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The tree property at both $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$

by

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Abstract. We force from large cardinals a model of ZFC in which $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$ both have the tree property. We also prove that if we strengthen the large cardinal assumptions, then in the final model $\aleph_{\omega+2}$ even satisfies the super tree property.

1. Introduction. Given a regular cardinal κ, we say that κ has the *tree property* when every κ-tree (i.e. every tree of height κ with levels of size less than κ) has a branch of length κ. The tree property provides a combinatorial characterisation of weakly compact cardinals.

THEOREM 1.1 (Erdős and Tarski [3]). *An inaccessible cardinal is weakly compact if and only if it satisfies the tree property.*

König's lemma establishes that the tree property holds at $\aleph_0$. On the other hand $\aleph_1$ does not satisfy the tree property (Aronszajn, 1934), and for larger regular cardinals whether they satisfy the tree property depends on the model we are considering, as the following theorems suggest:

- (Specker [16]) If $\tau^{<\tau} = \tau$, then the tree property fails at τ^+.
- (Mitchell [13]) Assume that τ is a regular cardinal such that $\tau^{<\tau} = \tau$ and λ is a weakly compact cardinal above τ. Then there is a forcing notion that preserves cardinals up to τ, turns λ into τ^{++} and forces the tree property at τ^{++}.

There is a wide literature concerning the construction of models of ZFC in which distinct regular small cardinals simultaneously satisfy the tree property. We list a few classical results of that sort:

- (Abraham [1]) Assume GCH and assume that κ is a supercompact cardinal and $\lambda > \kappa$ is a weakly compact cardinal. Then there is a forcing notion that makes $\kappa = \aleph_2$, $\lambda = \aleph_3$, and forces the tree property at $\aleph_2$ and $\aleph_3$.

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- (Cummings and Foreman [2]) Assume that infinitely many supercompact cardinals exist in a model of ZFC + GCH. Then there is a forcing notion that forces a model where the tree property holds simultaneously at every cardinal of the form $\aleph_{2+n}$ with $n < \omega$.
- (Sinapova [15] based on Magidor and Shelah [12]) Assume that ω-many supercompact cardinals exist in a model of GCH. Then there is a forcing notion that forces a model where the tree property holds at $\aleph_{\omega+1}$ and where $\aleph_\omega$ is strong limit.
- (Neeman [14]) Assume that infinitely many supercompact cardinals exist. Then one can force a model where the tree property holds at every $\aleph_{2+n}$ for finite n and at $\aleph_{\omega+1}$ with $\aleph_\omega$ strong limit.
- (Friedman and Halilović [5]) Assume that a weakly compact hypermeasurable cardinal exists. Then there is a forcing notion that forces the tree property at $\aleph_{\omega+2}$ with $\aleph_\omega$ strong limit.
- (Gitik [6]) Assume that there exists an increasing sequence $\langle\kappa_n\rangle_{n<\omega}$ of cardinals such that $o(\kappa_n) = \kappa_n^{+n+2}$, and a weakly compact cardinal λ above $\lim_{n<\omega}\kappa_n$. Then there is a forcing notion where the tree property holds at $\aleph_{\omega+2}$ with $\aleph_\omega$ strong limit.

All these results were oriented toward the construction of a model of ZFC where the tree property holds simultaneously at every regular cardinal; whether such a model can be found is still an open problem. The consistency of the tree property at $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$ is related to the more general problem of whether the successor and double successor of a singular cardinal can simultaneously satisfy the tree property. A partial answer to that problem has recently been provided by Unger [18] who proved the following:

Theorem 1.2 (Unger [18]). *Assume the existence of a supercompact cardinal λ and a weakly compact cardinal $\mu > \lambda$ in a model V of ZFC. Then there is a forcing extension of V where λ is singular strong limit of cofinality ω, the singular cardinal hypothesis fails at λ, there are no special Aronszajn trees at λ^+ and the tree property holds at λ^{++}.*

The property that there are no special Aronszajn trees is a weak version of the tree property. In this paper we show that one can force from large cardinals the (full) tree property simultaneously at $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$, provided we drop the requirement that $\aleph_\omega$ be strong limit. We also prove that if we strengthen the large cardinal assumptions, then in the final model we also have the *super tree property* at $\aleph_{\omega+2}$. The super tree property is a strong version of the tree property that provides a combinatorial characterisation of supercompact cardinals similar to the characterisation of weakly compact cardinals discussed above.

THEOREM 1.3 (Weiss [19], Jech [7] and Magidor [11]). *An inaccessible cardinal is supercompact if and only if it satisfies the super tree property.*

The definition of the super tree property is given in Section 6. Just as for the tree property, several consistency results have been proved for the super tree property at small cardinals:

- (Weiss [20]) Assume that κ is a supercompact cardinal. Then for every regular $\tau < \kappa$ such that $\tau^{<\tau} = \tau$, there is a forcing notion (Mitchell forcing) that preserves cardinals up to τ, turns κ into τ^{++} and forces the super tree property at τ^{++}. In particular, for every $n < \omega$, one can force from large cardinals the super tree property at $\aleph_{n+2}$.
- (Fontanella [4]) Assume that infinitely many supercompact cardinals exist in a model of ZFC + GCH. Then there is a forcing notion (due to Cummings and Foreman) that forces a model where the super tree property holds simultaneously at every cardinal of the form $\aleph_{n+2}$ with $n < \omega$.

Whether one can force the super tree property at $\aleph_{\omega+1}$ from large cardinals remains an open problem.

In Section 2 we introduce some notation and we list some basic results that will be used in the final proof. Section 3 is devoted to the main properties of our forcing construction. In Section 4 we show that in a generic extension the tree property holds at $\aleph_{\omega+1}$; then we show in Section 5 that $\aleph_{\omega+2}$ has the tree property in the same model. Finally, we prove in Section 6 that if we strengthen the large cardinal assumptions, then $\aleph_{\omega+2}$ even satisfies the super tree property.

2. Preliminaries and notation. The main reference for basic set theory is [8], while we will refer to [9] for large cardinal notions and to [10] for the forcing technique. Given a forcing $\mathbb{P}$ and conditions $p, q \in \mathbb{P}$, we use $p \leq q$ in the sense that p is stronger than q. Assume that $\mathbb{P}$ is a forcing notion in a model V; we will use $V^{\mathbb{P}}$ to denote the class of $\mathbb{P}$-names. If $G \subseteq \mathbb{P}$ is a generic filter over V, then $V[G]$ denotes the generic extension of V determined by G. If $a \in V^{\mathbb{P}}$ and $G \subseteq \mathbb{P}$ is generic over V, then a^G denotes the interpretation of a in $V[G]$. Every element x of the ground model V is represented in a canonical way by a name $\check{x}$. However, to simplify the notation, we will use just x instead of $\check{x}$ in forcing formulas.

Given a forcing $\mathbb{P}$ and a cardinal κ, we say that $\mathbb{P}$ has the *κ-covering property* if $\mathbb{P}$ preserves κ as a cardinal, and for every filter $G \subseteq \mathbb{P}$ generic over V, every set $X \subseteq V$ in $V[G]$ of cardinality less than κ is contained in a set $Y \in V$ of cardinality less than κ in V. We denote by $\mathrm{Coll}(\kappa, \lambda)$ the usual Levy collapse of λ to κ. Furthermore, $\mathrm{Add}(\kappa, \lambda)$ is the set of all partial functions $p : \lambda \to 2$ of size $< \kappa$, partially ordered by reverse inclusion.

Given two forcings $\mathbb{P}$ and $\mathbb{Q}$ in a model V of set theory, we recall that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent when

1. for every filter $G_{\mathbb{P}} \subseteq \mathbb{P}$ which is generic over V, there exists a filter $G_{\mathbb{Q}}$ which is generic over V and $V[G_{\mathbb{P}}] = V[G_{\mathbb{Q}}]$;
2. for every filter $G_{\mathbb{Q}} \subseteq \mathbb{Q}$ which is generic over V, there exists a filter $G_{\mathbb{P}}$ which is generic over V and $V[G_{\mathbb{Q}}] = V[G_{\mathbb{P}}]$.

To prove that two forcings $\mathbb{P}$ and $\mathbb{Q}$ are equivalent it is enough to define a dense embedding $i : \mathbb{P} \to \mathbb{Q}$, i.e. an order preserving map such that $\mathrm{Im}(i) \subseteq \mathbb{Q}$ is dense, and for every p, q in $\mathbb{P}$, if p and q are incompatible, then $i(p)$ and $i(q)$ are incompatible in $\mathbb{Q}$.

Definition 2.1. Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcings with greatest elements $1_{\mathbb{P}}$ and $1_{\mathbb{Q}}$ respectively. Then $\pi : \mathbb{P} \to \mathbb{Q}$ is a *projection* if

(1) π is order preserving;
(2) $\pi(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$;
(3) for every $p \in \mathbb{P}$ and for every $q \leq \pi(p)$ there exists $p' \leq p$ such that $\pi(p') \leq q$.

If $\pi : \mathbb{P} \to \mathbb{Q}$ is a projection, then we can factor forcing with $\mathbb{P}$ as forcing with $\mathbb{Q}$ followed by forcing with $\{p;\ \pi(p) \in H\}$ over the $\mathbb{Q}$-generic extension $V[H]$.

We now present a few lemmas that will be used in later sections.

Lemma 2.2 (Easton). *Let κ be regular. If $\mathbb{P}$ is a κ-c.c. forcing and $\mathbb{Q}$ is a κ-closed forcing, then the following hold:*

(1) $\Vdash_{\mathbb{Q}} \mathbb{P}$ *is κ-c.c.*;
(2) $\Vdash_{\mathbb{P}} \mathbb{Q}$ *is κ-distributive*;
(3) *if $G \subseteq \mathbb{P}$ is a generic filter over V and $H \subseteq \mathbb{Q}$ is a generic filter over V, then G and H are mutually generic over V*;
(4) $\mathbb{P} \times \mathbb{Q}$ *has the κ-covering property, i.e. for G and H as in claim* (3), *if $X \in V[G][H]$ is a set of ordinals of size $< \kappa$ in $V[G][H]$, then there is $Y \supseteq X$ in V of size less than κ in V*;
(5) *if $\mathbb{R}$ is a κ-closed forcing, then $\Vdash_{\mathbb{P}\times\mathbb{Q}}$ $\mathbb{R}$ is κ-distributive.*

Sketch of proof. (1) Let $\dot{A}$ be a $\mathbb{Q}$-name for an antichain of size κ in $\mathbb{P}$. We can build a decreasing sequence $\langle q_\alpha \rangle_{\alpha\in\kappa}$ of conditions in $\mathbb{Q}$ such that q_α decides the αth element of the antichain to be some element p_α of $\mathbb{P}$. Then the p_α's form an antichain of size κ in V, a contradiction.

(2) Let X be a sequence of ordinals of length $< \kappa$ in a forcing extension $V[G_{\mathbb{Q}} \times G_{\mathbb{P}}]$ where $G_{\mathbb{Q}}$ is generic for $\mathbb{Q}$ over V and $G_{\mathbb{P}}$ is generic for $\mathbb{P}$ over V. By the previous claim, X has a $\mathbb{P}$-name $\dot{X}$ of size $< \kappa$ in $V[G_{\mathbb{Q}}]$. As $\mathbb{Q}$ is κ-closed, $\dot{X}$ belongs to V, hence $X \in V[G_{\mathbb{P}}]$.

(3) It follows from the previous claims that every maximal antichain A of $\mathbb{P}$ in $V[H]$ has size $< \kappa$, so A is in V.

(4) By the distributivity of $\mathbb{Q}$ in $V[G]$, we have $X \in V[G]$. As $\mathbb{P}$ is κ-c.c., we get the result.

(5) By the previous claims, $\mathbb{R}\times\mathbb{Q}$ is κ-distributive in any generic extension $V[G_{\mathbb{P}}]$ by $\mathbb{P}$. ■

We say that κ is *indestructibly supercompact* when it is supercompact and its supercompactness is preserved in any forcing extension by a κ-directed closed forcing.

LEMMA 2.3 (Neeman [14] based on Abraham [1]). *Let $V \subseteq W$ be two models of set theory and let $\tau < \kappa$ be such that*

(1) *every set of ordinals of size $< \kappa$ in W is covered by a set of ordinals of size $< \kappa$ in V;*
(2) *in V, κ is inaccessible and τ is regular;*
(3) *κ, τ remain cardinals in W.*

Let $\mathbb{P}$ be a forcing notion in V whose conditions are functions of size $< \tau$ in V ordered by reverse inclusion. Then every family $\mathscr{F}$ of conditions of $\mathbb{P}$ of size κ in W can be refined to a family of the same size that forms a Δ-system.

Proof. Let $\mathscr{F} = \{p_\alpha\}_{\alpha<\kappa}$ be a family of conditions. We fix θ large enough for the following argument. For stationary many $M \prec H_\theta$ of size less than κ there exists a set $X_M \in M$ of size $< \kappa$ such that $p_{M\cap\kappa} \cap M \subseteq X_M$. By Fodor's theorem the function $M \mapsto X_M$ is constant on a stationary set S. Let X be such that $X_M = X$ for every $M \in S$. As X has size $< \kappa$, it is covered by a set $Y \in V$ of size $< \kappa$ in V. So for every $M \in S$, we have $p_{M\cap\kappa} \cap M \in [Y]^{<\tau}$. As $|Y| < \kappa$ and κ is inaccessible in V, the set $([Y]^{<\tau})^V$ has size $< \kappa$ in V, hence in W. It follows that there is a function $q \subseteq Y$ such that $p_{M\cap\kappa} \cap M = q$ for every M in a cofinal set $S' \subseteq S$. Let $\mathscr{F}' := \{p_{M\cap\kappa};\, M \in S'\}$. Then $\mathscr{F}'$ forms a Δ-system with root q. Indeed, given $M, M' \in S'$, we can find $M'' \in S'$ containing both $p_{M\cap\kappa}$ and $p_{M'\cap\kappa}$ as subsets; then $p_{M\cap\kappa} \cap p_{M'\cap\kappa} = p_{M\cap\kappa} \cap p_{M'\cap\kappa} \cap M'' = q$, hence $p_{M\cap\kappa}$ and $p_{M'\cap\kappa}$ are compatible. ■

LEMMA 2.4 (Magidor and Shelah [12]). *Assume that λ is a singular cardinal of countable cofinality and T is a λ^+-tree. If $\mathbb{R}$ is an ω_1-closed forcing, then $\mathbb{R}$ does not add cofinal branches to T.*

Proof. Suppose for a contradiction that $\dot{b}$ is an $\mathbb{R}$-name for a new cofinal branch. We inductively define a sequence $\langle r_\sigma\rangle_{\sigma\in\lambda^{<\omega}}$ of conditions in $\mathbb{R}$ such that:

- if $\sigma_1 \subseteq \sigma_2$ in $\lambda^{<\omega}$, then $r_{\sigma_2} \leq r_{\sigma_1}$ in $\mathbb{R}$;

- given $\sigma \in \lambda^{<\omega}$, for every pair σ_1, σ_2 of distinct immediate extensions of σ, there exists an ordinal $\alpha_{\sigma_1,\sigma_2}$ such that r_{σ_1} and r_{σ_2} force contradictory information about $\dot{b} \cap \mathrm{Lev}_{\alpha_{\sigma_1,\sigma_2}}$ (i.e. there are $a_1 \neq a_2$ in T such that $r_{\sigma_i} \Vdash a_i \in \dot{b}$ for $i \in \{1,2\}$).

Let β be the supremum of all the ordinals $\alpha_{\sigma_1,\sigma_2}$ so defined. By the closure of $\mathbb{R}$, we can find for every $f \in \lambda^\omega$ a condition r_f stronger than every condition in the sequence $\langle r_{f\restriction n}; n < \omega\rangle$ and such that r_f determines $\dot{b} \cap \mathrm{Lev}_\beta$. By construction, for any two distinct f, g in λ^ω, the conditions r_f and r_g force distinct values for $\dot{b} \cap \mathrm{Lev}_\beta$. It follows that $|\mathrm{Lev}_\beta| \geq \lambda^\omega$, contradicting the assumption that T is a λ^+-tree. ∎

Lemma 2.5 (Unger [17]). *Let $\mathbb{P}$ be a forcing notion such that $\mathbb{P} \times \mathbb{P}$ is κ-c.c. Then $\mathbb{P}$ has the κ-*approximation property*, i.e. given a set A of ordinals in a $\mathbb{P}$-generic extension $V[G]$, if $A \cap x \in V$ for every $x \in V$ of size $< \kappa$, then $A \in V$.*

Proof. Suppose for a contradiction that for some ordinal τ there exists a $\mathbb{P}$-name $\dot{A}$ such that

$$\Vdash_{\mathbb{P}} \dot{A} \subseteq \tau, \quad \forall x \in [\tau]^{<\kappa}(\dot{A} \cap x \in V) \text{ and } \dot{A} \notin V.$$

We inductively define conditions $\langle p_i^0, p_i^1\rangle_{i<\kappa}$ in $\mathbb{P}\times\mathbb{P}$, sets $\langle d_i^0, d_i^1\rangle_{i<\kappa}$ in $[\tau]^{<\kappa}$ and a $\subseteq$-strictly increasing sequence $\langle x_i\rangle_{i<\kappa}$ in $[\tau]^{<\kappa}$ such that

- for $\varepsilon \in \{0,1\}$, we have $p_i^\varepsilon \Vdash \dot{A} \cap x_i = d_i^\varepsilon$;
- $d_i^0 \neq d_i^1$ and $d_i^0 \cap \bigcup_{j<i} x_j = d_i^1 \cap \bigcup_{j<i} x_j$.

Suppose we have constructed $\langle p_i^0, p_i^1\rangle_{i<j}$, $\langle d_i^0, d_i^1\rangle_{i<j}$ and $\langle x_i\rangle_{i<j}$ successfully. Let $x := \bigcup_{i<j} x_i$, and let p be any condition in $\mathbb{P}$ deciding the value of $\dot{A} \cap x$ to be some set $d \in [\tau]^{<\kappa}$. As $\dot{A}$ does not belong to V, we can find $p_j^0, p_j^1 \leq p$, distinct d_j^0, d_j^1 and $x_j \supset x$ such that $p_j^\varepsilon \Vdash \dot{A} \cap x_j = d_j^\varepsilon$ for $\varepsilon \in \{0,1\}$. Then $p_j^\varepsilon \Vdash d = \dot{A} \cap x = d_j^\varepsilon \cap x$, hence $d_j^0 \cap x = d = d_j^1 \cap x$.

Now we claim that $\langle p_i^0, p_i^1\rangle_{i<\kappa}$ is an antichain, contradicting the κ-chain condition at $\mathbb{P}\times\mathbb{P}$. Suppose that for some $i < j$, the conditions (p_i^0, p_i^1) and (p_j^0, p_j^1) are compatible. Then $d_j^0 \cap x_i = d_i^0$ and $d_j^1 \cap x_i = d_i^1$. By construction, $d_j^0 \cap \bigcup_{l<j} x_l = d_j^1 \cap \bigcup_{l<j} x_l$, in particular $d_j^0 \cap x_i = d_j^1 \cap x_i$, contradicting $d_i^0 \neq d_i^1$. ∎

Lemma 2.6 (Silver). *For a regular cardinal λ let T be a λ-tree. Assume that $\mathbb{P}$ is a κ^+-closed forcing where $\kappa < \lambda$ and $2^\kappa \geq \lambda$. Then every cofinal branch through T in a generic extension $V[G]$ by $\mathbb{P}$ is already in V.*

Proof. We may assume that κ is minimal with $2^\kappa \geq \lambda$. Suppose for a contradiction that there exists a $\mathbb{P}$-name $\dot{b}$ for a new cofinal branch. We can build by induction, for each $s \in {}^{\leq\kappa}2$, conditions p_s and nodes x_s such that

- if s properly extends t, then $p_s \le p_t$ and $x_s >_T x_t$;
- $p_s \Vdash x_s \in \dot{b}$;
- for each α, the nodes $\{x_s;\ s \in {}^{\alpha}2\}$ are all on the same level, say η_α;
- for every $s \in {}^{<\kappa}2$, the nodes $x_{s^\frown 0}$ and $x_{s^\frown 1}$ are incomparable.

By the minimality of κ, for every $\alpha < \kappa$ the set $\{x_s;\ s \in {}^{\alpha}2\}$ has size less than λ, so we can choose $\eta_{\alpha+1}$. The closure of $\mathbb{P}$ guarantees that the construction works at limit stages. This leads to a contradiction, because the level η_κ of T must have fewer than λ elements, yet we have constructed 2^κ distinct ones. ∎

A generalisation of this lemma gives us the following result.

LEMMA 2.7 (Weiss [19, Proposition 2.1.12]). *Let κ be a regular cardinal and $\theta \ge \kappa$ be any ordinal. Assume that $\mathbb{Q}$ is an η^+-closed forcing with $\eta < \kappa \le 2^\eta$. Then $\mathbb{Q}$ has the* thin *κ-approximation property, i.e. if $\dot{b}$ is a $\mathbb{Q}$-name for a subset of θ and for every $x \in [\theta]^{<\kappa}$ we have $\Vdash_{\mathbb{Q}} \dot{b} \cap x \in V$ and $\{y \subseteq x;\ \exists q \in \mathbb{Q}(q \Vdash y = \dot{b} \cap x)\}$ has size less than κ, then $\Vdash_{\mathbb{Q}} b \in V$.*

The proof of such a lemma is obtained by a modification of the proof of Silver's lemma above. See Weiss [19] for more details.

3. The main forcing. We fix an increasing sequence $\langle \kappa_n \rangle_{n<\omega}$ of supercompact cardinals such that each κ_n is indestructible by κ_n-directed closed forcings. Let $\lambda := \lim_{n<\omega} \kappa_n$; we assume that a weakly compact cardinal μ exists above λ.

Consider the product $\mathbb{C} := \prod_{0<n<\omega} \mathrm{Coll}(\kappa_n, {<}\kappa_{n+1})$. Observe that for every $m < \omega$, we can write $\mathbb{C}$ as a product of a κ_m-Knaster forcing $\mathbb{C}_m := \prod_{n<m} \mathrm{Coll}(\kappa_n, {<}\kappa_{n+1})$ with a κ_m-directed closed forcing

$$\mathbb{C}^m := \prod_{n \ge m} \mathrm{Coll}(\kappa_n, {<}\kappa_{n+1}).$$

Assume that

$$I := \{\nu < \kappa_0;\ \nu \text{ is a singular strong limit cardinal of cofinality } \omega\}.$$

We let

$$\mathbb{S} := \sum_{\nu \in I} \mathrm{Coll}(\omega, \nu) \times \mathrm{Coll}(\nu^+, {<}\kappa_0),$$

that is, a condition of $\mathbb{S}$ is of the form (ν, a, b) where $\nu \in I$ and $(a, b) \in \mathrm{Coll}(\omega, \nu) \times \mathrm{Coll}(\nu^+, {<}\kappa_0)$, or it is the maximal element $\mathbf{1}_{\mathbb{S}}$. We have $(\nu, a, b) \le (\nu', a', b')$ if and only if $\nu = \nu'$, $a \le a'$ and $b \le b'$.

We denote by $\mathbb{A}$ the poset $\mathrm{Add}(\kappa_0, \mu)$, and for every $X \subseteq \mu$ we let $\mathbb{A}{\restriction}X$ be the set of all functions $f{\restriction}X$ for $f \in \mathbb{A}$. For α between λ^+ and μ, we let $\dot{Q}(\alpha)$ be an $\mathbb{A}{\restriction}\alpha \times \mathbb{C}_1 \times \mathbb{S}$-name for the poset that adds a λ^+-Cohen subset to λ^+. To force the tree property at $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$ we will use the following poset.

Definition 3.1. $\mathbb{M}$ is a forcing notion whose conditions are tuples (p, c, s, q) such that

(1) $p \in \mathbb{A}$;
(2) $c \in \mathbb{C}_1$;
(3) $s \in \mathbb{S}$;
(4) q is a function of size $\leq \lambda$ such that every $\alpha \in \mathrm{dom}(q)$ is a cardinal between λ^+ and μ, and $\Vdash_{\mathbb{A}\restriction\alpha\times\mathbb{C}_1\times\mathbb{S}} q(\alpha) \in \dot{Q}(\alpha)$.

We let $(p, c, s, q) \leq (p', c', s', q')$ iff $p \leq p'$, $c \leq c'$, $s \leq s'$, $\mathrm{dom}(q') \subseteq \mathrm{dom}(q)$ and for every $\alpha \in \mathrm{dom}(q')$, $(p\restriction\alpha, c, s) \Vdash q(\alpha) \leq q'(\alpha)$.

Definition 3.2. We define $\mathbb{Q}$ as the poset of all functions q such that $(0, 0, \mathbf{1}_\mathbb{S}, q)$ is a condition of $\mathbb{M}$. The ordering on $\mathbb{Q}$ is defined by

$$q \leq q' \Leftrightarrow (0, 0, \mathbf{1}_\mathbb{S}, q) \leq (0, 0, \mathbf{1}_\mathbb{S}, q').$$

We list some basic properties of $\mathbb{M}$:

Lemma 3.3. *The following hold for* $\mathbb{M}$*:*

(1) $\mathbb{M}$ *is a projection of* $\mathbb{A} \times \mathbb{C}_1 \times \mathbb{S} \times \mathbb{Q}$.
(2) $\mathbb{Q}$ *is* λ^+*-directed closed.*
(3) *Assume that* $G_A \subseteq \mathbb{A}$, $G_{C1} \subseteq \mathbb{C}_1$, $G_S \subseteq \mathbb{S}$ *and* $G_Q \subseteq \mathbb{Q}$ *are generic filters over* V*. Then every sequence of ordinals of length less than* κ_1 *in* $V[G_A \times G_{C1} \times G_S \times G_Q]$ *belongs to* $V[G_A \times G_{C1} \times G_S]$; *if the sequence has length less than* κ_0, *then it belongs to* $V[G_S]$.
(4) $\mathbb{M}$ *has the* κ_n*-covering property for every* $n < \omega$.
(5) $\mathbb{M}$ *preserves every* κ_n, *hence it preserves* λ.
(6) $\mathbb{M}$ *preserves* λ^+.
(7) $\mathbb{M}$ *is* μ*-c.c., hence it preserves* μ.
(8) $\mathbb{M}$ *collapses every cardinal between* λ^+ *and* μ, *and it makes* $2^{\kappa_0} = 2^\lambda = \lambda^{++} = \mu$ *hold.*
(9) $\mathbb{M}$ *turns* κ_0 *into* $\aleph_2$.

Proof. Similar properties are satisfied by the usual Mitchell forcing, and the same arguments apply to $\mathbb{M}$. So we give just a sketch of the proof of this lemma; for more details the reader can consult [13].

(1) The identity map is a projection of $\mathbb{A} \times \mathbb{C}_1 \times \mathbb{S} \times \mathbb{Q}$ on $\mathbb{M}$.

(2) Given a sequence $\langle q_i \rangle_{i<\lambda}$ of pairwise compatible conditions in $\mathbb{Q}$, we can define a lower bound q by letting $\mathrm{dom}(q) := \bigcup_{i<\lambda} \mathrm{dom}(q_i)$ and by taking, for every $\alpha \in \mathrm{dom}(q)$, an $\mathbb{A}\restriction\alpha$-name $q(\alpha)$ for the union of all $q_i(\alpha)$'s such that $\alpha \in \mathrm{dom}(q_i)$.

(3) As $\mathbb{A} \times \mathbb{C}_1 \times \mathbb{S}$ is κ_1-c.c., while $\mathbb{Q}$ is κ_1-closed, the first part of the statement follows directly from Easton's lemma. For the second part, assume

that X is a set of ordinals of size $< \kappa_0$ in $V[G_A \times G_{C1} \times G_S \times G_Q]$. The generic G_S selects some $\nu \in I$ such that the conditions in G_S all belong to $\mathbb{S}_\nu := \mathrm{Coll}(\omega, \nu) \times \mathrm{Coll}(\nu^+, <\kappa_0)$. Easton's lemma implies that $\mathbb{S}_\nu$ is κ_0-c.c. in $V[G_A \times G_{C1} \times G_Q]$, therefore X has an $\mathbb{S}_\nu$-name $\dot{X}$ of size $< \kappa_0$ in this model. By the closure of $\mathbb{A} \times \mathbb{C}_1 \times \mathbb{Q}$, we see that $\dot{X}$ belongs to V, hence X belongs to $V[G_S]$.

(4) As $\mathbb{M}$ is a projection of $\mathbb{A} \times \mathbb{C}_1 \times \mathbb{S} \times \mathbb{Q}$, it is enough to prove that this product has the κ_n-covering property for every $n < \omega$. If $n > 0$, then this follows immediately from Easton's lemma, as $\mathbb{A} \times \mathbb{C}_1 \times \mathbb{S}$ is κ_n-Knaster and $\mathbb{Q}$ is κ_n-closed. For $n = 0$, assume that $W := V[G_A \times G_{C1} \times G_S \times G_Q]$ is a generic extension of V via the product $\mathbb{A} \times \mathbb{C}_1 \times \mathbb{S} \times \mathbb{Q}$. In this model, we assume that $X \subseteq \tau$ is a set of ordinals of size $\gamma < \kappa_0$. By the previous claim, $X \in V[G_S]$. The filter G_S selects a $\nu \in I$ such that all conditions of G_S are in $\mathbb{S}_\nu := \mathrm{Coll}(\omega, \nu) \times \mathrm{Coll}(\nu^+, <\kappa_0)$. As this forcing is κ_0-c.c., we can find a set $Y \in V$ of size less than κ_0 such that X is covered by Y in $V[G_S]$.

(5) This is a direct consequence of claim (4).

(6) Suppose for a contradiction that forcing with $\mathbb{M}$ collapses λ^+ to have cofinality below λ. Then for some $n < \omega$, $\mathbb{M}$ adds a set of ordinals of size $< \kappa_n$ cofinal in λ^+. By claim (4), this set is covered by a cofinal set of size $< \kappa_n$ that lives in the ground model, contradicting the fact that λ^+ is regular in V.

(7) We show that $\mathbb{M}$ is even μ-Knaster. Let $\langle (p_i, c_i, s_i, q_i) \rangle_{i<\mu}$ be a sequence of conditions in $\mathbb{M}$. As μ is an inaccessible cardinal and the conditions of $\mathbb{M}$ are tuples of functions of size less than μ, a standard application of the Δ-system lemma gives us a subsequence $\langle (p_i, c_i, s_i, q_i) \rangle_{i \in I^*}$ of pairwise compatible conditions with $I^* \subseteq \mu$ cofinal.

(8) For every γ between λ^+ and μ, the forcing $\mathbb{A} \restriction \gamma$ makes $2^\lambda \geq \gamma$ (because $2^\lambda \geq 2^{\kappa_0}$ and the forcing adds γ-many subsets of κ_0), hence introducing a λ^+-Cohen subset of λ^+ over its generic extension collapses γ to λ^+. It follows that μ becomes the double successor of λ and $2^{\kappa_0} = 2^\lambda = \lambda^{++} = \mu$.

(9) Forcing with $\mathbb{S}$ collapses some ν to ω, and it collapses all cardinals between ν^+ and κ_0. ■

We want to force with $\mathbb{M} \times \mathbb{C}^1$. By Lemma 3.3, the κ_n's, λ and λ^+ are preserved by such a product. Moreover, if $V[G_M \times G_C]$ is an $\mathbb{M} \times \mathbb{C}^1$-generic extension, then in $V[G_M \times G_C]$ we have $\kappa_n = \aleph_{n+2}$, $\lambda = \aleph_\omega$, $\lambda^+ = \aleph_{\omega+1}$ and $\mu = \aleph_{\omega+2}$. We are going to prove that there exists an $\mathbb{M} \times \mathbb{C}^1$-generic extension in which the tree property holds simultaneously at $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$. Note that $\mathbb{M} \times \mathbb{C}^1$ is a projection of

$$\mathbb{R} := \mathbb{A} \times \mathbb{S} \times \mathbb{Q} \times \mathbb{C}.$$

We want to analyse the quotient $\mathbb{R}/(G_M \times G_C)$, where G_M and G_C are generic filters for $\mathbb{M}$ and $\mathbb{C}^1$ respectively.

REMARK 3.4. If G_M and G_C are generic filters for $\mathbb{M}$ and $\mathbb{C}^1$ respectively, $\mathbb{R}/(G_M \times G_C)$ is forcing equivalent to $\mathbb{Q}(G_M) := \{q \in \mathbb{Q};\ (0,0,\mathbf{1}_{\mathbb{S}},q) \in G_M\}$ ordered as a subposet of $\mathbb{Q}$.

LEMMA 3.5. *The quotient $\mathbb{R}/(G_M \times G_C)$ is forcing equivalent to an ω_1-closed forcing in $V[G_M \times G_C]$*

Proof. By Remark 3.4, the quotient is forcing equivalent to $\mathbb{Q}(G_M)$. We prove that such a forcing is ω_1-closed. Let $\langle q_n\rangle_{n<\omega}$ be a decreasing sequence of conditions of $\mathbb{Q}(G_M)$ in $V[G_M \times G_C]$. By Lemma 3.3(3) the sequence already exists in the $\mathbb{S}$-generic extension $V[G_S]$ determined by $G_M \times G_C$. We let $\dot{s}$ be an $\mathbb{S}$-name for such a sequence. Working in V, we define a condition $q \in \mathbb{Q}$ as follows. We let the domain of q be the set of all γ that potentially belong to the domain of some q_n. As $\mathbb{S}$ is λ^+-c.c., the domain of q will have size at most λ. For every $\gamma \in \mathrm{dom}(q)$,

$$\Vdash_{\mathbb{A}\restriction\gamma\times\mathbb{C}_1\times\mathbb{S}}\ \langle \dot{s}(n)(\gamma)\rangle_{n\in I_\gamma} \text{ is decreasing}$$

where I_γ is the set of all $n<\omega$ such that $\Vdash_{\mathbb{S}}\ \gamma \in \mathrm{dom}(\dot{s}(n))$. So we can fix an $\mathbb{A}\restriction\gamma \times \mathbb{C}_1 \times \mathbb{S}$-name $q(\gamma)$ such that

$$\Vdash_{\mathbb{A}\restriction\gamma\times\mathbb{C}_1\times\mathbb{S}}\ q(\gamma) = \bigcup_{n\in I_\gamma} \dot{s}(n)(\gamma).$$

In $V[G_M \times G_C]$, the condition q must belong to $\mathbb{Q}(G_M)$, because $(0,0,\mathbf{1}_{\mathbb{S}},q)$ is the weakest lower bound of the $(0,0,\mathbf{1}_{\mathbb{S}},q_n)$'s that all belong to G_M by hypothesis. ∎

We need to look at the forcing obtained when we factor $\mathbb{M}$ over one of its initial segments. Let β be between λ^+ and μ, and consider the projection $\pi_\beta : \mathbb{M} \to \mathbb{M}\restriction\beta$ given by restriction. π_β is a projection, so if G_β is generic for $\mathbb{M}\restriction\beta$ over V, then forcing with $\mathbb{M}$ can be regarded as first forcing with $\mathbb{M}\restriction\beta$, and then with

$$\mathbb{M}^\beta := \{(p,c,s,q) \in \mathbb{M};\ (p\restriction\beta, c, s, q\restriction\beta) \in G_\beta\},$$

ordered as a subposet of $\mathbb{M}$.

We let

$$\mathbb{Q}^\beta := \{q;\ (0,0,\mathbf{1}_{\mathbb{S}}, q\restriction\beta) \in G_\beta\}.$$

LEMMA 3.6. *Let β and G_β be as above, and let G_C be a generic filter for $\mathbb{C}^1$ over V. The following hold in $V[G_\beta \times G_C]$:*

(1) *$\mathbb{M}^\beta$ is a projection of $\mathbb{A}\restriction(\mu\setminus\beta)^V \times \mathbb{Q}^\beta$.*
(2) *$\mathbb{Q}^\beta$ is forcing equivalent to the forcing*

$$\bar{\mathbb{Q}}^\beta := \{q \in \mathbb{Q}^\beta;\ q\restriction\beta = \emptyset\}.$$

(3) *$\bar{\mathbb{Q}}^\beta$ is κ_0^+-closed.*

Proof. (1) The map $(p,q) \mapsto (p,0,\mathbf{1}_{\mathbb{S}},q)$ defines a projection of $\mathbb{A}\restriction(\mu\setminus\beta)^V \times \mathbb{Q}^\beta$ on $\mathbb{M}^\beta$.

(2) Assume H is generic for $\mathbb{Q}^\beta$, and let $\bar{H}$ be $H \cap \bar{\mathbb{Q}}^\beta$. Then for every dense set $\bar{D}$ on $\bar{\mathbb{Q}}^\beta$, the set D of all conditions $q \in \mathbb{Q}^\beta$ with $q\restriction(\mu\setminus\beta) \in D$ is a dense subset of $\mathbb{Q}^\beta$. Therefore $\bar{H}$ is generic for $\bar{\mathbb{Q}}^\beta$. Conversely, if $\bar{H}$ is generic for $\bar{\mathbb{Q}}^\beta$, then take H to be the set of all $q \in \mathbb{Q}^\beta$ such that $q\restriction(\mu\setminus\beta) \in \bar{H}$; if D is a dense subset of $\mathbb{Q}^\beta$, then the set $\bar{D}$ of all restrictions $q\restriction(\mu\setminus\beta)$ for $q \in D$ is a dense subset of $\bar{\mathbb{Q}}^\beta$, thus H is generic for $\mathbb{Q}^\beta$.

(3) Let $g_\beta \times h \times k$ be the $\mathbb{A}\restriction\beta \times \mathbb{C}_1 \times \mathbb{S}$-generic filter derived from G_β. As $\mathbb{Q}\times\mathbb{C}^1$ is κ_1-closed in V and $\mathbb{A}\times\mathbb{C}_1\times\mathbb{S}\times\mathbb{C}^1$ is κ_1-c.c., Easton's lemma implies that every κ_0-sequence of ordinals in $V[G_\beta \times G_C]$ belongs to $V[g_\beta \times h \times k]$. It follows that if $\langle q_\alpha\rangle_{\alpha<\kappa_0}$ is a decreasing κ_0-sequence of conditions of $\bar{\mathbb{Q}}^\beta$ in $V[G_\beta \times G_C]$, then such a sequence belongs to $V[g_\beta \times h \times k]$. Working in V we are going to define a condition $q^* \in \mathbb{Q}$ whose domain is the set of all $\gamma \geq \beta$ that are potential elements of the domain of some q_α; as $\mathbb{A} \times \mathbb{C}_1 \times \mathbb{S}$ has the λ^+-chain condition, the domain of q^* will have size at most λ. Let $\dot{s}$ be an $\mathbb{A}\restriction\beta \times \mathbb{C}_1 \times \mathbb{S}$-name for the sequence $\langle q_\alpha\rangle_{\alpha<\kappa_0}$. For every $\gamma \geq \beta$, the $\mathbb{A}\restriction\beta \times \mathbb{C}_1 \times \mathbb{S}$-name $\dot{s}$ can also be considered as an $\mathbb{A}\restriction\gamma \times \mathbb{C}_1 \times \mathbb{S}$-name. We let

$$I_\gamma := \{\alpha < \lambda;\ \Vdash_{\mathbb{A}\restriction\gamma\times\mathbb{C}_1\times\mathbb{S}} \gamma \in \mathrm{dom}(\dot{s}(\alpha))\}.$$

Assume that I_γ is non-empty. Then

$$\Vdash_{\mathbb{A}\restriction\gamma\times\mathbb{C}_1\times\mathbb{S}} \dot{s}\restriction I_\gamma \text{ is a decreasing sequence of conditions in } \mathrm{Add}(\lambda^+,1).$$

So we can find an $\mathbb{A}\restriction\gamma \times \mathbb{C}_1 \times \mathbb{S}$-name $q^*(\gamma)$ for the union of the $\dot{s}(\alpha)(\gamma)$'s for $\alpha \in I_\gamma$. Now we work in $V[G_\beta \times G_C]$. By construction, the condition q^* is a lower bound for the sequence $\langle q_\alpha\rangle_{\alpha<\kappa_0}$ and $(0,0,\mathbf{1}_{\mathbb{S}},q^*)\restriction\beta \in G_\beta$, thus q^* belongs to $\bar{\mathbb{Q}}^\beta$. ■

4. The tree property at $\aleph_{\omega+1}$. In this section we prove that there exists an $\mathbb{M}\times\mathbb{C}^1$-generic extension in which $\aleph_{\omega+1}$ has the tree property. So, first we are going to prove that there exists an $\mathbb{R}$-generic forcing extension of V in which $\aleph_{\omega+1}$ has the tree property, then we derive from this model an $\mathbb{M}\times\mathbb{C}^1$-generic extension in which the tree property holds at $\aleph_{\omega+1}$.

We will make use of the notion of *system* that was introduced by Magidor and Shelah [12].

Definition 4.1 (Magidor and Shelah [12]). Let D be a set and τ a cardinal. A *system* over $D \times \tau$ is a collection of transitive, reflexive relations $\{R_i\}_{i\in I}$ on $D \times \tau$ such that:

(1) if $(\alpha,\zeta)\, R_i\, (\beta,\eta)$ and $(\alpha,\zeta) \neq (\beta,\eta)$, then $\alpha < \beta$;
(2) if (α_0,ζ_0) and (α_1,ζ_1) are both below (β,η) in R_i, then (α_0,ζ_0) and (α_1,ζ_1) are comparable in R_i (by condition (1) this implies that

$(\alpha_0, \zeta_0)\, R_i\, (\alpha_1, \zeta_1)$ if $\alpha_0 < \alpha_1$, $(\alpha_1, \zeta_1)\, R_i\, (\alpha_0, \eta_0)$ if $\alpha_1 < \alpha_0$, and $\zeta_0 = \zeta_1$ if $\alpha_0 = \alpha_1$);

(3) for any $\alpha < \beta$ both in D, there is $i \in I$ and $\zeta, \eta \in \tau$ such that $(\alpha, \zeta)\, R_i\, (\beta, \eta)$.

For a system $\mathscr{R} := \{R_i\}_{i \in I}$ over $D \times \tau$, a *node* of $\mathscr{R}$ is an element of $D \times \tau$. For every $\alpha \in D$, the *αth level* of $\mathscr{R}$, denoted $\mathrm{Lev}_\alpha(\mathscr{R})$, is the set $\{\alpha\} \times \tau$.

DEFINITION 4.2 (Magidor and Shelah [12]). Let $\{R_i\}_{i \in I}$ be a system on $D \times \tau$. Then a *branch* through some R_i is a partial function $b : D \to \tau$ such that for any $\beta \in \mathrm{dom}(b)$ and any $\alpha < \beta$ in D, $\alpha \in \mathrm{dom}(b)$ if and only if there exists ζ such that $(\alpha, \zeta)\, R_i\, (\beta, b(\beta))$ and $b(\alpha)$ is equal to the unique ζ witnessing this (ζ is unique by condition (2) of the definition of system). We say that b is a *cofinal* if $\mathrm{dom}(b)$ is cofinal in D.

We are now ready to prove the main result of this section.

THEOREM 4.3. *There exists an $\mathbb{R}$-generic extension of V in which $\aleph_{\omega+1}$ has the tree property.*

Proof. We fix generic filters $G_A \subseteq \mathbb{A}$, $G_Q \subseteq \mathbb{Q}$ and $G_C \subseteq \mathbb{C}^1$ over V, and we let $W := V[G_A \times G_Q \times G_C]$. Suppose for a contradiction that there is no $\mathbb{R}$-generic extension where $\aleph_{\omega+1}$ has the tree property. Then we can fix an $\mathbb{S}$-name $\dot{T}$ such that

$$\Vdash^W_{\mathbb{S}} \ \dot{T} \text{ is a } \lambda^+\text{-Aronszajn tree}$$

($\mathbb{R}$ forces $\lambda^+ = \aleph_{\omega+1}$), and we can assume that $\dot{T}$ is a name for a subset of $\lambda^+ \times \lambda$. We prove the following.

CLAIM 4.4. *In W there exists a cofinal subset $J \subseteq \lambda^+$ and a natural number n such that for every $\alpha < \beta$ in J we can find $\zeta, \eta < \kappa_n$ and a condition of $\mathbb{S}$ that forces $(\alpha, \zeta) <_{\dot{T}} (\beta, \eta)$.*

Proof. The supercompactness of κ_0 is indestructible by directed closed forcings and $\mathbb{A} \times \mathbb{Q} \times \mathbb{C}^1$ is κ_0-directed closed, so κ_0 is supercompact in W. Let $j : W \to N$ be a λ^+-supercompact embedding with critical point κ_0. We work in W. By elementarity, $j(\dot{T})$ is a $j(\mathbb{S})$-name for a $j(\lambda)^+$-Aronszajn tree. In particular

$$(\lambda, \emptyset, \emptyset) \Vdash_{\mathbb{S}} \ j(\dot{T}) \text{ is a } j(\lambda)^+\text{-Aronszajn tree.}$$

Let γ^* be $\sup j[\lambda^+]$. Then γ^* is an ordinal below $j(\lambda^+)$. Using the closure of $\mathrm{Coll}(\lambda^+, {<}j(\kappa_0))$, we can inductively define, for every $\alpha < \lambda^+$, a condition $s_\alpha \in j(\mathbb{S})$ of the form $(\lambda, a_\alpha, b_\alpha)$, a natural number n_α and an ordinal $\zeta_\alpha < j(\kappa_{n_\alpha})$ such that $\langle b_\alpha \rangle_{\alpha < \lambda^+}$ is decreasing in $\mathrm{Coll}(\lambda^+, {<}j(\kappa_0))$ and

$$s_\alpha \Vdash \ (j(\alpha), \zeta_\alpha) <_{j(\dot{T})} (\gamma^*, 0).$$

The map $\alpha \mapsto n_\alpha$ must be constant on a cofinal subset $J \subseteq \lambda^+$, so there is $n < \omega$ such that $n_\alpha = n$ for every $\alpha \in J$. By shrinking J, we can assume that for some $a \in \mathrm{Coll}(\omega, \lambda)$, we have $a_\alpha = a$ for every $\alpha \in J$. We prove that J and n are as required. Given $\alpha < \beta$ in J we have $s_\beta = (\lambda, a, b_\beta) \leq (\lambda, a, b_\alpha) = s_\alpha$, hence s_β forces that both $(j(\alpha), \zeta_\alpha)$ and $(j(\beta), \zeta_\beta)$ are below $(\gamma^*, 0)$. This implies

$$s_\beta \Vdash (j(\alpha), \zeta_\alpha) <_{j(\dot{T})} (j(\beta), \zeta_\beta).$$

By elementarity, there exists a condition $s \in \mathbb{S}$ and two ordinals $\zeta, \eta < \kappa_n$ such that $s \Vdash (\alpha, \zeta) <_{\dot{T}} (\beta, \eta)$. ■

CLAIM 4.5. *In W there is a condition $s \in \mathbb{S}$ and a function $f : J' \to \kappa_n$ with $J' \subseteq J$ cofinal such that for $\alpha < \beta$ in J^*, we have $s \Vdash (\alpha, f(\alpha)) <_{\dot{T}} (\beta, f(\beta))$.*

Proof. Let $m := n + 2$. We can write G_C as a product $G_m \times G^m$ where G_m is generic for $\mathbb{C}_m$ over V and G^m is generic for $\mathbb{C}^m$ over V. As G^Q and G^m are generic for κ_m-directed closed forcings ($\mathbb{Q}$ is even λ^+-directed closed) and κ_m is indestructibly supercompact, we can fix a λ^+-supercompact embedding $j : V[G_Q \times G^m] \to N$ with critical point κ_m. An application of Lemma 2.3 shows that $\mathbb{A} \times \mathbb{C}_m$ is κ_m-Knaster in the model $V[G_Q \times G^m]$. It follows that $j{\restriction}(\mathbb{A} \times \mathbb{C}_m)$ is a complete embedding from $\mathbb{A} \times \mathbb{C}_m$ to $j(\mathbb{A} \times \mathbb{C}_m)$. So we can force with $\mathrm{Add}(\kappa_0, j(\mu) \setminus j[\mu]) \times \mathrm{Coll}(\kappa_{m-1}, {<}j(\kappa_m) \setminus \kappa_m)$ over W to get a $j(\mathbb{A} \times \mathbb{C}_m)$-generic filter $H_A \times H_m$ such that $j[G_A \times G_m] \subseteq H_A \times H_m$. Then j lifts to an embedding $j^* : W \to N[H_A \times H_m]$ that we rename j.

We consider an ordinal $\delta^* \in j(J)$ such that $\delta^* > \sup(j[\lambda^+])$. By elementarity, we can find, for every $\alpha \in J$, two ordinals $\zeta_\alpha, \eta_\alpha < \kappa_n$ and a condition $s_\alpha \in j(\mathbb{S}) = \mathbb{S}$ such that $s_\alpha \Vdash_{\mathbb{S}} (j(\alpha), \zeta_\alpha) <_{j(\dot{T})} (\delta^*, \eta_\alpha)$.

In W we can define, for every $s \in \mathbb{S}$, an order R_s on $J \times \kappa_n$ by letting

$$(\alpha, \zeta)\, R_s\, (\beta, \eta) \iff s \Vdash (\alpha, \zeta) <_{\dot{T}} (\beta, \eta).$$

Then $\{R_s\}_{s \in \mathbb{S}}$ is a system in W. Working in $N[H_A \times H_m]$, we define for every $s \in j(\mathbb{S})$ and $\eta < \kappa_n$ a set $b_{s,\eta} := \{(\alpha, \zeta) \in J \times \kappa_n;\ s \Vdash_{j(\mathbb{S})} (j(\alpha), \zeta) <_{j(\dot{T})} (\delta^*, \eta)\}$. Note that every $b_{s,\eta}$ is an R_s-branch. Consider the function $\alpha \mapsto (\zeta_\alpha, \eta_\alpha, s_\alpha)$. The number of possible tuples (ζ, η, s) in the range of this function is at most κ_n (because $\mathbb{S}$ has size κ_0 in $N[H_A \times H_m]$). Forcing with $\mathrm{Add}(\kappa_0, j(\mu) \setminus j[\mu]) \times \mathrm{Coll}(\kappa_{m-1}, {<}j(\kappa_m) \setminus \kappa_m)$ collapses λ^+ to have cofinality $\kappa_{m-1} > \kappa_n$, so we can find a cofinal set $J^* \subseteq J$, a condition s^* and ordinals $\zeta^*, \eta^* < \kappa_n$ such that for every $\alpha \in J^*$ we have $\zeta^* = \zeta_\alpha$, $\eta^* = \eta_\alpha$ and $s^* = s_\alpha$. Therefore $b^* := b_{s^*,\eta^*}$ is a cofinal R_{s^*}-branch for the system $\{R_s\}_{s \in \mathbb{S}}$. We prove that a cofinal branch for the system $\{R_s\}_{s \in \mathbb{S}}$ already existed in W. Note that b^* is a function from λ^+ to κ_n. Work in W. For every $\alpha \in J$, we can find a condition (p_α, c_α) of $\mathrm{Add}(\kappa_0, j(\mu) \setminus j[\mu]) \times \mathrm{Coll}(\kappa_{m-1}, {<}j(\kappa_m) \setminus \kappa_m)$, a condition $t_\alpha \in \mathbb{S}$ and an ordinal $f(\alpha) < \kappa_n$ such that $(p_\alpha, c_\alpha) \Vdash s^* = t_\alpha \wedge b^*(\alpha) = f(\alpha)$. As $\mathrm{Add}(\kappa_0, j(\mu) \setminus j[\mu]) \times \mathrm{Coll}(\kappa_{m-1}, {<}j(\kappa_m) \setminus \kappa_m)$ is λ^+-Knaster in W, there exists a cofinal set $J' \subseteq J$ such that the conditions in $\langle p_\alpha, c_\alpha\rangle_{\alpha\in J'}$ are pairwise compatible. As λ^+ is regular in W and $\mathbb{S}$ has size κ_0, by shrinking J' we can assume that there exists a condition $t \in S$ such that $t_\alpha = t$ for every $\alpha \in J'$. Then for every $\alpha < \beta$ in J', the condition $(p_\alpha, c_\alpha) \wedge (p_\beta, c_\beta)$ forces that

$$t \Vdash_{\mathbb{S}} (\alpha, f(\alpha)) <_{\dot{T}} (\beta, f(\beta)).$$

Hence we proved that in W there is a condition t that forces $(\alpha, f(\alpha)) <_{\dot{T}} (\beta, f(\beta))$. ■

So let $s \in \mathbb{S}$, and $f : J' \to \kappa_n$ be a condition as in the conclusion of the previous claim. Then s forces that f is a cofinal branch for $\dot{T}$, contradicting $\Vdash^W_{\mathbb{S}}$ $\dot{T}$ is Aronszajn. ■

Corollary 4.6. *There exists an $\mathbb{M}\times\mathbb{C}^1$-generic forcing extension of V in which $\aleph_{\omega+1}$ has the tree property.*

Proof. Apply Theorem 4.3 to get a generic filter G_R for $\mathbb{R}$ over V such that the tree property at $\aleph_{\omega+1}$ holds in $V[G_R]$. As $\mathbb{M}\times\mathbb{C}^1$ is a projection of $\mathbb{R}$, the filter G_R determines a generic filter $G_M \times G_C$ for $\mathbb{M} \times \mathbb{C}^1$ over V such that $V[G_M \times G_C] \subseteq V[G_R]$. Let T be an $\aleph_{\omega+1}$-tree in $V[G_M \times G_C]$. Then T has a cofinal branch b in $V[G_R]$. By Lemma 3.5, the quotient $\mathbb{R}/(G_M \times G_C)$ is ω_1-closed, so we can apply Lemma 2.4, hence b belongs to $V[G_M \times G_C]$. Therefore in the model $V[G_M \times G_C]$, the tree property holds at $\aleph_{\omega+1}$. ■

5. The tree property at $\aleph_{\omega+2}$. In this section we prove that in any $\mathbb{M} \times \mathbb{C}^1$-generic extension, $\aleph_{\omega+2}$ has the tree property.

Theorem 5.1. $\mathbb{M} \times \mathbb{C}^1$ *forces the tree property at* $\aleph_{\omega+2}$.

Proof. $\mathbb{M} \times \mathbb{C}^1$ makes $\mu = \aleph_{\omega+2}$. Suppose for a contradiction that there is a condition $r \in \mathbb{M} \times \mathbb{C}^1$ and a name $\dot{T}$ such that

$$r \Vdash \dot{T} \text{ is a } \mu\text{-Aronszajn tree.}$$

We can assume that the nodes of $\dot{T}$ are pairs of ordinals in $\mu \times \lambda^+$, therefore $\dot{T} \subseteq V_\mu$. Let D be the club of all ordinals $\alpha < \mu$ such that r forces $(\dot{T}\cap V_\alpha)^{\dot{G}} = \dot{T}\restriction\alpha$. The structure $\langle V_\mu, \in, \mathbb{M} \times \mathbb{C}^1, \Vdash, r, \dot{T}\rangle$ models the Π^1_1-statement "for all X, if X is an $\mathbb{M} \times \mathbb{C}^1$-name, then for all $s \le r$ in $\mathbb{M} \times \mathbb{C}^1$, s forces that X is not a cofinal branch for $\dot{T}$". Since μ is weakly compact, there exists an inaccessible cardinal $\alpha < \mu$ in D above λ such that $\langle V_\alpha, \in, (\mathbb{M} \times \mathbb{C}^1) \cap V_\alpha, \Vdash, r, \dot{T}\cap V_\alpha\rangle$ models the same statement. Note that $(\mathbb{M}\times\mathbb{C}^1)\cap V_\alpha = \mathbb{M}\restriction\alpha\times\mathbb{C}^1$. Let $G_M \times G_C$ be a generic filter for $\mathbb{M} \times \mathbb{C}^1$ that contains r, and let G_α be the generic filter for $\mathbb{M}\restriction\alpha$ derived from G_M. In $V[G_\alpha\times G_C]$ we have $\alpha = \lambda^{++}$ and, by our assumption on α, the tree $T_\alpha := (\dot{T} \cap V_\alpha)^{G_\alpha\times G_C}$ has no cofinal branches. In $V[G_M \times G_C]$ the tree $T := \dot{T}^{G_M\times G_C}$ is a μ-tree and $T\restriction\alpha = T_\alpha$. Therefore, if we consider a node $t \in T$ on level α, the set of all its predecessors determines a cofinal branch b for T_α. We show that b belongs to $V[G_\alpha \times G_C]$, which will lead us to a contradiction. $V[G_M \times G_C]$ is an $\mathbb{M}^\alpha$-generic extension of $V[G_\alpha \times G_C]$; we are going to prove that $\mathbb{M}^\alpha$ could not add cofinal branches to T_α. By Lemma 3.6 this forcing is a projection of $\mathrm{Add}(\kappa_0, \mu \setminus \alpha)^V \times \mathbb{Q}^\alpha$ where $\mathbb{Q}^\alpha$ is a κ_0^+-closed forcing in $V[G_\alpha \times G_C]$.

Let G_0^* be a generic filter for $\mathrm{Add}(\kappa_0, \mu \setminus \alpha)$ over $V[G_\alpha]$ and G_1^* a generic filter for $\mathbb{Q}^\alpha$ over $V[G_\alpha \times G_C]$ with $V[G_M \times G_C] \subseteq V[G_\alpha \times G_C][G_0^* \times G_1^*] = V[G_\alpha \times G_C][G_1^*][G_0^*]$.

An application of Lemma 2.3 shows that the poset $\mathrm{Add}(\kappa_0, \mu \setminus \alpha)^V \times \mathrm{Add}(\kappa_0, \mu \setminus \alpha)^V$ is κ_1-c.c. in $V[G_\alpha \times G_C][G_1^*]$. In particular, such a forcing is λ^+-c.c. in that model. The filter G_1^* collapses μ to have cofinality λ^+, and b is μ-approximated. It follows from Lemma 2.5 that $\mathrm{Add}(\kappa_0, \mu \setminus \alpha)^V$ could not add b, hence b belongs to $V[G_\alpha \times G_C][G_1^*]$.

In $V[G_\alpha \times G_C]$, we have $\alpha = \lambda^{++} = 2^{\kappa_0}$ and $\mathbb{Q}^\mu$ is κ_0^+-closed in that model. It follows from Lemma 2.6 that $b \in V[G_\alpha \times G_C]$, contradicting the fact that T_α is Aronszajn in that model. ■

This implies the main result of this paper.

COROLLARY 5.2. *There exists an $\mathbb{M} \times \mathbb{C}^1$-generic forcing extension of V in which both $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$ have the tree property.*

Proof. Apply Corollary 4.6 to get an $\mathbb{M} \times \mathbb{C}^1$-generic extension in which the tree property holds at $\aleph_{\omega+1}$. By Theorem 5.1 the tree property holds at $\aleph_{\omega+2}$ in that model. ■

6. The super tree property at $\aleph_{\omega+2}$. The super tree property concerns special objects that generalise the notion of κ-tree for a regular cardinal κ.

DEFINITION 6.1. Given a regular cardinal $\kappa \geq \omega_2$ and an ordinal $\theta \geq \kappa$, a *(κ, θ)-tree* is a set F satisfying the following properties:

(1) for every $f \in F$, we have $f : X \to 2$ for some $X \in [\theta]^{<\kappa}$;
(2) for all $f \in F$, if $X \subseteq \mathrm{dom}(f)$, then $f\restriction X \in F$;
(3) the set $\mathrm{Lev}_X(F) := \{f \in F;\ \mathrm{dom}(f) = X\}$ is non-empty for all $X \in [\theta]^{<\kappa}$;
(4) $|\mathrm{Lev}_X(F)| < \kappa$ for all $X \in [\theta]^{<\kappa}$.

As usual, when there is no ambiguity, we will simply write Lev_X instead of $\mathrm{Lev}_X(F)$. In a (κ, θ)-tree, levels are not indexed by ordinals, but by *sets of ordinals.* So the predecessors of a node in a (κ, θ)-tree are not (necessarily) well ordered and a (κ, θ)-tree is not a tree.

Definition 6.2. Given a regular cardinal $\kappa \geq \omega_2$, an ordinal $\theta \geq \kappa$ and a (κ, θ)-tree F,

(1) a *cofinal branch* for F is a function $b : \theta \to 2$ such that $b \restriction X$ is in $\mathrm{Lev}_X(F)$ for all $X \in [\theta]^{<\kappa}$;
(2) an *F-level sequence* is a function $D : [\theta]^{<\kappa} \to F$ such that $D(X)$ is in $\mathrm{Lev}_X(F)$ for every $X \in [\theta]^{<\kappa}$;
(3) given an F-level sequence D, an *ineffable branch* for D is a cofinal branch $b : \theta \to 2$ such that $\{X \in [\theta]^{<\kappa};\ b \restriction X = D(X)\}$ is stationary.

Definition 6.3. Given a regular cardinal $\kappa \geq \omega_2$ and an ordinal $\theta \geq \kappa$,

- (κ, θ)-ITP holds if for every (κ, θ)-tree F and every F-level sequence D there is an ineffable branch for D;
- we say that κ satisfies the *super tree property* if the (κ, θ')-ITP holds for all $\theta' \geq \kappa$.

For a more extensive presentation of this property, the reader can consult Weiss' PhD thesis [19]. Now we show that if μ is supercompact, then $\mathbb{M} \times \mathbb{C}^1$ forces a model of the super tree property at $\aleph_{\omega+2}$.

Theorem 6.4. *In the situation of Section* 3 *assume that μ is supercompact. Then $\mathbb{M} \times \mathbb{C}^1$ forces the super tree property at $\aleph_{\omega+2}$.*

Proof. Let $G_M \subseteq \mathbb{M}$ and G_C be generic filters over V. In $V[G_M \times G_C]$ we have $\mu = \aleph_{\omega+2}$, so we want to show that the super tree property holds at μ. Let F be a (μ, θ)-tree in $V[G_M \times G_C]$ and let D be an F-level sequence. As μ is supercompact in V, there exists an elementary embedding $j : V \to N$ with critical point μ, with $j(\mu) > \mu^\theta$ and such that N is closed under sequences of length μ^θ. The product $\mathbb{C}^1$ has size less than μ, so $j(\mathbb{C}^1) = \mathbb{C}^1$ and $j(G_C) = G_C$. Note that $j(\mathbb{M}) \restriction \mu = \mathbb{M}$. As $\mathbb{M}$ satisfies the μ-chain condition, $j \restriction \mathbb{M}$ is a complete subforcing of $j(\mathbb{M})$. Therefore, forcing with $j(\mathbb{M})$ over V, we can find a generic filter G_M^* such that $j[G_M] \subseteq G_M^*$. Then we can lift j to an embedding $\bar{j} : V[G_M \times G_C] \to N[G_M^* \times G_C]$ that we rename j.

We show that D has an ineffable branch in $V[G_M^* \times G_C]$. Now $j(F)$ is a $(j(\mu), j(\theta))$-tree and $j(D)$ is a $j(F)$-level sequence. We have $j[\theta] \in [j(\theta)]^{<j(\mu)}$, so $j(D)(j[\theta])$ is defined and we denote it by f. Let $b : \theta \to 2$ be the function defined by $b(\alpha) := f(j(\alpha))$. Then b is an ineffable branch for D, otherwise there is a club $C \subseteq [\theta]^{<\mu}$ in $V[G_M^* \times G_C]$ such that $b \restriction X \neq D(X)$ for all $X \in C$. By elementarity,

$$j(b) \restriction X \neq j(D)(X)$$

for all $X \in j(C)$. However, $j[\theta] \in j(C)$ and $j(b) \restriction j[\theta] = f = j(D)(j[\theta])$, and that leads us to a contradiction.

So an ineffable branch b for D exists in $V[G_M^* \times G_C]$. We want to show b belongs to $V[G_M \times G_C]$. For that we need to prove that $j(\mathbb{M})/(G_M \times G_C)$

could not add this branch. By Lemma 3.6 this forcing is a projection of $\mathrm{Add}(\kappa_0, j(\mu)\setminus j[\mu])^V \times j(\mathbb{Q})^\mu$ where $j(\mathbb{Q})^\mu$ is forcing equivalent to a κ_0^+-closed forcing in $V[G_M \times G_C]$, denoted by $\overline{j(\mathbb{Q})}^\mu$. Let $G_0^* \subseteq \mathrm{Add}(\kappa_0, j(\mu) \setminus j[\mu])^V$ and $G_1^* \subseteq \overline{j(\mathbb{Q})}^\mu$ be generic filters over $V[G_M \times G_C]$ such that $V[G_M^* \times G_C] \subseteq V[G_M \times G_C][G_0^* \times G_1^*] = V[G_M \times G_C][G_1^*][G_0^*]$.

Lemma 2.3 implies that $\mathrm{Add}(\kappa_0, j(\mu) \setminus j[\mu])^V \times \mathrm{Add}(\kappa_0, j(\mu) \setminus j[\mu])^V$ is λ^+-c.c. in $V[G_M \times G_C][G_1^*]$. Moreover, the filter G_1^* collapses μ to have cofinality λ^+ and b is μ-approximated, so we can apply Lemma 2.5, thus $\mathrm{Add}(\kappa_0, j(\mu) \setminus j[\mu])^V$ could not add b. Hence b belongs to $V[G_M \times G_C][G_1^*]$.

In $V[G_M \times G_C]$, we have $\mu = \lambda^{++} = 2^{\kappa_0}$ and $\overline{j(\mathbb{Q})}^\mu$ is κ_0^+-closed. So we can apply Lemma 2.7, hence the branch b belongs to $V[G]$. ∎

COROLLARY 6.5. *Assume that ZFC is consistent with the existence of a supercompact cardinal. Then ZFC is consistent with the super tree property at $\aleph_{\omega+2}$ plus the tree property at $\aleph_{\omega+1}$.*

Proof. Apply Corollary 4.6 with μ supercompact to get an $\mathbb{M}\times\mathbb{C}^1$-generic extension in which the tree property holds at $\aleph_{\omega+1}$. By Theorem 6.4 the super tree property at $\aleph_{\omega+2}$ holds in that model. ∎

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