qquad \overset{(7.2)}{\le} C_p + 8^p\sum_{n=\lceil p\rceil+1}^{\infty}2^{-3n} \le C_p + 8^p\sum_{n=0}^{\infty}2^{-n} \le C_p + 8^p\cdot 2. \end{aligned}$$

STEP 5. For $p \ge 3$ we now conclude that

$$\begin{aligned} \mathcal{I}_p^1(F) &\le \sum_{k,l\in\mathbb{N}}\int_{B_k}\int_{B_l}\kappa_i^p(y,z)\,d\mathcal{H}^1_F(y)\,d\mathcal{H}^1_F(z) \\ &= \sum_{\substack{k,l\in\mathbb{N}\\ k\ne l}}\int_{B_k}\int_{B_l}\kappa_i^p(y,z)\,d\mathcal{H}^1_F(y)\,d\mathcal{H}^1_F(z) \\ &\qquad + \sum_{n\in\mathbb{N}}\int_{B_n}\int_{B_n}\kappa_i^p(y,z)\,d\mathcal{H}^1_F(y)\,d\mathcal{H}^1_F(z) < \infty. \end{aligned}$$

Using $\mathcal{H}^1(F) \le 2$ together with Lemma 2.7 we have $\mathcal{I}_p^1(F) < \infty$ and $\mathcal{M}_p^1(F) < \infty$ for all $p \in (0,\infty)$. ■

Appendix

Lemma A.1 ($\mathrm{dist}(L_{x,y},0)$ in terms of $\measuredangle(x,0,y)$). *Let $x,y\in\mathbb{R}^n\setminus\{0\}$, $x\neq y$, be such that $\varepsilon := \arccos(x\cdot y/(\|x\|\|y\|))\in(0,\pi)$ and let $L_{x,y}$ denote the straight line connecting x and y. Then*

$$\mathrm{dist}(L_{x,y},0)\geq \frac{\sin(\varepsilon)}{2}\min\{\|x\|,\|y\|\}.$$

Proof. We can assume that $0,x,y\in\mathbb{R}^2$. Now we compute the area of the triangle given by $0,x,y$ as

$$\frac{1}{2}\sin(\varepsilon)\|x\|\,\|y\| = \frac{1}{2}\|x-y\|\,\mathrm{dist}(L_{x,y},0)$$

and obtain the proposition via $\|x-y\|\leq 2\max\{\|x\|,\|y\|\}$. ∎

Lemma A.2 (Integral I). *For $y,z>0$ and $p\geq 2$ we have*

$$\int_0^1 \frac{x^p}{(x^2+y^2)^{p/2}(x^2+z^2)^{p/2}}\,dx \leq \frac{\pi}{2^p}(zy)^{-(p-1)/2}.$$

Proof. We have

$$\begin{aligned}
\int_0^1 \frac{x^p}{(x^2+y^2)^{p/2}(x^2+z^2)^{p/2}}\,dx &= \int_0^1 \frac{x^p}{(x^4+(y^2+z^2)x^2+y^2z^2)^{p/2}}\,dx \\
&\overset{y^2+z^2\geq 2yz}{\leq} \int_0^1 \frac{x^p}{(x^4+2yzx^2+y^2z^2)^{p/2}}\,dx \\
&= \int_0^1 \frac{x^p}{(x^2+yz)^{2p/2}}\,dx = \int_0^1 \frac{x^p}{(x^2+yz)^p}\,dx \\
&= \int_0^1 \frac{1}{(x+yz/x)^p}\,dx = \int_0^1 \frac{1}{(x+yz/x)^2}\,\frac{1}{(x+yz/x)^{p-2}}\,dx \\
&\overset{x+zy/x\geq 2\sqrt{zy}}{\leq} \int_0^1 \frac{1}{(x+yz/x)^2}\,\frac{1}{(2\sqrt{zy})^{p-2}}\,dx \\
&\overset{\text{Lemma A.3}}{=} \frac{1}{2^{p-2}}\,\frac{1}{(zy)^{p/2-1}}\,\frac{1}{2}\Bigg(\frac{\arctan(1/\sqrt{zy})}{\sqrt{zy}} - \underbrace{\frac{1}{1+zy}}_{\geq 0}\Bigg) \\
&\leq \frac{1}{2^{p-2}}\,\frac{1}{(zy)^{p/2-1}}\,\frac{1}{2}\,\frac{\pi}{2}\,\frac{1}{\sqrt{zy}} = \frac{\pi}{2^p}(zy)^{-(p-1)/2}. \quad\blacksquare
\end{aligned}$$

Lemma A.3 (Integral II). *For $a>0$ we have*

$$\int_0^1 \frac{1}{(x+a/x)^2}\,dx = \frac{1}{2}\left(\frac{\arctan(1/\sqrt{a})}{\sqrt{a}} - \frac{1}{1+a}\right).$$

Proof. Indeed,

$$\left[\frac{1}{2}\left(\frac{\arctan(x/\sqrt{a})}{\sqrt{a}} - \frac{x}{x^2+a}\right)\right]'$$
$$= \frac{1}{2}\Bigg(\underbrace{\frac{1}{\sqrt{a}(1+(x/\sqrt{a})^2)}\frac{1}{\sqrt{a}}}_{=\frac{1}{a+x^2}} - \frac{1}{x^2+a} + \frac{2x^2}{(x^2+a)^2}\Bigg)$$
$$= \frac{x^2}{(x^2+a)^2} = \frac{1}{(x+a/x)^2}. \blacksquare$$

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