

Sets of nondifferentiability for conjugacies between expanding interval maps

by

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Dedicated to Michał Misiurewicz on the occasion of his 60th birthday

Abstract. We study differentiability of topological conjugacies between expanding piecewise $C^{1+\epsilon}$ interval maps. If these conjugacies are not C^1 , then their derivative vanishes Lebesgue almost everywhere. We show that in this case the Hausdorff dimension of the set of points for which the derivative of the conjugacy does not exist lies strictly between zero and one. Moreover, by employing the thermodynamic formalism, we show that this Hausdorff dimension can be determined explicitly in terms of the Lyapunov spectrum. These results then give rise to a “rigidity dichotomy” for the type of conjugacies under consideration.

1. Introduction and statement of results. In this paper we study aspects of nondifferentiability for conjugacy maps between expanding maps of the unit interval \mathcal{U} into itself which have precisely d increasing full inverse branches and each of these branches is a strictly contracting $C^{1+\epsilon}$ diffeomorphism on \mathcal{U} , for some fixed $\epsilon > 0$ and some fixed integer $d \geq 2$ (here, a map $f : \mathcal{U} \rightarrow f(\mathcal{U}) \subset \mathbb{R}$ is said to be a $C^{1+\epsilon}$ diffeomorphism if there exists an extension \tilde{f} of f to some open neighbourhood of \mathcal{U} which is a diffeomorphism such that $\tilde{f}|_{\mathcal{U}}$ is Hölder continuous with Hölder exponent equal to ϵ). Throughout, these expanding maps will be referred to as *expanding piecewise $C^{1+\epsilon}$ maps*. Clearly, each map of this type is naturally semi-conjugate to the full shift Σ over the alphabet $A := \{1, \dots, d\}$. Moreover, if S and T are two expanding piecewise $C^{1+\epsilon}$ maps, then the following diagram commutes, where σ refers to the usual shift map on Σ , and π_S and π_T denote the associated coding

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maps:

$$\begin{array}{ccccc}
 \mathcal{U} & \xleftarrow{\pi_T} & \Sigma & \xrightarrow{\pi_S} & \mathcal{U} \\
 T \downarrow & & \downarrow \sigma & & \downarrow S \\
 \mathcal{U} & \xleftarrow{\pi_T} & \Sigma & \xrightarrow{\pi_S} & \mathcal{U}
 \end{array}$$

The conjugacy map $\Theta : \mathcal{U} \rightarrow \mathcal{U}$ between the two systems (\mathcal{U}, S) and (\mathcal{U}, T) is then given by $T \circ \Theta = \Theta \circ S$ (see Figs. 1 and 4 for some examples), and one immediately verifies that Θ is continuous. The first main task of the paper will be to employ the thermodynamic formalism in order to give a detailed fractal analysis of the following three sets:

$$\begin{aligned}
 \mathcal{D}_\sim &= \mathcal{D}_\sim(S, T) := \{\xi \in \mathcal{U} : \Theta'(\xi) \text{ does not exist in the generalised sense}\}; \\
 \mathcal{D}_\infty &= \mathcal{D}_\infty(S, T) := \{\xi \in \mathcal{U} : \Theta'(\xi) = \infty\}; \\
 \mathcal{D}_0 &= \mathcal{D}_0(S, T) := \{\xi \in \mathcal{U} : \Theta'(\xi) = 0\}.
 \end{aligned}$$

Here, $\Theta'(\xi)$ *exists in the generalised sense* means that $\Theta'(\xi)$ either exists or else is equal to infinity (at the boundary points we interpret these quantities in terms of limits from the left or right, as appropriate). Note that we can trivially write $\mathcal{U} = \mathcal{D}_\sim \cup \mathcal{D}_\infty \cup \mathcal{D}_0 \cup \mathcal{D}$ where $\mathcal{D} := \{\xi \in \mathcal{U} : \Theta'(\xi) \in (0, \infty)\}$. However, as we will see, either $\mathcal{D} = \mathcal{U}$ or $\mathcal{D} = \emptyset$.

The second main task of this paper will be to find a necessary and sufficient condition for two expanding piecewise $C^{1+\epsilon}$ systems to be rigid in a certain sense.

To state our main results in greater detail, we first need to recall some concepts from the thermodynamic formalism. Let $\varphi, \psi : \Sigma \rightarrow \mathbb{R}_{<0}$ denote the Hölder continuous potential functions which are given for $x = (x_1 x_2 \dots) \in \Sigma$ by

$$\varphi(x) := \log (S_{x_1}^{-1})'(\pi_S(\sigma(x))) \quad \text{and} \quad \psi(x) := \log (T_{x_1}^{-1})'(\pi_T(\sigma(x))),$$

where S_a^{-1} and T_a^{-1} denote the inverse branches of S and T associated with $a \in A$. Also, let P denote the usual pressure function from the thermodynamic formalism (see [17] for the definition of P). Then there exists a function $\beta : \mathbb{R} \rightarrow \mathbb{R}$ given implicitly by the pressure equation

$$P(s\varphi + \beta(s)\psi) = 0 \quad \text{for all } s \in \mathbb{R}.$$

Note that β is well defined, since $\psi < 0$. With μ_s denoting the equilibrium measure associated with the potential function $s\varphi + \beta(s)\psi$, we have

$$\beta'(s) := \frac{-\int \varphi d\mu_s}{\int \psi d\mu_s} < 0.$$

Therefore, β is strictly decreasing, and one also immediately verifies that $\beta(1) = 0$ and $\beta(0) = 1$. If φ and ψ are *cohomologically independent*, that is, if there are no nontrivial choices of $b, c \in \mathbb{R}$ and $u \in C(\Sigma)$ such that

$b\varphi + c\psi = u \circ \sigma - u$ (in this situation, we will also say that S and T are cohomologically independent), then β is strictly convex (see e.g. [17]). In this situation the mean value theorem for derivatives gives that there exists a unique $s_0 \in (0, 1)$ such that $\beta'(s_0) = -1$. Let $\tilde{\beta} : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $\tilde{\beta}(s) := \beta(s) + s$. Note that $\tilde{\beta}$ is convex and has a unique minimum at s_0 . Also, $\tilde{\beta}(0) = \tilde{\beta}(1) = 1$ and $\tilde{\beta}(s_0) = \hat{\beta}(1)$, where $\hat{\beta}$ denotes the (concave) Legendre transform of β , given by $\hat{\beta}(s) := \inf_{t \in \mathbb{R}} (\beta(t) + st)$ for $s \in \mathbb{R}$. Finally, the level sets $\mathcal{L}(s)$ are defined by

$$\mathcal{L}(s) := \left\{ \xi \in \mathcal{U} : \lim_{n \rightarrow \infty} \frac{\mathcal{S}_n \varphi(\pi_S^{-1}(\xi))}{\mathcal{S}_n \psi(\pi_T^{-1}(\xi))} = s \right\},$$

where $\mathcal{S}_n f := \sum_{k=0}^{n-1} f \circ \sigma^k$ denotes the n th Birkhoff sum. By standard thermodynamic formalism (see e.g. [17]), we then have, for s in the closure $-\beta'(\mathbb{R})$ of the range of $-\beta'$,

$$\dim_{\mathbb{H}}(\mathcal{L}(s)) = \hat{\beta}(s)/s = \frac{1}{s} \inf_{t \in \mathbb{R}} (st + \beta(t)) = \inf_{t \in \mathbb{R}} (t + \beta(t)/s),$$

whereas for $s \notin -\beta'(\mathbb{R})$ we have $\mathcal{L}(s) = \emptyset$.

The first main result of this paper is now stated in the following theorem.

THEOREM 1.1. *Let S and T be two cohomologically independent expanding piecewise $C^{1+\epsilon}$ maps. Then*

$$0 < \dim_{\mathbb{H}}(\mathcal{D}_{\sim}) = \dim_{\mathbb{H}}(\mathcal{D}_{\infty}) = \dim_{\mathbb{H}}(\mathcal{L}(1)) = \tilde{\beta}(s_0) < 1.$$

Using the well known fact that both T and S have absolutely continuous invariant measures with C^{ϵ} densities (see e.g. [1, Theorem 2.1]), one immediately verifies that either $\Theta' = 0$ Lebesgue almost everywhere or Θ is $C^{1+\epsilon}$. The second main result of this paper is to give a refinement of this dichotomy, and this is summarized in the following theorem.

THEOREM 1.2. *Let S and T be two expanding piecewise $C^{1+\epsilon}$ maps. Then*

$$\text{the conjugacy } \Theta \text{ is a } C^{1+\epsilon} \text{ diffeomorphism} \Leftrightarrow \dim_{\mathbb{H}}(\mathcal{D}_{\sim}) = 0.$$

More precisely, the following ‘‘rigidity dichotomy’’ holds.

- (1) *If S and T are cohomologically dependent, then Θ is a $C^{1+\epsilon}$ diffeomorphism and hence absolutely continuous. Equivalently,*

$$\mathcal{D}_0 = \mathcal{D}_{\infty} = \mathcal{D}_{\sim} = \emptyset, \quad \text{and hence } \mathcal{U} = \{\xi \in \mathcal{U} : 0 < \Theta'(\xi) < \infty\}.$$

- (2) *If S and T are cohomologically independent, then Θ is singular, that is, $\lambda(\mathcal{D}_0) = 1$. Moreover, Θ is Hölder continuous with Hölder exponent equal to $(\sup_{x \in \mathbb{R}} -\beta'(x))^{-1}$, and*

$$0 < \dim_{\mathbb{H}}(\mathcal{D}_{\infty}) = \dim_{\mathbb{H}}(\mathcal{D}_{\sim}) < 1.$$

The latter theorem is closely related to classical work by Shub and Sullivan [21] on the smoothness of conjugacies between expanding maps of the unit circle \mathbb{S}^1 (see also e.g. [2, 9, 16, 22]). In [21] it was shown that if the conjugacy between two C^k expanding maps is absolutely continuous, for some $k \geq 2$, then the conjugacy is necessarily C^k too (also cf. [18]). Let us also mention a result by Cui [3] which states that the conjugacy map between two expanding $C^{1+\epsilon}$ circle endomorphisms is itself $C^{1+\epsilon}$, if it has finite, nonzero derivative at some point in \mathbb{S}^1 . So, to deduce Theorem 1.2 from Theorem 1.1, we need to adapt this result to the setting of interval maps.

In the case of circle maps we can combine our result on interval maps and the result of Cui to obtain a result for endomorphisms of \mathbb{S}^1 . For this one has to rewrite Theorem 1.1 in terms of orientation preserving expanding $C^{1+\epsilon}$ circle maps, and this then gives rise to the following result.

COROLLARY 1.3. *For the conjugacy map Φ between a given pair (\mathbb{S}^1, U) and (\mathbb{S}^1, V) of expanding $C^{1+\epsilon}$ endomorphisms of \mathbb{S}^1 , the following statements are equivalent.*

- (1) Φ is a $C^{1+\epsilon}$ circle map;
- (2) $\dim_{\mathbb{H}}(\{\xi \in \mathbb{S}^1 : \Phi'(\xi) \text{ does not exist in the generalised sense}\}) = 0$;
- (3) $\dim_{\mathbb{H}}(\{\xi \in \mathbb{S}^1 : 0 < \Phi'(\xi) < \infty\}) = 1$;
- (4) Φ is absolutely continuous;
- (5) Φ is bi-Lipschitz.

A natural question to ask is how the quantities $\dim_{\mathbb{H}}(\mathcal{D}_{\infty}(S, T))$ and $\dim_{\mathbb{H}}(\mathcal{D}_{\sim}(S, T))$ vary as S and T change. The next two results address this question.

PROPOSITION 1.4. *For a C^k family of expanding maps, the Hausdorff dimension of the set of nondifferentiability points has a C^{k-2} dependence.*

PROPOSITION 1.5. *There exists a pair of C^2 endomorphisms of \mathbb{S}^1 for which the set of nondifferentiability points for the associated conjugacy map has arbitrarily small Hausdorff dimension.*

The paper is organised as follows. In Sections 2 and 3 we give the proofs of Theorems 1.1 and 1.2. Section 4 discusses two basic examples, and one of these is then used in Section 5 for the proof of Proposition 1.5. In Section 5 we also give the proof of Proposition 1.4.

REMARK 1.6. (1) Note that

$$\mathcal{D}_{\sim}(S, T) \cup \mathcal{D}_{\infty}(S, T) = \{\xi \in \mathcal{U} : \Theta \text{ is not differentiable at } \xi\},$$

and hence Theorem 1.1 in particular implies that if S and T are cohomologically independent, then the Hausdorff dimension of the set of points for which Θ is not differentiable is equal to $\tilde{\beta}(s_0)$.

(2) There is also a variational formula for the Hausdorff dimension of \mathcal{D}_\sim . Namely, as we will see in Section 2.3,

$$\dim_{\mathbb{H}}(\mathcal{D}_\sim) = \sup \left\{ \frac{h(\mu)}{\int \varphi d\mu} : \frac{\int \varphi d\mu}{\int \psi d\mu} = 1 \right\},$$

where the supremum ranges over all σ -invariant probability measures on Σ . In particular, this variational formula shows that if we swap the roles of φ and ψ , then this has no effect on the Hausdorff dimension of the set of nondifferentiability. In other words, if instead of Θ we take the dual conjugacy $\widehat{\Theta}$, given by $S \circ \widehat{\Theta} = \widehat{\Theta} \circ T$, then the Hausdorff dimension of the set of points at which $\widehat{\Theta}'$ does not exist in the generalised sense coincides with $\dim_{\mathbb{H}}(\mathcal{D}_\sim)$, i.e.

$$\dim_{\mathbb{H}}(\mathcal{D}_\sim(S, T)) = \dim_{\mathbb{H}}(\mathcal{D}_\sim(T, S)).$$

(3) The conjugacy map Θ can also be viewed as the distribution function of the measure $m_\Theta := \lambda \circ \Theta$. This follows, since for $\xi \in \mathcal{U}$ we have

$$m_\Theta([0, \xi]) = \lambda([0, \Theta(\xi)]) = \Theta(\xi).$$

Hence, the investigations in this paper can also be seen as a study of singular distribution functions with support equal to \mathcal{U} . Note that there are strong parallels to the results in [12], where we used some of the outcomes of [13] to give a fractal analysis of nondifferentiability for Minkowski’s question mark function.

(4) Finally, let us mention that the statements in Theorems 1.1 and 1.2 can be generalised so that the derivative of Θ gets replaced by the s -Hölder derivative $\Delta_s \Theta$ of Θ , given for $s \in -\beta'(\mathbb{R})$ by

$$(\Delta_s \Theta)(\xi) := \lim_{\eta \rightarrow \xi} \frac{|\Theta(\eta) - \Theta(\xi)|}{|\eta - \xi|^s}.$$

For this more general derivative the relevant sets are

$$\begin{aligned} \mathcal{D}_\sim^{(s)} = \mathcal{D}_\sim^{(s)}(S, T) &:= \{\xi \in \mathcal{U} : (\Delta_s \Theta)(\xi) \text{ does not exist} \\ &\hspace{15em} \text{in the generalised sense}\}, \\ \mathcal{D}_\infty^{(s)} = \mathcal{D}_\infty^{(s)}(S, T) &:= \{\xi \in \mathcal{U} : (\Delta_s \Theta)(\xi) = \infty\}. \end{aligned}$$

Straightforward adaptations of the proofs in this paper then show that

$$\dim_{\mathbb{H}}(\mathcal{D}_\sim^{(s)}) = \dim_{\mathbb{H}}(\mathcal{D}_\infty^{(s)}) = \dim_{\mathbb{H}}(\mathcal{L}(s)).$$

In particular, this shows that on $-\beta'(\mathbb{R})$ the Lyapunov spectrum $s \mapsto \widehat{\beta}(s)/s$ coincides with the “spectrum of non- s -Hölder differentiability of Θ ”. Note that, by generalising results of [3, 7] and others, in [11] similar results were obtained for certain classes of Cantor-like sets.

2. Proof of Theorem 1.1

2.1. *The geometry of the derivative of Θ .* Let us first introduce some notations which will be used throughout.

DEFINITION. Let us say that $x = (x_1x_2\dots) \in \Sigma$ has an *i -block of length k at the n th level*, for $n, k \in \mathbb{N}$ and $i \in \{1, d\}$, if $x_{n+k+1} \in A \setminus \{i\}$ and $x_{n+m} = i$ for all $m \in \{1, \dots, k\}$. Moreover, we will say that $x = (x_1x_2\dots) \in \Sigma$ has a *strict i -block of length k at the n th level* if additionally $x_n \in A \setminus \{i\}$.

For ease of exposition, we define the function $\chi : \Sigma \rightarrow \mathbb{R}$ by $\chi := \psi - \varphi$. Also, let $D_\Theta(\xi, \eta)$ denote the differential quotient for Θ at ξ and η , that is,

$$D_\Theta(\xi, \eta) := \frac{\Theta(\xi) - \Theta(\eta)}{\xi - \eta}.$$

Moreover, we use \underline{a}_k to denote the word of length $k \in \mathbb{N}$ containing exclusively the letter $a \in A$, and we let \underline{a} denote the infinite word containing exclusively the letter $a \in A$. Also, $[x_1 \dots x_n]$ denotes the cylinder set associated with the finite word $(x_1, \dots, x_n) \in A^n$, that is,

$$[x_1 \dots x_n] := \{(y_1y_2\dots) \in \Sigma : y_i = x_i \text{ for all } i = 1, \dots, n\}.$$

Throughout, “ \asymp ” means that the ratio of the left hand side to the right hand side is uniformly bounded away from zero and infinity. Likewise, we use “ \ll ” to denote that the expression on the left hand side is uniformly bounded by the expression on the right hand side multiplied by some fixed positive constant.

Let us begin our discussion of the geometry of the derivative of Θ with the following crucial geometric observation.

PROPOSITION 2.1. *Let $x = (x_1x_2\dots), y = (y_1y_2\dots) \in \Sigma$ satisfy $y \in [x_1 \dots x_{n-1}]$ as well as $x_n = a$ and $y_n = b$ for some $n \in \mathbb{N}$ and $a, b \in A$ with $|a - b| = 1$ (note that for $n = 1$ we adopt the convention that $x_1 = a$ and $y_1 = b$). Moreover, assume that for some $k, l \in \mathbb{N}$, the word x has an i -block of length k at the n th level, and y has a j -block of length l at the n th level. Here, $i, j \in \{1, d\}$ are chosen such that if $a < b$ then $i = d$ and $j = 1$, whereas if $a > b$ then $i = 1$ and $j = d$. In this situation for $\xi := \pi_S(x)$ and $\eta := \pi_S(y)$,*

$$D_\Theta(\xi, \eta) \asymp e^{S_n \chi(x)} \frac{e^{k\psi((i))} + e^{l\psi((j))}}{e^{k\varphi((i))} + e^{l\varphi((j))}}.$$

Proof. We only consider the case $a = b + 1 > b$. The case $a < b$ is completely analogous and is left to the reader. In this situation, for some $p \in A \setminus \{1\}$ and $q \in A \setminus \{d\}$, the words x and y are of the form

$$x = (x_1 \dots x_{n-1} \underline{a}_k \underline{1}_k p \dots) \quad \text{and} \quad y = (x_1 \dots x_{n-1} \underline{b} \underline{d}_l q \dots).$$

Then consider the following cylinder sets:

$$\begin{aligned} I_1 &:= \pi_S([x_1 \dots x_{n-1} b \underline{d}_{l+1}]) & I_2 &:= \pi_S([x_1 \dots x_{n-1} a \underline{1}_{k+1}]), \\ J_1 &:= \pi_S([x_1 \dots x_{n-1} b \underline{d}_l]) & J_2 &:= \pi_S([x_1 \dots x_{n-1} a \underline{1}_k]). \end{aligned}$$

One immediately verifies that

$$I_1 \cup I_2 \subset [\eta, \xi] \subset J_1 \cup J_2.$$

Moreover, with $\eta' := \pi_T((x_1 \dots x_{n-1} b \underline{d})) = \pi_T((x_1 \dots x_{n-1} a \underline{1}))$ we have, using the bounded distortion property,

$$|\Theta(\xi) - \Theta(\eta)| = |\Theta(\xi) - \Theta(\eta')| + |\Theta(\eta') - \Theta(\eta)| \asymp e^{\mathcal{S}_n \psi(x)} (e^{k\psi((\underline{d}))} + e^{l\psi((\underline{1}))}).$$

Similarly, one obtains

$$\begin{aligned} |\xi - \eta| &\asymp \text{diam}(I_1) + \text{diam}(I_2) \asymp \text{diam}(J_1) + \text{diam}(J_2) \\ &\asymp e^{\mathcal{S}_n \varphi(x)} (e^{k\varphi((\underline{d}))} + e^{l\varphi((\underline{1}))}). \blacksquare \end{aligned}$$

Note that Proposition 2.1 does in particular contain *all* the cases in which, for given $x, y \in \Sigma$, $D_\Theta(\pi_S(x), \pi_S(y))$ can significantly deviate from $\exp(\mathcal{S}_n \chi(x))$. This is clarified by the following lemma, which addresses the cases not covered by Proposition 2.1.

LEMMA 2.2. *Let $x = (x_1 x_2 \dots), y = (y_1 y_2 \dots) \in \Sigma$ be such that $y \in [x_1 \dots x_{n-1}] \setminus [x_1 \dots x_n]$ and either $|x_n - y_n| > 1$, or if $|x_n - y_n| = 1$ then $\pi_S([x_1 \dots x_{n+1}]) \cap \pi_S([y_1 \dots y_{n+1}]) = \emptyset$. For $\xi := \pi_S(x)$ and $\eta := \pi_S(y)$,*

$$D_\Theta(\xi, \eta) \asymp e^{\mathcal{S}_n \chi(x)}.$$

Proof. We have either $\pi_S([x_1 \dots x_n]) \cap \pi_S([y_1 \dots y_n]) = \emptyset$, and hence there exists an interval separating these two sets, or if $\pi_S([x_1 \dots x_n]) \cap \pi_S([y_1 \dots y_n]) \neq \emptyset$ then $\pi_S([x_1 \dots x_{n+1}]) \cap \pi_S([y_1 \dots y_{n+1}]) = \emptyset$. Clearly, in both cases there exist $a, b \in A$ such that the interval $I_{ab} := \pi_S([x_1 \dots x_{n-1} ab])$ separates the intervals $\pi_S([x_1 \dots x_{n+1}])$ and $\pi_S([y_1 \dots y_{n+1}])$. Using this, we then obtain

$$\begin{aligned} e^{\mathcal{S}_n \psi(x)} &\ll \text{diam}(\Theta(I_{ab})) \ll |\Theta(\xi) - \Theta(\eta)| \\ &\ll \text{diam}(\Theta(\pi_S([x_1 \dots x_{n-1}])) \ll e^{\mathcal{S}_n \psi(x)}, \end{aligned}$$

and

$$e^{\mathcal{S}_n \varphi(x)} \ll \text{diam}(I_{ab}) \ll |\xi - \eta| \ll \text{diam}(\pi_S([x_1 \dots x_{n-1}])) \ll e^{\mathcal{S}_n \varphi(x)}. \blacksquare$$

LEMMA 2.3. *If $x = (x_1 x_2 \dots) \in \Sigma$ has an i -block of length k at the n th level, for some $n, k \in \mathbb{N}$ and $i \in \{1, d\}$, then for each $\eta \in \pi_S([x_1 \dots x_n] \setminus [x_1 \dots x_{n+1}])$, with $\delta := -\min\{\psi((\underline{1})), \psi((\underline{d}))\} > 0$ and $\xi := \pi_S(x)$,*

$$D_\Theta(\xi, \eta) \gg e^{\mathcal{S}_n \chi(x)} e^{-\delta k}.$$

Proof. Trivially, $|\xi - \eta| \ll \exp(\mathcal{S}_n \varphi(x))$. As in the proof of the previous lemma, one immediately verifies that

$$|\Theta(\xi) - \Theta(\eta)| \gg e^{\mathcal{S}_{n+k} \psi(x)} \gg e^{\mathcal{S}_n \psi(x)} e^{-\delta k}.$$

By combining these observations, the result follows. ■

LEMMA 2.4. *Let $x \in \Sigma$ and $\xi := \pi_S(x)$.*

- (1) *If $\limsup_{n \rightarrow \infty} e^{\mathcal{S}_n \chi(x)} = \infty$, then $\limsup_{\eta \rightarrow \xi} D_\Theta(\xi, \eta) = \infty$.*
- (2) *If $\liminf_{n \rightarrow \infty} e^{\mathcal{S}_n \chi(x)} = 0$, then $\liminf_{\eta \rightarrow \xi} D_\Theta(\xi, \eta) = 0$.*

Proof. Let $x = (x_1 x_2 \dots)$ and assume without loss of generality that

$$\xi \notin \pi_S(\{(x_1 x_2 \dots) \in \Sigma : \exists n \in \mathbb{N} \exists i \in \{1, d\} \forall k \geq n : x_k = i\}).$$

For $n \in \mathbb{N}$, the left and right boundary points of $\pi_S([x_1 \dots x_n])$ are given by $\xi_n := \pi_S((x_1 \dots x_n \underline{1}))$ and $\eta_n := \pi_S((x_1 \dots x_n \underline{d}))$. By assumption we have $\xi \notin \{\xi_n, \eta_n : n \in \mathbb{N}\}$. It then follows that

$$\min\{D_\Theta(\xi, \eta_n), D_\Theta(\xi, \xi_n)\} \leq D_\Theta(\xi_n, \eta_n) \leq \max\{D_\Theta(\xi, \eta_n), D_\Theta(\xi, \xi_n)\}.$$

Since $D_\Theta(\xi_n, \eta_n) \asymp e^{\mathcal{S}_n \chi(\xi)}$, the lemma follows. ■

We have the following immediate corollary.

COROLLARY 2.5. *Let $x \in \Sigma$ be such that*

$$\liminf_{n \rightarrow \infty} e^{\mathcal{S}_n \chi(x)} = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} e^{\mathcal{S}_n \chi(x)} = \infty.$$

Then $\pi_S(x) \in \mathcal{D}_\sim$.

For the remainder of this section we restrict the discussion to the following two cases:

- (2.1) CASE 1: $\psi(\underline{(1)})/\varphi(\underline{(1)}) < \min\{\psi(\underline{(d)})/\varphi(\underline{(d)}), 1\}$;
- CASE 2: $\psi(\underline{(d)})/\varphi(\underline{(d)}) < \min\{\psi(\underline{(1)})/\varphi(\underline{(1)}), 1\}$.

As we will see in Lemma 2.7, these two cases are in fact the only relevant cases for the purposes in this paper. In fact, without loss of generality, we will always assume that we are in the situation of Case 1. The discussion of Case 2 is completely analogous (essentially, one has to interchange the roles of 1 and d as well as of l and k), and will be left to the reader. Note that Cases 1 and 2 include the cases

$$e^{\chi(\underline{(1)})} > 1 > e^{\chi(\underline{(d)})} \quad \text{and} \quad e^{\chi(\underline{(1)})} < 1 < e^{\chi(\underline{(d)})},$$

which are for instance fulfilled in the Salem examples (Example 1) briefly discussed in Section 4.

On the basis of this assumption, we now make the following crucial observation.

LEMMA 2.6. Assume that we are in Case 1 of (2.1). Then for all $l \in \mathbb{N}$,

$$\frac{e^{l\psi(\underline{1})} + e^{k\psi(\underline{d})}}{e^{l\varphi(\underline{1})} + e^{k\varphi(\underline{d})}} \ll e^{\alpha k},$$

where $\alpha := \chi(\underline{1})\varphi(\underline{d})/\varphi(\underline{1}) > 0$. Moreover, if $l = \lfloor k\alpha/\chi(\underline{1}) \rfloor$ then

$$\frac{e^{l\psi(\underline{1})} + e^{k\psi(\underline{d})}}{e^{l\varphi(\underline{1})} + e^{k\varphi(\underline{d})}} \gg e^{\alpha k}.$$

Here, $\lfloor r \rfloor$ denotes the smallest integer greater than or equal to $r \in \mathbb{R}$.

Proof. First note that with $\alpha' := \varphi(\underline{d})/\varphi(\underline{1})$ the conditions in Case 1 immediately imply

$$e^{\varphi(\underline{1})} < e^{\psi(\underline{1})} \quad \text{and} \quad e^{\psi(\underline{d})} < e^{\alpha'\psi(\underline{1})}.$$

In particular, this implies that $\chi(\underline{1}) > 0$. Then for all $l \geq \alpha'k$,

$$\frac{e^{l\psi(\underline{1})} + e^{k\psi(\underline{d})}}{e^{l\varphi(\underline{1})} + e^{k\varphi(\underline{d})}} \leq \frac{e^{l\psi(\underline{1})} + e^{k\alpha'\psi(\underline{1})}}{e^{k\varphi(\underline{d})}} \leq 2 \frac{e^{k\alpha'\psi(\underline{1})}}{e^{k\varphi(\underline{d})}} = 2 \frac{e^{k\alpha'\psi(\underline{1})}}{e^{k\alpha'\varphi(\underline{1})}} = 2e^{\alpha k}.$$

If $l \leq \alpha'k$, then we obtain

$$\begin{aligned} \frac{e^{l\psi(\underline{1})} + e^{k\psi(\underline{d})}}{e^{l\varphi(\underline{1})} + e^{k\varphi(\underline{d})}} &\leq \frac{e^{l\psi(\underline{1})} + e^{k\alpha'\psi(\underline{1})}}{e^{l\varphi(\underline{1})}} \\ &\leq 2 \left(\frac{e^{\psi(\underline{1})}}{e^{\varphi(\underline{1})}} \right)^l \leq 2 \left(\frac{e^{\psi(\underline{1})}}{e^{\varphi(\underline{1})}} \right)^{\alpha'k} = 2e^{\alpha k}. \end{aligned}$$

Finally, if $l = \lfloor \alpha'k \rfloor$ then

$$\frac{e^{l\psi(\underline{1})} + e^{k\psi(\underline{d})}}{e^{l\varphi(\underline{1})} + e^{k\varphi(\underline{d})}} \asymp \frac{e^{l\psi(\underline{1})} + e^{k\psi(\underline{d})}}{2e^{k\varphi(\underline{d})}} \gg \frac{e^{k\alpha'\psi(\underline{1})}}{2e^{k\alpha'\varphi(\underline{1})}} \asymp e^{\alpha k}. \blacksquare$$

The next observation explains why we can restrict the discussion to the above two cases in (2.1). For this we define

$$\mathcal{D}_{\#} := \pi_S(\{x \in \Sigma : \lim_{n \rightarrow \infty} \mathcal{S}_n \chi(x) = -\infty\}),$$

$$\mathcal{D}_{\#}^* = \mathcal{D}_{\#} \setminus \pi_S(\{(x_1 x_2 \dots) \in \Sigma : \exists n \in \mathbb{N} \exists i \in \{1, d\} \forall k \geq n : x_k = i\}).$$

LEMMA 2.7. If we are in neither of the two cases in (2.1), then

$$\mathcal{D}_{\sim} \cap \mathcal{D}_{\#}^* = \emptyset.$$

Proof. Let $x = (x_1 x_2 \dots) \in \Sigma$ be such that $\xi := \pi_S(x) \in \mathcal{D}_{\#}^*$. Assume that x has a strict j -block of length k at the n th level, $j \in \{1, d\}$. We have to distinguish two cases. The first is

$$\frac{\psi(\underline{1})}{\varphi(\underline{1})} \geq 1 \quad \text{and} \quad \frac{\psi(\underline{d})}{\varphi(\underline{d})} \geq 1.$$

Then $e^{\psi(\underline{i})} \leq e^{\varphi(\underline{i})}$ for $i \in \{1, d\}$, and clearly $\frac{e^{l\psi(\underline{1})} + e^{k\psi(\underline{d})}}{e^{l\varphi(\underline{1})} + e^{k\varphi(\underline{d})}} \leq 1$ for all $k, l \in \mathbb{N}$. By combining this observation with Proposition 2.1 and Lemma 2.2, it follows that $\xi \in \mathcal{D}_0$, and hence $\xi \notin \mathcal{D}_{\sim}$.

The second case is

$$\frac{\psi((\underline{1}))}{\varphi((\underline{1}))} = \frac{\psi((\underline{d}))}{\varphi((\underline{d}))} < 1.$$

As in the proof of Lemma 2.6, it then follows that for each $i \in \{1, d\} \setminus \{j\}$ and for all $l, k \in \mathbb{N}$, we have

$$\frac{e^{l\psi((i))} + e^{k\psi((j))}}{e^{l\varphi((i))} + e^{k\varphi((j))}} \leq 2e^{k\chi((j))}.$$

Therefore, for each $\eta \in \pi_S([x_1 \dots x_{n-1}] \setminus [x_1 \dots x_n])$ we have

$$D_\Theta(\xi, \eta) \ll e^{\mathcal{S}_{n+k}\chi(x)}.$$

Using this observation and Lemma 2.2, we find that the derivative of Θ at ξ is equal to 0, and hence $\xi \notin \mathcal{D}_\sim$. ■

PROPOSITION 2.8. *Assume that we are in Case 1 of (2.1) and let $x = (x_1 x_2 \dots) \in \Sigma$ be such that $\xi := \pi_S(x) \in \mathcal{D}_\#^*$. Then $\xi \in \mathcal{D}_\sim$ if and only if there exist strictly increasing sequences $(n_m)_{m \in \mathbb{N}}$ and $(k_m)_{m \in \mathbb{N}}$ of positive integers such that x has a strict d -block of length k_m at the n_m th level for each $m \in \mathbb{N}$, and*

$$e^{\mathcal{S}_{n_m}\chi(x) + k_m\alpha} \gg 1 \quad \text{for all } m \in \mathbb{N}.$$

Proof. By Lemma 2.4 there exists a sequence $(\eta_m)_{m \in \mathbb{N}}$ such that

$$\lim_{n \rightarrow \infty} D_\Theta(\xi, \eta_n) = 0.$$

Now, for the “if” part, assume that ξ has strict d -blocks as specified in the proposition. For each $m \in \mathbb{N}$, choose η'_m in the interval $\pi_S([x_1 \dots x_{n_m}(x_{n_m+1} + 1)\underline{1}_{l_m} a])$, where $a \in A \setminus \{1\}$ and $l_m := k_m\alpha/\chi((\underline{1}))$. Combining Proposition 2.1, the second part of Lemma 2.6, and the fact that $\exp(\mathcal{S}_{n_m}\chi(x) + k_m\alpha) \gg 1$, we then obtain, for all $m \in \mathbb{N}$,

$$D_\Theta(\xi, \eta'_m) \asymp e^{\mathcal{S}_{n_m}\chi(x)} \frac{e^{k_m\psi((\underline{d}))} + e^{l_m\psi((\underline{1}))}}{e^{k_m\varphi((\underline{d}))} + e^{l_m\varphi((\underline{1}))}} \asymp e^{\mathcal{S}_{n_m}\chi(x) + k_m\alpha} \gg 1.$$

Combining this with the observation at the beginning of the proof, it follows that $\xi \in \mathcal{D}_\sim$.

For the “only if” part, let $x = (x_1 x_2 \dots) \in \Sigma$ with $\xi := \pi_S(x) \in \mathcal{D}_\sim \cap \mathcal{D}_\#^*$. Then there exists a sequence $(\eta_m)_{m \in \mathbb{N}}$ in \mathcal{U} and a strictly increasing sequence $(n_m)_{m \in \mathbb{N}}$ in \mathbb{N} such that $\eta_m \in \pi_S([x_1 \dots x_{n_m}])$ for all $m \in \mathbb{N}$ and

$$\liminf_{m \rightarrow \infty} D_\Theta(\xi, \eta_m) > 0.$$

Using Proposition 2.1 and Lemma 2.6, it follows that if x has a d -block of length k_m at the n_m th level, then for all $l, m \in \mathbb{N}$,

$$D_\Theta(\xi, \eta_m) \ll e^{\mathcal{S}_{n_m}\chi(x)} \frac{e^{l\psi((\underline{1}))} + e^{k_m\psi((\underline{d}))}}{e^{l\varphi((\underline{1}))} + e^{k_m\varphi((\underline{d}))}} \ll e^{\mathcal{S}_{n_m}\chi(x) + \alpha k_m}.$$

Since $\liminf_{m \rightarrow \infty} D_{\Theta}(\xi, \eta_m) > 0$, it follows that

$$\liminf_{m \rightarrow \infty} e^{\mathcal{S}_{nm}\chi(x) + \alpha k_m} > 0,$$

and therefore $e^{\mathcal{S}_{nm}\chi(x) + \alpha k_m} \gg 1$ for all $m \in \mathbb{N}$. ■

2.2. The upper bound. We start by observing that

$$\limsup_{n \rightarrow \infty} e^{\mathcal{S}_n \chi} > 0 \Rightarrow \limsup_{n \rightarrow \infty} \frac{\mathcal{S}_n \varphi}{\mathcal{S}_n \psi} \geq 1.$$

This implies that

$$\dim_{\text{H}}(\{\limsup_{n \rightarrow \infty} e^{\mathcal{S}_n \chi} > 0\}) \leq \dim_{\text{H}}\left(\left\{\limsup_{n \rightarrow \infty} \frac{\mathcal{S}_n \varphi}{\mathcal{S}_n \psi} \geq 1\right\}\right) = \widehat{\beta}(1).$$

Here, the final equality holds since the Lyapunov dimension spectrum $s \mapsto \widehat{\beta}(s)/s$ is decreasing in a neighbourhood of 1. Since the sets \mathcal{D}_{∞} and $\mathcal{D}_{\sim} \cap \{\limsup_{n \rightarrow \infty} e^{\mathcal{S}_n \chi} > 0\}$ are contained in $\{\limsup_{n \rightarrow \infty} e^{\mathcal{S}_n \chi} > 0\}$, the observation above gives the upper bound $\widehat{\beta}(1)$ for the Hausdorff dimension of each of these two sets.

Since $\lim_{n \rightarrow \infty} \exp(\mathcal{S}_n \chi(x)) = 0$ implies $\pi_S(x) \in \mathcal{D}_{\#}^*$, except for the countable set of end points of all refinements of the Markov partition, it is sufficient to show that

$$\dim_{\text{H}}(\mathcal{D}_{\sim} \cap \mathcal{D}_{\#}^*) \leq \widehat{\beta}(1).$$

This part of the proof is inspired by the arguments given in [11]. First note that it is sufficient to show that

$$\dim_{\text{H}}(\mathcal{D}_{\sim} \cap \mathcal{D}_{\#}^*) \leq \widetilde{\beta}(s) \quad \text{for all } s \leq 1.$$

In a nutshell, the idea is to show that for each $s \leq 1$ there is a suitable covering of $\mathcal{D}_{\sim} \cap \mathcal{D}_{\#}^*$ which will then be used to deduce that the $\widetilde{\beta}(s)$ -dimensional Hausdorff measure of $\mathcal{D}_{\sim} \cap \mathcal{D}_{\#}^*$ is finite.

If we are in neither of the two cases of (2.1), we have $\mathcal{D}_{\sim} \cap \mathcal{D}_{\#}^* = \emptyset$ by Lemma 2.7. Hence for ease of exposition, throughout the remaining part of this section we will again assume that we are in Case 1 of (2.1). Clearly, the considerations for Case 2 are completely analogous, and will therefore be omitted. Let us first introduce the stopping time τ_t with respect to χ on $\pi_S^{-1}(\mathcal{D}_{\#}^*)$ by

$$\tau_t(x) := \inf\{k \in \mathbb{N} : \mathcal{S}_k \chi(x) < -t\} \quad \text{for all } t > 0, x \in \pi_S^{-1}(\mathcal{D}_{\#}^*).$$

For each $n \in \mathbb{N}$ fix a partition \mathcal{C}_n of $\pi_S^{-1}(\mathcal{D}_{\#}^*)$ consisting of cylinder sets $[\omega]$ with the following property:

$$|\mathcal{S}_{\tau_n(x)} \chi(x) + n| \ll 1 \quad \text{for all } x \in [\omega].$$

Moreover, for $\epsilon > 0$ we define

$$\mathcal{C}_n(\epsilon) := \{[\omega]_{n\epsilon} : [\omega] \in \mathcal{C}_n\},$$

where n_ϵ is given by $n_\epsilon := \lfloor n(1 - \epsilon)/\alpha \rfloor$. For $s \in (0, 1)$ we choose $\epsilon > 0$ such that

$$(1 - \epsilon)\tilde{\beta}(s) > (-\chi((\underline{1}))/\varphi((\underline{1})))\beta(s).$$

This is possible, since on the one hand we have $\tilde{\beta}(s) - \beta(s) = s > 0$ and hence $\tilde{\beta}(s) > \beta(s)$ for all $s \in (0, 1)$. On the other hand, the fact that $\psi < 0$ immediately implies that $-\chi((\underline{1}))/\varphi((\underline{1})) < 1$. Recall that we are assuming that Case 1 of (2.1) holds, and therefore $-\chi((\underline{1}))/\varphi((\underline{1})) = 1 - \psi((\underline{1}))/\varphi((\underline{1})) > 0$. It then follows that

$$\begin{aligned} \sum_{n \in \mathbb{N}} \sum_{C \in \mathcal{C}_n(\epsilon)} (\text{diam}(C))^{\tilde{\beta}(s)} &\asymp \sum_{n \in \mathbb{N}} \sum_{C \in \mathcal{C}_n(\epsilon)} e^{\sup_{x \in C} \tilde{\beta}(s) \mathcal{S}_{\tau_n(x) + n_\epsilon} \varphi(x)} \\ &\ll \sum_{n \in \mathbb{N}} e^{n(1-\epsilon)\tilde{\beta}(s)\varphi((\underline{d}))/\alpha} \sum_{C \in \mathcal{C}_n} e^{\tilde{\beta}(s) \sup_{x \in C} \mathcal{S}_{\tau_n(x)} \varphi(x)} \\ &\asymp \sum_{n \in \mathbb{N}} e^{n(1-\epsilon)\tilde{\beta}(s)\varphi((\underline{d}))/\alpha + n\beta(s)} \sum_{C \in \mathcal{C}_n} e^{\sup_{x \in C} \mathcal{S}_{\tau_n(x)} (\tilde{\beta}(s)\varphi(x) + \beta(s)\chi(x))} \\ &\ll \sum_{n \in \mathbb{N}} (e^{(1-\epsilon)\tilde{\beta}(s)\varphi((\underline{d}))/\alpha + \beta(s)})^n < \infty. \end{aligned}$$

Here we have used the Gibbs property

$$\sum_{C \in \mathcal{C}_n} e^{\sup_{x \in C} \mathcal{S}_{\tau_n(x)} (\tilde{\beta}(s)\varphi(x) + \beta(s)\chi(x))} = \sum_{C \in \mathcal{C}_n} e^{\sup_{x \in C} \mathcal{S}_{\tau_n(x)} (s\varphi(x) + \beta(s)\psi(x))} \ll 1$$

of the Gibbs measure μ_s and the fact that

$$(1 - \epsilon)\tilde{\beta}(s)\varphi((\underline{d}))/\alpha + \beta(s) = (1 - \epsilon)\tilde{\beta}(s)\varphi((\underline{1}))/\chi((\underline{1})) + \beta(s) < 0.$$

Thus, for the limsup-set

$$C_\infty(\epsilon) := \{\xi \in \mathcal{U} : \xi \in \pi_S(\mathcal{C}_n(\epsilon)) \text{ for infinitely many } n \in \mathbb{N}\}$$

we now have

$$\dim_{\mathbb{H}}(C_\infty(\epsilon)) \leq \min_{s \in (0, 1)} \tilde{\beta}(s) = \hat{\beta}(1).$$

Hence, it remains to show that

$$\mathcal{D}_\sim \cap \mathcal{D}_\sharp^* \subset C_\infty(\epsilon) \quad \text{for all } \epsilon > 0.$$

For this, let $x \in \Sigma$ be such that $\xi := \pi_S(x) \in \mathcal{D}_\sim \cap \mathcal{D}_\sharp^*$. By Proposition 2.8, there exist strictly increasing sequences $(n_m)_{m \in \mathbb{N}}$ and $(k_m)_{m \in \mathbb{N}}$ of positive integers such that x has a d -block of length k_m at the n_m th level and

$$e^{\mathcal{S}_{n_m} \chi(x) + k_m \alpha} \gg 1 \quad \text{for each } m \in \mathbb{N}.$$

By setting $\ell(n_m) := \lfloor \mathcal{S}_{n_m} \chi(x) \rfloor$, it follows $\exp(k_m) \gg \exp(-\ell(n_m)/\alpha)$. Hence, for each $\epsilon > 0$ and for each m sufficiently large, we have $k_m \geq -\ell(n_m)(1 - \epsilon)/\alpha$. It follows that $\xi \in C_\infty(\epsilon)$, which finishes the proof of the upper bound.

2.3. The lower bound. In this section we show that the Hausdorff dimension of each of the sets \mathcal{D}_\sim and \mathcal{D}_∞ is bounded below by $\tilde{\beta}(s_0)$. Clearly, combining this with the results of the previous section will then complete the proof of Theorem 1.1. Let us begin by showing that

$$\dim_{\mathbb{H}}(\mathcal{D}_\sim) \geq \tilde{\beta}(s_0).$$

Recall that μ_s refers to the equilibrium measure for the potential $s\varphi + \beta(s)\psi$, and that s_0 is chosen so that

$$\beta'(s_0) = -\frac{\int \varphi d\mu_{s_0}}{\int \psi d\mu_{s_0}} = -1.$$

This implies that

$$0 = \int \psi d\mu_{s_0} - \int \varphi d\mu_{s_0} = \int \chi d\mu_{s_0}.$$

By the variational principle, we have

$$h(\mu_{s_0}) + s_0 \int \varphi d\mu_{s_0} + \beta(s_0) \int \psi d\mu_{s_0} = 0,$$

and hence

$$\frac{h(\mu_{s_0})}{-\int \varphi d\mu_{s_0}} = \beta(s_0) + s_0 = \tilde{\beta}(s_0).$$

Since we are in the expanding case, we can use Young's formula (see [14, 23]) to deduce that $\dim_{\mathbb{H}}(\pi_S(\mu_{s_0})) = \tilde{\beta}(s_0)$. The lower bound for the Hausdorff dimension of \mathcal{D}_\sim now follows from combining Corollary 2.5 with the following lemma.

LEMMA 2.9. *For μ_{s_0} -almost every $x \in \Sigma$ we have*

$$\liminf_{n \rightarrow \infty} e^{\mathcal{S}_n \chi(x)} = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} e^{\mathcal{S}_n \chi(x)} = \infty.$$

Proof. Note that $\int \chi d\mu_{s_0} = 0$. Thus, by the law of the iterated logarithm [6] there exists a constant $C > 0$ such that for μ_{s_0} -almost all $x \in \Sigma$ we have

$$\liminf_{n \rightarrow \infty} \frac{\mathcal{S}_n \chi(x)}{\sqrt{n \log \log n}} = -C \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{\mathcal{S}_n \chi(x)}{\sqrt{n \log \log n}} = C.$$

From this the conclusion follows. ■

Lemma 2.9 implies that $\pi_S(x) \in \mathcal{D}_\sim$ for μ_{s_0} -almost every $x \in \Sigma$, and hence

$$\dim_{\mathbb{H}}(\mathcal{D}_\sim) \geq \dim_{\mathbb{H}}(\pi_S(\mu_{s_0})) = \tilde{\beta}(s_0).$$

Therefore, it remains to show that

$$\dim_{\mathbb{H}}(\mathcal{D}_\infty) \geq \tilde{\beta}(s_0).$$

For this, we consider the set of the equilibrium measures $\{\mu_s : s > s_0\}$.

LEMMA 2.10. *For $s > s_0$, we have*

$$\int \chi d\mu_s > 0.$$

Proof. Since β is strictly convex, $s > s_0$ implies that $\beta'(s) > -1$. This gives

$$\frac{\int \varphi d\mu_s}{\int \psi d\mu_s} = -\beta'(s) < 1,$$

and hence $\int \chi d\mu_s > 0$. ■

Lemma 2.10 implies that for μ_s -almost every $x \in \Sigma$ we have (recall that we are assuming that $s > s_0$)

$$\lim_{n \rightarrow \infty} e^{\mathcal{S}_n \chi(x)} = \infty.$$

For the following lemma we introduce the following notations. For $x = (x_1 x_2 \dots) \in \Sigma$, $k, n \in \mathbb{N}$ and $i \in \{1, d\}$, let $k_n(x) := k$ if x has an i -block of length k at the n th level, and set $k_n(x) := 0$ if $x_{n+1} \notin \{1, d\}$. We then have the following routine Khinchin-type estimate, where $\kappa_{i,s} := -(s\varphi(\underline{i}) + \beta(s)\psi(\underline{i}))^{-1} > 0$ and $\kappa_s := \min\{\kappa_{i,s} : i = 1, d\}$.

LEMMA 2.11. *For μ_s -almost every $x \in \Sigma$, we have*

$$\limsup_{n \rightarrow \infty} \frac{k_n(x)}{\log n} \leq \kappa_s.$$

Proof. Let $\mathcal{C}_n^* := \{[\omega] : \omega \in A^n\}$ and recall

$$\sum_{C \in \mathcal{C}_n^*} \exp(\sup_{x \in C} \mathcal{S}_n(s\varphi + \beta(s)\psi)(x)) \asymp 1$$

for all $n \in \mathbb{N}$. For $\epsilon > 0$, let $k_{\epsilon,i,n} := \lfloor (1 + \epsilon)\kappa_{i,s} \log n \rfloor$. Then

$$\sum_{n \in \mathbb{N}} \sum_{[x_1 \dots x_n] \in \mathcal{C}_n^*} e^{\sup_{x \in [x_1 \dots x_n]_{k_{\epsilon,i,n}}} \mathcal{S}_{n+k_{\epsilon,i,n}}(s\varphi + \beta(s)\psi)(x)} \ll \sum_{n \in \mathbb{N}} n^{-(1+\epsilon)}.$$

Hence, by the Borel–Cantelli Lemma, the set of elements in Σ which lie in cylinder sets of the form $[x_1 \dots x_n]_{k_{\epsilon,i,n}}$ for infinitely many $n \in \mathbb{N}$ has μ_s -measure equal to zero. By passing to the complement of this limsup-set, the statement in the lemma follows. ■

We can now complete the proof of Theorem 1.1 as follows. By Lemma 2.3, there exists a constant $c > 0$ such that for each $x = (x_1 x_2 \dots) \in \Sigma$ and for each sequence $(\eta_n)_n$ in \mathcal{U} tending to $\xi := \pi_S(x)$,

$$\liminf_{n \rightarrow \infty} D_\Theta(\xi, \eta_n) \geq c \liminf_{n \rightarrow \infty} e^{\mathcal{S}_n \chi(x)} e^{-kn\delta}.$$

Moreover, using Lemma 2.10 and the ergodicity of μ_s , it follows that for μ_s -almost every $x \in \Sigma$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathcal{S}_n \chi(x) = \int \chi d\mu_s =: c_\chi(x) > 0.$$

Combining this with Lemma 2.11, it follows that for μ_s -almost every $x \in \Sigma$, with $\xi = \pi_S(x)$, we have

$$\liminf_{n \rightarrow \infty} D_\Theta(\xi, \eta_n) \geq c \liminf_{n \rightarrow \infty} e^{\mathcal{S}_n \chi(x)} e^{-\delta k_n(x)} \geq c \liminf_{n \rightarrow \infty} e^{nc_\chi(x)} n^{-\delta \kappa_s} = \infty.$$

This implies

$$\lim_{n \rightarrow \infty} D_{\Theta}(\xi, \eta_n) = \infty \quad \mu_s\text{-almost everywhere.}$$

Since π_S is bijective except on a countable number of points, we now conclude that for all $s > s_0$ we have

$$\dim_{\mathbb{H}}(\mathcal{D}_{\infty}) \geq \dim_{\mathbb{H}}(\pi_S(\mu_s)) = -\widehat{\beta}(-\beta'(s))/\beta'(s).$$

To complete the proof, simply note that $-\widehat{\beta}(-\beta'(s))/\beta'(s) \nearrow \widetilde{\beta}(s_0)$ for $s \searrow s_0$. This finishes the proof of Theorem 1.1.

3. Proof of Theorem 1.2. If (U, S) and (U, T) are $C^{1+\epsilon}$ conjugate, then clearly $\mathcal{D}_{\sim}(S, T) = \emptyset$, and hence $\dim_{\mathbb{H}}(\mathcal{D}_{\sim}(S, T)) = 0$. This gives one direction of the equivalence in Theorem 1.2.

For the other direction, assume that $\dim_{\mathbb{H}}(\mathcal{D}_{\sim}(S, T)) = 0$. Then Theorem 1.1 implies that φ and ψ are cohomologically dependent. That is, there exist $b, c \in \mathbb{R} \setminus \{0\}$ and a Hölder continuous function $u : \Sigma \rightarrow \mathbb{R}$ such that

$$b\varphi + c\psi = u - u \circ \sigma.$$

Then for all $s \in \mathbb{R}$,

$$P(s\varphi - (b/c)\beta(s)\varphi) = P((s - (b/c)\beta(s))\varphi) = 0,$$

and hence $\beta(s) = (s - 1)c/b$. Combining this with $\beta(0) = 1$ shows that $b/c = -1$, and therefore

$$\psi - \varphi = \chi = v - v \circ \sigma$$

for some Hölder continuous function $v : \Sigma \rightarrow \mathbb{R}$. Note that in particular $\psi(\underline{i}) = \varphi(\underline{i})$ for each $i \in \{1, d\}$. Combining this with Proposition 2.1, it follows that uniformly for all $\xi, \eta \in \mathcal{U}$ we have

$$D_{\Theta}(\xi, \eta) \asymp 1.$$

This shows that there exists a constant $c_0 > 1$ such that for all $\xi \in \mathcal{U}$,

$$c_0^{-1} < \liminf_{\eta \rightarrow \xi} D_{\Theta}(\xi, \eta) \leq \limsup_{\eta \rightarrow \xi} D_{\Theta}(\xi, \eta) < c_0.$$

Since the derivative of Θ exists Lebesgue almost everywhere, it follows that $\Theta'(\xi)$ is uniformly bounded away from zero and infinity for Lebesgue almost every $\xi \in \mathcal{U}$. We can now complete the proof by an argument similar to the one in [3] (see the introduction for a statement of the main result of [3]). We have split the discussion into four steps. Here, for $c \in \mathbb{R}$, we let $f_c : \mathbb{R} \rightarrow \mathbb{R}$ denote the multiplication map given by $x \mapsto cx$, and we set $\sigma_0 := S'(0) = T'(0)$. Note that, since $\psi(\underline{1}) = \varphi(\underline{1})$, we clearly have $S'(0) = T'(0)$.

Linearisation. For each $n \in \mathbb{N}$, let S_1^{-n} and T_1^{-n} denote the inverse branches of S^n and T^n , respectively, such that 0 is contained in $S_1^{-n}(\mathcal{U})$ and

$T_1^{-n}(\mathcal{U})$. Using the bounded distortion property and the fact that $(S_1^{-n})'$ and $(T_1^{-n})'$ are uniformly Hölder continuous, we see, by Arzelà–Ascoli, that there exist subsequences of $(f_{\sigma_0^n} \circ S_1^{-n})_{n \in \mathbb{N}}$ and $(f_{\sigma_0^n} \circ T_1^{-n})_{n \in \mathbb{N}}$ which converge uniformly on \mathcal{U} to $C^{1+\epsilon}$ diffeomorphisms γ_S and γ_T respectively. Note that clearly $\gamma_S \circ S = f_{\sigma_0} \circ \gamma_S$ and $\gamma_T \circ T = f_{\sigma_0} \circ \gamma_T$.

Differentiation. The uniform Hölder continuity of $(S_1^{-n})'$ and $(T_1^{-n})'$ and the fact that the conjugacy Θ is bi-Lipschitz imply that the right derivative of Θ at zero exists and that it has a finite and positive value.

Localisation. We have

$$\begin{aligned} f_{\sigma_0} \circ \gamma_T \circ \Theta \circ \gamma_S^{-1} &= \gamma_T \circ T \circ \Theta \circ \gamma_S^{-1} = \gamma_T \circ \Theta \circ S \circ \gamma_S^{-1} \\ &= \gamma_T \circ \Theta \circ \gamma_S^{-1} \circ \gamma_S \circ S \circ \gamma_S^{-1} \\ &= \gamma_T \circ \Theta \circ \gamma_S^{-1} \circ f_{\sigma_0} \circ \gamma_S \circ \gamma_S^{-1} \\ &= \gamma_T \circ \Theta \circ \gamma_S^{-1} \circ f_{\sigma_0}, \end{aligned}$$

which shows that $\gamma_T \circ \Theta \circ \gamma_S^{-1}$ commutes with f_{σ_0} . Using this and the differentiability of Θ at 0, we deduce that $\gamma_T \circ \Theta \circ \gamma_S^{-1} = (f_{\sigma_0})^n \circ \gamma_T \circ \Theta \circ \gamma_S^{-1} \circ (f_{1/\sigma_0})^n$ on the domain of γ_S . Therefore, for ξ in this domain,

$$\gamma_T \circ \Theta \circ \gamma_S^{-1}(\xi) = \frac{\gamma_T \circ \Theta \circ \gamma_S^{-1}(\sigma_0^{-n} \cdot \xi)}{\sigma_0^{-n} \cdot \xi} \cdot \xi \rightarrow \kappa_0 \cdot \xi \quad \text{as } n \rightarrow \infty,$$

where $\kappa_0 > 0$ denotes the right derivative of $\gamma_T \circ \Theta \circ \gamma_S^{-1}$ at zero. It now follows that there exists $\delta > 0$ such that $\Theta|_{[0, \delta]}$ is a $C^{1+\epsilon}$ diffeomorphism.

Globalisation. Let $n \in \mathbb{N}$ be chosen such that $S_1^{-n}(\mathcal{U}) \subset [0, \delta]$. Since $\Theta = T^n \circ \Theta \circ S_1^{-n}$, it follows that $\Theta : \mathcal{U} \rightarrow \mathcal{U}$ is a $C^{1+\epsilon}$ diffeomorphism. This completes the proof of the main part of Theorem 1.2.

In order to prove the Hölder regularity of Θ , as claimed in part (2) of Theorem 1.2, let $\xi, \eta \in \mathcal{U}$ and put $\rho := \sup_{s \in \mathbb{R}} (-\beta'(s)) > 0$. Clearly, $\mathcal{S}_n \psi(x) / \mathcal{S}_n \varphi(x) > 1/\rho$ for all $x \in \Sigma$ and $n \in \mathbb{N}$. Without loss of generality, we can assume that $\xi = \pi_S(x_1 x_2 \dots) < \eta = \pi_S(y_1 y_2 \dots)$ and $\eta \in \pi_S([x_1 \dots x_n] \setminus [x_1 \dots x_{n+1}])$. Moreover, let us only consider the case where x has a strict d -block of length k at the n th level and y has a strict 1-block of length l at the $(n + 1)$ th level, for some $k, l, n \in \mathbb{N}$. Then there exists a uniform constant $C > 0$ such that

$$\begin{aligned} |\Theta(\xi) - \Theta(\eta)| &\leq C(e^{\mathcal{S}_{n+k}\psi(x)} + e^{\mathcal{S}_{n+l}\psi(y)}) \\ &= C\left(e^{\frac{\mathcal{S}_{n+k}\psi(x)}{\mathcal{S}_{n+k}\varphi(x)} \mathcal{S}_{n+k}\varphi(x)} + e^{\frac{\mathcal{S}_{n+l}\psi(y)}{\mathcal{S}_{n+l}\varphi(y)} \mathcal{S}_{n+l}\varphi(y)}\right) \\ &\leq C(e^{\rho^{-1} \mathcal{S}_{n+k}\varphi(x)} + e^{\rho^{-1} \mathcal{S}_{n+l}\varphi(y)}) \\ &\leq 2C(e^{\mathcal{S}_{n+k}\varphi(x)} + e^{\mathcal{S}_{n+l}\varphi(y)})^{1/\rho} \leq 2C|\xi - \eta|^{1/\rho}. \end{aligned}$$

Note that if one of the blocks has infinite word length, then one has to use approximations of this block by words of finite lengths.

It remains to show that if S and T are cohomologically independent, then Θ has to be singular with respect to the Lebesgue measure λ . For this, note that on the unit interval without the boundary points of all refinements of the Markov partition we have $\Theta = \pi_T \circ \pi_S^{-1}$. Therefore, it is sufficient to show that the measure $\lambda \circ \Theta$, whose distribution function is equal to Θ , is singular with respect to λ . Since $\mu_\psi \circ \pi_T^{-1}$, $\mu_\varphi \circ \pi_S^{-1}$ and λ are all in the same measure class, it follows that $\lambda \circ \Theta$ is absolutely continuous with respect to $\mu_\psi \circ \pi_T^{-1} \circ \pi_T \circ \pi_S^{-1} = \mu_\psi \circ \pi_S^{-1}$. On the other hand, since S and T are cohomologically independent, $\mu_\psi \circ \pi_S^{-1}$ is singular with respect to $\mu_\varphi \circ \pi_S^{-1}$. This finishes the proof of Theorem 1.2.

4. Examples. In this section we consider two families of examples: the Salem family and the sine family. For the Salem family we will see in Section 5 that it gives rise to conjugacies whose sets of nondifferentiability have Hausdorff dimensions arbitrarily close to zero.

EXAMPLE 1 (the Salem family). Let us consider a class of examples studied by Salem in [20]. Namely, we consider the family of conjugacy maps $\{\Theta_\tau : \tau \in (0, 1) \setminus \{1/2\}\}$ which arises from the following endomorphisms of \mathcal{U} . For $\xi \in \mathcal{U}$, we define

$$T(\xi) := 2\xi \bmod 1 \quad \text{and} \quad S_\tau(\xi) := \begin{cases} \xi/\tau & \text{if } 0 \leq \xi \leq \tau, \\ (\xi - \tau)/(1 - \tau) & \text{if } \tau < \xi \leq 1. \end{cases}$$

The maps $\Theta_\tau : [0, 1] \rightarrow [0, 1]$ are then given by $T \circ \Theta_\tau = \Theta_\tau \circ S_\tau$. One

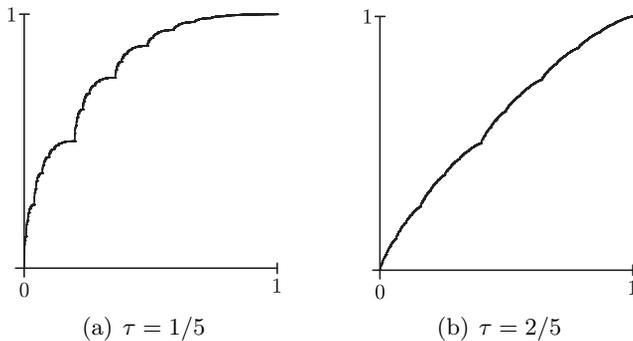


Fig. 1. The conjugating map Θ_τ for the Salem case

immediately verifies that Θ_τ is strictly monotone and has the property that $\Theta'_\tau(\xi) = 0$ for Lebesgue almost every $\xi \in \mathcal{U}$. Note that the conjugacies considered in [20] are in fact dual to the ones which we consider here. However, this has no effect on the Hausdorff dimension of $\mathcal{D}_\sim(S_\tau, T)$ (see Re-

mark 1.6(2)), and our conjugacies have the advantage that they allow us to determine β_τ and $\dim_{\mathbb{H}}(\mathcal{D}_\sim(S_\tau, T))$ rather explicitly. For this, first note that in the current situation the potential functions φ_τ and ψ are given for $x = (x_1 x_2 \dots) \in \Sigma$ by

$$\psi(x) = -\log 2 \quad \text{and} \quad \varphi_\tau(x) = \begin{cases} \log \tau & \text{if } x_1 = 1, \\ \log(1 - \tau) & \text{if } x_1 = 2. \end{cases}$$

The function β_τ is defined implicitly by $P(s\varphi_\tau + \beta(t)\psi) = 0$. Since we have $\exp(s \log \tau - \beta_\tau(s) \log 2) + \exp(s \log(1 - \tau) - \beta_\tau(s) \log 2) = 1$, an elementary calculation gives that β_τ is given explicitly by

$$\beta_\tau(s) = \frac{P(s\varphi_\tau)}{\log 2} = \log_2(\tau^s + (1 - \tau)^s) \quad \text{for each } s \in \mathbb{R}.$$

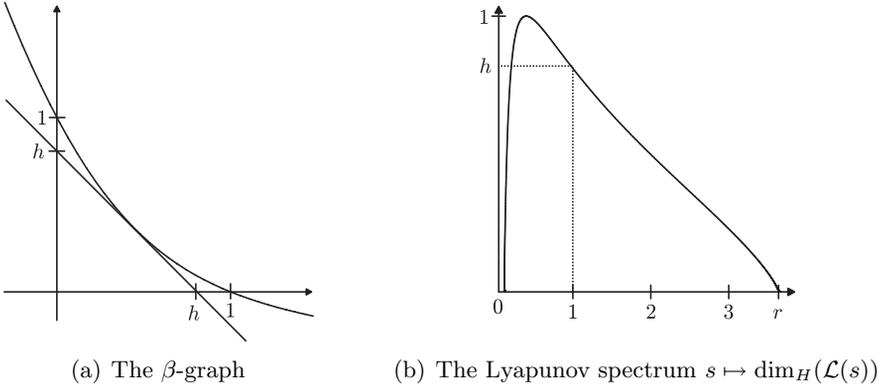


Fig. 2. The β -graph and the graph of the Lyapunov dimension spectrum in the Salem case for $\tau = 0.08$; in both figures $h = 0.8107\dots$ denotes the Hausdorff dimension of \mathcal{D}_\sim . The conjugacy Θ_τ is $1/r$ -Hölder regular.

In order to compute $\dim_{\mathbb{H}}(\mathcal{D}_\sim(S_\tau, T))$, let ν_τ be the $(p_\tau, 1 - p_\tau)$ -Bernoulli measure such $\int \psi d\nu_\tau / \int \varphi_\tau d\nu_\tau = 1$. We then have

$$1 = \frac{\int \psi d\nu_\tau}{\int \varphi_\tau d\nu_\tau} = -p_\tau \log_2 \tau - (1 - p_\tau) \log_2(1 - \tau),$$

and hence

$$p_\tau = \frac{1 + \log_2(1 - \tau)}{\log_2 \tau - \log_2(1 - \tau)}.$$

One then immediately verifies that the supremum in Remark 1.6(2) is attained for $\mu = \nu_\tau$, and hence

$$\dim_{\mathbb{H}}(\mathcal{D}_\sim(S_\tau, T)) = -p_\tau \log_2 p_\tau - (1 - p_\tau) \log_2(1 - p_\tau).$$

The graphs of β_τ and of the corresponding dimension spectrum are given in Fig. 2. Also, Fig. 3(b) shows $\dim_{\mathbb{H}}(\mathcal{D}_\sim(S_\tau, T))$ in dependence on τ .

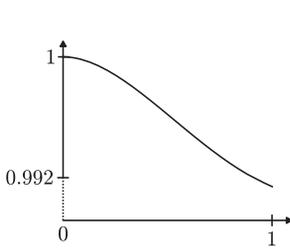
Finally, let us mention that one can also explicitly calculate the number $s_0(S_\tau)$ which is determined by $\beta'_\tau(s_0(S_\tau)) = -1$. A straightforward calculation gives

$$s_0(S_\tau) = (\log(\tau^{-1} - 1))^{-1} \log\left(\frac{\log(2\tau)}{\log(2/(1-\tau))}\right).$$

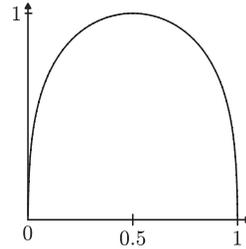
EXAMPLE 2 (the sine family). Let T be given as in the previous example, and for each $\tau \in (0, 1)$ let the map $R_\tau : \mathcal{U} \rightarrow \mathcal{U}$ be defined by

$$R_\tau(\xi) := 2\xi + \frac{\tau}{2\pi} \sin(2\pi\xi) \pmod 1 \quad \text{for each } \xi \in \mathcal{U}.$$

The associated conjugacies Ψ_τ are given by $\Psi_\tau \circ R_\tau = T \circ \Psi_\tau$ (see Fig. 4). We can then use Theorem 1.1 to compute the Hausdorff dimension of the set $\mathcal{D}_\sim(R_\tau, T)$ of points at which Ψ_τ is not differentiable in the generalised sense. This is plotted as a graph in Fig 3. (Note that taking the conjugacy in the other direction would yield exactly the same result.)

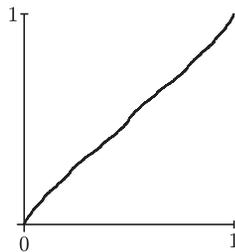


(a) The graph of $\tau \mapsto \dim D_\sim(R_\tau, T)$ for the sine family with parameter τ

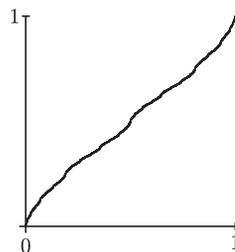


(b) The graph of $\tau \mapsto \dim D_\sim(S_\tau, T)$ for the Salem family with parameter τ

Fig. 3. The two dimension spectra



(a) $\tau = 0.4$



(b) $\tau = 0.8$

Fig. 4. The graphs of the conjugating maps Ψ_τ for $\tau = 0.4$ and $\tau = 0.8$ in the sine-family example

5. Proofs of Propositions 1.4 and 1.5

Proof of Proposition 1.4. We start by observing that the Hausdorff dimension of the set \mathcal{D}_\sim depends regularly on the expanding maps. Let $T_\tau : \mathcal{U} \rightarrow \mathcal{U}$ be elements of the Banach manifold of the C^k family of expanding maps, with a C^k dependence on $\tau \in (-\epsilon, \epsilon)$, say, and assume that T_0 is the usual d -to-1 linear expanding map. Let $0 = a_0^{(\tau)} < a_1^{(\tau)} < \dots < a_d^{(\tau)} = 1$ denote the T_τ -preimages of zero. For each $\alpha > 0$, we then define the operator $\mathcal{T}_\tau : C^\alpha(\mathcal{U}, \mathbb{R}) \rightarrow C^\alpha(\mathcal{U}, \mathbb{R})$ on the space of α -Hölder continuous functions (see e.g. [8]) by

$$(\mathcal{T}_\tau h)(\xi) := \frac{1}{d} h(\{T_\tau(\xi)\}) + \frac{j}{d} \quad \text{for each } \xi \in [a_j^{(\tau)}, a_{j+1}^{(\tau)}], j \in \{0, \dots, d-1\}.$$

Also, with $\|h\|_\infty$ denoting the usual supremum norm, we define a norm $\|\cdot\|$ on $C^\alpha(\mathcal{U}, \mathbb{R})$ by

$$\|h\| := \sup_{\xi \neq \eta} \frac{|h(\xi) - h(\eta)|}{|\xi - \eta|^\alpha} + \|h\|_\infty.$$

We observe that on each of the intervals $[a_j^{(\tau)}, a_{j+1}^{(\tau)}]$ we have

$$|\mathcal{T}_\tau h_1(\xi) - \mathcal{T}_\tau h_2(\xi)| \leq \frac{1}{d} |h_1(T_\tau(\xi)) - h_2(T_\tau(\xi))| \leq \frac{1}{d} \|h_1 - h_2\|_\infty$$

and

$$\begin{aligned} |\mathcal{T}_\tau(h_1 - h_2)(\xi) - \mathcal{T}_\tau(h_1 - h_2)(\eta)| &\leq \frac{1}{d} \|h_1 - h_2\|_{C^\alpha} |T_\tau(\xi) - T_\tau(\eta)|^\alpha \\ &\leq \frac{1}{d} \|h_1 - h_2\|_{C^\alpha} |T_\tau(\xi) - T_\tau(\eta)|^\alpha \\ &\leq \left(\frac{1}{d} \|h_1 - h_2\|_{C^\alpha} \|T_\tau\|_{C^1}^\alpha \right) |\xi - \eta|^\alpha. \end{aligned}$$

In particular, for $\alpha > 0$ sufficiently small, \mathcal{T}_τ is a contraction with respect to $\|\cdot\|$. Moreover, $I - \mathcal{T}_\tau : C^\alpha(\mathcal{U}, \mathbb{R}) \rightarrow C^\alpha(\mathcal{U}, \mathbb{R})$ is invertible, and by the Implicit Function Theorem there exists a C^k family $\{h_\tau \in C^\alpha(\mathcal{U}, \mathcal{U}) : \tau \in (-\epsilon, \epsilon)\}$ such that h_0 is the identity map and $\mathcal{T}_\tau h_\tau = h_\tau$.

Let us consider the map $H_\tau : (-\epsilon, \epsilon) \rightarrow C^{k-1}(\mathcal{U}) \times C^\alpha(\mathcal{U})$ given by $H_\tau(\tau) := (\log |T'_\tau|, h_\tau)$. Clearly, it is C^{k-1} as a map on Banach spaces. Also, we define the composition operator $\mathcal{O} : C^{k-1}(\mathcal{U}) \times C^\alpha(\mathcal{U}) \rightarrow C^\alpha(\mathcal{U})$ by $\mathcal{O}(f, g) := f \circ g$, which is C^{k-2} , by a result of [5]. We then consider the image of H_τ under \mathcal{O} , that is,

$$\mathcal{O} \circ H_\tau : \tau \mapsto (\log |T'_\tau|, h_\tau) \mapsto \mathcal{O}(\log |T'_\tau|, h_\tau) = \log |T'_\tau| \circ h_\tau \in C^\alpha,$$

which is again C^{k-2} [5]. (Note that if instead we would consider $\tilde{\mathcal{O}} : C^{k-1}(\mathcal{U}) \times C^0(\mathcal{U}) \rightarrow C^0(\mathcal{U})$, then $\tilde{\mathcal{O}}(H_\tau)$ would be C^{k-1} ; but we need to work with Hölder functions, which causes the loss of an extra derivative.)

Now consider the potential function $\varphi_\tau : \xi \mapsto -\log |T'_\tau(h_\tau(\pi_{T_\tau}(\xi)))|$ defined on \mathcal{U} , and then let $\beta_\tau(s)$ be defined implicitly by

$$P(-s \log |T'_\tau| + \beta_\tau(s) \log |T'_0|) = 0.$$

Since the pressure function is analytic, the Implicit Function Theorem implies that the function $\tau \mapsto \beta_\tau$ is analytic. Also, it follows that $\tau \mapsto \dim_{\mathbb{H}}(\mathcal{D}_\sim(T_\tau, T_0))$ is a C^{k-2} function (for an example see Fig. 3). This completes the proof of Proposition 1.4.

Proof of Proposition 1.5. The aim is to show that there exists a conjugacy between two elements of the space $C^2(\mathbb{S}^1)$ of C^2 expanding circle maps such that the Hausdorff dimension of the set of points at which this conjugacy is nondifferentiable in the generalised sense is arbitrarily close to 0. We start by considering the Salem case but where the maps are defined on the circle \mathbb{S}^1 . For ease of exposition, we use the same notation and let $T : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ and $S_\tau : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ refer to the circle maps which correspond to the interval maps defined in Example 1. The corresponding conjugacy Θ_τ is given as before by $T \circ \Theta_\tau = \Theta_\tau \circ S_\tau$. From our analysis in Example 1 it is clear that $\dim_{\mathbb{H}}(\mathcal{D}_\sim(S_\tau, T))$ tends to zero for τ tending to zero (see Fig. 2). However, whereas T is a C^2 map of the circle, S_τ is clearly not (although it is always piecewise expanding C^2 when viewed as a map of \mathcal{U} into itself). So, in order to find a C^2 example, we have to apply some perturbations to S_τ . For this, let β_τ and $\psi, \varphi_\tau : \Sigma \rightarrow \mathbb{R}$ be given as in Example 1. As before we choose $s_0(S_\tau)$ satisfying $\beta'_\tau(s_0(S_\tau)) = -1$. For the remaining part of the proof, let $\tau \in (0, 1) \setminus \{1/2\}$ be fixed.

C^2 denseness. We use the metric d_{KL} considered by Keller and Liverani in [10]. This metric is given, for expanding piecewise C^2 maps F and G of the unit interval \mathcal{U} into itself, by

$$d_{\text{KL}}(F, G) := \inf\{\gamma > 0 \mid \exists X \subseteq \mathcal{U} \exists \text{ a diffeomorphism } H : \mathcal{U} \rightarrow \mathcal{U} \text{ such that} \\ \lambda(X) > 1 - \gamma, G|_X = F \circ H|_X \text{ and} \\ \forall \xi \in \mathcal{U} : |H(\xi) - \xi| < \gamma, |1 - (H^{-1})'(H(\xi))| < \gamma\}.$$

One immediately verifies that there exists a sequence $(S^{(n)})_{n \in \mathbb{N}}$ of functions in $C^2(\mathbb{S}^1)$ such that $\lim_{n \rightarrow \infty} d_{\text{KL}}(S^{(n)}, S_\tau) = 0$, where the $S^{(n)}$ are viewed as interval maps.

Norms and operators. Let $B_0(\mathcal{U}) := \{f : \mathcal{U} \rightarrow \mathbb{R} \mid \|f\|_0 < \infty\}$ be the Banach space with the combined norm $\|\cdot\|_0$ given by $\|f\|_0 := \|f\|_1 + \|f\|_{BV}$, where $\|\cdot\|_1$ denotes the L^1 norm and $\|\cdot\|_{BV}$ the bounded variation seminorm, given by $\sup\{\sum_{i=1}^n |f(\xi_{i+1}) - f(\xi_i)| \mid 0 \leq x_1 < \dots < x_n \leq 1, n \in \mathbb{N}\}$. Also, let the weak operator norm $\|\cdot\|_W$ be given by $\|\mathcal{L}\|_W := \sup\{\|\mathcal{L}(g)\|_1 \mid g \in B_0(\mathcal{U}), \|g\|_{BV} \leq 1\}$. Finally, for an expanding map $S : \mathcal{U} \rightarrow \mathcal{U}$ we define the transfer operator $\mathcal{L}_S : B_0(\mathcal{U}) \rightarrow B_0(\mathcal{U})$, for

$g \in B_0(\mathcal{U})$ and $\xi \in \mathcal{U}$, by

$$\mathcal{L}_S g(\xi) = \sum_{S(\eta)=\xi} |S'(\eta)|^{-1} g(\eta).$$

Continuity. Firstly, note that it can be shown that $\lim_{n \rightarrow \infty} \|\mathcal{L}_{S^{(n)}} - \mathcal{L}_{S_\tau}\|_W = 0$ (see comment (a) on page 143 of [10]). Furthermore, by [10, Corollary 1], for each $t \in \mathbb{R}$ fixed, the leading eigenvalues of the operators $\mathcal{L}_{S^{(n)}}$ converge to the leading eigenvalue of \mathcal{L}_{S_τ} . That is,

$$\lim_{n \rightarrow \infty} P(t \log |(S^{(n)})'|) = P(t \log |S'_\tau|).$$

Local uniform convergence. Recall that the map given by

$$t \mapsto \beta_{S^{(n)}}(t) := P(-t \log(S^{(n)})') / \log 2$$

is differentiable and convex. Using the above ‘Continuity’, we then have $\lim_{n \rightarrow \infty} \beta_{S^{(n)}}(t) = \beta_{S_\tau}(t)$ for each $t \in \mathbb{R}$ fixed. Since pointwise convergence of sequences of differentiable convex functions implies local uniform convergence (see [19, Theorem 10.8]), we now conclude that

$$\lim_{n \rightarrow \infty} \beta_{S^{(n)}}(s_0(S^{(n)})) + s_0(S^{(n)}) = \beta_{S_\tau}(s_0(S_\tau)) + s_0(S_\tau).$$

Since $\beta_S(s_0(S)) + s_0(S) = \dim_{\text{H}}(\mathcal{D}_{\sim}(S, T))$, this finishes the proof of the proposition.

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