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Classification of homotopy classes of equivariant gradient maps

by

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Abstract. Let V be an orthogonal representation of a compact Lie group G and let $S(V), D(V)$ be the unit sphere and disc of V, respectively. If $F : V \to \mathbb{R}$ is a G-invariant C^1-map then the G-equivariant gradient C^0-map $\nabla F : V \to V$ is said to be admissible provided that $(\nabla F)^{-1}(0) \cap S(V) = \emptyset$. We classify the homotopy classes of admissible G-equivariant gradient maps $\nabla F : (D(V), S(V)) \to (V, V \setminus \{0\})$.

1. Introduction. The purpose of this paper is to understand the topological invariants associated with equivariant gradient maps.

More precisely, we assume V is a real finite-dimensional orthogonal representation of a compact Lie group G. In other words, V is a real finite-dimensional linear space with a scalar product and there is an orthogonal action of G on V denoted by gx for $g \in G,\ x \in V$. We are then interested in the homotopy classes of G-equivariant gradient mappings from V into V, which are non-zero on the unit sphere $S(V)$ of V. Here a map $\nabla F : V \to V$ is said to be *G-equivariant* if $\nabla F(gx) = g\nabla F(x)$ for $g \in G,\ x \in V$. We completely classify the homotopy classes of such maps in terms of a G-equivariant gradient degree defined here. Our gradient degree is a natural generalization of the classical Brouwer degree with the ring of integers replaced by a ring $U(G)$ determined by the orbit types of G (see [4] for the definition of $U(G)$).

Results of this type are of considerable interest because they show that there are no extra invariants to be found. Note that the study of the homotopy classes of G-equivariant gradient maps is a natural problem as these maps arise in many applications (see [3, 10, 11, 12] for example). The stronger homotopy invariance properties of our degree are reflected in the possibility

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to obtain connected sets of bifurcation solutions as in [3, 10, 11, 12]. The Conley index is not suitable for this.

The analogous problem of homotopy classification of G-equivariant mappings when there is no gradient structure is well known in G-equivariant topology. In this case classifying our homotopies is equivalent to classifying the equivariant maps of $S(V)$ into itself. It turns out that in this case the classical Brouwer degree, which assigns an integer to each homotopy class, can be replaced by a more refined invariant which assigns to a homotopy class an element of the Burnside ring $A(G)$ of G. In most (but not all) cases, this new invariant is determined by the Brouwer degrees of the induced maps on fixed point subspaces of isotropy subgroups (see [4, 9] and the references therein).

In the case $G = \{e\}$, $V = \mathbb{R}^n$ Parusiński proved that the classical Brouwer degree is the only invariant (see [8]). More precisely, he showed that two gradient maps non-vanishing on the unit sphere S^{n-1} are homotopic as gradient maps if and only if they are homotopic in the ordinary sense.

On the other hand, the first named author essentially showed in [3] that for the group $SO(2)$, the analogue of the Parusiński result for $SO(2)$-equivariant gradient maps does not hold and he constructed an extra invariant. The work was further improved in [10] for $SO(2)$-actions and in [6] for general compact Lie group actions.

In the proof of the main result of [8] the following construction plays a fundamental role: assume that $F = \nabla\varphi$ is a gradient C^0-map on $\mathbb{R}^n$. For $x \in S^{n-1}$ we have $F(x) = F_0(x) + f(x)x$, where $F_0 = \nabla(\varphi_{|S^{n-1}})$. Then F is non-vanishing on S^{n-1} if and only if $F_0^{-1}(0) \cap f^{-1}(0) = \emptyset$. Thus to an admissible gradient vector field F on $\mathbb{R}^n$ there corresponds the gradient pair (F_0, f) which "lives" on the $(n-1)$–dimensional sphere. In this way Parusiński reduces dimension.

In our paper, generalizing the above construction, we introduce the notion of equivariant gradient pair. More precisely, given a smooth Riemannian manifold M with a smooth action of a compact Lie group G we consider pairs (X, f), where X is a G-equivariant gradient vector field on M and f is a G-invariant function on M. There is also a natural notion of homotopy between G-equivariant gradient pairs.

In this article we consider smooth Riemannian G-manifolds M with a finite number of orbit types, i.e. $\#\Phi(M) < \infty$ (see Section 2), and notice that if $\Phi(M) = \{(H)\}$ then there is a bijection between the homotopy classes of G-equivariant gradient pairs on $M = M_{(H)}$ and the homotopy classes of gradient pairs on the quotient manifold $M_{(H)}/G$. Next combining Corollary 3.1 and Corollary 6.2 we classify the homotopy classes of equivariant gradient pairs on M.

In Section 2 we recall some definitions of equivariant topology and give the precise definitions used in our paper. Section 3 contains the basic technical results. The main results of this section are Theorem 3.1 and Corollary 3.1. Section 4 contains the main results of this paper. Theorem 4.1 gives the classification of homotopy classes of G-equivariant gradient pairs on M and Corollary 4.1 gives the classification of homotopy classes of admissible (non-vanishing on $S(V)$) G-equivariant gradient maps. In Section 5 we hint at some relation between the G-equivariant Conley index theory and the G-equivariant gradient degree. In the Appendix we classify the homotopy classes of gradient pairs on N, where N is a connected smooth Riemannian manifold with $\dim N \geq 1$. We discuss Parusiński's proof of the above mentioned result (see [8]). Essentially we rewrite this proof using the language of gradient pairs. We are convinced that understanding this approach may result in a better understanding of the technicalities of our paper.

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2. Equivariant gradient maps and pairs. In this section we set up notation and terminology. Throughout this article G stands for a compact Lie group, and V denotes a real, finite-dimensional, orthogonal representation of G with G-invariant scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. Set $D(V) = \{v \in V : |v| \leq 1\}$, $B(V) = \{v \in V : |v| < 1\}$ and $S(V) = \{v \in V : |v| = 1\}$. Since the representation V is orthogonal, these sets are G-invariant.

Moreover, we will denote by M a Riemannian smooth G-manifold. Since the group G is compact, without loss of generality one can assume that M is equipped with a G-invariant metric $\langle \cdot, \cdot \rangle$. The action of the group G on $M \times [0,1]$ is given by $g(v,t) = (gv,t)$ for any $(g,v,t) \in G \times M \times [0,1]$. The tangent bundle $T(M)$ is considered as a G-bundle with the induced orthogonal G-action.

We will denote by $\nabla\varphi : M \to T(M)$ the gradient of a C^1-map $\varphi : M \to \mathbb{R}$. The class of G-invariant C^k-maps $\varphi : M \to \mathbb{R}$ will be denoted by $C^k_G(M, \mathbb{R})$. It is known that if $\varphi \in C^1_G(M, \mathbb{R})$, then $\nabla\varphi : M \to T(M)$ is a G-equivariant continuous vector field.

The set of conjugacy classes of closed subgroups of G will be denoted by $\Phi(G)$ and the conjugacy class of a closed subgroup $H \subset G$ will be denoted by (H). A subgroup $H \subset G$ is *subconjugate* to a subgroup $K \subset G$ (written $(H) < (K)$) if H is conjugate to a subgroup of K. Subconjugation defines a partial ordering on $\Phi(G)$.

Let $G_v = \{g \in G : gv = v\}$ be the isotropy subgroup of $v \in M$. For $(H) \in \Phi(G)$ we set $M_{(H)} = \{v \in M : (G_v) = (H)\}$ and $\Phi(M) = \{(H) \in \Phi(G) : M_{(H)} \neq \emptyset\}$. Notice that $M_{(H)}$ is a submanifold of M and the quotient map $\pi_{(H)} : M_{(H)} \to M_{(H)}/G$ is smooth. Moreover, the orbit $G \cdot v$ is a submanifold

of $M_{(G_v)}$ diffeomorphic to G/G_v. We will denote by $\Psi_{(H)}(M)$ the family of connected components of $M_{(H)}/G$ and $\Psi(M) = \bigcup_{(H)\in\Phi(M)} \Psi_{(H)}(M)$.

DEFINITION 2.1. Let $F \in C^1_G(V,\mathbb{R})$. A map $\nabla F \in C^0_G(V,V)$ is called a *G-equivariant admissible gradient map* (written $\nabla F \in \mathcal{G}(V)$) if $(\nabla F)^{-1}(0) \cap S(V) = \emptyset$.

DEFINITION 2.2. Let $H \in C^1_G(V\times[0,1],\mathbb{R})$. The map $\nabla_v H : V\times[0,1] \to V$ is called a $\mathcal{G}(V)$-*homotopy* if $(\nabla_v H)^{-1}(0)\cap(S(V)\times[0,1]) = \emptyset$. We say that maps $\nabla F_0, \nabla F_1 \in \mathcal{G}(V)$ are $\mathcal{G}(V)$-*homotopic* if there is a $\mathcal{G}(V)$-homotopy $\nabla_v H : V\times[0,1] \to V$ such that $\nabla_v H(\cdot,i) = \nabla F_i(\cdot)$ for $i = 0,1$.

Let $\nabla F \in \mathcal{G}(V)$. For $v \in S(V)$ we have the unique decomposition

$$\nabla F(v) = (\nabla F(v) - \langle \nabla F(v), v\rangle \cdot v) + \langle \nabla F(v), v\rangle \cdot v = X_{\nabla F}(v) + f_{\nabla F}(v)\cdot v,$$

where $X_{\nabla F} = \nabla(F_{|S(V)})$ and $f_{\nabla F} \in C^0_G(S(V),\mathbb{R})$.

Similarly, if ∇H is a $\mathcal{G}(V)$-homotopy then for $v \in S(V)$ we have the decomposition

$$\begin{aligned}\nabla_v H(v,t) &= (\nabla_v H(v,t) - \langle \nabla_v H(v,t), v\rangle \cdot v) + \langle \nabla_v H(v,t), v\rangle \cdot v \\ &= X_{\nabla H}(v,t) + f_{\nabla H}(v,t)\cdot v,\end{aligned}$$

where $X_{\nabla H}(\cdot,t) = \nabla(H(\cdot,t)_{|S(V)})$ and $f_{\nabla H} \in C^0_G(S(V)\times[0,1],\mathbb{R})$.

The above leads us to the following definitions.

DEFINITION 2.3. We say that (X,f) is an *equivariant gradient pair* on M if

- $f \in C^0_G(M,\mathbb{R})$,
- $N_-(f) = \{x \in M : f(x) \le 0\}$ is a compact subset of M,
- $X : M \to TM$ and there is $\varphi \in C^1_G(M,\mathbb{R})$ such that $X = \nabla\varphi$,
- $X^{-1}(0) \cap f^{-1}(0) = \emptyset$.

The class of equivariant gradient pairs on M will be denoted by $\mathcal{P}(M)$.

DEFINITION 2.4. We say that (X,f) is an *equivariant gradient pair homotopy* on M if

- $X : M\times[0,1] \to TM$ and there is $\varphi \in C^1_G(M\times[0,1],\mathbb{R})$ such that $X(\cdot,t) = \nabla_x\varphi(\cdot,t)$ for $t \in [0,1]$,
- $f \in C^0_G(M\times[0,1],\mathbb{R})$,
- $N_-(f) = \{x \in M\times[0,1] : f(x,t) \le 0\}$ is a compact subset of $M\times[0,1]$,
- $X^{-1}(0) \cap f^{-1}(0) = \emptyset$.

Equivariant gradient pair homotopies will be called $\mathcal{P}(M)$-*homotopies*.

From now on the set of $\mathcal{G}(V)$-homotopy classes will be denoted by $\mathcal{G}[V]$. Moreover, the set of $\mathcal{P}(M)$-homotopy classes in $\mathcal{P}(M)$ will be denoted by $\mathcal{P}[M]$. The principal significance of the next theorem is that it allows us to

reduce the classification of $\mathcal{G}[V]$ to the classification of $\mathcal{P}[S(V)]$. In fact it is the first step in the classification of $\mathcal{G}[V]$.

LEMMA 2.1. *The map* $\Upsilon_V : \mathcal{G}[V] \to \mathcal{P}[S(V)]$ *defined by* $\Upsilon_V([\nabla F]) = [(X_{\nabla F}, f_{\nabla F})]$ *is a bijection.*

Lemma 2.1 is an equivariant version of Lemma 1 of [8] with the same proof, therefore we omit it.

LEMMA 2.2. *Let* M_0, M_1 *be Riemannian smooth* G*-manifolds with* G*-invariant metrics such that there exists a* G*-equivariant diffeomorphism* $\Phi : M_0 \to M_1$. *Then the map* $\Phi^* : \mathcal{P}(M_1) \to \mathcal{P}(M_0)$ *given by* $\Phi^*(\nabla\varphi, f) = (\nabla(\varphi \circ \Phi), f \circ \Phi)$ *induces a bijection* $[\Phi^*] : \mathcal{P}[M_1] \to \mathcal{P}[M_0]$.

The easy proof of this lemma is left to the reader.

DEFINITION 2.5. We denote by $\mathcal{K}(M)$ the collection of pairs (U, X) such that

- $U \subset M$ is open invariant;
- there is $\varphi \in C^1_G(U)$ such that $X = \nabla\varphi$;
- $X^{-1}(0)$ is a compact subset of U.

DEFINITION 2.6. We say that (U, X) is a $\mathcal{K}(M)$-*homotopy* if

- $U \subset M \times [0,1]$ is open invariant;
- there is $\varphi \in C^1_G(U)$ such that $X = \nabla_x\varphi$;
- $X^{-1}(0)$ is a compact subset of U.

Two pairs $(U_0, X_0), (U_1, X_1) \in \mathcal{K}(M)$ are said to be $\mathcal{K}(M)$-*homotopic* if there exists a $\mathcal{K}(M)$-homotopy (U, X) such that, for $i = 0, 1$,

$$U_i = \{x \in M : (x, i) \in U\}, \quad X_i(x) = X(x, i).$$

In this case (U, X) is called a $\mathcal{K}(M)$-homotopy connecting (U_0, X_0) and (U_1, X_1). We denote by $\mathcal{K}[M]$ the set of all $\mathcal{K}(M)$-homotopy classes.

The following lemma will be useful in our considerations.

LEMMA 2.3. *Let* $(U, X) \in \mathcal{K}(M)$. *If* $W \subset U$ *is an open invariant subset and* $X^{-1}(0) \subset W$ *then*

(a) $(W, X_{|W}) \in \mathcal{K}(M)$;
(b) (U, X) *and* $(W, X_{|W})$ *are* $\mathcal{K}(M)$*-homotopic.*

Proof. (a) Obvious.
(b) Set $V = U \times [0,1] \setminus (U \setminus W) \times \{1\}$ and $Y(x,t) = X(x)$ for $(x,t) \in V$. Evidently (Y, V) is a $\mathcal{K}(M)$-homotopy connecting (U, X) and $(W, X_{|W})$. ∎

Let (X, f) be any pair in $\mathcal{P}(M)$. The assignment

$$(X, f) \mapsto \mathcal{K}(X, f) = (f^{-1}((-\infty, 0)), X_{|U})$$

defines a transformation $\mathcal{K} : \mathcal{P}(M) \to \mathcal{K}(M)$.

It is obvious from the definition of $\mathcal{K}$ that if $(X_0, f_0), (X_1, f_1) \in \mathcal{P}(M)$ are homotopic in $\mathcal{P}(M)$, then $\mathcal{K}(X_0, f_0)$ and $\mathcal{K}(X_1, f_1)$ are homotopic in $\mathcal{K}(M)$. Therefore $\mathcal{K}$ induces a transformation $\mathfrak{K} : \mathcal{P}[M] \to \mathcal{K}[M]$.

LEMMA 2.4. *The mapping* $\mathfrak{K} : \mathcal{P}[M] \to \mathcal{K}[M]$ *is bijective.*

Proof. Surjectivity. Let $(U, X) \in \mathcal{K}(M)$. Choose φ such that $X = \nabla\varphi$. Choose an open invariant set $U_0 \subset U$ such that $\overline{U}_0$ is compact and

$$X^{-1}(0) \subset U_0 \subset \overline{U}_0 \subset U.$$

Consider a smooth invariant function $f : M \to [-1, 1]$ such that

$$f(x) = \begin{cases} -1 & \text{for } x \in X^{-1}(0), \\ 1 & \text{for } x \in M \setminus U_0. \end{cases}$$

Choose an open invariant $U_1 \subset U$ such that $\overline{U}_0 \subset U_1 \subset \overline{U}_1 \subset U$. Consider a smooth invariant function $\mu : M \to [0, 1]$ such that

$$\mu(x) = \begin{cases} 1 & \text{for } x \in \overline{U}_0, \\ 0 & \text{for } x \in M \setminus U_1. \end{cases}$$

Set

$$\psi(x) = \begin{cases} \mu(x)\varphi(x) + (1 - \mu(x)) & \text{for } x \in X_1, \\ 1 & \text{for } x \in M \setminus U_1. \end{cases}$$

Let $Y = \nabla\psi$ and $W = f^{-1}((-\infty, 0))$. Since $N_-(f) \subset U_0$ and $\psi(x) = \varphi(x)$ for $x \in U_0$, $(Y, f) \in \mathcal{P}(M)$. By the definition of $\mathcal{K}$, $\mathcal{K}(Y, f) = (W, X_{|W})$. According to Lemma 2.3, (U, X) and $(W, X_{|W})$ are homotopic in $\mathcal{K}(M)$. This completes the proof of surjectivity.

The proof of injectivity is similar and is omitted. ∎

3. Normality conditions. Throughout this section, (**H**) stands for a maximal element in $\Phi(M)$. The goal of this section is to establish a bijection

$$\mathcal{P}[M] \approx \mathcal{P}[M_{(\mathbf{H})}] \times \mathcal{P}[M \setminus M_{(\mathbf{H})}]. \tag{3.1}$$

In what follows, the tangent bundle $T(M_{(\mathbf{H})})$ will be considered as a subbundle of the restriction $T(M)_{|M_{(\mathbf{H})}}$. Since $M_{(\mathbf{H})}$ is a G-invariant, closed, smooth submanifold of M, $N = T(M_{(\mathbf{H})})^{\perp}$ is a G-invariant subbundle of $T(M)_{|M_{(\mathbf{H})}}$, called the *normal bundle* of $M_{(\mathbf{H})}$. We denote by $\pi : N \to M_{(\mathbf{H})}$ the bundle projection.

By the tubular neighbourhood theorem (see Theorem 2.2 of [2]) there exists an open invariant $\Omega \subset M$ and a smooth G-equivariant diffeomorphism $\Lambda : N \to \Omega$ which maps the zero section of N onto $M_{(\mathbf{H})}$. From now

on we identify Ω with N. Using this convention, for $r > 0$, we denote by $D_r(N), B_r(N), S_r(N)$ the corresponding bundles of closed discs, open discs and spheres of radius r, respectively. We also put $M_\infty = M \setminus D_1(N)$. Define $X_\nu : N \to T(N)$ by $X_\nu = \nabla\nu$, where $\nu(x) = \frac{1}{2}|x|^2$. Clearly X_ν is continuous and G-equivariant. For $x \in N$ consider the decomposition $T_xM = V_x \oplus H_x$, where $V_x = \mathrm{Ker}(D\pi(x))$ and $H_x = (V_x)^\perp$.

The following result is a simple consequence of the definition of X_ν.

REMARK 3.1. *If $x \in N \setminus M_{(\mathbf{H})}$ then $X_\nu(x) \notin H_x$.*

Let $(U_0, \nabla\varphi_0)$ be any pair in $\mathcal{K}(M_{(\mathbf{H})})$ and let $(U_\infty, \nabla\varphi_\infty)$ be any pair in $\mathcal{K}(M_\infty)$. Define (U, X) by

$$U := \widehat{U}_0 \cup U_\infty, \quad \text{where} \quad \widehat{U}_0 = \pi^{-1}(U_0) \cap B_1(N),$$

$$X(x) = \begin{cases} \nabla\varphi_0(\pi(x)) + X_\nu(x) & \text{for } x \in \widehat{U}_0, \\ \nabla\varphi_\infty(x) & \text{for } x \in U_\infty. \end{cases}$$

Note that $(D\pi(x))^*$ maps $T_{\pi(x)}(M_{(\mathbf{H})})$ isomorphically onto H_x. Since

$$\nabla(\varphi_0 \circ \pi)(x) = (D\pi(x))^*(\nabla\varphi_0(\pi(x))), \tag{3.2}$$

Remark 3.1 implies that if $x \in \widehat{U}_0$ then $X(x) = 0$ if and only if $x \in M_{(\mathbf{H})}$ and $\nabla\varphi_0(x) = 0$. Thus $(U, X) \in \mathcal{K}(M)$ and the assignment

$$((U_0, \nabla\varphi_0), (U_\infty, \nabla\varphi_\infty)) \mapsto (U, X)$$

defines a transformation $\mathcal{E} : \mathcal{K}(M_{(\mathbf{H})}) \times \mathcal{K}(M_\infty) \to \mathcal{K}(M)$. Clearly $\mathcal{E}$ preserves $\mathcal{K}(M_{(\mathbf{H})}) \times \mathcal{K}(M_\infty)$-homotopy classes. Therefore it induces a transformation $\mathfrak{E} : \mathcal{K}[M_{(\mathbf{H})}] \times \mathcal{K}[M_\infty] \to \mathcal{K}[M]$.

We can now state a theorem which plays an important role in our considerations.

THEOREM 3.1. *The mapping $\mathfrak{E} : \mathcal{K}[M_{(\mathbf{H})}] \times \mathcal{K}[M_\infty] \to \mathcal{K}[M]$ is bijective.*

Before proceeding to the proof, we introduce some notation and prove three lemmas.

LEMMA 3.1. *Let $\varphi \in C^1_G(M, \mathbb{R})$. If $x_0 \in M_{(\mathbf{H})}$ then $\nabla\varphi(x_0) \in T_{x_0}(M_{(\mathbf{H})})$.*

Proof. Since $(\mathbf{H})$ is a maximal element in $\Phi(M)$, $M_{(\mathbf{H})}$ is a closed, G-invariant, smooth submanifold of M. Let $\Lambda : N(M_{(\mathbf{H})}) \to \Lambda(N(M_{(\mathbf{H})}))$ be an open, G-invariant, tubular neighbourhood of $M_{(\mathbf{H})}$ in M. Since Λ is a G-equivariant diffeomorphism, it preserves isotropy groups and therefore $(N(M_{(\mathbf{H})}))_{(\mathbf{H})} = M_{(\mathbf{H})}$. Notice that $G_{(x_0,v)} \subset G_{x_0}$ for any $(x_0, v) \in T_{x_0}(M)$.

Fix $x_0 \in M_{(\mathbf{H})}$. We claim that $(T_{x_0}(M))_{(\mathbf{H})} \subset T_{x_0}(M_{(\mathbf{H})})$. In fact since $(N(M_{(\mathbf{H})}))_{(\mathbf{H})} = M_{(\mathbf{H})}$, it is clear that for $(x_0, v) \in N_{x_0}(M_{(\mathbf{H})})$ we have $G_{(x_0,v)} = G_{x_0}$ iff $v = 0$. Fix $(x_0, v, w) \in T_{x_0}(M) = T_{x_0}(N(M_{(\mathbf{H})})) =$

$N_{x_0}(M_{(\mathbf{H})}) \oplus T_{x_0}(M_{(\mathbf{H})})$ and notice that $G_{(x_0,v,w)} \subset G_{x_0}$. Moreover, notice that if $G_{(x_0,v,w)} = G_{x_0}$ then $v = 0$. Summing up, we obtain

$$(T_{x_0}(M))_{(\mathbf{H})} = (N_{x_0}(M_{(\mathbf{H})}) \oplus T_{x_0}(M_{(\mathbf{H})}))_{(\mathbf{H})} = (T_{x_0}(M_{(\mathbf{H})}))_{(\mathbf{H})} \subset T_{x_0}(M_{(\mathbf{H})}). \tag{3.3}$$

Fix $\varphi \in C^1_G(M,\mathbb{R})$. It is clear that $\nabla\varphi : M \to TM$ is a continuous G-equivariant map. Taking into account (3.3) we obtain

$$(\nabla\varphi)_{|M_{(\mathbf{H})}} : M_{(\mathbf{H})} \to (T(M)_{|M_{(\mathbf{H})}})_{(\mathbf{H})} \subset T(M_{(\mathbf{H})}),$$

which completes the proof. ∎

Consider a function $\theta \in C^1(\mathbb{R},\mathbb{R})$ satisfying the following conditions:

- $\theta(\tau) = 0$ for $\tau \le 2$,
- $\theta'(\tau) > 0$ for $2 < \tau < 3$,
- $\theta(\tau) = 1$ for $\tau \ge 3$.

Define $\Theta : M \to M$ by

$$\Theta(x) = \begin{cases} \theta(|x|)x & \text{for } x \in N, \\ x & \text{for } x \in M \setminus N. \end{cases}$$

REMARK 3.2. If $|x| \le 2$ then $\Theta(x) = \pi(x)$ and $(D\Theta(x))^*$ maps $T_x M_{(\mathbf{H})}$ isomorphically onto H_x. If $|x| > 2$ then $D\Theta(x) : T_x M \to T_{\Theta(x)} M$ is an isomorphism.

Now, consider $(U,X) \in \mathcal{K}(M)$. Choose $\varphi \in C^1_G(U,\mathbb{R})$ such that $X = \nabla\varphi$. Define $\Theta^*(U,X) = (\Theta^{-1}(U), \nabla(\varphi \circ \Theta)_{|\Theta^{-1}(U)})$.

LEMMA 3.2. *For any $(U,X) \in \mathcal{K}(M)$ we have*

(a) $\Theta^*(U,X) \in \mathcal{K}(M)$,
(b) $\Theta^*(U,X)$ *and* (U,X) *are* $\mathcal{K}(M)$*-homotopic.*

Proof. (a) Let $Y = \nabla(\varphi \circ \Theta)$. We claim that for $x \in \Theta^{-1}(U)$,

$$Y(x) = 0 \;\Leftrightarrow\; X(\Theta(x)) = 0. \tag{3.4}$$

To prove this we consider the following two cases.

1. $x \in D_2(N) \cap \Theta^{-1}(U)$. According to the definition of Θ we have $D\Theta(x) = D\pi(x)$. Thus $D\Theta(x)$ maps H_x isomorphically onto $T_{\Theta(x)}(M_{(\mathbf{H})})$, and $D\Theta^*(x)$ maps $T_{\Theta(x)}(M_{(\mathbf{H})})$ isomorphically onto H_x. Since

$$\nabla(\varphi \circ \Theta)(x) = (D\Theta(x))^*(\nabla\varphi(\Theta(x))) \tag{3.5}$$

and, according to Lemma 3.1, $\nabla\varphi(\Theta(x)) \in T_{\Theta(x)}(M_{(\mathbf{H})})$, the proof of this case is complete.

2. $x \in \Theta^{-1}(U) \setminus D_2(N)$. In this case $D\Theta(x)$ is an isomorphism, which implies (3.4).

Thus $Y^{-1}(0) = \Theta^{-1}(X^{-1}(0))$. Clearly $Y^{-1}(0)$ is a compact subset of $\Theta^{-1}(U)$ and the proof of (a) is complete.

(b) For $t \in [0,1]$ define $\Theta_t : M \to M$ by

$$\Theta_t(x) = \begin{cases} [t\theta(|x|) + 1 - t]x & \text{for } x \in N, \\ x & \text{for } x \in M \setminus N. \end{cases}$$

Set $V = \{(x,t) \in M \times [0,1] : x \in \Theta_t^{-1}(U)\}$ and $Z(x,t) = \nabla(\varphi \circ \Theta_t)(x)$. Evidently (V,Z) is a $\mathcal{K}(M)$-homotopy connecting (U,X) and $\Theta^*(U,X)$. ∎

Consider a function $\mu \in C^1(\mathbb{R},\mathbb{R})$ satisfying the following conditions:

- $\mu(\tau) = 0$ for $\tau \le 0$,
- $\mu'(\tau) > 0$ for $0 < \tau < 2$,
- $\mu(\tau) = 1$ for $\tau \ge 2$.

Define $\widehat{\mu} \in C^1_G(M,\mathbb{R})$ by

$$\widehat{\mu}(x) = \begin{cases} \mu\big(\frac{1}{2}|x|^2\big) & \text{for } x \in N, \\ 1 & \text{for } x \in M \setminus N. \end{cases}$$

Define $X_\mu : M \to T(M)$ by $X_\mu = \nabla\widehat{\mu}$. Clearly X_μ is continuous and G-equivariant.

Let $(U,X) \in \mathcal{K}(M)$. Define

$$\mathcal{N}(U,X) = (\Theta^{-1}(U), \nabla(\varphi \circ \Theta)_{|\Theta^{-1}(U)} + X_{\mu|\Theta^{-1}(U)}).$$

LEMMA 3.3. *For any $(U,X) \in \mathcal{K}(M)$ we have*

(a) $\mathcal{N}(U,X) \in \mathcal{K}(M)$,

(b) $\mathcal{N}(U,X)$ *and* (U,X) *are $\mathcal{K}(M)$-homotopic.*

Proof. (a) Let $Y = \nabla(\varphi \circ \Theta)$. We claim that for $x \in \Theta^{-1}(U)$,

$$(Y + X_\nu)^{-1}(0) = Y^{-1}(0) \cap X_\nu^{-1}(0). \tag{3.6}$$

To prove this we consider the following two cases.

- $x \in D_2(N) \cap \Theta^{-1}(U)$. By (3.5), $Y(x) \in H_x$. Applying Remark 3.1 we obtain (3.6).
- $x \in \Theta^{-1}(U) \setminus D_2(N)$. In this case $X_\nu(x) = 0$, which implies (3.6).

Thus $(Y + X_\nu)^{-1}(0) \subset Y^{-1}(0)$, hence $(Y + X_\nu)^{-1}(0)$ is compact and the proof of (a) is complete.

(b) For $(x,t) \in \Theta^{-1}(U) \times [0,1]$ set $Z(x,t) = Y(x) + (1-t)X_\nu(x)$. Then $(\Theta^{-1}(U) \times [0,1], Z)$ is a $\mathcal{K}(M)$-homotopy connecting $\mathcal{N}(U,X)$ and $\Theta^*(U,X)$. By Lemma 3.2(b), $\Theta^*(U,X)$ and (U,X) are $\mathcal{K}(M)$-homotopic and the proof is complete. ∎

Proof of Theorem 3.1. Let $(U,\nabla\varphi) \in \mathcal{K}(M)$. Set $U_0 = U \cap M_{(\mathbf{H})}$, $\varphi_0 = \varphi_{|U_0}$, $U_\infty = \Theta^{-1}(U) \cap M_\infty$, $\mathcal{R}_0(U,X) = (U_0, \nabla\varphi_0)$, $\mathcal{R}_\infty(U,X) = \mathcal{N}(U,X)_{|U_\infty}$.

According to Lemma 3.1, $\mathcal{R}_0(U,X) \in \mathcal{K}(M_{(\mathbf{H})})$ and by the definition of $\mathcal{N}(U,\nabla\varphi)$, $\mathcal{R}_\infty(U,\nabla\varphi) \in \mathcal{K}(M_\infty)$.

Define a map $\mathcal{R} : \mathcal{K}(M) \to \mathcal{K}(M_{(\mathbf{H})}) \times \mathcal{K}(M_\infty)$ by

$$\mathcal{R}(U,X) = (\mathcal{R}_0(U,X), \mathcal{R}_\infty(U,X)).$$

It follows directly from the definition of $\mathcal{E}$ that $\mathcal{E}(\mathcal{R}(U,X))$ and $\mathcal{N}(U,X)$ are $\mathcal{K}(M)$-homotopic. Note that $\mathcal{R}$ preserves $\mathcal{K}(M)$-homotopy classes. Thus $\mathcal{R}$ induces a map $\mathfrak{R} : \mathcal{K}[M] \to \mathcal{K}[M_{(\mathbf{H})}] \times \mathcal{K}[M_\infty]$. Hence Lemma 3.3 implies that $\mathfrak{E} \circ \mathfrak{R}$ is the identity map of $\mathcal{K}[M]$.

It remains to show that $\mathfrak{R} \circ \mathfrak{E}$ equals the identity.

Let $(U_0, \nabla\varphi_0) \in \mathcal{K}(M_{(\mathbf{H})})$ and $(U_\infty, \nabla\varphi_\infty) \in \mathcal{K}(M_\infty)$. Moreover, define $(U,X) = \mathcal{R}_\infty(\mathcal{E}((U_0,\nabla\varphi_0),(U_\infty,\nabla\varphi_\infty)))$. Set $U_1 = \Theta^{-1}(U_\infty)$, $X_1 = X_{|U_1}$, $U_2 = \Theta^{-1}(\pi^{-1}(U_0) \cap B_1(N)) \cap M_\infty$ and $X_2 = X_{|U_2}$. Observe that

(a) $U = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$,

(b) $(U_1, X_1), (U_2, X_2) \in \mathcal{K}(M_\infty)$,

(c) $X_2(x) = \nabla(\varphi_0 \circ \pi)(x) + \alpha(x) X_\nu(x)$ for $x \in U_2$, where $\alpha(x) \geq 0$.

Setting $W = U_2 \times [0,1]$ and $Y_t(x) = (1-t)X_2(x) + tX_\nu(x)$ for $(x,t) \in W$ we obtain a $\mathcal{K}(M_\infty)$-homotopy (W,Y) connecting (U_2,X_2) and (U_2,Y_1). Clearly $Y_1(x) \neq 0$ for all $x \in U_2$. Thus, in view of (a), Lemma 2.3 implies that (U,X) is homotopic to (U_1,X_1) in $\mathcal{K}(M_\infty)$. Since $(U_1,X_1) = \mathcal{N}(U_\infty,\nabla\varphi_\infty)$, by a simple repetition of the arguments given in the proof of Lemma 3.3, we see that (U_1,X_1) is homotopic to $(U_\infty,\nabla\varphi_\infty)$ in $\mathcal{K}(M_\infty)$. Thus we have shown that $\mathcal{R}_\infty \circ \mathcal{E}((U_0,\nabla\varphi_0),(U_\infty,\nabla\varphi_\infty))$ is homotopic to $(U_\infty,\nabla\varphi_\infty)$. Since $\mathcal{R}_0 \circ \mathcal{E}((U_0,\nabla\varphi_0),(U_\infty,\nabla\varphi_\infty)) = (U_0,\nabla\varphi_0)$, $\mathcal{R} \circ \mathcal{E}$ induces the identity on $\mathcal{K}[M_{(\mathbf{H})}] \times \mathcal{K}[M_\infty]$. Thus the proof of Theorem 3.1 is complete. ∎

We finish this section with a corollary which allows us to reduce the classification of $\mathcal{P}[M]$ to the classification of $\mathcal{P}[M_{(H)}]$ and $\mathcal{P}[M \setminus M_{(H)}]$, where $(H) \in \Phi(M)$. The corollary is a direct consequence of Lemmas 2.2, 2.4 and Theorem 3.1.

COROLLARY 3.1. *Assume additionally that* $\#\Phi(M) < \infty$, *i.e.* $\Phi(M) = \{(H_1), \ldots, (H_k)\}$. *Then there is a bijection*

$$\Gamma_M : \mathcal{P}[M] \to \mathcal{P}[M_{(H_1)}] \times \cdots \times \mathcal{P}[M_{(H_k)}].$$

It is worth pointing out that if the manifold M is compact, then $\#\Phi(M) < \infty$ (see [2], [5]).

4. Main results. In this section we prove the main result of this article. Throughout this section we assume additionally that $\#\Phi(M) < \infty$. Namely, we classify the homotopy classes $\mathcal{P}[M]$ (see Theorem 4.1). Moreover, as a corollary we obtain a classification of $\mathcal{G}[V]$.

We start with the definition of a group $U(G,M)$. The degree we are going to construct will be an element of $U(G,M)$. Fix $(H) \in \Phi(M)$. Since $M_{(H)}$ is a smooth Riemannian G-manifold, $M_{(H)}/G$ is a smooth Riemannian manifold and the projection $\pi_{(H)} : M_{(H)} \to M_{(H)}/G$ is a smooth G-invariant map. Define

$$I((H),M) = \{W : W \text{ is a connected component of } M_{(H)}/G \text{ with } \dim W \geq 1\},$$

$$U_{(H)}(G,M) = \bigoplus_{W \in I((H),M)} \mathbb{Z}, \quad U(G,M) = \bigoplus_{(H)\in\Phi(M)} U_{(H)}(G,M).$$

It is understood that if $I((H),M) = \emptyset$, then $U_{(H)}(G,M) = \{0\}$.

Fix $[(X,f)] \in \mathcal{P}[M]$. We will define an element

$$\mathrm{DEG}([(X,f)],M) = \sum_{(H)\in\Phi(M)} \mathrm{DEG}_{(H)}([(X,f)],M) \in U(G,M),$$

where $\mathrm{DEG}_{(H)}([(X,f)],M) \in U_{(H)}(G,M)$.

Suppose first that $\#\Phi(M) = 1$, i.e. $\Phi(M) = \{(H)\}$. In this situation

- $M = M_{(H)}$,
- $U(G,M) = U_{(H)}(G,M_{(H)})$,
- $\mathrm{DEG}([(X,f)],M) = \mathrm{DEG}_{(H)}([(X,f)],M_{(H)}) \in U_{(H)}(G,M_{(H)})$.

Fix a pair $(X,f) = (\nabla\varphi, f) \in \mathcal{P}(M)$. Since $\Phi(M) = \{(H)\}$, M/G is a smooth manifold and the quotient map $\pi_{(H)} : M \to M/G$ is smooth. Define functions $\varphi_0, f_0 \in C^2(M/G,\mathbb{R})$ such that $\varphi(x) = (\varphi_0 \circ \pi_{(H)})(x)$ and $f(x) = (f_0 \circ \pi_{(H)})(x)$.

Notice that if $W \in I((H),M)$, then the group G acts trivially on W.

It is clear that if $W \in I((H),M)$, then $[(\nabla\varphi_0, f_0)_{|W}] \in \mathcal{P}[W]$. Moreover, it is easy to check that the map

$$P_{(H)} : \mathcal{P}[M] \to \mathcal{P}[M/G] = \prod_{W\in I((H),M)} \mathcal{P}[W]$$

defined by $P_{(H)}([(X,f)]) = \prod_{W\in I((H),M)}[(\nabla\varphi_0, f_0)_{|W}]$ is a bijection. On the other hand, we have a bijection

$$\gamma_{(H)} = \sum_{W\in I((H),M)} \gamma_W : \prod_{W\in I((H),M)} \mathcal{P}[W] \to U(G,M) = \bigoplus_{W\in I((H),M)} \mathbb{Z},$$

where the bijections $\gamma_W : \mathcal{P}[W] \to \mathbb{Z}$ are given by Corollary 6.2. Finally, we define a bijection $\mathrm{DEG}_{(H)}(\cdot,M) : \mathcal{P}(M) \to U(G,M)$ by

$$\mathrm{DEG}_{(H)}([(X,f)],M) := (\gamma_{(H)} \circ P_{(H)})([(X,f)]).$$

Suppose now that $\#\Phi(M) > 1$, i.e. $\Phi(M) = \{(H_1),\ldots,(H_k)\}$, $k > 1$. In this situation we define a bijection $\mathrm{DEG}(\cdot,M) : \mathcal{P}[M] \to U(G,M)$ as

follows:

$$\mathrm{DEG}([(X,f)],M) = ((\mathrm{DEG}_{(H_1)}(\cdot,M_{(H_1)}),\dots,\mathrm{DEG}_{(H_k)}(\cdot,M_{(H_k)}))\circ\Gamma_M)([(X,f)]),$$

where $\Gamma_M : \mathcal{P}[M] \to \mathcal{P}[M_{(H_1)}]\times\cdots\times\mathcal{P}[M_{(H_k)}]$ is the bijection given by Corollary 3.1.

Notice that we have just proved the following theorem.

THEOREM 4.1. *Let G be any compact Lie group and let M be a smooth Riemannian G-manifold such that $\#\Phi(M) < \infty$. Then the above defined map $\mathrm{DEG}(\cdot,M) : \mathcal{P}[M] \to U(G,M)$ is a bijection.*

Since $S(V) \subset V$ is a compact, Riemannian G-manifold, $\#\Phi(S(V)) < \infty$. Therefore combining Lemma 2.1 with Theorem 4.1 we obtain the following corollary.

COROLLARY 4.1. *Let V be an orthogonal representation of a compact Lie group G. Then the map $\nabla_G\text{-deg} : \mathcal{G}[V] \to U(G,S(V))$ defined by*

$$\nabla_G\text{-}\deg([\nabla F],B(V)) = \mathrm{DEG}(\Upsilon_V([\nabla F],S(V)))$$

is a bijection.

REMARK 4.1. It is easy to check that $\nabla_G\text{-deg} = \mathrm{DEG}(\cdot,S(V))\circ\Upsilon_V$ is nothing other than the degree for G-equivariant gradient maps defined in [6].

5. Final remarks. Let $F_0, F_1 \in C^2_G(V,\mathbb{R})$ be such that $(\nabla F_i)^{-1}(0)\cap D(V) = \{0\}$ and $\det\nabla^2 F_i(0) \neq 0$ for $i = 0,1$. Moreover, assume that

$$\nabla_G\text{-}\deg([\nabla F_0],B(V)) = \nabla_G\text{-}\deg([\nabla F_1],B(V)).$$

From Corollary 4.1 it follows that there exists a family of potentials $F_t \in C^2_G(V,\mathbb{R})$, $t\in[0,1]$, such that the maps $\nabla F_0, \nabla F_1$ are joined by a $\mathcal{G}(V)$-homotopy $\nabla_v F_t : (D(V),S(V)) \to (V, V\setminus\{0\})$, $t\in[0,1]$. Let $\phi \in C^2_G(V,\mathbb{R})$ be such that there are $0 < \alpha,\varepsilon < 1$ such that

- $D_\alpha(V) \subset \phi^{-1}(1)$,
- $\mathrm{cl}(D_{1+\varepsilon}(V)\setminus D_{1-\varepsilon}(V)) \subset \phi^{-1}(0)$.

Finally, define a family of potentials $H_t \in C^2_G(V,\mathbb{R})$, $t\in[0,1]$, as follows:

$$H_t(v) = (1-\phi(v))\cdot F_t(v) + \frac{1}{2}\cdot\phi(v)\cdot\langle\nabla^2 F_t(0)v,v\rangle.$$

Notice that $\nabla_v H_t : V \to V$, $t\in[0,1]$, is a $\mathcal{G}(V)$-homotopy such that

- $\nabla_v H_t(0) = 0$ for any $t\in[0,1]$,
- $(\nabla_v H_i(0))^{-1}(0)\cap D(V) = \{0\}$ for $i = 0,1$,
- $\nabla^2_v H_t(0)$ is a continuous family of self-adjoint G-equivariant linear maps joining $\nabla^2_v F_0(0)$ and $\nabla^2_v F_1(0)$.

Thus there is no global bifurcation of connected sets of non-trivial solutions of the equation $\nabla_v H_t(v) = 0$ off $\{0\} \times [0,1]$. However, there is local bifurcation off $\{0\} \times [0,1]$ if $i_G(\nabla_v^2 F_0(0), 0) \neq i_G(\nabla_v^2 F_1(0), 0)$, i.e. the G-equivariant Conley indices of the origin are different. This can be proved by the Conley index techniques. Thus the degree for G-equivariant gradient maps is the natural invariant to study global bifurcation of zeros for G-equivariant gradient mappings.

It would be of interest to know if there is an example where there are no non-trivial connected sets of non-trivial solutions intersecting $\{0\} \times [0,1]$ as in the work of Böhme [1].

We need to be careful in the interpretation of this phenomenon. For gradient maps, it is well known that the Conley index provides stronger invariants for homotopy classes of gradient maps. But this only applies when not only the critical points but also the connecting orbits joining them do not cross the boundary of the neighbourhood we are working in. We can actually do a little better, but this does not matter for our present discussion. In particular, suppose that $\nabla F_0, \nabla F_1 : V \to V$ are gradient G-equivariant mappings such that $\nabla F_0(v) \neq 0$ and $\nabla F_1(v) \neq 0$ if $\|v\| \geq 1$.

If $\nabla_G\text{-deg}([\nabla F_0], B(V)) = \nabla_G\text{-deg}([\nabla F_1], B(V))$ then, by Corollary 4.1, $[\nabla F_0] = [\nabla F_1]$, i.e. $\nabla F_0, \nabla F_1$ are homotopic through a G-equivariant gradient homotopy ∇F_t, $t \in [0,1]$, such that $\nabla F_t(v) \neq 0$ if $\|v\| = 1$.

Now suppose that ∇F_0 has the property that all the connecting orbits of ∇F_0 joining zeros of ∇F_0 lie in $\|x\| < 1$ and ∇F_1 has the same property. If ∇F_0 and ∇F_1 have different G-equivariant Conley indices on the unit ball, it follows that for any equivariant homotopy F_t, $t \in [0,1]$, joining ∇F_0 and ∇F_1 we have no uniform bound on the zeros of F_t and connecting orbits (otherwise the Conley indices on large balls would be the same). Hence we see that the homotopy theory of gradient maps is quite different when we bound only the zeros, or both the zeros and the connecting orbits.

6. Appendix. In this section we classify the homotopy classes of gradient pairs on a connected smooth Riemannian manifold N. Unless otherwise stated, throughout this section we assume that $\dim N > 1$. The proof is similar in spirit to that of [8].

In the first step we reduce the classification of homotopy classes of gradient pairs on a manifold N to the classification of admissible gradient maps on $\mathbb{R}^{\dim N}$ (see Theorem 6.1).

In the second step, for the convenience of the reader, we briefly sketch the classification of admissible gradient maps on $\mathbb{R}^{\dim N}$, due to Parusiński [8] (see Theorem 6.4).

Finally, in Corollary 6.2 we classify the homotopy classes of gradient pairs on a connected smooth Riemannian manifold N.

We keep the notation of the previous sections, although in this section $G = \{e\}$.

Let $B^n_\alpha = \{x \in \mathbb{R}^n : |x| < \alpha\}$, $D^n_\alpha = \mathrm{cl}(B^n_\alpha)$ and $S^{n-1}_\alpha = \partial B^n_\alpha$. We write B^n, D^n, S^{n-1} instead of B^n_1, D^n_1, S^{n-1}_1, respectively.

The principal significance of Theorem 6.1 is that it allows us to reduce the classification of $\mathcal{P}[N]$ to the classification of $\mathcal{G}[\mathbb{R}^{\dim N}]$.

In order to prove this theorem we first need two technical lemmas. We leave their proofs to the reader.

LEMMA 6.1. *Let $(\nabla\varphi, f) \in \mathcal{P}(N)$. If (U, h) is a chart on N such that*

- $h^{-1}(D^{\dim N}) \subset U$,
- $\mathrm{cl}(U) \cap N_-(f) = \emptyset$,
- *φ is a Morse function on $h^{-1}(D^{\dim N})$,*

then $(\nabla\varphi, f)$ is $\mathcal{P}(N)$-homotopic to a pair $(\nabla\psi, f) \in \mathcal{P}(N)$ such that

- $(\nabla\psi)^{-1}(0) \cap h^{-1}(D^{\dim N}) = \emptyset$,
- *$\nabla\varphi = \nabla\psi$ on $N_-(f)$.*

LEMMA 6.2. *Let $(\nabla\varphi, f) \in \mathcal{P}(N)$. If (U, h) is a chart on N such that*

- $h^{-1}(D^{\dim N}) \subset U$,
- $\mathrm{cl}(U) \cap N_-(f) = \emptyset$,
- $(\nabla\varphi)^{-1}(0) \cap h^{-1}(D^{\dim N}) = \emptyset$,
- *φ is a Morse function on $N_-(f)$,*

then the pair $(\nabla\varphi, f)$ is $\mathcal{P}(N)$-homotopic to a pair $(\nabla\psi, g) \in \mathcal{P}(N)$ such that

- $N_-(g) = h^{-1}(D^{\dim N})$,
- *ψ is a Morse function on $N_-(g)$,*
- $g^{-1}(0) = h^{-1}(S^{\dim N-1})$.

The following theorem is the first step in the classification of $\mathcal{P}[N]$.

THEOREM 6.1. *There is a bijection $j_N : \mathcal{P}[N] \to \mathcal{G}[\mathbb{R}^{\dim N}]$.*

Proof. Fix $[(\nabla\varphi, f)] \in \mathcal{P}(N)$. It is easy to show that there is $\varphi_1 \in C^2(N, \mathbb{R})$ such that

- φ_1 is a Morse function on $N_-(f) \setminus f^{-1}(0)$,
- $\varphi_1(x) = \varphi(x)$ on $N \setminus (N_-(f) \setminus f^{-1}(0))$.

It is evident that $(\nabla(t \cdot \varphi_1 + (1-t) \cdot \varphi), f)$ is a well defined $\mathcal{P}(N)$-homotopy joining $(\nabla\varphi, f)$ to $(\nabla\varphi_1, f)$. Let (U, h) be a chart on N such that $\mathrm{cl}(U) \cap N_-(f) = \emptyset$. There is no loss of generality in assuming that $h^{-1}(D^{\dim N}) \subset U$. It is clear that there is $\varphi_2 \in C^2(N, \mathbb{R})$ such that

- $\varphi_2(x) = \varphi_1(x)$ for $x \in N \setminus U$,
- φ_2 is regular on $N_-(f) \cup h^{-1}(D^{\dim N})$.

It is easy to verify that $(\nabla(t\cdot\varphi_2+(1-t)\cdot\varphi_1), f)$ is a well defined $\mathcal{P}(N)$-homotopy joining $(\nabla\varphi_1, f)$ to $(\nabla\varphi_2, f)$. By Lemma 6.1 the gradient pair $(\nabla\varphi_2, f)$ is $\mathcal{P}(N)$-homotopic to a pair $(\nabla\varphi_3, f)$ such that

- $X_3^{-1}(0)\cap h^{-1}(D^{\dim N})=\emptyset$,
- φ_3 is a Morse function on $N_-(f)$.

From Lemma 6.2 it follows that there is a gradient pair $(\nabla\varphi_4, f_4)$ $\mathcal{P}(N)$-homotopic to the pair $(\nabla\varphi_3, f)$ such that

- $N_-(f_4)=h^{-1}(D^{\dim N})$,
- $f_4^{-1}(0)=h^{-1}(S^{\dim N-1})$,
- φ_4 is a Morse function on $N_-(f_4)$.

Let $\phi:\mathbb{R}^{\dim N}\to\mathbb{R}$ be any C^1-extension of $\varphi_4\circ h^{-1}: D^{\dim N}\to\mathbb{R}$. We define a map $j_N:\mathcal{P}[N]\to\mathcal{G}[\mathbb{R}^{\dim N}]$ by $j_N([(\nabla\varphi, f)])=[\nabla\phi]$. Directly from the definition it follows that j_N is well defined. What is left is to show that j_N is a bijection.

Define a map $k_N:\mathcal{G}[\mathbb{R}^{\dim N}]\to\mathcal{P}[N]$ in the following way. Let (U,h) be a chart on N such that $h^{-1}(D^{\dim N}_{1+\varepsilon})\subset U$. Fix $\nabla\varphi\in\mathcal{G}(\mathbb{R}^{\dim N})$ and denote by $\psi_{\nabla\varphi}\in C^1(N,\mathbb{R})$ any extension of $\varphi\circ h: h^{-1}(D^{\dim N}_{1+\varepsilon})\to\mathbb{R}$. Finally, choose any $f_{\nabla\varphi}\in C^0(N,\mathbb{R})$ such that $N_-(f_{\nabla\varphi})=h^{-1}(D^{\dim N})$ and $f^{-1}_{\nabla\varphi}(0)=h^{-1}(S^{\dim N-1})$. Define $k_N:\mathcal{G}[\mathbb{R}^{\dim N}]\to\mathcal{P}[N]$ by $k_N[\nabla\varphi]=[(\nabla\psi_{\nabla\varphi}, f_{\nabla\varphi})]$. It is clear that k_N is well defined. We check at once that k_N is the inverse of j_N, which completes the proof. ∎

Let $\varphi\in C^2(N,\mathbb{R})$ be a Morse function such that $\nabla\varphi(p)=0$. Then we denote by $m^-(p,\nabla\varphi)\in\mathbb{N}\cup\{0\}$ the Morse index of p. If $\Omega\subset\mathbb{R}^n$ is an open and bounded subset, and moreover $q\in C^0(\mathrm{cl}(\Omega),\mathbb{R}^n)$ is such that $q^{-1}(0)\cap\partial\Omega=\emptyset$ then $\deg(q,\Omega,0)\in\mathbb{Z}$ denotes the Brouwer degree.

LEMMA 6.3. *Let $(\nabla\varphi, f)\in\mathcal{P}(N)$ be such that $\varphi\in C^2(N,\mathbb{R})$ is a Morse function on $N_-(f)$. Then*

$$\deg(j_N([(\nabla\varphi, f)]), B^n, 0)=\sum_{p\in N_-(f)\cap(\nabla\varphi)^{-1}(0)}(-1)^{m^-(p,\nabla\varphi)}.$$

Proof. The easy proof is left to the reader. ∎

From Theorem 6.1 it follows that to classify $\mathcal{P}[N]$ it is enough to classify $\mathcal{G}[\mathbb{R}^{\dim N}]$. In fact $\mathcal{G}[\mathbb{R}^{\dim N}]$ has been classified in [8]. Below we recall the main steps of this proof.

Let $\nabla F\in\mathcal{G}(\mathbb{R}^n)$. For $x\in S^{n-1}$ we have the unique decomposition

$$\begin{aligned}\nabla F(x)&=(\nabla F(x)-\langle\nabla F(x),x\rangle\cdot x)+\langle\nabla F(x),x\rangle\cdot x\\&=X_{\nabla F}(x)+f_{\nabla F}(x)\cdot x.\end{aligned}\tag{6.1}$$

It is clear that $(X_{\nabla F}, f_{\nabla F})=[(\nabla(F_{|S^{n-1}}), f_{\nabla F})]\in\mathcal{P}(S^{n-1})$.

LEMMA 6.4. *The map* $i_n : \mathcal{G}[\mathbb{R}^n] \to \mathcal{P}[S^{n-1}]$ *defined by* $i_n([\nabla F]) = [(X_{\nabla F}, f_{\nabla F})]$ *is a bijection.*

LEMMA 6.5. *If* $F_{|S^{n-1}} \in C^2(S^{n-1}, \mathbb{R})$ *is a Morse function, then*

$$\deg([\nabla F], B^n, 0) = 1 - \sum_{p \in N_-(f_{\nabla F}) \cap X_{\nabla F}^{-1}(0)} (-1)^{m^-(p, X_{\nabla F})}.$$

As a direct consequence of Theorem 6.1 and Lemmas 6.4, 6.5 we obtain the following corollary.

COROLLARY 6.1. *Fix any* $n \geq 2$. *Then*

(1) *the map* $\mathcal{G}[\mathbb{R}^{n+1}] \xrightarrow{i_{n+1}} \mathcal{P}[S^n] \xrightarrow{j_{S^n}} \mathcal{G}[\mathbb{R}^n]$ *is a bijection,*
(2) *for* $\nabla F \in \mathcal{G}[\mathbb{R}^{n+1}]$,

$$\deg((j_{S^n} \circ i_{n+1})[\nabla F], B^n, 0) = 1 - \deg([\nabla F], B^{n+1}, 0).$$

From Corollary 6.1 it follows that to classify $\mathcal{G}[\mathbb{R}^{n+1}]$ it is enough to classify $\mathcal{P}[S^1]$.

THEOREM 6.2. *There is a bijection* $j_{S^1} : \mathcal{P}[S^1] \to \mathbb{Z}$. *Moreover, if* $(\varphi', f) \in \mathcal{P}(S^1)$ *and* $\varphi \in C^2(S^1, \mathbb{R})$ *is a Morse function then*

$$j_{S^1}([(\varphi', f)]) = \sum_{p \in N_-(f) \cap (\varphi')^{-1}(0)} (-1)^{m^-(p, \varphi')}.$$

Furthermore, $j_{S^1}([(\sin kt, \cos kt)]) = k$.

We can also classify the gradient pairs on an open interval.

THEOREM 6.3. *There is a bijection* $j_{(0,2\pi)} : \mathcal{P}[(0, 2\pi)] \to \mathbb{Z}$. *If* $(\varphi', f) \in \mathcal{P}((0, 2\pi))$ *and* $\varphi \in C^2((0, 2\pi), \mathbb{R})$ *is a Morse function then*

$$j_{(0,2\pi)}([(\varphi', f)]) = \sum_{p \in N_-(f) \cap (\varphi')^{-1}(0)} (-1)^{m^-(p, \varphi')}.$$

Moreover, $j_{(0,2\pi)}([(\sin kt, \cos kt)]) = k$.

The following theorem is a direct consequence of Lemma 6.3.

THEOREM 6.4. *For* $k > 1$ *the bijection*

$$\delta_k \circ j_{S^1} \circ i_2 \circ j_{S^2} \circ i_3 \circ \ldots \circ j_{S^{k-1}} \circ i_k : \mathcal{G}[\mathbb{R}^k] \to \mathbb{Z}$$

is given by

$$(\delta_k \circ j_{S^1} \circ i_2 \circ j_{S^2} \circ i_3 \circ \ldots \circ j_{S^{2k-1}} \circ i_{2k})([\nabla F]) = \deg([\nabla F], B^k, 0),$$

where the bijection $\delta_k : \mathbb{Z} \to \mathbb{Z}$ *is defined by*

$$\delta_k(m) = \begin{cases} 1 - m & \text{if } k \text{ is even,} \\ m & \text{otherwise.} \end{cases}$$

Finally, we classify the homotopy classes of gradient pairs on a connected smooth Riemannian manifold N.

COROLLARY 6.2. *Let N be a connected smooth Riemannian manifold with $\dim N \geq 1$. Then there is a bijection $\Upsilon_N : \mathcal{P}[N] \to \mathbb{Z}$ given by*

$$\Upsilon_N([(\nabla\varphi, f)]) = \sum_{p \in N_-(f) \cap (\nabla\varphi)^{-1}(0)} (-1)^{m^-(p, \nabla\varphi)},$$

where $\varphi \in C^2(N, \mathbb{R})$ is a Morse function.

Proof. We define $\Upsilon_N : \mathcal{P}[N] \to \mathbb{Z}$ as follows:

$$\Upsilon_N = \begin{cases} \delta_{\dim N} \circ j_{S^1} \circ i_2 \circ j_{S^2} \circ i_3 \circ \ldots \circ j_{S^{\dim N - 1}} \circ i_{\dim N} \circ j_N & \\ & \text{if } \dim N > 1, \\ j_{S^1} & \text{if } N \approx S^1, \\ j_{(0,2\pi)} & \text{if } N \approx (0, 2\pi). \end{cases}$$

Fix $[(\nabla\varphi, f)] \in \mathcal{P}(N)$. Without loss of generality one can assume that $\varphi \in C^2(N, \mathbb{R})$ is a Morse function. The rest of the proof is a consequence of Lemma 6.3 and Theorems 6.2–6.4. ∎

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