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Group-theoretic conditions under which closed aspherical manifolds are covered by Euclidean space

by

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Abstract. Hass, Rubinstein, and Scott showed that every closed aspherical (irreducible) 3-manifold whose fundamental group contains the fundamental group of a closed aspherical surface, is covered by Euclidean space. This theorem does not generalize to higher dimensions. However, we provide geometric tools with which variations of this theorem can be proved in all dimensions.

1. Introduction and statement of results. Given a closed aspherical manifold M, one is interested in conditions on its fundamental group which ensure that M is covered by Euclidean space.

Employing least area techniques, Hass, Rubinstein, and Scott [9] showed that this is the case when M is a P^2-irreducible 3-manifold whose fundamental group contains a subgroup isomorphic to the fundamental group of a closed surface other than S^2 or P^2. It is a long-standing conjecture that *all* irreducible closed aspherical 3-manifolds are covered by Euclidean space.

Davis [3] constructed examples that answered the higher-dimensional conjecture in the negative. In fact, Davis's exotic manifolds illustrate that the Hass–Rubinstein–Scott Theorem does not generalize to higher dimensions. His open contractible manifolds (of any given dimension greater than three) are not homeomorphic to Euclidean space, although each of them covers a closed manifold M whose fundamental group contains a subgroup isomorphic to the fundamental group of a closed codimension-one manifold N which is covered by Euclidean space.

Independently, Houghton [10] and Jackson [11] proved the following theorem (see Sections 2 and 3 for definitions):

Theorem 1. *Let*

$$1 \to H \to G \to Q \to 1$$

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be a short exact sequence of finitely presented infinite groups. If either H or Q is one-ended, then G is simply connected at infinity.

The universal covering space of a closed aspherical manifold M is homeomorphic to Euclidean space if it is simply connected at infinity (provided we assume irreducibility if the manifold is 3-dimensional). Since this is the case precisely if its fundamental group $G = \pi_1(M)$ is simply connected at infinity, Theorem 1 implies that M will be covered by Euclidean space if one can exhibit a finitely presented infinite normal subgroup H of G of infinite index such that either H or G/H is one-ended. (Note that the fundamental group of a compact manifold is always finitely presented.)

In this article, we will develop tools that yield a geometric proof of the Houghton–Jackson Theorem. We will also show how one can use these same tools to prove related results. Some of the theorems, for which we provide alternative proofs, are known. However, our techniques allow us to establish results that were not previously known.

One of the theorems we shall prove in this way is

THEOREM 2. *Let M be a closed aspherical n-manifold (irreducible if $n = 3$). Suppose the fundamental group of M contains a non-trivial cyclic normal subgroup. Then M is covered by Euclidean space.*

REMARK. We note that such a subgroup must, in fact, be infinite cyclic.

In dimensions $n \geq 5$, Theorem 2 has been proved by Lee and Raymond [12], using algebraic techniques. In dimension $n = 3$, Theorem 2 can also be deduced from the Seifert fiber space conjecture, whose proof was completed only recently by Gabai [8], and, independently, by Casson and Jungries [2].

Combining Theorems 1 and 2 (see Section 6), one obtains a Hass–Rubinstein–Scott-like result:

COROLLARY A. *Let N and M be closed aspherical manifolds of dimension k and n, respectively, with $k < n$ (and M irreducible if $n = 3$). If $\pi_1(N)$ is isomorphic to a normal subgroup of $\pi_1(M)$, then M is covered by Euclidean space.*

We will then relax the normality condition and establish

THEOREM 3. *Let H be a finitely presented subgroup of a finitely presented group G. Suppose that the index of H in its normalizer $\mathbb{N}_G(H)$ in G is infinite. If both H and G are one-ended and the pair (G, H) is two-ended, then G is simply connected at infinity.*

Applied to the setting of aspherical manifolds, Theorem 3 implies the following alternative to Corollary A.

COROLLARY B. *Let N and M be closed orientable aspherical manifolds of dimension $n-1$ and n, respectively (with M irreducible if $n = 3$). If*

$\pi_1(N)$ is isomorphic to a subgroup of $\pi_1(M)$ which has infinite index in its normalizer in $\pi_1(M)$, then M is covered by Euclidean space.

The fact that the above assumptions ensure the two-endedness of the pair $(\pi_1(M), \pi_1(N))$ follows from a theorem by Swarup [17].

In the last section of this article, we will analyze Davis's examples from the viewpoint of Corollary B. We will show that the situation is prototypical of the obstruction which one encounters, by verifying that $\pi_1(N)$ *equals* its normalizer in $\pi_1(M)$ in these examples.

2. Definitions. We begin by reviewing some basic definitions. Recall that if $p : \overline{X} \to X$ is a covering map (of connected, locally path connected topological spaces), then the group $\mathrm{Aut}(\overline{X} \to X)$ of covering transformations is isomorphic to $\mathbb{N}_G(H)/H$, where $H = p_\#(\pi_1(\overline{X}))$, $G = \pi_1(X)$, and $\mathbb{N}_G(H)$ denotes the normalizer of H in G. We will always suppress base points. The action of $\mathrm{Aut}(\overline{X} \to X)$ on $\overline{X}$ is properly discontinuous and fixed-point free. (Since all our spaces will be locally compact and Hausdorff, we will call the action of a group Q on a topological space Y *properly discontinuous* if $\{g \in Q \mid g(C) \cap C \neq \emptyset\}$ is finite for every compact subset $C \subseteq Y$.) If X has a universal covering space, we will denote it by $\widetilde{X}$.

Conversely, if a group G acts on a connected, locally path connected topological space Y properly discontinuously and fixed-point free, then the quotient map $Y \to Y/G$ is a regular covering projection with automorphism group isomorphic to G.

We will call the action of a group H on a topological space X *cocompact* if there is a compact subset $C \subseteq X$ such that $H(C) = X$. Here, and later, $H(E)$ is defined to be $\bigcup\{h(E) \mid h \in H\}$ for subsets $E \subseteq X$.

A non-compact topological space Y is called *one-ended* if for every compact set $A \subseteq Y$ there is a compact set $B \subseteq Y$ such that $A \subseteq B$ and every pair of points in $Y \setminus B$ is joined by a path in $Y \setminus A$. A one-ended space Y is called *simply connected at infinity* if for every compact set $A \subseteq Y$ there is a compact set $B \subseteq Y$ such that $A \subseteq B$ and loops in $Y \setminus B$ contract in $Y \setminus A$. In the next section, we will extend these definitions to groups. Two-endedness of pairs of groups will be defined in Section 7.

A topological space Y is called *locally simply connected* if for every $y \in Y$ and every neighborhood U of y in Y there is a neighborhood V of y in Y such that $V \subseteq U$ and loops in V contract in U.

A manifold is called *aspherical* if its universal covering space is contractible. We note that all open contractible manifolds of dimension at least two are one-ended. Moreover, if an open contractible n-manifold is simply connected at infinity, then it is homeomorphic to Euclidean space, provided we assume that the manifold is irreducible in case $n = 3$. (For $n = 3$, see

Wall [19] and Brown [1]; for $n = 4$, this is due to Freedman [7]; for $n \geq 5$ we have the result of Stallings [16] and its strengthening by Siebenmann [15].) Clearly, all one-dimensional and two-dimensional closed aspherical manifolds are covered by Euclidean space.

We add to this list of definitions some relative notions of connectivity. Let a triple $C \subseteq D \subseteq Y$ of topological spaces be given. We will from now on say that C is *path connected in* D if every pair of points in C is joined by a path in D. Similarly, if all loops in C contract in D, we will call C *simply connected in* D. We will say that C is *one-ended in* D *with respect to* Y if C is not contained in a compact subset of Y and for every compact set $A \subseteq Y$ there is a compact set $B \subseteq Y$ such that $A \subseteq B$ and every pair of points in $C \setminus B$ is joined by a path in $D \setminus A$. Similarly, C is *simply connected at infinity in* D *with respect to* Y if C is not contained in a compact subset of Y and for every compact set $A \subseteq Y$ there is a compact set $B \subseteq Y$ such that $A \subseteq B$ and loops in $C \setminus B$ contract in $D \setminus A$. Whenever the ambient space Y is understood, we drop the reference to it.

3. Some tools. For this section, we fix two topological spaces X and Y which are connected, locally path connected (and hence path connected), locally compact, and Hausdorff. Suppose H is a subgroup of a group G and assume that H and G act properly discontinuously on the spaces X and Y, respectively. Suppose, further, that the action of H on X is cocompact. We will also assume that Y is locally simply connected (although this is irrelevant for Lemmas 1, 3, and 4).

Remark. The existence of an action of H on a space X which satisfies the above hypotheses is equivalent to the fact that H is finitely generated.

We state the following lemma for the record, its proof is immediate.

Lemma 1. *For every compact set $C \subseteq Y$ there is a compact set $D \subseteq Y$ such that $C \subseteq D$ and C is path connected in D.*

Lemma 2. *Suppose Y is simply connected. Then for every compact set $C \subseteq Y$ there is a compact set $D \subseteq Y$ such that $C \subseteq D$ and C is simply connected in D.*

Proof. Choose open subsets $U_0, U_1, \ldots, U_k, V_0, V_1, \ldots, V_k$ of Y such that $C \subseteq \bigcup\{U_i \mid i = 0, 1, \ldots, k\}$ and, for each i, U_i is simply connected in V_i, and V_i has compact closure. One can then find a finite collection $\mathcal{W}$ of open path connected subsets of Y such that $C \subseteq \bigcup \mathcal{W}$ and with the property that for each pair $W_1, W_2 \in \mathcal{W}$ with $W_1 \cap W_2 \neq \emptyset$, there is a U_i with $W_1 \cup W_2 \subseteq U_i$.

For example, one could use the following partition of unity to find $\mathcal{W}$: choose a compact set $E \subseteq Y$ with $C \subseteq \operatorname{int} E \subseteq E \subseteq \bigcup\{U_i \mid i = 0, 1, \ldots, k\}$ and continuous functions $\phi_i : E \to [0, 1]$ such that $\phi_i^{-1}((0, 1]) \subseteq U_i$ for all

$i = 0, 1, \ldots, k$ and $\sum_{i=0}^{k} \phi(x) = 1$ for each $x \in E$. Define a map f from E to a k-simplex $\sigma_k = \langle v_0, v_1, \ldots, v_k\rangle$ by defining $f(x) = \sum_{i=0}^{k} \phi_i(x)v_i$ for $x \in E$. Let $\mathcal{W}'$ be a covering of σ_k by finitely many open sets so that for each pair $W'_1, W'_2 \in \mathcal{W}'$ with $W'_1 \cap W'_2 \neq \emptyset$, $W'_1 \cup W'_2$ lies in the open star S_i of v_i in σ_k for some i. Since $f^{-1}(S_i) \subseteq U_i$, one can now select the desired collection $\mathcal{W}$ from the path components of the sets $f^{-1}(W') \cap \operatorname{int} E$, $W' \in \mathcal{W}'$.

Define a finite graph Γ as follows. For each $W \in \mathcal{W}$ take a vertex $v(W)$. Join two distinct vertices $v(W)$ and $v(W')$ by an edge $e(W, W')$ whenever $W \cap W' \neq \emptyset$. Choose a map $\mu : \Gamma \to Y$ such that $\mu(v(W)) \in W$ and $\mu(e(W, W')) \subseteq W \cup W'$ for all $W, W' \in \mathcal{W}$. Since Y is simply connected, there is a homotopy from μ to a constant map. Choose a compact set D such that it contains the closure of each V_i and the image of this homotopy. A loop α in C can now be subdivided into paths α_i so that each α_i lies in an element $W_i \in \mathcal{W}$. If we connect the endpoints of each α_i to $\mu(v(W_i))$ with a path in W_i, we produce a bootstrap pattern between α and Γ whose loops lie alternately in a member of $\mathcal{W}$ and in the union of two intersecting members of $\mathcal{W}$. This allows us to homotope α into $\mu(\Gamma)$ within D. From there we can contract it to a point within D. ∎

LEMMA 3. *For every compact set $C \subseteq Y$ there is a compact set $D \subseteq Y$ such that $C \subseteq D$ and $H(C)$ is path connected in $H(D)$.*

The proof is similar to but simpler than the proof of

LEMMA 4. *If X is one-ended, then for every compact set $C \subseteq Y$ there is a compact set $D \subseteq Y$ such that $C \subseteq D$ and $H(C)$ is one-ended in $H(D)$.*

Proof. Let a compact set $C \subseteq Y$ be given. Choose a compact set $E \subseteq X$ so that $H(\operatorname{int} E) = X$. Choose a compact set $D' \subseteq Y$ such that $C \subseteq D'$ and $g_1(D') \cap g_2(D') \neq \emptyset$ whenever $g_1, g_2 \in H$ and $g_1(E) \cap g_2(E) \neq \emptyset$. (This is possible since the set $\{g \in H \mid E \cap g(E) \neq \emptyset\}$ is finite.) Choose a compact set $D \subseteq Y$ such that $D' \subseteq D$ and D' is path connected in D.

Now, let $A \subseteq Y$ be compact. Choose a compact set $L \subseteq X$ such that $X \setminus L$ is path connected in $X \setminus \bigcup\{g(E) \mid g \in H,\ g(D) \cap A \neq \emptyset\}$. Define the compact set $B = \bigcup\{g(D) \mid g \in H \text{ and either } g(E) \cap L \neq \emptyset \text{ or } g(D) \cap A \neq \emptyset\}$.

If $a, b \in H(C) \setminus B$, then there are $g_a, g_b \in H$ such that $a \in g_a(D')$ and $b \in g_b(D')$. Hence, $g_a(D) \cap A = \emptyset$, $g_a(E) \cap L = \emptyset$, $g_b(D) \cap A = \emptyset$, and $g_b(E) \cap L = \emptyset$. Pick a point $a' \in g_a(E)$, a point $b' \in g_b(E)$, and choose a path $\gamma' : [0, 1] \to X \setminus \bigcup\{g(E) \mid g \in H, g(D) \cap A \neq \emptyset\}$ with $\gamma'(0) = a'$ and $\gamma'(1) = b'$. Choose $n \in \mathbb{N}$ such that for each $i \in \{0, 1, \ldots, n-1\}$ there is a $g_i \in H$ such that $\gamma'\big(\big[\frac{i}{n}, \frac{i+1}{n}\big]\big) \subseteq g_i(\operatorname{int} E)$. Then we have $g_a(D') \cap g_0(D') \neq \emptyset$, $g_b(D') \cap g_{n-1}(D') \neq \emptyset$, and $g_i(D') \cap g_{i+1}(D') \neq \emptyset$ for all $i \in \{0, 1, \ldots, n-2\}$, but $g_i(D) \cap A = \emptyset$ for all $i \in \{0, 1, \ldots, n-1\}$. Since D' is path connected in D, there is a path $\gamma : [0, 1] \to H(D) \setminus A$ with $\gamma(0) = a$ and $\gamma(1) = b$. ∎

Lemma 5. *Suppose both X and Y are simply connected. Then for every compact set $C \subseteq Y$ there is a compact set $D \subseteq Y$ such that $C \subseteq D$ and $H(C)$ is simply connected in $H(D)$.*

Remark. The existence of an action of H on a *simply connected* space X which satisfies the additional hypotheses stated at the beginning of this section is equivalent to the fact that H is finitely presented.

Proof. Let $C \subseteq Y$ be compact. Choose a compact set $C' \subseteq Y$ such that $C \subseteq \operatorname{int} C'$. Choose a compact set $E \subseteq X$ such that $H(\operatorname{int} E) = X$ and $g_1(E) \cap g_2(E) \neq \emptyset$ whenever $g_1, g_2 \in H$ and $g_1(C') \cap g_2(C') \neq \emptyset$. Choose a compact set $E' \subseteq X$ such that $E \subseteq E'$ and E is path connected in E'. Choose a compact set $F \subseteq Y$ such that $C' \subseteq F$ and $\bigcap\{g(F) \mid g \in S\} \neq \emptyset$ whenever $S \subseteq H$ and $\bigcap\{g(E') \mid g \in S\} \neq \emptyset$. Choose a compact set $F' \subseteq Y$ such that $F \subseteq F'$ and F is path connected in F'. Put $F'' = \bigcup\{g(F') \mid g \in H,$ $F' \cap h(F') \neq \emptyset$, and $h(F') \cap g(F') \neq \emptyset$ for some $h \in H\}$.

By Lemma 2, there is a compact subset $D \subseteq Y$ such that $F'' \subseteq D$ and F'' is simply connected in D. For $n \in \mathbb{N}$ and $x \in [0,1]^2$ define the sets

$$\mathcal{G}(n) = \left\{ \left\{\frac{i}{n}\right\} \times \left[\frac{j}{n}, \frac{j+1}{n}\right] \,\middle|\, i \in \{0,1,\dots,n\},\, j \in \{0,1,\dots,n-1\} \right\}$$

$$\cup \left\{ \left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left\{\frac{j}{n}\right\} \,\middle|\, i \in \{0,1,\dots,n-1\},\, j \in \{0,1,\dots,n\} \right\},$$

$$\mathcal{B}(n) = \{P \in \mathcal{G}(n) \mid P \subseteq \partial[0,1]^2\}, \qquad \mathcal{I}(n) = \mathcal{G}(n) \setminus \mathcal{B}(n),$$

$$\mathcal{N}(x,n) = \{P \in \mathcal{G}(n) \mid x \in P\},$$

$$\mathcal{D}(n) = \left\{\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\right\} \times \left\{\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\right\}.$$

Let $\gamma : \partial[0,1]^2 \to H(C)$ be a loop. Since $\gamma(\partial[0,1]^2) \subseteq H(\operatorname{int} C')$, there is an $n \in \mathbb{N}$ such that for all $P \in \mathcal{B}(n)$ there is a $g_P \in H$ with $\gamma(P) \subseteq g_P(\operatorname{int} C')$. Choose $\gamma' : \partial[0,1]^2 \to X$ such that $\gamma'(P) \subseteq g_P(E')$ for all $P \in \mathcal{B}(n)$. Since X is simply connected, we can extend γ' to $f' : [0,1]^2 \to X$. Since $H(\operatorname{int} E') = X$, there is an $m \in \mathbb{N}$ ($m \geq 2$) such that for all $P \in \mathcal{G}(nm)$ there is a $g_P \in H$ with $f'(P) \subseteq g_P(\operatorname{int} E')$.

Extend γ to a map $f : \partial[0,1]^2 \cup \mathcal{D}(nm) \to H(F)$ such that $f(x) \in \bigcap\{g_P(F) \mid P \in \mathcal{N}(x,nm)\}$ for all $x \in \mathcal{D}(nm)$. Next, extend f to a map $\bigcup \mathcal{G}(nm) \to H(F')$ such that for all $P \in \mathcal{I}(nm)$ with $P \subseteq \operatorname{int}[0,1]^2$ we have $f(P) \subseteq g_P(F')$, and for all $P \in \mathcal{I}(nm)$ with $P \not\subseteq \operatorname{int}[0,1]^2$ we have $f(P) \subseteq g_P(F') \cup g_Q(F')$ for some $Q \in \mathcal{B}(n)$ with $P \cap Q \neq \emptyset$. Finally, since for all $i,j \in \{0,1,\dots,nm-1\}$ there is a $g \in H$ such that $f\big(\partial\big(\big[\frac{i}{nm}, \frac{i+1}{nm}\big] \times \big[\frac{j}{nm}, \frac{j+1}{nm}\big]\big)\big) \subseteq g(F'')$, we can, by choice of D, extend f to a map $[0,1]^2 \to H(D)$. Hence, γ contracts in $H(D)$. ∎

Similarly, we have

LEMMA 6. *Suppose X is simply connected at infinity and Y is simply connected. Then for every compact set $C \subseteq Y$ there is a compact set $D \subseteq Y$ such that $C \subseteq D$ and $H(C)$ is simply connected at infinity in $H(D)$.*

Proof. We have to change the proof of Lemma 5 only slightly. Using the same setup, let $A \subseteq Y$ be compact. Choose a compact set $L \subseteq X$ so that $X \setminus L$ is simply connected in $X \setminus \bigcup\{g(E') \mid g \in H,\, g(D) \cap A \neq \emptyset\}$. Set $B = A \cup \bigcup\{g(C') \mid g \in H,\, g(E') \cap L \neq \emptyset\}$. Now, if $\gamma : \partial[0,1]^2 \to H(C) \setminus B$ is a loop, then $\gamma'(\partial[0,1]^2) \subseteq X \setminus L$ (where γ' is as before) and γ' extends to a map $f' : [0,1]^2 \to X \setminus \bigcup\{g(E') \mid g \in H,\, g(D) \cap A \neq \emptyset\}$. This yields an extension $f : [0,1]^2 \to H(D) \setminus A$ of γ. ■

Let us recall the following standard terminology. The group H is called *one-ended* if X is one-ended, and *simply connected at infinity* if X is locally simply connected, simply connected and simply connected at infinity. It is well known that these definitions do not depend on the choice of X, but rather are invariants of the group H. Note that admissible choices for X include (appropriate) locally finite CW-complexes and topological manifolds.

REMARK. Lemmas 4 and 6 also provide an alternative to the standard proofs of the fact that the above definitions are independent of the choice of X. For if we further assume the action of H on Y to be cocompact, the conclusions of Lemmas 4 and 6 are equivalent to Y being one-ended and simply connected at infinity, respectively.

LEMMA 7. *Suppose both X and Y are simply connected, that the action of G on Y is also cocompact, and that H is a normal subgroup of G. Then for every compact set $A \subseteq Y$ there is a compact set $B \subseteq Y$ with $A \subseteq B$ such that if γ is a loop in $Y \setminus H(B)$ and $h \in H$, we can homotope γ to $h \circ \gamma$ by a homotopy missing $H(A)$.*

Proof. Let the compact set $A \subseteq Y$ be given. Choose a compact set $C \subseteq Y$ such that $G(\operatorname{int} C) = Y$. By Lemma 3, there is a compact set $D \subseteq Y$ such that $C \subseteq D$ and $H(C)$ is path connected in $H(D)$. Define $E = \bigcup\{g(D) \mid g \in G,\, D \cap g(D) \neq \emptyset\}$. Then E is a compact subset of Y containing D. Using Lemma 5, we may choose a compact set $F \subseteq Y$ such that $E \subseteq F$ and $H(E)$ is simply connected in $H(F)$. Put $B = \bigcup\{g(F) \mid g \in G,\, A \cap g(F) \neq \emptyset\}$.

Now, let $\gamma : [0,1] \to Y \setminus H(B)$ be a map with $\gamma(0) = \gamma(1)$ and let $h \in H$. Choose $n \in \mathbb{N}$ such that for all $i \in \{0,1,\ldots,n-1\}$ there is a $g_i \in G$ with $\gamma\big(\big[\frac{i}{n}, \frac{i+1}{n}\big]\big) \subseteq g_i(\operatorname{int} C)$. Then both $\gamma\big(\big[\frac{i}{n}, \frac{i+1}{n}\big]\big)$ and $h \circ \gamma\big(\big[\frac{i}{n}, \frac{i+1}{n}\big]\big)$ are contained in $g_i H(C)$ for all $i \in \{0,1,\ldots,n-1\}$. For each $i \in \{0,1,\ldots,n-1\}$ we connect $\gamma\big(\frac{i}{n}\big)$ to $h \circ \gamma\big(\frac{i}{n}\big)$ with a path in $g_i H(D)$. For $i \in \{0,1,\ldots,n-1\}$ and $j = i+1$, we have $g_i(D) \cap g_j(D) \neq \emptyset$, where we put $g_n = g_0$. Therefore, for all $h' \in H$ there is an $h'' \in H$ such that $g_i h''(D) \cap g_j h'(D) \neq \emptyset$, because H is normal in G, so that $g_j h'(D) \subseteq g_i h''(E)$ by definition of E. We conclude that

$g_jH(D) \subseteq g_iH(E)$. Consequently, the two paths joining $\gamma(\frac{i}{n})$ to $h \circ \gamma(\frac{i}{n})$ and $\gamma(\frac{j}{n})$ to $h \circ \gamma(\frac{j}{n})$, respectively, both lie in $g_iH(E)$, as do the paths $\gamma|_{[\frac{i}{n},\frac{j}{n}]}$ and $h \circ \gamma|_{[\frac{i}{n},\frac{j}{n}]}$. By choice of F, these loops can be contracted in $g_iH(F)$.

This fills in a homotopy from γ to $h \circ \gamma$ missing $H(A)$. For if $g_iH(F) \cap H(A) \neq \emptyset$ for some $i \in \{0, 1, \ldots, n-1\}$, we would have $g_ih'(F) \cap A \neq \emptyset$ for some $h' \in H$, so that $g_ih'(F) \subseteq B$, by the definition of B, implying that $\gamma([\frac{i}{n}, \frac{i+1}{n}]) \subseteq g_i(C) \subseteq g_i(F) \subseteq H(B)$, contrary to our assumption on γ. ∎

Later, we will also need

LEMMA 8. *Let L be a locally finite simplicial complex, $r : \widetilde{L} \to L$ the universal covering, U a connected subcomplex of L such that the inclusion induced homomorphism $\pi_1(U) \to \pi_1(L)$ is surjective, and P a finite subcomplex of L. Then for every compact set $A \subseteq \widetilde{L}$ there is a compact set $B \subseteq \widetilde{L}$ with $A \subseteq B$ and such that loops in $r^{-1}(U \cup P) \setminus B$ can be homotoped into $r^{-1}(U)$ with a homotopy in $\widetilde{L}$ missing A.*

Proof. Since $\pi_1(U) \to \pi_1(L)$ is surjective and U is connected, there is a homotopy H that takes the 1-skeleton of L into U leaving the 1-simplices of U fixed. For each 1-simplex σ of the finitely many 1-simplices of P which do not lie in U, choose a compact subset E_σ of $\widetilde{L}$ which contains a lift of the given homotopy that takes σ into U. Put $E = \bigcup E_\sigma$. Let a compact set $A \subseteq \widetilde{L}$ be given. Choose a compact set $B' \subseteq \widetilde{L}$ with $A \subseteq B'$ such that every translate of E (under a covering translation) which intersects A, lies in B'. Choose a compact set $B \subseteq \widetilde{L}$ with $B' \subseteq B$ such that every simplex of $\widetilde{L}$ which meets B', lies in B. Let α be a loop in $r^{-1}(U \cup P) \setminus B$. We can homotope $r \circ \alpha$ to a loop that lies in the union of U and the 1-skeleton of P such that during the homotopy points do not leave the top-dimensional simplex containing them. The lift of this homotopy lies in $\widetilde{L} \setminus B'$; call its end α'. Now, $r \circ \alpha'$ can be homotoped into U using the homotopy H. We lift this homotopy to a homotopy of α' and call its end α''. If a point is moved during this final homotopy, then it must lie in a translate of E. Hence, the track of such a point must miss A. ∎

4. Proof of Theorem 2.

From [20] we quote

LEMMA 9 (The Orbit Lemma I). *Suppose W is an open contractible n-manifold, $n \geq 3$. Let h be a non-trivial homeomorphism of W onto itself so that the group H of homeomorphisms generated by h acts without fixed points and properly discontinuously on W. If C is a compact subset of W, then loops of W can be homotoped off $H(C)$. Furthermore, given a compact set A there is a compact set B which contains A so that loops in $W \setminus B$ can be homotoped off $H(C)$ by a homotopy that lies in $W \setminus A$.*

Proof of Theorem 2. Let $H = \langle h \rangle$ be a non-trivial cyclic normal subgroup of $\pi_1(M)$. Note that h must have infinite order. (Otherwise $s = |\langle h \rangle| < \infty$ and $\widetilde{M}/\langle h \rangle$ is a finite-dimensional $K(\mathbb{Z}_s, 1)$, which contradicts the fact that $\mathbb{Z}_s$ has infinite cohomological dimension.) We assume that $n \geq 3$ and wish to show that $\widetilde{M}$ is simply connected at infinity.

Let $C \subseteq \widetilde{M}$ be compact. Use Lemma 7 to choose a compact set $E \subseteq \widetilde{M}$ such that $C \subseteq E$ and loops in $\widetilde{M} \setminus H(E)$ can be homotoped to any H-translate via a homotopy missing $H(C)$. By Lemma 9, there is a compact set $D \subseteq \widetilde{M}$ such that $E \subseteq D$ and loops in $\widetilde{M} \setminus D$ can be homotoped into $\widetilde{M} \setminus H(E)$ by a homotopy missing E (and hence C). Now, let γ be a loop in $\widetilde{M} \setminus D$. By choice of D we may already assume that γ lies in $\widetilde{M} \setminus H(E)$. Since $\widetilde{M}$ is simply connected, γ can be contracted to a point in $\widetilde{M}$. Let η be the image of this contraction. Choose $n \in \mathbb{N}$ such that $h^n(\eta) \cap C = \emptyset$. We then homotope γ to $h^n \circ \gamma$ with a homotopy missing $H(C)$, where it contracts missing C. ■

5. A geometric proof of Theorem 1. We now want to use our tools of Section 3 to give a geometric proof of the Houghton–Jackson Theorem.

To this end, let K be a finite connected simplicial complex with fundamental group G. We identify H with a subgroup of G and Q with G/H. Let $p : \widetilde{K} \to K$ be the universal covering and identify $G \equiv \mathrm{Aut}(\widetilde{K} \to K)$. Put $\overline{K} = \widetilde{K}/H$ with quotient (and covering) map $q : \widetilde{K} \to \overline{K}$, so that $H \equiv \mathrm{Aut}(\widetilde{K} \to \overline{K})$. We also identify $Q = G/H$ with $\mathrm{Aut}(\overline{K} \to K)$. Note that H is isomorphic to the fundamental group of a finite simplicial complex, because it is finitely presented. Since H acts properly discontinuously and cocompactly on the universal cover of this complex, we are in the setting of Section 3 with two simply connected spaces.

Choose a finite connected subcomplex C of $\widetilde{K}$ such that $G(\mathrm{int}\, C) = \widetilde{K}$, $H(C)$ is path connected, and the inclusion induced homomorphism $\pi_1(q(C)) \to \pi_1(\overline{K}) \simeq H$ is surjective.

We inductively define the following subsets of G/H. Put $B_0 = \{H\}$ and $B_n = \{gH \in G/H \mid gH(C) \cap g'H(C) \neq \emptyset \text{ for some } g'H \in B_{n-1}\}$. Then each B_n is finite and $G/H = \bigcup B_n$. Finally, put $T_n = \bigcup\{gH(C) \mid gH \in B_n \setminus B_{n-1}\}$.

Lemma 10. *G is one-ended.*

Proof. Let $D \subseteq \widetilde{K}$ be a compact set. Choose $n \in \mathbb{N}$ such that $D \subseteq \bigcup_{j<n} T_j$ and $D \cap T_n = \emptyset$.

We claim that we may choose a compact set $E \subseteq \widetilde{K}$ with $D \subseteq E$ and such that for any $g_1H, g_2H \in B_n$, every point of $g_1H(C) \setminus E$ can be joined to some point of $g_2H(C)$ by a path missing D. To see how, for each of the finitely many pairs $g_1H, g_2H \in B_n$, connect C to $g_1^{-1}g_2(C)$ by some path α in $\widetilde{K}$. Then for every $h \in H$, $g_1h \circ \alpha$ connects $g_1h(C)$ to $g_1hg_1^{-1}g_2(C) \subseteq g_2H(C)$. Hence, for any $g_1H, g_2H \in B_n$, every point of $g_1H(C)$ can be connected to some point of $g_2H(C)$ by a concatenation of a translate of a path β in C and a translate of some finite collection of paths (α_i). Since only finitely many translates of C and only finitely many translates of each α_i intersect D, we can choose E as claimed.

Now, let $a, b \in \widetilde{K} \setminus E$. These points may or may not lie in $\bigcup_{j\le n} T_j$. Without loss of generality, say $a \notin \bigcup_{j\le n} T_j$ and $b \in g_1H(C)$ for some $g_1H \in B_n$. Connect a to a point $p_1 \in g_2H(C)$ for some $g_2H \in B_n \setminus B_{n-1}$ with a path in $\widetilde{K} \setminus \bigcup_{j<n} T_j$. Connect b to a point $p_2 \in g_2H(C)$ by a path missing D. Finally, connect p_1 and p_2 in $g_2H(C)$. This yields a path from a to b missing D. ∎

LEMMA 11. *Suppose Q is one-ended and a compact set $E \subseteq \widetilde{K}$ is given. Then for every compact set $A \subseteq \widetilde{K}$ there is a compact set $B \subseteq \widetilde{K}$ such that $A \subseteq B$ and loops in $\widetilde{K} \setminus B$ can be homotoped off $H(E)$ by a homotopy that lies in $\widetilde{K} \setminus A$.*

Proof. Since $\overline{K}$ is one-ended, there is a finite subcomplex P of $\overline{K}$ and a connected subcomplex U of $\overline{K}$ such that $U \cap q(E) = \emptyset$ and $\overline{K} = U \cup P$. Now, Q is infinite so that $v(q(C)) \cap P = \emptyset$ for some $v \in Q$. Hence, $v(q(C)) \subseteq U$. Since $\pi_1(q(C)) \to \pi_1(\overline{K})$ is surjective, so is $\pi_1(v(q(C))) \to \pi_1(\overline{K})$. Hence $\pi_1(U) \to \pi_1(\overline{K})$ is surjective. The result now follows at once from Lemma 8. ∎

LEMMA 12. *If Q is one-ended, then G is simply connected at infinity.*

Proof. Since H is infinite, we can repeat the argument used in the last paragraph of the proof of Theorem 2, substituting Lemma 11 for Lemma 9 and an appropriate element of H for h^n. ∎

LEMMA 13. *If $a, b \in T_n$ lie in the same component of $\widetilde{K} \setminus \bigcup_{j<n} T_j$, then a and b can be joined by a path in T_n.*

Proof. Join a to b via two paths, one lying in $\widetilde{K} \setminus \bigcup_{j<n} T_j$ and the other in $\bigcup_{j\le n} T_j$. Since $\widetilde{K}$ is simply connected this loop contracts. The result now follows from the fact that T_n separates $\bigcup_{j>n} T_j$ from $\bigcup_{j<n} T_j$. ∎

If H is one-ended we can choose a finite subcomplex D of $\widetilde{K}$ such that $C \subseteq D$ and $H(C)$ is one-ended in $H(D)$, by Lemma 4. Let us also arrange for $H(D)$ to be path connected. We then put $T'_n = \bigcup\{gH(D) \mid gH \in B_n \setminus B_{n-1}\}$.

LEMMA 14. *Suppose H is one-ended. Then for every compact set $A \subseteq \widetilde{K}$ there is a compact subset $B \subseteq \widetilde{K}$ with $A \subseteq B$ such that if $a, b \in T_n \setminus B$ lie in the same component of T_n, then a and b can be joined by a path in $T'_n \setminus A$.*

Proof. By Lemma 4, there is a compact set $B' \subseteq \widetilde{K}$ with $A \subseteq B'$ such that for all the finitely many $gH \in B_n \setminus B_{n-1}$, $gH(C) \setminus B'$ is path connected in $gH(D) \setminus A$. Choose a compact set $B \subseteq \widetilde{K}$ with $B' \subseteq B$ such that for any $g_1H, g_2H \in B_n \setminus B_{n-1}$ with $g_1H(C)$ and $g_2H(C)$ in the same component of T_n, every point of $g_1H(C) \setminus B$ can be joined to some point of $g_2H(C)$ by a path in $T_n \setminus B'$.

Now suppose that $a, b \in T_n \setminus B$ lie in the same component of T_n. Say $b \in gH(C)$ with $gH \in B_n \setminus B_{n-1}$. Join a to a point $p_0 \in gH(C)$ by a path in $T_n \setminus B'$. Then join p_0 to b by a path in $gH(D) \setminus A$. This yields a path from a to b in $T'_n \setminus A$. ∎

LEMMA 15. *If H is one-ended, then G is simply connected at infinity.*

Proof. Let a compact set $A \subseteq \widetilde{K}$ be given. By Lemma 7, there is a compact set $E \subseteq \widetilde{K}$ with $A \subseteq E$ so that loops in $\widetilde{K} \setminus H(E)$ can be homotoped to any H-translate via a homotopy missing $H(A)$. Choose $n \in \mathbb{N}$ such that $E \subseteq \bigcup_{j<n} T'_j$ and $E \cap T'_n = \emptyset$. Pick any $sH \in B_n \setminus B_{n-1}$. Since $\pi_1((sH)(q(C))) \to \pi_1(\overline{K})$ is surjective, we can use Lemma 8 to find a compact set $F \subseteq \widetilde{K}$ with $E \subseteq F$ such that loops in $\bigcup_{j\le n} T'_j \setminus F$ can be homotoped into $sH(D)$ by a homotopy missing E (and hence missing A). By Lemma 14, there is a compact set $B \subseteq \widetilde{K}$ with $F \subseteq B$ such that points in $T_n \setminus B$ that are in the same component of T_n can be joined by paths in $T'_n \setminus F$.

Now, let γ be a loop in $\widetilde{K} \setminus B$. We wish to show that γ contracts in $\widetilde{K} \setminus A$. We may assume that γ is an edge path in the 1-skeleton of $\widetilde{K}$. First we argue that we may assume, without loss of generality, that γ lies either in $\bigcup_{j\le n} T'_j \setminus F$ or in $\widetilde{K} \setminus H(E)$. For if γ intersects $\bigcup_{j<n} T_j$ and intersects $\bigcup_{j>n} T_j$ we can cut it into finitely many subpaths that lie either in $\bigcup_{j\le n} T_j \setminus B$ or in $\bigcup_{j\ge n} T_j \setminus B$ and whose endpoints are in $T_n \setminus B$. The latter kind has its endpoints in the same component of T_n, by Lemma 13. We join these endpoints by paths in $T'_n \setminus F$. This leaves us with the problem of contracting finitely many loops that lie either in $\bigcup_{j\le n} T'_j \setminus F$ or in $\widetilde{K} \setminus H(E)$ with homotopies that miss A. Since the first kind can be homotoped into $sH(D) \subseteq \widetilde{K} \setminus H(E)$ with a homotopy that misses A, we are actually left with only loops of the second kind.

So, we now assume that γ is a loop in $\widetilde{K} \setminus H(E)$ to be contracted missing A. We do this as before. Since $\widetilde{K}$ is simply connected, we can contract γ to a point in $\widetilde{K}$. Let η be the image of that contraction. Since H is infinite, there is an $h \in H$ with $h(\eta) \cap A = \emptyset$. By choice of E we can homotope γ to $h \circ \gamma$ by a homotopy missing $H(A)$, where it contracts missing A. ∎

6. Proof of Corollary A. Since we may assume by Theorem 2 that N is at least 2-dimensional, we see that $\widetilde{N}$ is one-ended. Then $H = \pi_1(N)$

is one-ended and infinite. Finally, both $\widetilde{M}/H$ and N are $K(H,1)$'s and are thus homotopy equivalent. Then $H_n(\widetilde{M}/H;\mathbb{Z}_2) = H_n(N;\mathbb{Z}_2) = 0$. Therefore $\widetilde{M}/H$ is not compact. If we denote $\pi_1(M)$ by G, this implies that G/H, whose cardinality equals the number of sheets of the covering $\widetilde{M}/H \to M$, is infinite. Now apply Theorem 1 to complete the proof. ∎

7. Proof of Theorem 3. Let H be a subgroup of a group G. Suppose L is a finite simplicial complex with regular covering $\widehat{L} \to L$ whose automorphism group is isomorphic to G. Denote the quotient $\widehat{L}/H$ by $\overline{L}$. We say that the pair (G,H) is *two-ended* if for every compact set $A \subseteq \overline{L}$ there is a compact set $B \subseteq \overline{L}$ with $A \subseteq B$ such that $\overline{L} \setminus B$ has two components both of which are unbounded. It can be shown that this notion is independent of the choice of $\widehat{L} \to L$. (See [13] and [14] for a more general discussion of ends of pairs of groups.)

Proof of Theorem 3. Let $\widetilde{K}, \overline{K}, K, p, q$ be as in Section 5. Let $A \subseteq \widetilde{K}$ be a compact set. We will find a compact set $B \subseteq \widetilde{K}$ such that $A \subseteq B$ and $\widetilde{K} \setminus B$ is simply connected in $\widetilde{K} \setminus A$. Since (G,H) is two-ended, we may choose a finite subcomplex $C_1 \subseteq \widetilde{K}$ such that $A \subseteq C_1$ and $\overline{K} \setminus q(C_1)$ has two components both of which are unbounded. We also arrange for the inclusion induced map $\pi_1(q(C_1)) \to \pi_1(\overline{K})$ to be surjective and for $H(C_1)$ to be path connected. (Note that $\pi_1(\overline{K}) \simeq H$ is finitely generated.) Use Lemmas 4 and 5 to choose a finite subcomplex $C_2 \subseteq \widetilde{K}$ such that $C_1 \subseteq C_2$, $H(C_1)$ is one-ended in $H(C_2)$, and $H(C_1)$ is simply connected in $H(C_2)$. Again, we may assume that $\overline{K} \setminus q(C_2)$ has two components both of which are unbounded. Since the infinite group $\mathbb{N}_G(H)/H \simeq \mathrm{Aut}(\overline{K} \to K)$ acts properly discontinuously on $\overline{K}$ and $(gH)(q(T)) = q(g(T))$ for all $g \in \mathbb{N}_G(H)$ and $T \subseteq \widetilde{K}$, there are elements $g_1, \ldots, g_5 \in \mathbb{N}_G(H)$ such that the collection $\{q(g_i(C_2)) \mid i = 1, \ldots, 5\}$ is pairwise disjoint and such that $q(g_i(C_2))$ lies in the bounded component of $\overline{K} \setminus (q(g_{i-1}(C_2)) \cup q(g_{i+1}(C_2)))$ for $i = 2,3,4$. We take $g_3 = 1$.

Let D be a finite subcomplex of $\widetilde{K}$ such that $q(D)$ equals the complement of the two unbounded components of $\overline{K} \setminus (q(g_1(C_2)) \cup q(g_5(C_2)))$. By Lemma 8, there is a compact set $C_3 \subseteq \widetilde{K}$ with $A \subseteq C_3$ and such that loops in $q^{-1}(q(D)) \setminus C_3 = H(D) \setminus C_3$ can be homotoped into $q^{-1}(q(g_2(C_1))) = g_2H(C_1)$ missing A. Finally, we choose a compact set $B \subseteq \widetilde{K}$ with $C_3 \subseteq B$ such that $g_1H(C_1) \setminus B$ and $g_5H(C_1) \setminus B$ are path connected in $g_1H(C_2) \setminus C_3$ and $g_5H(C_2) \setminus C_3$, respectively.

Now, let γ be a loop in $\widetilde{K} \setminus B$. We may assume that γ is an edge path in the 1-skeleton of $\widetilde{K}$. If γ lies outside of $H(D)$, we first contract it in $\widetilde{K}$. Since $g_iH(C_1)$ is simply connected in $g_iH(C_2)$, we can cut off this singular disk at $g_2H(C_1) \cup g_4H(C_1)$ and cap it off at $g_2H(C_2) \cup g_4H(C_2)$. Hence, in this case, γ contracts missing A.

If γ has subpaths which lie outside of $H(D)$ and whose endpoints are in $g_1H(C_1)$ or $g_5H(C_1)$, we connect the endpoints of each such subpath by a path in $g_1H(C_2) \setminus C_3$ or $g_5H(C_2) \setminus C_3$, respectively. Since we can deal with these newly formed loops as in the previous case, we may now assume that γ lies entirely in $H(D) \setminus C_3$.

If γ lies in $H(D) \setminus C_3$, we can homotope it into $g_2H(C_1)$ missing A, where it contracts within $g_2H(C_2)$, still missing A. ∎

8. Davis's examples. In this section we will analyze the examples of Davis mentioned in the introduction, and discover that $\pi_1(N)$ *equals* its normalizer in $\pi_1(M)$ in these examples.

A Coxeter system $\Gamma = \langle V \mid v^2 = 1,\ (uv)^{m(u,v)} = 1 \text{ for all } u, v \in V\rangle$ (a group defined in terms of finitely many generators and specific relations) is called *right-angled* if $m(u,v) \in \{\infty, 2\}$ for all $u \neq v$. Its *nerve* is defined to be the abstract simplicial complex $N(\Gamma, V)$ consisting of all non-empty subsets of V which generate a finite subgroup of Γ, where incidence is by inclusion.

Let P be the first barycentric subdivision of a non-simply connected PL-homology $(n-1)$-sphere. (Such examples exist in all dimensions 3 and higher.) Then there is exactly one right-angled Coxeter system (Γ, V) whose nerve $N(\Gamma, V)$ is isomorphic to P, namely, the Coxeter group Γ which is generated by the vertex set V of P and whose only relations are of the form $(uv)^2 = 1$ whenever $\{u, v\} \in P$ (cf. [4]). Let C be the unique compact contractible n-manifold with boundary $N(\Gamma, V)$; it will serve as a *basic chamber*. Denote the dual cell of a vertex v in $N(\Gamma, V)$ by C_v (i.e. C_v is the star of v in a further barycentric subdivision of $N(\Gamma, V)$.) Put $\mathcal{M}(\Gamma) = \Gamma \times C/{\sim}$ where $(g, x) \sim (h, y) \Leftrightarrow x = y$ and $g^{-1}h \in \langle v \mid x \in C_v\rangle$. Then Γ acts properly discontinuously and cocompactly on $\mathcal{M}(\Gamma)$ by left multiplication on the first coordinate.

In [3], Davis shows that $\mathcal{M}(\Gamma)$ is an open contractible manifold which is not homeomorphic to Euclidean space. It is well known that the commutator subgroup $[\Gamma, \Gamma]$ of Γ is torsion free and of finite index in Γ. (To see this, consider the canonical epimorphism $\phi : \Gamma \to \mathbb{Z}_2^{|V|}$. Clearly, $[\Gamma, \Gamma] \subseteq \ker\phi$. Conversely, if $g \in \ker\phi$, then g is the product of generators each appearing an even number of times. Hence, $g[\Gamma, \Gamma] = [\Gamma, \Gamma]$ in the abelian group $\Gamma/[\Gamma, \Gamma]$ so that $g \in [\Gamma, \Gamma]$. The index is consequently given by $[\Gamma : [\Gamma, \Gamma]] = [\Gamma : \ker\phi] = |\mathbb{Z}_2^{|V|}| = 2^{|V|}$. The fact that $[\Gamma, \Gamma]$ is torsion free can be proved using the right-angled reduction scheme below, as is done for example in Lemma 1.5 of [5].) Therefore $[\Gamma, \Gamma]$ acts fixed-point free and cocompactly on $\mathcal{M}(\Gamma)$. The quotient $\mathcal{M}(\Gamma)/[\Gamma, \Gamma]$ is our manifold M.

Fix an element $v \in V$. Then v acts as a reflection on $\mathcal{M}(\Gamma)$ through the fixed-point set $\mathrm{Fix}(v) = \{p \in \mathcal{M}(\Gamma) \mid v(p) = p\}$. Put $\widetilde{V} = V \cap \mathrm{lk}(v, N(\Gamma, V))$ and let $(\widetilde{\Gamma}, \widetilde{V})$ be the induced right-angled Coxeter system. Then the nerve of $(\widetilde{\Gamma}, \widetilde{V})$ is the PL-sphere $\mathrm{lk}(v, N(\Gamma, V))$ and $\mathrm{Fix}(v) = \{gC_v \mid g \in \widetilde{\Gamma}\}$. In fact, we can identify $\mathrm{Fix}(v)$ with $\mathcal{M}(\widetilde{\Gamma})$, where C_v takes on the role of the basic chamber. (See, for example, [6].) Now, C_v is a ball, so that $\mathrm{Fix}(v) = \mathcal{M}(\widetilde{\Gamma})$ is homeomorphic to $(n-1)$-dimensional Euclidean space. Since the commutator subgroups, being the kernels of the respective abelianization homomorphisms, satisfy $[\widetilde{\Gamma}, \widetilde{\Gamma}] = [\Gamma, \Gamma] \cap \widetilde{\Gamma}$, the covering map $p : \mathcal{M}(\Gamma) \to \mathcal{M}(\Gamma)/[\Gamma, \Gamma] = M$ restricts to $p|_{\mathrm{Fix}(v)} : \mathrm{Fix}(v) \to \mathrm{Fix}(v)/[\widetilde{\Gamma}, \widetilde{\Gamma}]$. The quotient $\mathcal{M}(\widetilde{\Gamma})/[\widetilde{\Gamma}, \widetilde{\Gamma}]$ is our manifold N. Clearly $\pi_1(N) = [\widetilde{\Gamma}, \widetilde{\Gamma}] \leq [\Gamma, \Gamma] = \pi_1(M)$. We will now verify that in these examples $\mathbb{N}_{[\Gamma,\Gamma]}([\widetilde{\Gamma}, \widetilde{\Gamma}]) = [\widetilde{\Gamma}, \widetilde{\Gamma}]$.

Recall that Coxeter groups have a very simple solution to the word problem [18]: a word (finite sequence of generators) is reduced (minimal in length) if and only if it cannot be shortened by a combination of the following two operations: (i) the obvious cancellation of a subword of the form uu, and (ii) replacement of a subword of the form $uwuwuw\ldots$ (of length m) by $wuwuwu\ldots$ (of length m), where m is the order of the element uw in the group. This becomes especially easy to check in a right-angled Coxeter group. Specifically, if an element g of a right-angled Coxeter group is expressed as a product of generators, say $g = u_1 \ldots u_q$, then it can be brought into reduced form by repeated application of the following operation: deletion of some $u_i = u_j$ $(1 \leq i < j \leq q)$ which commute with all of $u_{i+1}, \ldots, u_{j-1}$ in between, so that $g = u_1 u_2 \ldots \widehat{u}_i \ldots \widehat{u}_j \ldots u_q$, where the hat denotes omission.

Let $g \in [\Gamma, \Gamma]$ with $g[\widetilde{\Gamma}, \widetilde{\Gamma}]g^{-1} = [\widetilde{\Gamma}, \widetilde{\Gamma}]$. Express $g = u_1 \ldots u_q$ reduced with all $u_i \in V$. We will show that $u_i \in \widetilde{V}$ for all i, so that $g \in [\widetilde{\Gamma}, \widetilde{\Gamma}]$. Choose the maximal index i_0 with $u_{i_0} \notin \widetilde{V}$ (if there is such an index). Since $[\widetilde{\Gamma}, \widetilde{\Gamma}]$ is a normal subgroup of $\widetilde{\Gamma}$, we have $u_1 u_2 \ldots u_{i_0}[\widetilde{\Gamma}, \widetilde{\Gamma}]u_{i_0} \ldots u_2 u_1 = [\widetilde{\Gamma}, \widetilde{\Gamma}]$. We now show that $u_{i_0} x = x u_{i_0}$ for all $x \in \widetilde{V}$. Let $x \in \widetilde{V}$. Choose $y \in \widetilde{V}$ with $xyxy \neq 1$. (The easiest way of seeing that such a y always exists is to take v to be a barycenter of a top-dimensional simplex in the original triangulation of the homology sphere.) Then

$$u_1 \ldots u_{i_0} xyxy u_{i_0} \ldots u_1 s_1 \ldots s_p = 1$$

for some $s_i \in \widetilde{V}$, where we may assume $s_1 \ldots s_p$ to be reduced. Applying the above right-angled reduction scheme to this equation, we conclude that $u_{i_0} x = x u_{i_0}$, because otherwise only five types of deletions would be possible:

(i) $u_1 \ldots u_{i_0} xyx\widehat{y} u_{i_0} \ldots \widehat{u}_j \ldots u_1 s_1 \ldots s_p$ with $i_0 > j \geq 1$;
(ii) $u_1 \ldots u_{i_0} xyx\widehat{y} u_{i_0} \ldots u_1 s_1 \ldots \widehat{s}_j \ldots s_p$ with $1 \leq j \leq p$;

(iii) $u_1 \dots \widehat{u}_i \dots u_{i_0} xyx(y) u_{i_0} \dots \widehat{u}_j \dots u_1 s_1 \dots s_p$ with $1 \le i < i_0$, $i_0 > j \ge 1$;
(iv) $u_1 \dots \widehat{u}_i \dots u_{i_0} xyx(y) u_{i_0} \dots u_1 s_1 \dots \widehat{s}_j \dots s_p$ with $1 \le i < i_0$, $1 \le j \le p$;
(v) $u_1 \dots u_{i_0} xyx(y) u_{i_0} \dots \widehat{u}_i \dots u_1 s_1 \dots \widehat{s}_j \dots s_p$ with $i_0 > i \ge 1$, $1 \le j \le p$.

(The parentheses around y in (iii)–(v) denote the possibility of y no longer occurring in this position, due to a previous deletion of type (i) or (ii).) However, no finite combination of these five reductions would ever cancel the word. (Notice that repeated application of any of these five types of deletions across the individually reduced expressions $u_1 \dots u_{i_0}$, $xyxy$, $u_{i_0} \dots u_1$, and $s_1 \dots s_p$ will leave each expression *individually* reduced, because any reduction of the thus shortened expression could have been carried out in the original expression, where all relevant, but now missing, generators would have commuted with the would-be deletion pair.)

Since $u_{i_0} x = x u_{i_0}$ for all $x \in \widetilde{V}$, we must have $u_{i_0} = v$, because v is the only vertex of $N(\Gamma, V)$ which is joined to all vertices of its link. Inductively, we conclude that all $u_i \in \{v\} \cup \widetilde{V}$. Since the word $u_1 \dots u_q$ is reduced and contains every generator an even number of times, we have in fact $u_i \in \widetilde{V}$ for all i. ∎

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