Dynamics of circle maps with flat spots

by

Jacek Graczyk (Orsay)

For M. Misiurewicz's 60th birthday

Abstract. We study a certain class of weakly order preserving, non-invertible circle maps with irrational rotation numbers and exactly one flat interval. For this class of circle maps we explain the geometric and dynamic structure of orbits. In particular, we formulate the so called upper and lower scaling rules which show an asymmetric and double exponential decay of geometry.

1. Introduction

1.1. Motivation. The objective of our presentation is to give a possibly complete description of the dynamics of a certain class of weakly order preserving, non-invertible circle maps which appear in a natural way (upper maps) while studying the dependence of the rotation interval on the parameter value for one-parameter families of endomorphisms of the circle (see [4]). A prototype family of this type is the sine family which constitutes one of the most famous models of transition to chaos:

$$(1) \qquad g_t(x) = x + (c/2\pi)\sin(2\pi x) + t \pmod 1.$$

Family (1) models a variety of physical phenomena and was studied intensively by mathematicians as well as physicists.

For $c \leq 1$ the maps g_t are homeomorphisms of the circle to itself and the rotation number $\rho(g_t)$ of g_t constitutes a topological invariant that measures the rate at which the orbit of a point wraps around the circle.

In general, for a weakly order preserving circle map the concept originated with Poincaré and is defined in terms of a lift f to the real line as:

$$\rho = \lim_{n \to \infty} \frac{f^n(x)}{n}$$

[267]

where x is any initial point. Both f and the rotation number are defined uniquely up to shifts by integers.

When $c > 1$ the maps g_t are not invertible and have no rotation number (chaos). However, for these maps the notion of rotation interval can be introduced as a natural extension of the concept of rotation number for homeomorphisms [2]. There is an important result of [3] which says that the upper endpoint of the rotation interval is equal to the rotation number of a certain weakly order preserving circle map (upper map) which satisfies all our assumptions.

Assumptions and notation. We fix counterclockwise orientation of the circle. The metric on S^1 is given by the natural projection

$$\Pi(x) = \exp(2\pi i x)$$

from the real line. Continuous circle maps can be lifted to the universal cover and written as continuous real maps obeying $f(x+1) - f(x) \in \mathbb{Z}$ for every real x. In this paper we will not make a strong distinction between objects defined on the circle and their lifts to the real line using both concepts according to our needs.

We denote by $|J|$ the length of the interval $J \subset S^1$ and by $\mathrm{dist}(x, y)$ or simply $|x - y|$ the length of the shortest closed interval which contains both x and y.

The distance $\mathrm{dist}(X, Y)$ between two sets X and Y is defined as

$$\inf\{\mathrm{dist}(x, y) : x \in X,\, y \in Y\}.$$

The left and the right endpoints of an interval J will be denoted by $l(J)$ and $r(J)$ respectively.

Here is a list of the hypotheses.

HYPOTHESES.

1. We consider weakly order preserving circle maps f, at least twice continuously differentiable except for at most two points (endpoints of a flat interval).
2. The first derivative of f is everywhere positive except for the closure of an open non-degenerate interval I (the flat interval) on which it is equal to zero.
3. The right-sided derivative of f at $r(I)$ is different from zero ($r(I)$ is not critical).
4. There exists a left-sided neighborhood of $l(I)$ and a real number $\ell > 1$ so that for every x from this neighborhood f can be written as $f(x) = -|\phi(x)|^\ell + f(I)$, where ϕ is a C^2 diffeomorphism on a neighborhood of $l(I)$.
5. The rotation number $\rho \in (0, 1)$ of f is irrational.

1.2. Preliminaries

Uniform constants. We will mean by a *uniform constant* a function on our class of maps which depends continuously on the quasi-symmetric norm of the map on $S^1 \setminus I$, the logarithm of the size of I, the order ℓ of the singularity of f at $l(I)$, the C^2 norm of f on $S^1 \setminus I$ and the lower bound of the first derivative on $S^1 \setminus (I \cup U)$, where U (see Fact 2.1 in the next subsection) is a left-sided neighborhood of $l(I)$ on which the second derivative is negative. Uniform positive constants will be usually denoted by the letters K, A, M with suitable subscripts, and we always assume $0 < \lambda < 1$.

Superexponential convergence. In our estimates we will often use a convergence which is faster than exponential. Typically, it will be double exponential convergence given by sequences of the form $\exp(-\exp(Kn))$ where K is a uniform constant.

DEFINITION 1.1. We say that a sequence of real numbers c_n tends to $c \in \mathbb{R}$ *uniformly superexponentially* if there exist uniform positive constants M and K such that for every $n \geq M$,

$$|c_n - c| \leq \exp(-\exp(Kn)).$$

Continued fractions and dynamics. The rotation number ρ can be written as an infinite continued fraction

$$\rho = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}},$$

where a_i are positive integers. If we cut off the portion of the continued fraction beyond the nth position, and write the resulting fraction in lowest terms as p_n/q_n then the numbers q_n for $n \geq 1$ satisfy the recurrence relation

$$q_{n+1} = a_{n+1}q_n + q_{n-1}, \qquad q_0 = 1, \, q_1 = a_1.$$

The number q_n is the iterate of the rotation by ρ for which the orbit of any point makes the *closest return* so far to the point itself.

By the Poincaré Theorem, maps from our class are semi-conjugate to a rotation. In particular, this implies the same order of orbits for both f and the rotation by ρ. The numbers q_n are called *closest returns*.

Continued fractions and partitions. We will use the preimages of the flat interval I to define a system of partitions of the circle.

Consider all $q_n + q_{n+1} - 1$ preimages of I and I itself together with all holes between successive preimages of I. Set, for n even,

$$\square_i^n = f^{-i}(r(f^{-q_n}(I)), l(I)) \quad \text{and} \quad \square_i^{n+1} = f^{-i}(r(I), l(f^{-q_{n+1}}(I))),$$

and for n odd,

$$\square_i^n = f^{-i}(r(I), l(f^{-q_n}(I))) \quad \text{and} \quad \square_i^{n+1} = f^{-i}(r(f^{-q_{n+1}}(I)), l(I)).$$

Thus, all holes of the nth partition can be divided into two groups:

- q_{n+1} "long" holes $\square_0^n, \ldots, \square_{q_{n+1}-1}^n$,
- q_n "short" holes $\square_0^{n+1}, \ldots, \square_{q_n-1}^{n+1}$.

The partition obtained by the above construction will be denoted by $\mathcal{A}_n$ and called the nth *dynamical partition* of the circle.

We will briefly explain the structure of the partitions. Take two subsequent dynamical partitions of order n and $n+1$. The latter is clearly a refinement of the former. All "short" holes of $\mathcal{A}_n$ become "long" holes of $\mathcal{A}_{n+1}$ while all "long" holes of $\mathcal{A}_n$ split into a_{n+2} preimages of I and a_{n+2} "long" holes and one "short" hole of the next partition $\mathcal{A}_{n+1}$:

$$\square_i^n = \bigcup_{j=1}^{a_{n+2}} f^{-i-q_n-jq_{n+1}}(I) \cup \bigcup_{j=0}^{a_{n+2}-1} \square_{i+q_n+jq_{n+1}}^{n+1} \cup \square_i^{n+2}.$$

Several of the proofs in the following will strongly depend on the relative positions of the points and intervals of the dynamical partitions. When reading the proofs the reader is advised to keep the following picture in mind, which shows some of these objects near the flat interval I. To denote the relevant objects in the picture we use temporary notation: $\underline{i}$ stands for $f^i(I)$ and $\underline{0}$ stands for I.
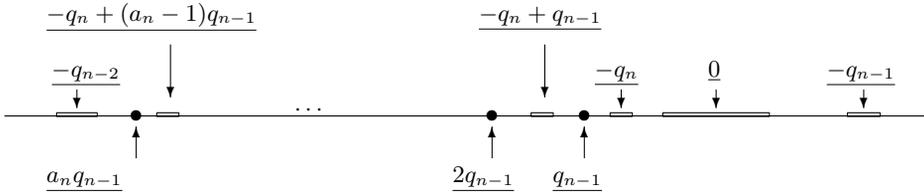
REMARK. Observe that $f^{q_n}(I)$ lies to the right of I for n even, and to the left of I for n odd. If we do not want to specify holes of the nth partition then we denote them briefly by $\square(n)$.

1.3. Presentation of results

Description and statement of results. We begin by recalling the definition of the set Ω.

DEFINITION 1.2. A point x belongs to the *set of non-wandering points* $\Omega(f)$ if and only if for any open neighborhood $V \ni x$ there exists an integer $n > 0$ such that the intersection of V and $f^n(V)$ is non-empty, where f^n stands for the nth composition of f.

Our goal is to describe both the behavior of high iterates of the map f and the geometric structure of the set Ω of non-wandering points.

Upper scaling rules. The crucial technical observation is stated as the First Basic Lemma (**FBL**) below. The scaling rules formulated there, by

which we mean the ratio of the distance between the flat interval and its two successive closest returns from the same side, give the upper bounds of the geometry of the Cantor set Ω. We prove that the scaling rules are at least superexponentially decreasing and do not depend on *even* coefficients in the continued fraction expansion of the rotation number. This reflects the hybrid dynamics of f which is topologically expansive, i.e. maps a proper interval $(S^1 \setminus I)$ onto the whole circle, critical of polynomial type at the left side of I, and non-degenerate at the right side of I.

Let p_n/q_n be the nth convergent to the continued fraction representation of the rotation number $\rho = [0, a_1, \ldots, a_n, \ldots]$. Put $\gamma(1) = 1$ and for all $n > 1$,

$$\gamma(n) := \gamma(\rho, n) = \prod_{0 < i \leq [n/2]} \left(1 + \frac{a_{2i+1}}{\ell}\right).$$

LEMMA 1.1 (First Basic Lemma). *Suppose that f satisfies the Hypotheses. There is a uniform constant $P > 0$ and a constant $0 < \lambda < 1$ such that for every $n \geq 1$:*

$$\text{dist}(f^{q_n}(I), I) \leq \lambda^{\gamma(n)}, \qquad \frac{|f^{-q_n}(I)|}{\text{dist}(f^{-q_n}(I), I)} \geq P\lambda^{-\gamma(n)}.$$

Further, there exist uniform constants $N, A > 0$ such that for every integer $n \geq N$, the constant λ can be chosen as $(\text{dist}(f^{q_N}(I), I))^{A/\gamma(N)}$.

A uniform version of **FBL** is stated in Section 4. As an immediate corollary of the upper scaling rules we obtain Theorem 1 which says that independently of the rotation number the flat interval I is approximated at least superexponentially by successive closest returns $f^{q_n}(I)$.

THEOREM 1. *If f satisfies the Hypotheses then there is a positive constant $0 < \lambda < 1$ such that $\text{dist}(f^{q_n}(I), I)$ tends to zero faster than $\lambda^{(\sqrt{1+1/\ell})^n}$.*

Lower scaling rules. The upper scaling rules allow us to formulate distortion and convexity lemmas in Sections 5 and 6. These analytic estimates lead to the Second Basic Lemma (**SBL**) setting lower bounds for scaling rules.

Put $\overline{\gamma}(1) = 1$ and for every $n > 1$ define

$$\overline{\gamma}(n) := \overline{\gamma}(\rho, n) = \prod_{0 < i \leq [n/2]} \left(1 + \frac{\ell + 1}{\ell - 1} a_{2i+1}\right).$$

LEMMA 1.2 (Second Basic Lemma). *Let f satisfy the Hypotheses. There is a uniform constant N and a constant $0 < \lambda < 1$ such that for every $n \geq N$,*

$$\text{dist}(f^{q_n}(I), I) \geq \lambda^{\overline{\gamma}(n)}, \qquad \frac{|f^{-q_n}(I)|}{\text{dist}(f^{-q_n}(I), I)} \leq \lambda^{-\overline{\gamma}(n)}.$$

The constant λ can be chosen as $(\text{dist}(f^{q_N}(I), I))^{A/\overline{\gamma}(N)}$, where A is a uniform constant.

REMARKS. Similar but somewhat different scaling rules were considered in [5] for rotation numbers of bounded type; that is, where all coefficients a_i of the continued fraction expansion of ρ are supposed to be bounded. A further study of scaling rules for circle maps with a flat interval and rotation numbers of bounded type can be found in [1].

Scaling rules of Fact 3.3 below imply that the set Ω of non-wandering points coincides with $S^1 \setminus \bigcup_{n=0}^{\infty} f^{-n}(I)$ and is of zero Lebesgue measure for every map satisfying the Hypotheses (Proposition 1). For rotation numbers of bounded type, the Hausdorff dimension of Ω was studied in [1].

2. Technical tools. Our main technical tools are right-ratio and cross-ratio inequalities.

The cross-ratio inequality (CRI). We shall adapt the cross-ratio inequality introduced in [4] to our needs. For four points a, b, c, d arranged according to the standard orientation of the circle so that $a < b < c < d$ and $b, c \in (a, d)$ we define their *cross-ratio* as

$$\mathbf{Cr}(a, b, c, d) = \frac{|a - b|\,|c - d|}{|a - c|\,|b - d|}.$$

By the distortion of the cross-ratio we mean

$$\mathcal{D}\mathbf{Cr}(a, b, c, d) = \frac{\mathbf{Cr}(f(a), f(b), f(c), f(d))}{\mathbf{Cr}(a, b, c, d)}.$$

Let us consider a set of quadruples $\{a_i, b_i, c_i, d_i\}$ with the following properties:

1. Each point of the circle belongs to at most k intervals (a_i, d_i).
2. The intervals (b_i, c_i) do not intersect I.

Then

$$\prod_{i=0}^{n} \mathcal{D}\mathbf{Cr}(a_i, b_i, c_i, d_i) \leq C_k$$

and the constant C_k does not depend on the set of quadruples.

The right-ratio inequality (RRI). Suppose we have three points a, b, c so that $a < b < c$. The *right-ratio* and its distortion are defined as follows:

$$\mathbf{R}(a, b, c) = \frac{|b - c|}{|a - b|}, \qquad \mathcal{D}\mathbf{R}(a, b, c) = \frac{\mathbf{R}(f(a), f(b), f(c)}{\mathbf{R}(a, b, c)}.$$

Consider now a set of triples $\{a_i, b_i, c_i\}$, $0 \leq i \leq n$, which satisfy the following conditions:

1. Each point of the circle belongs to at most k intervals (a, c).
2. The intervals (a_i, b_i) do not intersect the flat interval I and I is not contained in (b_i, c_i).

Then

$$\prod_{i=0}^{n} \mathcal{D}\mathbf{R}(a_i, b_i, c_i) \leq R_k$$

and the constant R_k does not depend on the set of triples.

REMARK. In this paper we will consider only sets of triples and quadruples formed by taking iterations either of an initial triple or quadruple. Therefore, to characterize a given set of triples or quadruples we will indicate the initial object in braces followed after a semicolon by the number of iterates one must perform to get all of them.

Proof of the right-ratio inequality. We start with a simple fact from calculus.

FACT 2.1. *There is a left-sided neighborhood U of the point $l(I)$ such that for all $y \in U$ the second derivative of f is negative. We can also assume that $|U| < 1/2$ is so small that $|f(U)| < 1/2$.*

Next we list some properties of the right-ratio which immediately yield the desired result.

1. Upper convexity of f restricted to U implies that the distortion of the right-ratio is strictly less than 1 for triples contained in $U \cup I$.
2. Let U' be a left-sided neighborhood of $l(I)$ strictly contained in U. A total distortion of the right-ratio for triples contained in the complement of $I \cup U'$ is bounded by the variation of $\log \frac{df}{dx}$ over this set.
3. There are at most k triples which are not covered by any of the above two cases. For these triples the distortion of the right-ratio is also bounded, as can be checked by a direct calculation.

3. Topological description of the set Ω. There is the following characterization of the set Ω.

PROPOSITION 1. *The set Ω is equal to $S^1 \setminus \bigcup_{i=0}^{\infty} f^{-i}(I)$ and is of zero Lebesgue measure.*

We start the proof of Proposition 1 with two simple observations. Let U be the neighborhood from Fact 2.1.

FACT 3.1. *There are constants K_1, K_2 such that if $y \in U$ then*
$$K_1 |l(I) - y|^{\ell} \leq |f(l(I)) - f(y)| \leq K_2 |l(I) - y|^{\ell},$$
$$K_1 |l(I) - y|^{\ell-1} \leq \frac{df}{dx}(y) \leq K_2 |l(I) - y|^{\ell-1}.$$

FACT 3.2. *There is a constant K_3 such that for every $y, z \in U$,*
$$\frac{|f(y) - f(z)|}{|f(y) - f(I)|} \leq K_3 \frac{|y - z|}{|y - l(I)|}$$
provided $z \in (y, l(I))$.

LEMMA 3.1. *There is a uniform constant $K_4 > 0$ such that*

$$\frac{|f^{-q_n}(I)|}{\mathrm{dist}(f^{-q_n}(I), I)} \geq K_4.$$

The sequence $\mathrm{dist}(f^{q_n}(I), I)$ *tends to zero uniformly and at least exponentially.*

Proof. First, we prove that the length of $f^{-q_n}(I)$ cannot be too small in comparison to the distance to the flat interval I. Assume that n is *even*, i.e. $f^{-q_n}(I)$ is to the left of I.

Apply **CRI** to

$$\{l(f^{-q_n+1}(I)), r(f^{-q_n+1}(I)), l(f^{-q_{n+1}+1}(I)), r(f^{-q_{n+1}+1}(I)); q_n - 1\}$$

to obtain

(2) $\quad \dfrac{|I|}{|l(I) - l(f^{-q_{n+1}+q_n}(I))|} \dfrac{|f^{-q_{n+1}+q_n}(I)|}{|r(I) - r(f^{-q_{n+1}+q_n}(I))|}$

$$\leq C_1 \frac{|f^{-q_n+1}(I)|}{|l(f^{-q_n+1}(I)) - l(f^{-q_{n+1}+1}(I))|} \frac{|f^{-q_{n+1}+1}(I)|}{|r(f^{-q_n+1}(I)) - r(f^{-q_{n+1}+1}(I))|}.$$

Fact 3.2 yields

$$\frac{|f^{-q_n+1}(I)|}{|l(f^{-q_n+1}(I)) - f(I)|} \leq K_3 \frac{|f^{-q_n}(I)|}{|l(f^{-q_n}(I)) - l(I)|}.$$

The first factor of the left-hand side of (2) is bounded away from zero while the second factor on the right-hand side is less than 1. Therefore,

(3) $\quad \dfrac{|f^{-q_n}(I)|}{|l(f^{-q_n}(I)) - l(I)|} \geq K_5 \dfrac{|f^{-q_{n+1}+q_n}(I)|}{|r(I) - r(f^{-q_{n+1}+q_n}(I))|}.$

The interval $f^{-q_{n+1}+q_n}(I)$ lies between $r(I)$ and $r(f^{-q_{n+1}+q_n}(I))$. Therefore, **RRI** written for $\{r(I), l(f^{-q_{n+1}+q_n}(I)), r(f^{-q_{n+1}+q_n}(I)); q_{n+1} - q_n\}$ gives a lower bound of (3), which yields the inequality from Lemma 3.1.

For n *odd*, the considerations are much simpler. To get the same estimates as above it is enough to write **RRI** for $\{r(I), l(f^{-q_n}(I)), r(f^{-q_n}(I)); q_n\}$.

Finally for both cases, since $f^{q_{n+1}}(I)$ lies between $f^{-q_n}(I)$ and I we obtain

$$\mathrm{dist}(f^{q_{n+1}}(I), I) \leq (1/(1 + K_4)) \, \mathrm{dist}(f^{q_{n-1}}(I), I),$$

which completes the proof of Lemma 3.1. ∎

First estimates. We write down four simple inequalities which will set a basis for the proof of the main estimates in the next section. We will also use them to prove Proposition 1.

Consider the nth partition of the circle. We form a set of initial triples by taking the endpoints of all holes $\square(n) \in \mathcal{A}_n$ together with the right endpoint

of the preimage $I(\square(n)) \in \mathcal{A}_n$ of the flat interval I lying immediately to the right of $\square(n) \in \mathcal{A}_n$. **RRI** implies:

FACT 3.3. *The ratio*

$$\frac{|\square(n)|}{|I(\square(n))|}$$

is bounded from above by either

- $(R_1/|I|) \operatorname{dist}(f^{q_n}(I), I))$ *or* $(R_1/|I|) \operatorname{dist}(f^{-q_{n+1}}(I), I))$ *for n odd,*
- $(R_1/|I|) \operatorname{dist}(f^{-q_n}(I), I))$ *or* $(R_1/|I|) \operatorname{dist}(f^{q_{n+1}}(I), I))$ *for n even.*

Proof of Proposition 1. The inclusion $\Omega \subset S^1 \setminus \bigcup_{i=0}^{\infty} f^{-i}(I)$ is evident. So, we have to prove that each point in $S^1 \setminus \bigcup_{i=0}^{\infty} f^{-i}(I)$ is non-wandering. By Fact 3.3 the complement of all preimages of I does not contain any density point with respect to the Lebesgue measure. Hence, by the Lebesgue Density Lemma the set of non-wandering points is of zero Lebesgue measure. In particular, $\bigcup_{i=0}^{\infty} f^{-i}(I)$ is dense in S^1. Moreover, $S^1 \setminus \bigcup_{i=0}^{\infty} f^{-i}(I)$ contains no isolated points, because such an isolated point would lie on the boundary of two inverse images of I and therefore would imply the existence of a periodic point. Since f is order preserving, we see that $\{f^i(I) : i > 0\}$ is dense in $S^1 \setminus \bigcup_{i=0}^{\infty} f^{-i}(I)$. Hence, every point in $S^1 \setminus \bigcup_{i=0}^{\infty} f^{-i}(I)$ has a dense orbit, by the property of f being order preserving.

4. Scaling rules—upper bounds. We recall that p_n/q_n is the nth convergent of the continued fraction representation of the rotation number $\rho = [0, a_1, \ldots, a_n, \ldots]$.

Let $1 \leq m < n$ and define

$$\gamma(n, m) := \gamma(\rho, n, m) = \prod_{[m/2] < i \leq [n/2]} \left(1 + \frac{a_{2i+1}}{\ell}\right).$$

In the above definition, we adopt the rule that the product over an empty set of indices is equal to 1. For example, if n is odd then $\gamma(n, n-1) = 1$. Also, $\gamma(n, n) = 1$. We see that $\gamma(n) = \gamma(n, 1)$. By the definition, for any $m \leq k \leq n$, $\gamma(n, m) = \gamma(n, k)\gamma(k, m)$.

Our main technical task in this section is to prove **FBL**.

We want to compare the length of $f^{-q_n}(I)$ with $\operatorname{dist}(f^{-q_n}(I), I)$ and show that the latter is much smaller than the former. Our considerations fall into two parts:

4.1. Proof of FBL when $f^{-q_n}(I)$ is to the left of I

LEMMA 4.1. *There are uniform constants n_0, K_6 such that*

$$|r(I) - l(f^{-q_{n-1}}(I))|^{1/\ell} \leq K_6 |f^{-q_n}(I)| \quad \text{for all } n \geq n_0.$$

Proof. From Lemma 3.1 there is n_0 such that for every $n \geq n_0$ even, $f^{-q_{n-2}}(I) \subset U$, where U is the neighborhood from Fact 2.1.

We will show that the opposite inequality

(4) $$|r(I) - l(f^{-q_{n-1}}(I))|^{1/\ell} \geq (1/\alpha)|f^{-q_n}(I)|$$

leads to a contradiction if only α is small enough.

Since the right-sided derivative of f at $r(I)$ is positive, the lengths $|f(I) - l(f^{-q_{n-1}+1}(I))|$ and $|r(I) - l(f^{-q_{n-1}}(I))|$ are comparable. By the inequality (4) we get

$$|f(I) - l(f^{-q_{n-1}+1}(I))|^{1/\ell} \geq (K_7/\alpha)|f^{-q_n}(I)|$$

and consequently, by Lemma 3.1 and Fact 3.1,

$$\geq (K_8/\alpha)|l(f^{-q_n+1}(I)) - f(I)|^{1/\ell}.$$

Thus

(5) $$|f(I) - l(f^{-q_{n-1}+1}(I))| \geq (K_8/\alpha)^\ell |l(f^{-q_n+1}(I)) - f(I)|.$$

For $a_n \geq i \geq 0$ set

$$\beta_i = |l(f^{-q_n+iq_{n-1}}(I)) - f^{iq_{n-1}}(I)|, \qquad S_i = |l(f^{-q_n+iq_{n-1}}(I)) - l(I)|.$$

We shall show that

(6) $$\frac{\beta_{i+1}}{S_{i+1}} \leq \left(K_9\alpha \frac{S_i}{\beta_0}\right)^\ell \frac{\beta_i}{S_i}$$

for all $0 \leq i \leq a_n - 1$.

Writing **CRI** for

$$\{l(f^{-q_n+iq_{n-1}+1}(I)), f^{iq_{n-1}+1}(I), l(f^{-q_{n-1}+1}(I)), r(f^{-q_{n-1}+1}(I)); q_{n-1} - 1\},$$

we obtain

(7) $$\frac{|I|}{|f^{(i+1)q_{n-1}}(I) - r(I)|} \frac{\beta_{i+1}}{S_{i+1}}$$

$$\leq C_1 \frac{|l(f^{-q_n+iq_{n-1}+1}(I)) - f^{iq_{n-1}+1}(I)|}{|l(f^{-q_n+iq_{n-1}+1}(I)) - l(f^{-q_{n-1}+1}(I))|} \frac{|f^{-q_{n-1}+1}(I)|}{|f^{iq_{n-1}+1}(I) - r(f^{-q_{n-1}+1}(I))|}.$$

By Facts 3.2 and 3.1 we have

(8) $$\frac{|l(f^{-q_n+iq_{n-1}+1}(I)) - f^{iq_{n-1}+1}(I)|}{|l(f^{-q_n+iq_{n-1}+1}(I)) - f(I)|} \leq K_3 \frac{\beta_i}{S_i},$$

(9) $$\frac{|l(f^{-q_n+iq_{n-1}+1}(I)) - f(I)|}{|l(f^{-q_n+1}(I)) - f(I)|} \leq (K_2/K_1)\left(\frac{S_i}{\beta_0}\right)^\ell.$$

The first factor on the left-hand side of (7) is greater than a uniform constant. We can also neglect the last factor on the right-hand side of (7) as it is smaller than 1. Further, we substitute (8) and (9) into the resulting inequality to obtain

(10) $$\frac{\beta_{i+1}}{S_{i+1}} \leq K_{10}\left(\frac{S_i}{\beta_0}\right)^\ell \frac{|f(I) - l(f^{-q_n+1}(I))|}{|l(f^{-q_n+iq_{n-1}+1}(I)) - l(f^{-q_{n-1}+1}(I))|} \frac{\beta_i}{S_i}.$$

Note that

$$(f(I), l(f^{-q_{n-1}+1}(I))) \subset (l(f^{-q_n+iq_{n-1}+1}(I)), l(f^{-q_{n-1}+1}(I))).$$

Therefore, substituting (5) into (10) we finally obtain (6).

We pass to estimating the rate of growth of β_i/S_i. To simplify the notation we set *new* α to be equal to $K_9\alpha$.

We begin by showing that $S_{i+1} - S_i \leq \kappa\beta_{i+1}$ for a uniform constant $\kappa > 1$. To do this we define $\epsilon_{i+1} = |f^{(i+1)q_{n-1}}(I) - l(f^{-q_n+iq_{n-1}}(I))|$. Observe that $S_{i+1} - S_i = \beta_{i+1} + \epsilon_{i+1}$. The interval $(f^{(i+1)q_{n-1}}(I), l(f^{-q_n+iq_{n-1}}(I)))$ is contained in the hole $(r(f^{-q_n+(i+1)q_{n-1}}(I)), l(f^{-q_n+iq_{n-1}}(I)))$ of the dynamical partition $\mathcal{A}_{n-1}$, and therefore the error term ϵ_{i+1} can be estimated using the cross-ratio inequality written for the endpoints of the intervals $f^{-q_n+(i+1)q_{n-1}}(I)$ and $f^{-q_n+iq_{n-1}}(I)$ and the number of iterates $q_n - (i+1)q_{n-1}$ (see the figure below with a temporary notation $\underline{i} := f^i(I)$ and $\underline{0} := I$).
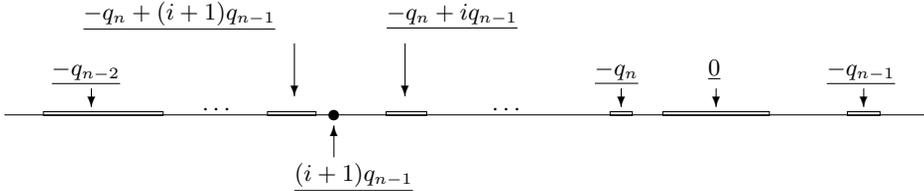
Indeed, using simple algebra and Lemma 3.1, we have

$$\frac{\beta_{i+1}}{\epsilon_{i+1} + \beta_{i+1}} \geq \frac{1}{C_1} \frac{|I|}{|l(I) - l(f^{-q_{n-1}}(I))|} \frac{|f^{-q_{n-1}}(I)|}{|r(I) - r(f^{-q_{n-1}}(I))|}$$
$$\geq \frac{K_4|I|}{C_1(1+K_4)}.$$

In particular, there exists a uniform constant $\kappa > 1$ such that $\beta_{i+1} + \epsilon_{i+1} \leq \kappa\beta_{i+1}$ for every $0 \leq i \leq a_n - 1$.

We shall show that for all $0 \leq i \leq a_n$ the following inequalities are true if only α is small enough:

(i) $S_i \leq \beta_0(1 + 2\alpha^{\ell-1})(1 + \alpha^{\ell-1})(1 + \alpha^{\ell-1}/2)\cdots(1 + \alpha^{\ell-1}/2^{i-1})$,
(ii) $\beta_i/S_i \leq \alpha^{i(l-1)}$.

Induction. For $i = 0$ our inequalities trivially hold. Suppose that the induction hypothesis is true for some $i \leq a_n - 1$ and α is chosen so that $\alpha \exp(4\ell\alpha^{\ell-1}) \leq 1$. Further, using (6) with the new α, we obtain

$$\beta_{i+1}/S_{i+1} \leq (\alpha \exp(4\alpha^{\ell-1}))^\ell(\beta_i/S_i) \leq \alpha^{\ell-1}(\beta_i/S_i) \leq \alpha^{(\ell-1)(i+1)}.$$

To prove the first inequality we assume additionally that $\alpha^{\ell-1} < 1/4\kappa$ and use the fact that the function $x \mapsto x/(1-x)$ is increasing for $x < 1$.

Then

$$S_{i+1} = S_i + \beta_{i+1} + \epsilon_{i+1} \leq \left(1 + \frac{(\epsilon_{i+1} + \beta_{i+1})/S_{i+1}}{1 - (\epsilon_{i+1} + \beta_{i+1})/S_{i+1}}\right)S_i$$
$$= \left(1 + \frac{\kappa\alpha^{(i+1)(\ell-1)}}{1 - \kappa\alpha^{(i+1)(\ell-1)}}\right)S_i \leq \left(1 + \frac{\kappa\alpha^{i(\ell-1)}}{1 - \kappa\alpha^{i(\ell-1)}}\alpha^{\ell-1}\right)S_i$$
$$\leq \beta_0(1 + 2\alpha^{\ell-1})\cdots(1 + \alpha^{\ell-1}/2^i).$$

Conclusion of the proof of Lemma 4.1. As a consequence of (ii), we obtain

$$\beta_{a_n}/S_{a_n} \leq \alpha^{(\ell-1)a_n}.$$

Let us recall that $-q_{n-2} = -q_n + a_nq_{n-1}$. Fact 3.2 and **CRI** written for

$$\{l(f^{-q_{n-2}+1}(I)), f^{a_nq_{n-1}+1}(I), l(f^{-q_{n-1}+1}(I)), r(f^{-q_{n-1}+1}(I)); q_{n-2} - 1\}$$

imply that

$$\frac{\beta_{a_n}}{S_{a_n}} \geq \frac{1}{K_3}\frac{|l(f^{-q_{n-2}+1}(I)) - f^{a_nq_{n-1}+1}(I)|}{|l(f^{-q_{n-2}+1}(I)) - l(f^{-q_{n-1}+1}(I))|}$$
$$\geq \frac{|I|}{C_1K_3}\frac{|f^{-q_{n-1}+q_{n-2}}(I)|}{|f^{q_n}(I) - r(f^{-q_{n-1}+q_{n-2}}(I))|}.$$

By **RRI**, the right-hand side of the above inequality is bounded from below by a constant K_{11}. Choosing $\alpha^{\ell-1}$ smaller than $|I|K_{11}/C_1K_1$, we obtain a contradiction. ∎

Conclusion of the proof of FBL. By Lemma 4.1 and Fact 3.1 we obtain

$$(11) \qquad \frac{|f^{-q_n}(I)|}{|r(f^{-q_n}(I)) - l(I)|} \geq K_{12}\left(\frac{|f(I) - l(f^{-q_{n-1}+1}(I))|}{|r(f^{-q_n+1}(I)) - f(I)|}\right)^{1/\ell}.$$

Let $0 \leq i \leq a_{n+1} - 2$ (we temporarily assume that $a_{n+1} > 1$). **RRI** written for

$$\{f(I), l(f^{-q_{n-1}-(i+1)q_n+1}(I)), l(f^{-q_{n-1}-iq_n+1}(I)); q_{n-1} + (i+1)q_n - 1\}$$

yields

$$(12) \qquad \frac{|f(I) - l(f^{-q_{n-1}-(i+1)q_n+1}(I))|}{|f(I) - l(f^{-q_{n-1}-iq_n+1}(I))|} \leq (R_1/|I|)\,\mathrm{dist}(f^{q_{n-1}}(I), I).$$

Multiply through the inequalities (12) for $0 \leq i \leq a_{n+1} - 2$ to get

$$(13) \qquad \frac{|f(I) - l(f^{-q_{n+1}+q_n+1}(I))|}{|f(I) - l(f^{-q_{n-1}+1}(I))|} \leq ((R_1/|I|)\,\mathrm{dist}(f^{q_{n-1}}(I), I))^{a_{n+1}-1}.$$

The next inequality will relate estimates (13) and (11). **RRI** written for $\{r(f^{-q_n+1}(I)), f(I), l(f^{-q_{n+1}+q_n+1}(I)); q_{n+1} - 1\}$ implies

$$(14) \quad \frac{|f(I) - l(f^{-q_{n+1}+q_n+1}(I))|}{|f(I) - r(f^{-q_{n+1}+1}(I))|} \geq (1/R_1) \frac{|f^{q_n}(I) - f^{q_{n+1}}(I)|}{|f^{q_{n+1}}(I) - f^{q_{n+1}-q_n}(I)|}$$

$$\geq \frac{|I|/R_1}{|l(I) - f^{q_{n+1}-q_n}(I)|} \geq (|I|/R_1) \operatorname{dist}(f^{q_{n-1}}(I), I)^{-1}.$$

Combining the inequalities (11), (13) and (14) we obtain

$$(15) \quad \frac{|f^{-q_n}(I)|}{|r(f^{-q_n}(I)) - l(I)|} \geq K_{12}((R_1/|I|) \operatorname{dist}(f^{q_{n-1}}(I), I))^{-a_{n+1}/\ell}$$

and consequently

$$(16) \quad \operatorname{dist}(f^{q_{n+1}}(I), I)$$

$$\leq K_{13}((R_1/|I|) \operatorname{dist}(f^{q_{n-1}}(I), I))^{a_{n+1}/\ell} \operatorname{dist}(f^{q_{n-1}}(I), I)$$

$$\leq (K_{14} \operatorname{dist}(f^{q_{n-1}}(I), I))^{1+a_{n+1}/\ell}.$$

If $a_{n+1} = 1$ then (15) and (16) follow directly from (14) and (11).

Let $1 \leq m \leq n$ be even numbers. Then

$$\gamma(n, m) = \prod_{m/2 < i \leq n/2} (1 + a_{2i+1}/\ell).$$

The recursive estimate (16) implies that

$$\operatorname{dist}(f^{q_{n+1}}(I), I) \leq K_{14}^{\gamma(n,n-2)+\gamma(n,n-4)+\cdots+\gamma(n,m)} \operatorname{dist}(f^{q_{m+1}}(I), I)^{\gamma(n,m)}.$$

Observe that

$$\sum_{i=0}^{(n-m)/2} \gamma(n, m + 2i) = \gamma(n, m) \sum_{i=0}^{(n-m)/2} \gamma(m + 2i, m)^{-1}$$

$$\leq \gamma(n, m) \sum_{i=0}^{\infty} (1 + 1/\ell)^{-i} \leq (\ell + 1) \cdot \gamma(n, m).$$

Consequently,

$$(17) \quad \operatorname{dist}(f^{q_{n+1}}(I), I) \leq (K_{15} \operatorname{dist}(f^{q_{m+1}}(I), I))^{\gamma(n,m)}.$$

By Lemma 3.1, there is a uniform constant $0 < \lambda_1 < 1$ such that for every $n \in \mathbb{N}$, $\operatorname{dist}(f^{q_n}(I), I) < \lambda_1^n$. Choose n_0 even so that $K_{15} \operatorname{dist}(f^{q_{n_0}+1}(I), I) \leq 1/2$. The constant n_0 is uniform and, from (17), if $n \geq n_0$ is even then

$$(18) \quad \operatorname{dist}(f^{q_{n+1}}(I), I) \leq (1/2)^{\gamma(n,n_0)}.$$

Let $\lambda_2 = (1/2)^{1/\gamma(n_0)}$. Using (17) and (18), we see that for every $n \geq n_0$ even,

$$(19) \quad \operatorname{dist}(f^{q_{n+1}}(I), I) \leq (1/2)^{\gamma(n,n_0)} = (1/2)^{\gamma(n,1)/\gamma(n_0,1)} = \lambda_2^{\gamma(n)} = \lambda_2^{\gamma(n+1)}.$$

The constant λ_2 is not uniform and it depends on the rotation number. Since $\operatorname{dist}(f^{q_{n+1}}(I), I)$ are all smaller than 1, we obtain the first formula of **FBL**

for all n *odd*. The second formula of **FBL** for n *even* follows from the first
for n odd and (15).

Let $N \geq n_0 + 1$ be an odd number chosen so that $K_{15}(\text{dist}(f^{q_N}(I), I))^{1/2}$
< 1. The explicit formula for λ stated in **FBL** can be derived from (17).
Let $A = 1/2$ and denote $\theta = (\text{dist}(f^{q_N}(I), I))^{A/\gamma(N)}$. If $n + 1 \geq N$ then the
inequality (17) with $m + 1 := N$ implies that $\text{dist}(f^{q_{n+1}}(I), I)$ is bounded by

$$(\text{dist}(f^{q_N}(I), I))^{\gamma(n, N-1)/2} = (\text{dist}(f^{q_N}(I), I))^{\gamma(n, N)/2} = \theta^{\gamma(n+1)}.$$

4.2. Proof of FBL when $f^{-q_n}(I)$ is to the right of I. The claims
of **FBL** in this case follow easily from the estimates of the last subsec-
tion. Indeed, note that $f^{q_{n+1}}(I)$ lies between $f^{-q_{n+2}}(I)$ and $f^{-q_{n+2}+q_{n+1}}(I)$.
Therefore,

$$(20) \qquad \text{dist}(f^{q_{n+1}}(I), I) \leq |r(I) - l(f^{-q_{n+2}+q_{n+1}}(I))|.$$

Apply **RRI** for

$$\{r(I), l(f^{-q_n - iq_{n+1}}(I)), r(f^{-q_n - iq_{n+1}}(I)); q_n + iq_{n+1}\},$$

where $0 \leq i \leq a_{n+2} - 1$, and a simple algebra to get

$$(21) \qquad \frac{|r(I) - l(f^{-q_n - iq_{n+1}}(I))|}{|f^{-q_n - iq_{n+1}}(I)|} \leq (R_1/|I|) \, \text{dist}(f^{q_n}(I), I).$$

Multiplying through (21) for $0 \leq i \leq a_{n+2} - 1$, we obtain

$$(22) \qquad \frac{|r(I) - l(f^{-q_{n+2}+q_{n+1}}(I))|}{|r(I) - r(f^{-q_n}(I))|} \leq \prod_{i=0}^{a_{n+2}-1} \frac{|r(I) - l(f^{-q_n - iq_{n+1}}(I))|}{|r(I) - r(f^{-q_n - iq_{n+1}}(I))|}$$
$$\leq ((R_1/|I|) \, \text{dist}(f^{q_n}(I), I))^{a_{n+2}}.$$

The inequality (20) and $|r(I) - r(f^{-q_n}(I))| \leq \text{dist}(f^{q_{n-1}}(I), I)$ allow us to
write (21) in the recursive form

$$(23) \qquad \text{dist}(f^{q_{n+1}}(I), I) \leq \text{dist}(f^{q_{n-1}}(I), I)((R_1/|I|) \, \text{dist}(f^{q_n}(I), I))^{a_{n+2}}.$$

We recall that n_0 is a uniform constant defined in the discussion leading
to the estimate (18). Suppose that $R_1/|I| > 2$ and choose $n_1 > n_0$ so that
$(|I|/R_1)^2 > (1/2)^{\gamma(n_1, n_0)}$. The constant n_1 is uniform since it can be defined
as a number for which

$$(2/\log 2) \log(R_1/|I|) < (1 + 1/\ell)^{(n_1 - n_0)/2}.$$

Further, for every $1 \leq m \leq n$,

$$(24) \qquad a_{n+2} \cdot \gamma(n, m) \geq \gamma(n + 1, m)/(\ell + 1).$$

We use the estimate (19). Consequently, for every $n \geq n_1$ there exists $\lambda_3 \in (0,1)$ such that

$$
\begin{aligned}
(25) \qquad \operatorname{dist}(f^{q_{n+1}}(I), I) &\leq ((R_1/|I|)(1/2)^{\gamma(n,n_0)})^{a_{n+2}} \\
&\leq (1/2)^{(\gamma(n,n_0)-(1/2)\gamma(n_1,n_0))a_{n+2}} \\
&\leq (1/2)^{a_{n+2}\cdot\gamma(n,n_1)} \\
&\leq (1/\sqrt[\ell+1]{2})^{\gamma(n+1,n_1)} \leq \lambda_3^{\gamma(n+1)}.
\end{aligned}
$$

This proves the first estimate of **FBL**. The constant λ_3 is not uniform as it is defined by

$$
\lambda_3 = (1/2)^{1/((\ell+1)\gamma(n_1))}.
$$

The second inequality of **FBL** follows directly from the first and **RRI** written for $\{r(I), l(f^{-q_n}(I)), r(f^{-q_n}(I)); q_{n-1}\}$. Indeed,

$$
(26) \qquad \frac{|f^{-q_n}(I)|}{|r(I) - l(f^{-q_n}(I))|} \geq (|I|/R_1)(\operatorname{dist}(f^{q_n}(I), I))^{-1}.
$$

The explicit formula for λ in the statement of **FBL** is a consequence of (23) combined with the estimates (24) and (17). Indeed, choose $N \geq n_0$ *odd* so that $(K_{15}R_1/|I|)(\operatorname{dist}(f^{q_N}(I), I))^{1/2} < 1$. We can also drop $\operatorname{dist}(f^{q_{n-1}}(I), I)$ from the inequality (23). Set $A = 1/(2(\ell+1))$ and $\theta = (\operatorname{dist}(f^{q_N}(I), I))^{A/\gamma(N)}$. Then, for all $n \geq N$ *odd*,

$$
\begin{aligned}
\operatorname{dist}(f^{q_{n+1}}(I), I) &\leq ((R_1/|I|)\operatorname{dist}(f^{q_n}(I), I))^{a_{n+2}} \\
&\leq (K_{15}(R_1/|I|)\operatorname{dist}(f^{q_N}(I), I))^{\gamma(n+1,N-1)/(\ell+1)} \\
&\leq (\operatorname{dist}(f^{q_N}(I), I))^{\gamma(n+1,N)/2(\ell+1)} = \theta^{\gamma(n+1)}.
\end{aligned}
$$

4.3. Uniform version of FBL

LEMMA 4.2. *There is a uniform constant $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$,*

$$
\operatorname{dist}(f^{q_n}(I), I) \leq 2^{-\gamma(n,n_0)}, \qquad \frac{|f^{-q_n}(I)|}{\operatorname{dist}(f^{-q_n}(I), I)} \geq 2^{\gamma(n,n_0)}.
$$

Proof. The first inequality follows directly from the estimate (19) when n is odd and from (23) when n is even. The second inequality is a consequence of the first and the inequalities (15) and (26). ∎

4.4. Proof of Theorem 1. The coefficients a_i in the statement of **FBL** are positive integers.

5. Analytic description of the dynamics

5.1. Distortion. The crucial observation made in this subsection is that the logarithm of the total distortion of an iterate f^i on an interval J with the property that $J, \ldots, f^{i-1}(J)$ are contained in the holes of the nth partition

$\mathcal{A}_n$ tends to 0 uniformly superexponentially provided $J, \ldots, f^{i-1}(J)$ do not intersect the hole $\Delta \in \mathcal{A}_n$ adjacent to the critical point $l(I)$. The explanation of this phenomenon is the following: by **FBL** the holes are substantially shorter than the neighboring preimages of the flat interval I which separate the holes from the critical point.

LEMMA 5.1 (Distortion Lemma). *Let* $J, \ldots, f^{i-1}(J)$, $i \geq 1$, *be a collection of intervals contained in the holes of* $\mathcal{A}_n$ *which do not intersect the hole* $\Delta \in \mathcal{A}_n$ *adjacent to* $l(I)$. *Then for any* $z_1, z_2 \in J$ *the logarithm of the distortion*

$$\left| \log \frac{df^i}{dx}(z_1) - \log \frac{df^i}{dx}(z_2) \right|$$

tends to 0 *uniformly superexponentially when n goes to infinity.*

Proof. Let z_1, z_2 be in the same hole of $\mathcal{A}_n$ different from Δ. By the Mean Value Theorem,

$$(27) \quad \mathcal{D} := \left| \log \frac{df^i}{dx}(z_1) - \log \frac{df^i}{dx}(z_2) \right| \leq \sum_{j=0}^{i-1} \left(\left| \frac{d^2 f}{dx^2}(\xi_j) \right| \Big/ \left| \frac{df}{dx}(\xi_j) \right| \right) |f^j(J)|$$

$$\leq M_0 \sum_{\square(n) \in \mathcal{A}_n \setminus \{\Delta\}} \frac{|\square(n)|}{\text{dist}(\square(n), I)},$$

where $\xi_j \in f^j(J)$. Let $I(\square(n)) \in \mathcal{A}_n$ be a preimage of the flat interval I lying immediately to the right of $\square(n)$. By Fact 3.3, the ratio $|\square(n)|/|I(\square(n))|$ is bounded from above by

$$(28) \qquad\qquad (R_1/|I|) \, \text{dist}(f^{q_n}(I), I) \quad \text{if } n \text{ is odd},$$
$$(29) \qquad\qquad (R_1/|I|) \, \text{dist}(f^{-q_n}(I), I) \quad \text{if } n \text{ is even}.$$

If n is odd then Δ coincides with $(r(f^{-q_{n+1}}(I)), l(I))$ and $f^{q_n}(I)$ lies in the hole $\Delta_1 \in \mathcal{A}_n$ between $f^{-q_{n+1}+q_n}(I)$ and $f^{-q_{n+1}}(I)$, i.e. Δ_1 is to the left of $f^{-q_{n+1}}(I)$. By **RRI** written for $\{f^{q_n}(I), l(f^{-q_{n+1}}(I)), r(f^{-q_{n+1}}(I)); q_{n+1}\}$,

$$(30) \quad \frac{|f^{q_n}(I) - l(f^{-q_{n+1}}(I))|}{|f^{-q_{n+1}}(I)|} \leq (R_1/|I|) \, \text{dist}(f^{q_{n+1}+q_n}(I), I) \leq R_1/|I|$$

and

$$\text{dist}(f^{q_n}(I), I) = \text{dist}(\Delta_1, I) + |f^{q_n}(I) - l(f^{-q_{n+1}}(I))|$$
$$\leq \text{dist}(\Delta_1, I) \left(1 + \frac{|f^{q_n}(I) - l(f^{-q_{n+1}}(I))|}{|f^{-q_{n+1}}(I)|} \right)$$
$$\leq \text{dist}(\Delta_1, I)(1 + R_1/|I|).$$

Hence, for n odd,

$$
(31) \qquad \mathcal{D} \leq M_1 \sum_{\square(n) \in \mathcal{A}_n \setminus \{\Delta\}} \frac{|\square(n)|}{|I(\square(n))|} \frac{|I(\square(n))|}{\operatorname{dist}(\square(n), I)}
$$

$$
\leq M_2 \sum_{\square(n) \in \mathcal{A}_n \setminus \{\Delta, \Delta_1\}} \operatorname{dist}(\Delta_1, I) \left(1 + \frac{|I(\square(n))|}{\operatorname{dist}(\square(n), I)} \right)
$$

$$
\leq M_3 \operatorname{dist}(\Delta_1, I) \int_{\operatorname{dist}(\Delta_1, I)}^{1} \frac{1}{x} \, dx
$$

$$
= M_3 \operatorname{dist}(\Delta_1, I) |\log(\operatorname{dist}(\Delta_1, I))|.
$$

For n even, Δ coincides with $(r(f^{-q_n}(I)), l(I))$. Therefore,

$$
\frac{|\square(n)|}{|I(\square(n))|} \leq (R_1/|I|)|\Delta|
$$

and

$$
(32) \qquad \mathcal{D} \leq M_4 \sum_{\square(n) \in \mathcal{A}_n \setminus \{\Delta\}} |\Delta| \frac{|I(\square(n))|}{\operatorname{dist}(\square(n), I)}
$$

$$
\leq M_5 |\Delta| \int_{|\Delta|}^{1} \frac{1}{x} \, dx = M_5 |\Delta| \, |\log(|\Delta|)|.
$$

In view of **FBL**, the proof of Lemma 5.1 is complete. ∎

In the next step we will estimate distortion on the holes of $\mathcal{A}_n$ adjacent to the flat interval I. We recall that the rotation number ρ of f has a continued fraction representation $\rho = [0, a_1, \ldots, a_n, \ldots]$.

LEMMA 5.2. *Denote by $\Delta \in \mathcal{A}_n$ the hole adjacent to I from the left and by Δ' the whole adjacent to I from the right.*

- *For every $n \in \mathbb{N}$ even and every $0 \leq k < q_{n+1}$, the logarithm of the distortion of f^k on $f(\Delta)$ tends uniformly and superexponentially to 0.*
- *For every $n \in \mathbb{N}$ odd and every $0 \leq i < a_{n+1}$, the logarithm of the distortion of $f^{q_n - 1}$ on $f^{iq_n + 1}(\Delta')$ tends uniformly and superexponentially to 0.*

Proof. For n even, $\Delta = \square_0^n$. Observe that the intervals $\{f^i(\Delta) : 0 \leq i < q_{n+1}\}$ and $\{f^{-j}(I) : 0 < j \leq q_n\}$ are pairwise disjoint. This means that the intervals $\{f^i(\Delta) : 0 \leq i < q_{n+1}\}$ are contained in the holes of $\mathcal{A}_{n-2}$ as $q_{n-1} + q_{n-2} \leq q_n$. Consider now those intervals of $\{f^i(\Delta) : 0 \leq i < q_{n+1}\}$ which lie in the hole $\Delta_1 := \square_0^{n-2} \in \mathcal{A}_{n-2}$ adjacent to $l(I)$. Every such $f^i(\Delta)$ is disjoint from $\{f^{-q_{n-2} - mq_{n-1}}(I) : 0 \leq m \leq a_n\}$ and thus it is contained in the union of the holes of $\mathcal{A}_{n-1} \setminus \{\Delta\}$.

Denote by $\mathcal{B}_{n-1}$ the set of all holes of $\mathcal{A}_{n-1}$ contained in $\Delta_1 \setminus \Delta$. As in the proof of Lemma 5.1 (see (27)), for every $z_1, z_2 \in f(\Delta)$ we have

$$
(33) \qquad \mathcal{D} := \left| \log \frac{df^k}{dx}(z_1) - \log \frac{df^k}{dx}(z_2) \right|
$$

$$
\leq \sum_{j=0}^{k-1} \left(\left| \frac{d^2 f}{dx^2}(\xi_j) \right| \Big/ \left| \frac{df}{dx}(\xi_j) \right| \right) |f^{j+1}(\Delta)|
$$

$$
\leq M_6 \left(\sum_{\square \in A_{n-2} \setminus \{\Delta_1\}} \frac{|\square|}{\operatorname{dist}(\square, I)} + \sum_{\square \in \mathcal{B}_{n-1}} \frac{|\square|}{\operatorname{dist}(\square, I)} \right),
$$

where $\xi_j \in f^{j+1}(\Delta)$. Since $\mathcal{B}_{n-1} \subset \mathcal{A}_{n-1} \setminus \{\Delta\}$, we apply the estimates (31) and (32) to conclude the proof.

For n *odd*, $\Delta' = \square_0^n$. Fix $0 \leq i < a_{n+1}$ and observe that the intervals $\{f^{iq_n+j}(\Delta') : 1 \leq j < q_n\}$ are pairwise disjoint and contained in the union of the holes of $\mathcal{A}_{n-1} \setminus \{\square_0^{n-1}\}$. The claim follows by Lemma 5.1. ∎

We conclude this section with a corollary that the graph of $f^{q_{n+1}}$ for n even and large in "the small scale" is similar to the graph of x^ℓ up to an affine change of coordinates.

COROLLARY 5.1. *There are uniform constants* n_0, M_7 *such that for every* $n \geq n_0$ *even and any two points* $z_1, z_2 \in \square_i^n$, $0 \leq i < q_{n+1}$,

$$
\frac{1}{M_7} \left(\frac{|z_1 - r(\square_i^n)|}{|z_2 - r(\square_i^n)|} \right)^{\ell-1} \leq \frac{df^{q_{n+1}}}{dx}(z_1) \Big/ \frac{df^{q_{n+1}}}{dx}(z_2) \leq M_7 \left(\frac{|z_1 - r(\square_i^n)|}{|z_2 - r(\square_i^n)|} \right)^{\ell-1}.
$$

Proof. Combine the estimates of Lemma 5.1, Lemma 5.2, and Fact 3.1. ∎

5.2. Miscelaneous estimates. Now, we are in a position to show that the holes of the nth partition are very small not only in comparison to the preimages of I which lie immediately to their right (see **FBL**) but also to their left.

LEMMA 5.3. *There are uniform constants* n_0, M_8 *such that for every* $n \geq n_0$ *even and every* $0 \leq i < q_{n+1}$,

$$
\frac{|f^{-i-q_n}(I)|}{|\square_i^n|} \geq (M_8 \operatorname{dist}(f^{q_{n-1}}(I), I))^{-a_{n+1}/\ell}.
$$

Proof. By **RRI** and the uniform version of **FBL**, the interval $f^{-i-q_n}(I)$ is very small in comparison to $f^{-i}(I)$. Hence, the ratio $|f^{-q_n-i}(I)|/|\square_i^n|$ is greater than a uniform constant multiple of $|f^{-q_n}(I)|/|\square_0^n|$. The lemma follows by the estimate (15). ∎

We will show that the map $f^{q_{n+1}}(z)$ is expanding with a large derivative provided $z \in \square_i^n$ is not too close to $l(\square_i^n)$.

LEMMA 5.4. *There are uniform constants n_0, M_9 with the following property.*

- *For every $z \in \square_i^n$, $0 \le i \le q_{n+1}$, and $n \ge n_0$ even,*

$$\frac{df^{q_{n+1}}}{dx}(z) \ge (M_9 \operatorname{dist}(f^{q_{n-1}}(I), I))^{-a_n/\ell} \left(\frac{|z - r(\square_i^n)|}{|\square_i^n|} \right)^{\ell-1}.$$

- *For every $z \in (l(\square_i^n), l(f^{-q_{n+1}-q_n-i}(I)), 0 \le i < q_{n+1}$, and $n \ge n_0$ odd,*

$$\frac{df^{q_{n+1}}}{dx}(z) \ge \frac{M_9}{\operatorname{dist}(f^{q_n}(I), I)}.$$

Proof. Assume that n is even. Note that $f^{-q_n-i}(I) \subset f^{q_{n+1}}(\square_i^n)$. By the Mean Value Theorem and the Convexity Lemma, there exists $\xi \in \square_i^n$ such that

$$\frac{df^{q_{n+1}}}{dx}(l(\square_i^n)) \ge \frac{df^{q_{n+1}}}{dx}(\xi) \ge \frac{|f^{-i-q_n}(I)|}{|\square_i^n|}.$$

Substitute the above inequality in the formula of Corollary 5.1 with z_1 and z_2 replaced, respectively, by z and $l(\square_i^n)$. The estimate follows by Lemma 5.3.

Assume now that n is odd. Since $f^{-q_{n+1}-q_n-i}(I) \subset \square_i^n$, there is $\xi \in f^{-q_{n+1}-q_n-i}(I)$ such that

$$\frac{df^{q_{n+1}}}{dx}(\xi) \ge \frac{|f^{-i-q_n}(I)|}{|\square_i^n|}.$$

By Lemma 5.1, the graph of $f^{q_{n+1}}$ on $(l(\square_i^n), l(f^{-q_{n+1}-q_n-i}(I))$ is almost linear. Hence, for every z from that interval,

$$\frac{df^{q_{n+1}}}{dx}(z) > M_{10} \frac{|f^{-i-q_n}(I)|}{|\square_i^n|}.$$

The second estimate of the lemma follows from (26). ∎

5.3. Nonlinearity. The one-form $\mathcal{N}f = (f''/f') \, dx$, called the *nonlinearity of f*, measures the infinitesimal change of the distortion of f at the point x. We will use this quantity to get convexity estimates of the second derivative of f^{q_n}.

FACT 5.1. *There exist uniform positive constants H_1 and H_2 with the following properties. For every x outside the flat interval I, we have*

$$|\mathcal{N}f(x)| \operatorname{dist}(x, I) \le H_1.$$

Additionally, if x is close enough to $l(I)$ then

$$\mathcal{N}f(x) \operatorname{dist}(x, I) \le -H_2.$$

Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a C^2 function. We say that f is *upper convex* if $f''(x) \le 0$ for every $x \in I$.

The *Convexity Lemma* states that $f^{q_{n+1}}$ on $\square_i^n$ is upper convex for n even and large.

LEMMA 5.5 (Convexity Lemma). *There are uniform constants $n_0, H > 0$ with the following property. If $n \geq n_0$ is even then for every $\square_i^n$ of the partition $\mathcal{A}_n$,*

$$|\square_i^n| \mathcal{N} f^{q_{n+1}}(x) \leq -H.$$

In particular, the graph of $f^{q_{n+1}}$ restricted to $\square_i^n$ is upper convex.

Proof. We choose n_0 so that for every n even and greater than n_0, the second derivative of f is negative on $\square_0^n$.

By Lemma 5.1, there is H_4 such that for every $z \in \square_i^n$,

$$\frac{df^i}{dx}(x) \geq H_4 \frac{|\square_0^n|}{|\square_i^n|}.$$

There is a uniform constant H_5 such that $\mathcal{N}^+ f := \max(0, \mathcal{N} f) \leq H_5$. After these preparations we are ready to estimate the rescaled nonlinearity $|\square_i^n| \mathcal{N} f^{q_{n+1}}(x)$ on the interval $\square_i^n$. By the chain rule, Lemmas 5.1, 5.2, and Fact 5.1,

$$|\square_i^n| \mathcal{N} f^{q_{n+1}}(x) = |\square_i^n| \sum_{j=0}^{q_{n+1}-1} \mathcal{N} f(f^j(x)) \frac{df^j}{dx}(x)$$

$$\leq H_6 |\square_0^n| \mathcal{N} f(f^i(x)) + H_7 \sum_{j=0, j \neq i}^{q_{n+1}-1} \mathcal{N}^+ f(f^j(x)) \frac{df^j}{dx}(x) |\square_i^n|$$

$$\leq -H_8 \frac{|\square_0^n|}{\operatorname{dist}(f^i(x), I)} + H_9 \sum_{j=0, j \neq i}^{q_{n+1}-1} |f^j(\square_i^n)|$$

$$\leq -H_8 + H_9 \sum_{\square \in \mathcal{A}_{n-1} \cup \mathcal{A}_{n-2}} |\square| \leq -H_{10},$$

as the total length of the holes of $\mathcal{A}_{n-1}$ and $\mathcal{A}_{n-2}$ tends to 0 uniformly superexponentially. The estimate

$$\frac{df^j}{dx}(x) |\square_i^n| \leq H_{10} |f^j(\square_i^n)| \quad \text{for every } 0 \leq j < q_{n+1}$$

also uses Fact 3.1. ∎

6. Scaling rules—lower bounds. The proof of **SBL** falls into two parts.

6.1. Proof of SBL when $f^{-q_n}(I)$ is to the left of I. Throughout the proof of **SBL** we will usually use the letter A with suitable subscripts to denote subsequent uniform constants appearing in the estimates. By Lemma 4.1,

(34) $\quad$ $\mathrm{dist}(f^{q_{n+1}}(I), I) \geq |f^{-q_{n+2}}(I)| \geq K_6 |r(I) - l(f^{-q_{n+1}}(I))|^{1/\ell}.$

Since the first derivative of f is bounded,

(35) $\quad$ $|r(I) - l(f^{-q_{n+1}}(I))| \geq A_0 |f(I) - l(f^{-q_{n+1}+1}(I))|.$

By Fact 3.1,

(36) $\quad$ $|r(f^{-q_n+1}(I)) - f(I)| \geq K_1 |r(f^{-q_n}(I)) - l(I)|^{\ell}.$

Write **RRI** for $\{r(f^{-q_n+1}(I)), f(I), l(f^{-q_{n+1}+1}(I)); q_{n+1} - 1\}$ and then combine it with (35), (36) to obtain

(37)
$$\frac{|r(I) - l(f^{-q_{n+1}}(I))|}{|r(f^{-q_n}(I)) - l(I)|^l} \geq \frac{A_0 K_1}{R_1} \frac{|f^{q_{n+1}}(I) - l(I)|}{|f^{-q_n+q_{n+1}}(I) - f^{q_{n+1}}(I)|}$$
$$\geq A_1 \frac{|f^{q_{n+1}}(I) - l(I)|}{|l(f^{-q_n}(I)) - l(I)|},$$

the last inequality being a consequence of the fact that $f^{-q_n+q_{n+1}}(I) \in \square_{q_n}^n$ (Fact 3.3 and **FBL** imply that $|f^{-q_n+q_{n+1}}(I) - l(f^{-q_n}(I))|$ is much smaller than $|f^{-q_n}(I)|$).

Substituting (34) into (37) and then taking the ℓth root of both sides, we get

(38)
$$\frac{|f^{q_{n+1}}(I) - l(I)|}{|r(f^{-q_n}(I)) - l(I)|} \geq A_2 \frac{|f^{q_{n+1}}(I) - l(I)|^{1/\ell}}{|l(f^{-q_n}(I)) - l(I)|^{1/\ell}}.$$

Let $b_n = |f^{q_n}(I)) - l(I)|$. By **FBL** (see also Lemma 4.2), b_{n-1} and $|l(f^{-q_n}(I)) - l(I)|$ are equal up to a superexponentially small error when n goes to infinity. Also, $|r(f^{-q_n}(I)) - l(I)| \geq b_{n+1}$. Therefore, (38) becomes

$$\left(\frac{b_{n+1}}{b_{n-1}}\right)^{(\ell-1)/\ell} \geq A_3 \frac{|r(f^{-q_n}(I)) - l(I)|}{|l(f^{-q_n}(I)) - l(I)|}.$$

Applying **RRI** for $\{l(f^{-q_n+1}(I)), r(f^{-q_n+1}(I)), f(I); q_n - 1\}$ and using the fact that $|\square_0^n|$ is superexponentially smaller than $|f^{-q_n}(I)|$, we get

(39)
$$\left(\frac{|r(f^{-q_n}(I)) - l(I)|}{|l(f^{-q_n}(I)) - l(I)|}\right)^{\ell} \geq A_4 b_n,$$

and consequently

$$\left(\frac{b_{n+1}}{b_{n-1}}\right)^{\ell-1} \geq A_5 b_n.$$

Our objective now is to find a recurrence formula for b_{n+1}/b_{n-1}. To this end we prove the following lemma.

LEMMA 6.1. *For every* $2 \leq i \leq a_{n+1}$,

$$\frac{|r(I) - f^{(i-1)q_n}(I)|}{|r(I) - f^{iq_n}(I)|} \geq A_6 |r(I) - l(f^{-q_{n-1}}(I))|.$$

Proof. By Lemmas 4.1 and 3.1,

(40) $$|f(I) - l(f^{-q_{n-1}+1}(I))| \leq A_7|f(I) - l(f^{-q_n+1}(I))|.$$

By (40) and **RRI** written for $\{l(f^{-q_n+1}(I)), f(I), f^{(i-1)q_n+1}(I); q_n - 1\}$,

$$\frac{|r(I) - f^{(i-1)q_n}(I)|}{|r(I) - l(f^{-q_{n-1}}(I))|} \geq A_8 \frac{|f(I) - f^{(i-1)q_n+1}(I)|}{|f(I) - l(f^{-q_n+1}(I))|}$$

$$\geq A_9 \frac{|f^{q_n}(I) - f^{iq_n}(I)|}{|f^{q_n}(I) - l(I))|}$$

$$\geq A_{10}|f^{q_n}(I) - f^{iq_n}(I)|.$$

Since $(f^{q_n}(I), f^{iq_n}(I)) \supset f^{-q_{n+1}+q_n}(I)$, the lemma follows by Fact 3.3. ∎

Multiplying through the inequalities of Lemma 6.1 for $2 \leq i \leq a_{n+1}$, we obtain

(41) $$b_n \geq |r(I) - f^{a_{n+1}q_n}(I)|(A_{10}|r(I) - l(f^{-q_{n-1}}(I))|)^{a_{n+1}-1}.$$

Note that $f^{-q_n-q_{n-1}}(I) \subset (r(I), f^{a_nq_n}(I))$. Hence, Lemma 5.3 implies that the intervals $(r(I), f^{a_nq_n}(I))$ and $(r(I), l(f^{-q_{n-1}}(I)))$ are of the same size up to a superexponentially small error. Hence,

(42) $$b_n \geq (A_{11}|r(I) - l(f^{-q_{n-1}}(I))|)^{a_{n+1}}.$$

The inequality (42) is also true for $a_{n+1} = 1$ and follows directly from Lemma 5.3 and the inequality

$$r(f^{-q_{n+1}}(I)) = r(f^{-q_n-q_{n-1}}(I)) \leq f^{q_n}(I) \leq l(f^{-q_{n-1}}(I)).$$

Indeed, $|f^{-q_{n-1}-q_n}(I)| \geq M_8|\square_{q_{n-1}}^n| = M_8\,|r(f^{-q_{n+1}}(I)) - l(f^{-q_{n-1}}(I))|$ and hence b_n and $|r(I) - l(f^{-q_{n-1}}(I))|$ are uniformly comparable.

The inequality

(43) $$\frac{|r(I) - l(f^{-q_{n-1}}(I))|}{b_{n-1}^\ell} \geq A_{12}b_{n-1}(\mathrm{dist}(f^{2q_{n-1}}(I), f^{q_{n-1}}(I)))^{-1}$$

$$\geq A_{12}\frac{b_{n-1}}{b_{n-3}}$$

is a simple consequence of Fact 3.1 and **RRI** written for $\{f^{q_{n-1}+1}(I), f(I), l(f^{-q_{n-1}}(I)); q_{n-1} - 1\}$. By **FBL**, the quantity A_{12}/b_{n-3} tends uniformly to ∞. The inequality (43) becomes

(44) $$|r(I) - l(f^{-q_{n-1}}(I))| \geq A_{13}(b_{n-1})^{\ell+1},$$

where A_{13} can be made arbitrarily large by increasing n.

Substituting (44) into (42), we obtain

(45) $$b_n \geq (A_{14}b_{n-1})^{(\ell+1)a_{n+1}},$$

which together with (39) gives the recurrence formula

$$b_{n+1} \geq (A_{15}b_{n-1})^{1+a_{n+1}\frac{\ell+1}{\ell-1}}.$$

There is a uniform constant N such that $A_{14}, A_{15} > 1$ for every $n \geq N$. We may assume that N is odd. Let $\omega = b_N^{1/\overline{\gamma}(N)}$. The recurrence formula yields for every $n + 1 \geq N$,

$$(46) \qquad b_{n+1} \geq A_{15} b_N^{\overline{\gamma}(n+1)/\overline{\gamma}(N)} = A_{15}\, \omega^{\overline{\gamma}(n+1)} \geq \omega^{\overline{\gamma}(n+1)}.$$

This proves the first formula of **SBL** for n *odd*. The second formula of **SBL** for n *even* is an immediate consequence of the first and the inequalities (39) and (45) if only A_{14} is chosen large enough.

6.2. Proof of SBL when f^{-q_n} is to the right of I. The first inequality follows directly from (45) and (46). We adjust the indices in these formulas to reflect the fact that n is assumed to be odd in Subsection 6.2. Since $A_{14} > 1$, there is a uniform constant A_{16} such that for every $n \geq N$ *odd*,

$$b_{n+1} \geq A_{14} b_n^{(\ell+1)a_{n+2}} \geq A_{14}\omega^{(\ell+1)a_{n+2}\overline{\gamma}(n)} \geq \omega^{A_{16}\overline{\gamma}(n+1)}.$$

The second inequality can be obtained from **CRI** written for

$$\{r(I), l(f^{-q_n}(I)), r(f^{-q_n}(I)), l(f^{-q_n - q_{n-1}}(I)); q_n\}.$$

Indeed, using $b_n \geq A_{15}\omega^{\overline{\gamma}(n)}$ and $b_{n-1} \geq \omega^{A_{16}\overline{\gamma}(n-1)}$, we deduce that for every $n \geq N$ *odd*,

$$\frac{|r(I) - l(f^{-q_n}(I))|}{|f^{-q_n}(I)|} \geq A_{10} b_n b_{n-1} \geq (A_{10}A_{15})\omega^{\overline{\gamma}(n) + A_{16}\overline{\gamma}(n-1)} \geq \omega^{A_{17}\overline{\gamma}(n)},$$

where A_{17} is a uniform constant and A_{15} is chosen so that $A_{15}A_{10} > 1$.

References

[1]   J. Graczyk, L. B. Jonker, G. Świątek, F. M. Tangerman and J. J. P. Veerman, *Differentiable circle maps with a flat interval*, Comm. Math. Phys. 173 (1995), 599–622.
[2]   R. Ito, *Rotation sets are closed*, Math. Proc. Cambridge Philos. Soc. 89 (1981), 107–111.
[3]   M. Misiurewicz, *Rotation interval for a class of maps of the real line into itself*, Ergodic Theory Dynam. Systems 6 (1986), 117–132.

[4] G. Świątek, *Rational rotation numbers for maps of the circle*, Comm. Math. Phys. 119 (1988), 109–128.

[5] J. J. P. Veerman and F. M. Tangerman, *Scalings in circle maps I*, ibid. 134 (1990), 89–107.

Jacek Graczyk
Université de Paris-Sud, Mathématique
91405 Orsay, France
E-mail: jacek.graczyk@math.u-psud.fr