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Inscribing compact non-σ-porous sets into analytic non-σ-porous sets

by

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Abstract. The main aim of this paper is to give a simpler proof of the following assertion. Let A be an analytic non-σ-porous subset of a locally compact metric space E. Then there exists a compact non-σ-porous subset of A. Moreover, we prove the above assertion also for σ-$\mathbf{P}$-porous sets, where $\mathbf{P}$ is a porosity-like relation on E satisfying some additional conditions. Our result covers σ-$\langle g\rangle$-porous sets, σ-porous sets, and σ-symmetrically porous sets.

1. Introduction. There are many papers in real analysis and Banach space theory dealing with different kinds of porosity and σ-porosity. We refer to survey papers [Za_2] and [Za_5] for definitions and applications of these notions. This paper is a continuation of [ZP], [ZZ_1], and [ZZ_2], where structural properties of σ-porosity are studied. Namely, we consider, for several types of porosity, the following question:

(Q) Let A be an analytic non-σ-porous subset of a metric space E. Does there exist a closed non-σ-porous set $K \subset A$?

For ordinary (Denjoy–Dolzhenko) porosity this natural question was mentioned in [Za_2] and an affirmative answer was given independently by J. Pelant (in the case of a topologically complete metric space E) and by M. Zelený (in the case of a compact metric space E) in an unpublished manuscript [Ze]. In the present article, we prove the same result for $\langle g\rangle$-porosity (in a locally compact metric space E), for symmetrical porosity (in $\mathbb{R}$), and also for several other types of porosity. Especially the notion of σ-symmetrical porosity has found interesting applications (see [Za_2] and [Za_5] for references).

Pelant's proof giving an explicit construction of a closed non-σ-porous set was rather complicated; its modification can be found in [ZP]. The method

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of construction of non-σ-porous sets from [ZP] works in general complete metric spaces and is applicable to a number of problems but it is more complicated than the method of the present article and it uses some special properties of the ordinary (Denjoy–Dolzhenko) porosity. One can probably use that approach also for some other types of porosity (the case of σ-$\langle g\rangle$-porosity is promising) but then the construction would become extraordinarily technical.

The proof in [Ze] is non-constructive; it uses a version of the well known "overspill method" (see [K, p. 290]). No explicit construction of a compact non-σ-porous set is given; instead, a coanalytic rank is defined for certain systems of compact sets and "very complicated" compact σ-porous subsets of A are constructed by transfinite induction to show that the rank is unbounded. Then the existence of the desired non-σ-porous set is deduced by applying a theorem of descriptive set theory.

The present paper is based on the idea of [Ze]. In fact, this idea was already used in [ZZ_1] and the present article uses a number of concepts and results of [ZZ_1]. In particular, we do not use a version of the overspill method here but it is used in [ZZ_1] in the proof of the basic lemma to which we refer (Lemma 2.24 below). However, the transfinite construction presented here is more complicated than that of [ZZ_1] and requires new ideas.

In [ZZ_1], we proved that the set of all compact σ-porous subsets of an arbitrary separable locally compact metric space E is a coanalytic non-Borel subset of the "hyperspace" of all compact subsets of E. To prove this result not only for ordinary porosity but also for a number of other types of porosity, we introduced some new concepts concerning point-set relations (recalled in Section 2). This abstract setting enables us to prove a general theorem, which easily implies an affirmative answer to Question (Q) not only for ordinary porosity, but also for $\langle g\rangle$-porosity (in a locally compact metric space E), symmetrical porosity in $\mathbb{R}$, and several other types of porosity (see Remark 5.5). On the other hand (in contrast to the "complexity problem" in [ZZ_1]) our methods do not work for strong porosity (see e.g. [ZZ_1] for the definition), for which (Q) remains open.

Our abstract setting has also another important advantage. It enables us to prove the affirmative answer to Question (Q) in a transparent way in two steps: In Section 3, we prove it (for some types of "abstract porosities") in the case of a G_δ set $A \subset E$, and in Section 4 we show that the general case is a relatively easy consequence of this special case, if we apply it to a suitable "abstract porosity" in $E \times C$, where C is the Cantor set. This proof is not standard even for ordinary porosity since we have to apply the G_δ case in $E \times C$ not to the ordinary porosity but to another suitably defined porosity on $E \times C$.

Section 5 is devoted to applications of the abstract result to $\langle g\rangle$-porosity, ordinary porosity, and symmetrical porosity.

Further, note that Question (Q) has an affirmative answer for "σ-lower porosity" (i.e., for σ-very porous sets in the terminology of [Za_1]) and for similar types of porosity. In these cases, the proof is significantly easier (see [ZZ_2]).

2. Notation and definitions. Let (X,ϱ) be a metric space. The open ball with center $x\in X$ and radius $r>0$ is denoted by $B(x,r)$. Let $A\subset X$, $A\neq\emptyset$, and $\varepsilon>0$. Then the symbol $B(A,\varepsilon)$ stands for the set $\{y\in X;\ \varrho(y,A)<\varepsilon\}$.

We say that $\mathbf{R}$ is a *point-set relation on* X if it is a relation between points of X and subsets of X. Thus a point-set relation $\mathbf{R}$ is a subset of $X\times 2^X$. The symbol $\mathbf{R}(x,A)$, where $x\in X$ and $A\subset X$, means that $(x,A)\in\mathbf{R}$, i.e. $\mathbf{R}$ holds for the pair (x,A).

We consider the following properties of a point-set relation $\mathbf{R}$ on X.

(A1) If $A\subset B\subset X$, $x\in X$, and $\mathbf{R}(x,B)$, then $\mathbf{R}(x,A)$.
(A2) $\mathbf{R}(x,A)$ if and only if there is $r>0$ such that $\mathbf{R}(x,A\cap B(x,r))$.
(A3) $\mathbf{R}(x,A)$ if and only if $\mathbf{R}(x,\bar{A})$.

We say that a point-set relation $\mathbf{P}$ on X is a *porosity-like relation* if P satisfies the "axioms" **(A1)–(A3)**. Note that virtually all types of porosity satisfy **(A1)–(A3)**.

Let $\mathbf{P}$ be a porosity-like relation on X. We say that $A\subset X$ is

- $\mathbf{P}$-*porous at* $x\in X$ if $\mathbf{P}(x,A)$,
- $\mathbf{P}$-*porous* if $\mathbf{P}(x,A)$ for every $x\in A$,
- σ-$\mathbf{P}$-*porous* if A is a countable union of $\mathbf{P}$-porous sets.

NOTATION 2.1. Let (X,ϱ) be a metric space and $\mathbf{R}$ be a point-set relation on X. If $A\subset X$ and $B\subset X$, then $\mathbf{R}(A,B)\overset{\text{def}}{\Longleftrightarrow}\forall a\in A:\mathbf{R}(a,B)$. Let moreover I be a nonempty index set and $\mathbf{R}_\iota$, $\iota\in I$, be point-set relations on X. Then the point-set relations $\neg\mathbf{R}$, $\bigcup_{\iota\in I}\mathbf{R}_\iota$, $\bigcap_{\iota\in I}\mathbf{R}_\iota$ on X are defined in the natural way; namely $(\neg\mathbf{R})(x,A)\overset{\text{def}}{\Longleftrightarrow}\neg(\mathbf{R}(x,A))$, $(\bigcup_{\iota\in I}\mathbf{R}_\iota)(x,A)\overset{\text{def}}{\Longleftrightarrow}\exists\iota\in I:\mathbf{R}_\iota(x,A)$ and $(\bigcap_{\iota\in I}\mathbf{R}_\iota)(x,A)\overset{\text{def}}{\Longleftrightarrow}\forall\iota\in I:\mathbf{R}_\iota(x,A)$.

LEMMA 2.2. *Let Ξ be a nonempty countable set, and $\mathbf{Q}$ be a porosity-like relation on a metric space X such that $\mathbf{Q}=\bigcup_{\xi\in\Xi}\mathbf{U}^\xi$, where $\mathbf{U}^\xi$'s satisfy* **(A1)**. *If $A\subset X$ is a σ-$\mathbf{Q}$-porous set, then A can be written as $A=\bigcup_{n=1}^\infty B_n$, where each B_n satisfies $\mathbf{U}^\xi(B_n,B_n)$ for some $\xi\in\Xi$.*

Proof. We can write $A=\bigcup_{n=1}^\infty A_n$, where each A_n is $\mathbf{Q}$-porous, and each A_n can be written as $A_n=\bigcup_{\xi\in\Xi}C_n^\xi$, where $C_n^\xi=\{x\in A_n;\ \mathbf{U}^\xi(x,A_n)\}$. Using property **(A1)** of $\mathbf{U}^\xi$ we have $\mathbf{U}^\xi(C_n^\xi,C_n^\xi)$ and we are done. ∎

Let X be a metric space. The symbol $\mathcal{C}_{\mathrm{b}}(X)$ denotes the space of all nonempty bounded closed subsets of X with the Hausdorff metric

$$h(F,C)=\max\{\sup\{\varrho(x,F);\, x\in C\},\sup\{\varrho(y,C);\, y\in F\}\},$$

and $\mathcal{K}^*(X)$ is its subspace of all nonempty compact subsets of X. The symbol $\mathcal{K}(X)$ denotes the space of *all* compact subsets of X. The space $\mathcal{K}(X)$ is also equipped with the Hausdorff metric; the empty set is considered to be an isolated point of $\mathcal{K}(X)$. (To have a fixed metric on $\mathcal{K}(X)$, we choose $a\in X$, and the distance of $\emptyset$ to a nonempty compact subset $K\subset X$ is defined as $h(\emptyset,K):=\mathrm{dist}(a,K)+\mathrm{diam}(K)+1$.)

The next lemma is easy to prove and therefore the proof will be omitted.

LEMMA 2.3. *Let X be a compact metric space.*

(i) *Let $K_i\in\mathcal{K}(X)$, $i\in\mathbb{N}$, and $K_1\subset K_2\subset\cdots$. Then $K_i\to\overline{\bigcup_{j=1}^{\infty}K_j}$ in $\mathcal{K}(X)$.*
(ii) *Let $K\in\mathcal{K}(X)$, $K_i\in\mathcal{K}(X)$, $i\in\mathbb{N}$, and $K_i\to K$ in $\mathcal{K}(X)$. Then $K\cup\bigcup_{i=1}^{\infty}K_i\in\mathcal{K}(X)$.*
(iii) *If $G\subset X$ is a G_δ set, then $\{K\in\mathcal{K}(X);\, K\subset G\}$ is a G_δ set in $\mathcal{K}(X)$.*
(iv) *Let $L\in\mathcal{K}(X)$ be clopen. Then $K\mapsto K\cap L$ is a continuous mapping of $\mathcal{K}(X)$ to itself.*
(v) *Let f be a continuous mapping of X to a metric space Y. Then $K\mapsto f(K)$ is a continuous mapping of $\mathcal{K}(X)$ to $\mathcal{K}(Y)$.*

DEFINITION 2.4. Let X be a metric space. Let $\mathbf{R}$ be a point-set relation on X and $\mathbf{P}$ be a porosity-like relation on X.

- If $A\subset X$, then we define

$$\mathbf{N}(\mathbf{R},A)=\{x\in A;\, (\neg\mathbf{R})(x,A)\}.$$

- If the set

$$\{(x,K)\in X\times\mathcal{C}_{\mathrm{b}}(X);\, \mathbf{R}(x,K)\}$$

  is open, then we say that $\mathbf{R}$ is *stable*.
- If $A\subset X$, then we define

$$\ker_{\mathbf{P}}(A)=A\setminus\bigcup\{O;\, O\subset X \text{ is open}, A\cap O \text{ is } \sigma\text{-}\mathbf{P}\text{-porous}\}.$$

The next property was introduced in [ZZ$_1$], where it is called **(D2)**. It is a technical notion which is necessary for the proof of the basic lemma of [ZZ$_1$] (see Lemma 2.24 below). Although virtually no porosity-like relation is stable, many of them are obtained from stable relations using countable unions and countable intersections. Stability of a relation is an important notion in proofs that some point-set relations have **(D)**.

DEFINITION 2.5. Let X be a metric space. Let $\mathbf{R}$ be a point-set relation on X. We say that $\mathbf{R}$ satisfies the condition **(D)** if the set

$$\{(L,K)\in\mathcal{K}^*(X)\times\mathcal{K}^*(X);\ \exists O\subset X \text{ open}: L\cap O\neq\emptyset,\ (\neg\mathbf{R})(L\cap O, K\cap G)\}$$

is analytic for every $G\subset X$ open.

LEMMA 2.6. *Let* $\mathbf{P}$ *be a porosity-like relation on a metric space* X *and* $A\subset X$. *Then*

(i) $\ker_{\mathbf{P}}(A)$ *is closed in* A.
(ii) *If* A *is not* σ-$\mathbf{P}$-*porous*, *then* $\ker_{\mathbf{P}}(A)$ *is nonempty and* $\ker_{\mathbf{P}}(A)\cap G$ *is non-*σ-$\mathbf{P}$-*porous for every open* $G\subset X$ *intersecting* $\ker_{\mathbf{P}}(A)$.
(iii) $\ker_{\mathbf{P}}(\ker_{\mathbf{P}}(A))=\ker_{\mathbf{P}}(A)$.
(iv) *If* $\ker_{\mathbf{P}}(A)=A$, *then* $\ker_{\mathbf{P}}(A\cap G)=A\cap G$ *whenever* $G\subset X$ *is open.*

Proof. Assertions (i) and (iv) follow immediately from the definition of ker. Assertion (ii) follows easily from the fact (see [Za$_4$, Lemma 3]) that a set $M\subset X$ is σ-$\mathbf{P}$-porous if and only if for each $y\in M$ there exists $r>0$ such that $B(y,r)\cap M$ is σ-$\mathbf{P}$-porous. Having (ii) it is easy to infer (iii). ∎

LEMMA 2.7. *Let* X *be a metric space*, *and* $\mathbf{R}$ *be a stable point-set relation on* X *with* **(A3)**. *Let* $A\subset X$ *be bounded. Then* $\mathbf{N}(\mathbf{R},A)$ *is closed in* A.

Proof. Since $\mathbf{R}$ is stable, the set $\{x\in X;\ \mathbf{R}(x,\bar{A})\}$ is open. Using also property **(A3)** of $\mathbf{P}$ we conclude that $\mathbf{N}(\mathbf{R},A)=A\setminus\{x\in X;\ \mathbf{R}(x,A)\}=A\setminus\{x\in X;\ \mathbf{R}(x,\bar{A})\}$ is closed in A. ∎

LEMMA 2.8 ([ZZ$_1$, Lemma 3.7]). *Let* E *be a separable complete metric space. Let* $\mathbf{R}_k$, $k\in\mathbb{N}$, *be stable point-set relations on* E *with* **(A1)** *and* **(A3)**. *Then the relations* $\mathbf{V}_1:=\bigcap_{k=1}^{\infty}\mathbf{R}_k$ *and* $\mathbf{V}_2:=\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\mathbf{R}_k$ *have the properties* **(A1)**, **(A3)**, *and* **(D)**.

For technical reasons and only for the purposes of this paper we define the following notions.

DEFINITION 2.9. Let $\varepsilon>0$, $n\in\mathbb{N}$, E be a locally compact metric space, and $(\mathbf{R}_k^1)$, $(\mathbf{R}_k^2)$ $(k\in\mathbb{N})$ be two sequences of point-set relations on E. We say that the condition $\mathbf{C}(\varepsilon,(\mathbf{R}_k^1),(\mathbf{R}_k^2),n)$ is satisfied if, for every $H\subset E$, $k\in\mathbb{N}$, $k\geq n$, and every compact set $K\subset E$ with $(\neg\mathbf{R}_k^2)(K,H)$, there exists a finite system $\mathcal{S}$ of open sets in E such that

(i) each element of $\mathcal{S}$ intersects H,
(ii) $\bigcup\mathcal{S}\subset B(K,\varepsilon)$,
(iii) if $J\subset E$ intersects each element of $\mathcal{S}$, then $(\neg\mathbf{R}_k^1)(K,J)$.

Perhaps a few words about this definition are in order. Suppose that $(\neg\mathbf{R}_k^2)(K,H)$ for $k\geq n$. Roughly speaking, it would be helpful in further constructions if we could replace H by a suitable set $J \subset H$ so that $(\neg\mathbf{R}_k^2)(K, J)$ for $k \geq n$. In the applications of interest, this is not possible in general. But if we know that $\mathbf{C}(\varepsilon, (\mathbf{R}_k^1), (\mathbf{R}_k^2), n)$ holds, then we get $(\neg\mathbf{R}_k^1)(K, J)$ (for $k \geq n$ and any suitable J) and it makes the desired constructions still possible.

Definition 2.10. Let $\mathcal{I}$ be a σ-ideal of subsets of a locally compact metric space E. We say that $\mathcal{I}$ is *good* if there exists a porosity-like relation $\mathbf{P}$ on E such that $\mathcal{I}$ is the σ-ideal of all σ-$\mathbf{P}$-porous sets and there exist a nonempty countable set $\varXi$ and point-set relations $\mathbf{R}_n^\xi$ on E, $\xi \in \varXi$, $n \in \mathbb{N}$, such that

(G1) $\mathbf{P} = \bigcup_{\xi\in\varXi} \mathbf{V}^\xi$, where $\mathbf{V}^\xi = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty \mathbf{R}_k^\xi$,

(G2) $\mathbf{R}_n^\xi$ satisfies **(A1)**, **(A3)**, and is stable for every $\xi \in \varXi$, $n \in \mathbb{N}$,

(G3) for every $\varepsilon > 0$ and every $\xi_1 \in \varXi$ there exist $\xi_2 \in \varXi$ and $n \in \mathbb{N}$ such that $\mathbf{C}(\varepsilon, (\mathbf{R}_k^{\xi_1}), (\mathbf{R}_k^{\xi_2}), n)$ holds.

The upper limit of $\mathbf{R}_k^\xi$'s in **(G1)** is motivated by the fact that $\langle g\rangle$-porosity, symmetrical porosity, and some other types of porosity can be naturally expressed in this form (see Section 5). Condition **(G3)** is a type of condition which naturally appears in constructions of non-porous (or even non-σ-porous sets). Simpler conditions of this type are e.g. **(A5)** from [ZZ$_1$] and that of [ZZ$_2$, Definition 2.2].

Our aim is to prove that Question (Q) has an affirmative answer for σ-$\mathbf{P}$-porosity if the σ-$\mathbf{P}$-porous sets form a good σ-ideal. Since we have to prove several lemmas before the proof of the main result, we fix the meaning of the symbols E, ϱ, $\mathbf{P}$, $\varXi$, $\mathbf{V}^\xi$, $\mathbf{R}_n^\xi$, and ξ^k in the rest of this section and in the next section.

Setting 2.11. Let (E, ϱ) be a fixed *compact* metric space, $\mathbf{P}$ be a fixed porosity-like relation on E, $\varXi$ be a nonempty countable set, $\mathbf{V}^\xi$, $\xi \in \varXi$, be fixed point-set relations on E, and $\mathbf{R}_n^\xi$, $\xi \in \varXi$, $n \in \mathbb{N}$, be fixed point-set relations on E such that **(G1)**–**(G3)** are satisfied. Let $\{\xi^k\}_{k=1}^\infty$ be a fixed sequence containing each element of $\varXi$ infinitely many times.

Let $\varepsilon > 0$, $\xi_1, \xi_2 \in \varXi$, $n \in \mathbb{N}$. We employ the symbol $\mathbf{C}(\varepsilon, \xi_1, \xi_2, n)$ to denote that the condition $\mathbf{C}(\varepsilon, (\mathbf{R}_k^{\xi_1}), (\mathbf{R}_k^{\xi_2}), n)$ holds.

Lemma 2.12. *Let $A \subset E$. Suppose $\mathbf{C}(\varepsilon, \xi_1, \xi_2, n)$, and let B be a subset of E. Let $x \in E$ be such that $B(x, \varepsilon) \subset B$ and $(\neg\mathbf{V}^{\xi_2})(x, A)$. Then $(\neg\mathbf{V}^{\xi_1})(x, A \cap B)$.*

Proof. There exists $m \in \mathbb{N}$ such that $m \geq n$ and $(\neg\mathbf{R}_j^{\xi_2})(x, A)$ for every $j \geq m$. Fix $j \geq m$. According to **(G3)** there exists a finite system $\mathcal{S}$ of open sets such that

- each element of $\mathcal{S}$ intersects A,
- $\bigcup \mathcal{S} \subset B(x,\varepsilon)$,
- if $J \subset E$ intersects each element of $\mathcal{S}$, then $(\neg \mathbf{R}_j^{\xi_1})(x,J)$.

Since $A \cap B$ intersects each element of $\mathcal{S}$, we have $(\neg \mathbf{R}_j^{\xi_1})(x, A\cap B)$. Thus $(\neg \mathbf{R}_j^{\xi_1})(x, A\cap B)$ for every $j \ge m$ and, consequently, $(\neg \mathbf{V}^{\xi_1})(x, A\cap B)$. ∎

Note that $\mathbf{V}^{\xi}$ need not be a porosity-like relation. However, the following lemma holds.

LEMMA 2.13. *The relations* $\mathbf{V}^{\xi}$ *satisfy* **(A1)**, **(A3)**, **(D)**, *and* $\mathbf{P}$ *is necessarily a porosity-like relation.*

Proof. Lemma 2.8, **(G1)**, and **(G2)** imply that $\mathbf{V}^{\xi}$'s satisfy **(A1)**, **(A3)**, **(D)**, and $\mathbf{P}$ satisfies **(A1)** and **(A3)**. We check **(A2)** for $\mathbf{P}$. Let $\varepsilon > 0$, $x \in E$, and $A \subset E$ be such that $\mathbf{P}(x, A \cap B(x,\varepsilon))$. Then for some $\xi_1 \in \Xi$ we have $\mathbf{V}^{\xi_1}(x, A\cap B(x,\varepsilon))$. Find $\xi_2 \in \Xi$ and $n \in \mathbb{N}$ with $\mathbf{C}(\varepsilon, \xi_1, \xi_2, n)$. Using Lemma 2.12 we obtain $\mathbf{V}^{\xi_2}(x, A)$. ∎

DEFINITION 2.14. Let $\mu \in \mathbb{N}^{\mathbb{N}}$, $k \in \mathbb{N}$, and $s = (s(1),\dots,s(k))$ be a finite sequence of elements of $\mathbb{N}$. Then we define

$$\begin{aligned} \mu(k) &= k\text{th member of } \mu, \\ \mu|k &= (\mu(1),\dots,\mu(k)), \\ s \diamond \mu &= (s(1),\dots,s(k),\mu(k+1),\mu(k+2),\dots). \end{aligned}$$

Let us note that the next definition uses the sequence $\{\xi^n\}$, which has already been fixed in Setting 2.11.

DEFINITION 2.15. Let $m,n,l \in \mathbb{N}$, $A \subset E$, $\mu \in \mathbb{N}^{\mathbb{N}}$, and $s = (s(1),\dots$ $\dots, s(m))$ be a finite sequence of natural numbers. Then we define

$$g_l^n(A) = \ker_{\mathbf{P}} \Big(\bigcap_{j=l}^{\infty} \mathbf{N}(\mathbf{R}_j^{\xi^n}, A) \Big), \qquad f_s(A) = g_{s(m)}^m \circ \dots \circ g_{s(1)}^1(A).$$

Let $A \subset E$, $B \subset E$. Then the symbol $B \overset{\mu}{\hookrightarrow} A$ stands for $B \subset \bigcap_{k=1}^{\infty} f_{\mu|k}(A)$.

Observe that $g_l^n(A) \subset A$ by the definition of ker and $\mathbf{N}(\mathbf{R}, A)$ (Definition 2.4).

DEFINITION 2.16. Let $\mu, \nu \in \mathbb{N}^{\mathbb{N}}$. Then the symbol $\mu \le \nu$ means that $\mu(k) \le \nu(k)$ for every $k \in \mathbb{N}$.

The next observations are easy to prove.

OBSERVATION 2.17. (i) *Let s be a finite sequence of natural numbers, $\tau \in \mathbb{N}^{\mathbb{N}}$, and A, B be subsets of E such that $A \overset{\tau}{\hookrightarrow} f_s(B)$. Then $A \overset{s\diamond\tau}{\hookrightarrow} B$.*

(ii) *Let $\tau \in \mathbb{N}^{\mathbb{N}}$ and A, B be subsets of E such that $A \overset{\tau}{\hookrightarrow} B$. If $\nu \in \mathbb{N}^{\mathbb{N}}$ and $\tau \leq \nu$, then $A \overset{\nu}{\hookrightarrow} B$.*

DEFINITION 2.18. We will define systems $\mathcal{C}_\alpha$, $\alpha \leq \omega_1$, of nonempty compact subsets of E inductively. We put $\mathcal{C}_0 = \mathcal{K}^*(E)$. The system $\mathcal{C}_\alpha$, $0 < \alpha \leq \omega_1$, is defined by

$$K \in \mathcal{C}_\alpha \overset{\text{def}}{\iff} (K \in \mathcal{K}^*(E) \;\&\; (\forall \beta < \alpha \;\forall \xi \in \varXi \;\forall B \subset E \text{ open}, K \cap B \neq \emptyset \;\exists L \in \mathcal{C}_\beta : L \subset K \cap B, (\neg \mathbf{V}^\xi)(L, K \cap B))).$$

The last definition enables us to define a rank needed for an application of the overspill method (via Lemma 2.24). Note that we do not know whether it is a coanalytic rank. However, we will not need this information.

DEFINITION 2.19. Let $K \in \mathcal{K}^*(E)$. Then $\mathrm{rk}(K) = \sup\{\alpha;\, K \in \mathcal{C}_\alpha\}$.

REMARK 2.20. It is not difficult to see that $\mathrm{rk}(K) \geq \alpha$ if and only if $K \in \mathcal{C}_\alpha$ and thus $\mathrm{rk}(K) = \max\{\alpha;\, K \in \mathcal{C}_\alpha\}$.

The next lemma follows directly from the above definition.

LEMMA 2.21. *Let $\alpha < \omega_1$. Let $\mathcal{K} \subset \mathcal{K}^*(E)$, $\mathcal{K} \neq \emptyset$ and $\mathrm{rk}(K) \geq \alpha$ for each $K \in \mathcal{K}$. Then $\mathrm{rk}(\overline{\bigcup \mathcal{K}}) \geq \alpha$.*

LEMMA 2.22. (i) *Let $K \in \mathcal{K}^*(E)$, $\mathrm{rk}(K) \geq \alpha$. If G is an open set intersecting K, then there exists $F \in \mathcal{K}^*(E)$ with $F \subset K \cap G$ and $\mathrm{rk}(F) \geq \alpha$.*

(ii) *Let $\{\alpha_i\}_{i=1}^\infty$ be a nondecreasing sequence of countable ordinals and $\lim(\alpha_i + 1) = \alpha$. Let $K_i \in \mathcal{K}^*(E)$, $i \in \mathbb{N}$, be such that $\mathrm{rk}(K_i) \geq \alpha_i$, $(\neg \mathbf{V}^{\xi^i})(K_i, K_{i+1})$, and $K_i \subset K_{i+1}$ for every $i \in \mathbb{N}$. Then $\mathrm{rk}(\overline{\bigcup_{i=1}^\infty K_i}) \geq \alpha$.*

Proof. (i) Find an open set H intersecting K with $\overline{H} \subset G$. Put $F = \overline{K \cap H}$. We have $F \subset K \cap G$.

Let $\beta < \alpha$, $\xi \in \varXi$, and $B \subset E$ be an open set intersecting F. Then $K \cap H \cap B \neq \emptyset$ and, consequently, there exists $L \in \mathcal{C}_\beta$ such that $L \subset K \cap H \cap B \subset F \cap B$ and $(\neg \mathbf{V}^\xi)(L, K \cap H \cap B)$. Since $\mathbf{V}^\xi$ satisfies **(A1)** by Lemma 2.13, we have $(\neg \mathbf{V}^\xi)(L, F \cap B)$. Thus $\mathrm{rk}(F) \geq \alpha$.

(ii) Define $K = \overline{\bigcup_{i=1}^\infty K_i}$. Let $\beta < \alpha$, $\xi_1 \in \varXi$, and B be an open set intersecting K. We find $i \in \mathbb{N}$ such that $K_i \cap B \neq \emptyset$ and $\mathrm{rk}(K_i) \geq \beta$. According to (i) there exists $L \in \mathcal{K}^*(E)$ such that $\mathrm{rk}(L) \geq \beta$ and $L \subset K_i \cap B$. Find $\varepsilon > 0$ with $\mathrm{dist}(L, E \setminus B) > \varepsilon$. There exist $\xi_2 \in \varXi$ and $n \in \mathbb{N}$ with

$\mathbf{C}(\varepsilon, \xi_1, \xi_2, n)$. For some $j \geq i$ we have $(\neg \mathbf{V}^{\xi_2})(K_j, K_{j+1})$. We also have $L \subset K_i \subset K_j \subset K_{j+1} \subset K$. Now **(A1)** for $\mathbf{V}^{\xi}$'s implies $(\neg \mathbf{V}^{\xi_2})(L, K)$, and so Lemma 2.12 gives $(\neg \mathbf{V}^{\xi_1})(L, K \cap B)$. This shows that $\mathrm{rk}(K) \geq \alpha$. ■

LEMMA 2.23. *Let $\mathcal{G}$ be an analytic subset of $\mathcal{K}^*(E)$ with*

$$\sup\{\mathrm{rk}(K);\ K \in \mathcal{G}\} = \omega_1.$$

*Then $\mathcal{G}$ contains a non-σ-**P**-porous set.*

The previous lemma follows immediately from Lemma 2.13 and from the next basic lemma of [ZZ$_1$].

LEMMA 2.24 ([ZZ$_1$, Lemma 5.2]). *Let X be a separable locally compact metric space and $\varXi$ be a nonempty countable set. Let $\mathbf{Q} = \bigcup_{\xi \in \varXi} \mathbf{W}^{\xi}$ be a porosity-like relation on X, where each $\mathbf{W}^{\xi}$ satisfies* **(A1)**, **(A3)**, *and* **(D)**. *Let* rk *be the rank corresponding to the relations $\mathbf{W}^{\xi}$, i.e., the one we obtain by replacing $\mathbf{V}^{\xi}$ by $\mathbf{W}^{\xi}$ in Definitions* 2.18 *and* 2.19. *Let $\mathcal{G}$ be an analytic subset of $\mathcal{K}^*(X)$ with* $\sup\{\mathrm{rk}(K);\ K \in \mathcal{G}\} = \omega_1$. *Then $\mathcal{G}$ contains a non-σ-**Q**-porous set.*

3. Inscribing into a G_δ set

LEMMA 3.1. *Let $G \subset E$ be a G_δ set with* $\ker_{\mathbf{P}}(G) = G \neq \emptyset$. *Then there exist $\mu \in \mathbb{N}^{\mathbb{N}}$ and $x \in G$ such that $\{x\} \overset{\mu}{\hookrightarrow} G$.*

Proof. First observe that if $A \subset E$ is not σ-**P**-porous, then for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $g^n_m(A)$ is not σ-**P**-porous. Indeed, the set

$$T := A \setminus \bigcup_{p=1}^{\infty} \bigcap_{j=p}^{\infty} \mathbf{N}(\mathbf{R}_j^{\xi^n}, A)$$

is **P**-porous, since $\mathbf{V}^{\xi^n}(T, T)$. Thus $\bigcup_{p=1}^{\infty} \bigcap_{j=p}^{\infty} \mathbf{N}(\mathbf{R}_j^{\xi^n}, A)$ is non-σ-**P**-porous. Hence we can find $m \in \mathbb{N}$ such that $\bigcap_{j=m}^{\infty} \mathbf{N}(\mathbf{R}_j^{\xi^n}, A)$ is non-σ-**P**-porous. Now Lemma 2.6(ii) shows that $g^n_m(A)$ is non-σ-**P**-porous.

Let τ be a complete metric on G equivalent to ϱ on G. We will find a sequence $\{k_n\}_{n=1}^{\infty}$ of natural numbers and a sequence $\{H_n\}_{n=0}^{\infty}$ of closed sets in (G, τ) such that $H_0 = G$ and for every $n \in \mathbb{N}$ we have

- $\mathrm{diam}_\tau H_n < 1/n$,
- $H_n \subset g^n_{k_n}(H_{n-1})$,
- H_n is non-σ-**P**-porous.

Suppose that we have defined H_n and k_n. Then there exists $k_{n+1} \in \mathbb{N}$ such that $g^{n+1}_{k_{n+1}}(H_n)$ is non-σ-**P**-porous. The set $g^{n+1}_{k_{n+1}}(H_n)$ is closed in H_n by Lemmas 2.6(i) and 2.7. Therefore $g^{n+1}_{k_{n+1}}(H_n)$ is closed also in (G, τ). Take an open ball B in (G,τ) with $\operatorname{diam}_\tau B < 1/(n+1)$ intersecting $g^{n+1}_{k_{n+1}}(H_n)$ and put $H_{n+1} = g^{n+1}_{k_{n+1}}(H_n) \cap \overline{B}^{(G,\tau)}$. Using Lemma 2.6(ii) we deduce that H_{n+1} is non-σ-**P**-porous. This finishes the construction of the desired sequences.

We have $\bigcap_{n=1}^\infty H_n = \{x\}$. Putting $\mu = \{k_n\}_{n=1}^\infty$ we obtain $\{x\} \overset{\mu}{\hookrightarrow} G$. ■

Roughly speaking, our aim is to show that inside a G_δ non-σ-**P**-porous set one can find a compact set with arbitrarily large countable rank. Having this we can apply Lemma 2.23. We use transfinite induction on the rank. The precise assertion is formulated in the next lemma. The previous lemma in fact verifies the first step of the induction, and Lemmas 3.3 and 3.4 help us to prove the general induction step.

LEMMA 3.2. *Let α be an ordinal number with $\alpha < \omega_1$. Let $G \subset E$ be a G_δ set with* $\ker_{\mathbf{P}}(G) = G \neq \emptyset$. *Then there exist $K \in \mathcal{K}^*(E)$ and $\mu \in \mathbb{N}^{\mathbb{N}}$ such that $K \overset{\mu}{\hookrightarrow} G$ and* $\operatorname{rk}(K) \geq \alpha$.

LEMMA 3.3. *Assume that the assertion of Lemma* 3.2 *holds for an ordinal number $\alpha < \omega_1$. Let*

- $\varepsilon > 0$,
- $n, k \in \mathbb{N}$, $n < k$,
- $\mu \in \mathbb{N}^{\mathbb{N}}$, $m \in \mathbb{N}$ *and $\xi \in \Xi$ be such that $\mu(k) \leq m$ and* $\mathbf{C}(\varepsilon, \xi, \xi^k, m)$,
- $F \subset \{j \in \mathbb{N};\, j \geq m\}$ *be a finite set*,
- $G \subset E$ *be a G_δ set with* $\ker_{\mathbf{P}}(G) = G \neq \emptyset$,
- $K \in \mathcal{K}^*(E)$ *and* $K \overset{\mu}{\hookrightarrow} G$.

Then there exist $L \in \mathcal{K}^*(E)$ and $\nu \in \mathbb{N}^{\mathbb{N}}$ such that

(a) $\mu|n = \nu|n$,
(b) $K \cup L \overset{\nu}{\hookrightarrow} G$,
(c) *for every $p \in F$ we have* $(\neg\mathbf{R}^\xi_p)(K, L)$,
(d) $\operatorname{rk}(L) \geq \alpha$,
(e) $h(L, K) < \varepsilon$.

Proof. Since $K \overset{\mu}{\hookrightarrow} G$ and $k > n$, we have $K \subset f_{\mu|k}(G) \subset g^k_{\mu(k)}(f_{\mu|n}(G))$. Thus $(\neg\mathbf{R}^{\xi^k}_p)(K, f_{\mu|n}(G))$ for every $p \in \mathbb{N}$ with $p \geq \mu(k)$. Using the condition $\mathbf{C}(\varepsilon, \xi, \xi^k, m)$ we find for every $p \in F$ a finite system $\mathcal{S}_p$ of open sets such that

(i) each element of $\mathcal{S}_p$ intersects $f_{\mu|n}(G)$,
(ii) $\bigcup \mathcal{S}_p \subset B(K, \varepsilon)$,
(iii) if $J \subset E$ intersects each element of $\mathcal{S}_p$, then $(\neg\mathbf{R}^\xi_p)(K, J)$.

Adding finitely many appropriate open sets to the system $\mathcal{S}_p$, if necessary, we may suppose that $\mathcal{S}_p$ also satisfies

(iv) if $J \subset B(K,\varepsilon)$ is a compact set intersecting each element of $\mathcal{S}_p$, then $h(J,K) < \varepsilon$.

Indeed, it is sufficient to add to $\mathcal{S}_p$ the sets of a finite system $\mathcal{Z}$ of open balls with radii less than $\varepsilon/2$ such that each ball from $\mathcal{Z}$ intersects K and $K \subset \bigcup \mathcal{Z}$.

Put $\mathcal{S} = \bigcup_{p\in F} \mathcal{S}_p$. We know that $f_{\mu|n}(G)$ is a G_δ set (according to Lemmas 2.6(i) and 2.7) with $\ker_{\mathbf{P}}(f_{\mu|n}(G)) = f_{\mu|n}(G) \neq \emptyset$ (Lemma 2.6(iii)). Thus $\ker_{\mathbf{P}}(f_{\mu|n}(G)\cap S) = f_{\mu|n}(G)\cap S \neq \emptyset$ for every $S \in \mathcal{S}$ (Lemma 2.6(iv)). Since we assume that the assertion of Lemma 3.2 holds for α, we can find for every $S \in \mathcal{S}$ a nonempty compact set K_S and $\nu_S \in \mathbb{N}^{\mathbb{N}}$ such that $K_S \overset{\nu_S}{\hookrightarrow} f_{\mu|n}(G) \cap S$ and $\mathrm{rk}(K_S) \geq \alpha$. According to Observation 2.17(i) we have $K_S \overset{\mu|n\diamond\nu_S}{\hookrightarrow} G$ for every $S \in \mathcal{S}$. We choose $\nu \in \mathbb{N}^{\mathbb{N}}$ so that $\mu|n = \nu|n$, $\mu \leq \nu$, and $\mu|n \diamond \nu_S \leq \nu$ for every $S \in \mathcal{S}$. The desired L is defined by $L := \bigcup\{K_S;\, S \in \mathcal{S}\}$.

Properties (a) and (e) are obviously satisfied. Property (b) follows from the definition of ν and Observation 2.17(ii), (c) follows from (iii), and (d) is a consequence of Lemma 2.21. ■

Lemma 3.4. *Assume that the assertion of Lemma* 3.2 *holds for an ordinal number* $\alpha < \omega_1$. *Let*

- $\varepsilon > 0$,
- $n \in \mathbb{N}$,
- $\mu \in \mathbb{N}^{\mathbb{N}}$, $\xi \in \Xi$,
- $G \subset E$ *be a* G_δ *set with* $\ker_{\mathbf{P}}(G) = G \neq \emptyset$,
- $K \in \mathcal{K}^*(E)$ *and* $K \overset{\mu}{\hookrightarrow} G$.

Then there exist $L \in \mathcal{K}^*(E)$ *and* $\nu \in \mathbb{N}^{\mathbb{N}}$ *such that*

(a) $\mu|n = \nu|n$,
(b) $L \overset{\nu}{\hookrightarrow} G$,
(c) $(\neg\mathbf{V}^{\xi})(K,L)$,
(d) $\mathrm{rk}(L) \geq \alpha$,
(e) $K \subset L \subset B(K,\varepsilon)$.

Proof. Using **(G3)** and the definition of the sequence $\{\xi^k\}_{k=1}^{\infty}$ we find sequences $\{k_i\}_{i=1}^{\infty}$, $\{n_i\}_{i=1}^{\infty}$ of natural numbers such that

- $\mathbf{C}(\varepsilon/i, \xi, \xi^{k_i}, n_i)$,
- $k_i > n + i$,
- $\mu(k_i) \leq n_i$,
- $\{n_i\}_{i=1}^{\infty}$ is an increasing sequence.

Put $F_i = \{j \in \mathbb{N};\, n_i \leq j < n_{i+1}\}$, $i \in \mathbb{N}$.

Using Lemma 3.3 we find for every $i \in \mathbb{N}$ a set $L_i \in \mathcal{K}^*(E)$ and $\nu_i \in \mathbb{N}^{\mathbb{N}}$ such that

(1) $\mu|(n+i) = \nu_i|(n+i)$,
(2) $K \cup L_i \overset{\nu_i}{\hookrightarrow} G$,
(3) for every $p \in F_i$ we have $(\neg \mathbf{R}_p^{\xi})(K, L_i)$,
(4) $\mathrm{rk}(L_i) \geq \alpha$,
(5) $h(L_i, K) < \varepsilon/i$.

Put $L := K \cup \bigcup_{i=1}^{\infty} L_i$. Since $\lim_{i\to\infty} \varepsilon/i = 0$, we see that L is a compact set (Lemma 2.3(ii)). Because of (1) we can find $\nu \in \mathbb{N}^{\mathbb{N}}$ such that $\mu|n = \nu|n$ and $\nu_i \leq \nu$ for every $i \in \mathbb{N}$. Thus condition (a) clearly holds. We now verify (b)–(e).

(b) follows from (2), Observation 2.17(ii), and the definition of ν.

(c) If $p \geq n_1$, then there is $i \in \mathbb{N}$ with $p \in F_i$ and $(\neg \mathbf{R}_p^{\xi})(K, L_i)$. Thus $(\neg \mathbf{R}_p^{\xi})(K, L)$. This implies $(\neg \mathbf{V}^{\xi})(K, L)$.

(d) follows from Lemma 2.21 since $L = \overline{\bigcup\{L_i;\, i \in \mathbb{N}\}}$ according to (5), and (e) clearly follows from the definition of L and (5). ∎

Proof of Lemma 3.2. We proceed by transfinite induction over countable ordinals.

Case $\alpha = 0$. According to Lemma 3.1 there exist $\mu \in \mathbb{N}^{\mathbb{N}}$ and $x \in E$ with $\{x\} \overset{\mu}{\hookrightarrow} G$. We put $K = \{x\}$ and we are done.

Case $\alpha > 0$. Suppose that the assertion holds for every $\beta < \alpha$. Find a nondecreasing sequence $\{\alpha_q\}_{q=1}^{\infty}$ of ordinal numbers such that $\alpha_q < \alpha$, $q \in \mathbb{N}$, and $\lim(\alpha_q + 1) = \alpha$.

Now we will construct a sequence $\{\mu_q\}_{q=1}^{\infty}$ of elements of $\mathbb{N}^{\mathbb{N}}$ and a sequence $\{K_q\}_{q=1}^{\infty}$ of elements of $\mathcal{K}^*(E)$ such that for every $q \in \mathbb{N}$ we have

(1) $\mu_{q+1}|q = \mu_q|q$,
(2) $K_q \overset{\mu_q}{\hookrightarrow} G$,
(3) $(\neg \mathbf{V}^{\xi_q})(K_q, K_{q+1})$,
(4) $\mathrm{rk}(K_q) \geq \alpha_q$,
(5) there is a complete metric h_q on $\mathcal{K}(f_{\mu_q|q}(G))$ equivalent to the Hausdorff metric such that $h_q(K_j, K_{j+1}) \leq 2^{-j}$ for every $j \in \mathbb{N}$, $j \geq q$,
(6) $K_q \subset K_{q+1}$.

By the induction hypothesis there exist K_1 and μ_1 satisfying (2) and (4) for $q = 1$. Since $f_{\mu_1|1}(G)$ is a G_δ subset of E, $\mathcal{K}(f_{\mu_1|1}(G))$ is a G_δ subset of $\mathcal{K}(E)$ (Lemma 2.3(iii)). We find a complete metric h_1 on $\mathcal{K}(f_{\mu_1|1}(G))$ equivalent to the Hausdorff metric. Now assume that we have constructed $K_1, \ldots, K_m$, $\mu_1, \ldots, \mu_m$, and metrics $h_1, \ldots, h_m$.

We have $f_{\mu_m|m}(G) \subset f_{\mu_{m-1}|m-1}(G) \subset \cdots \subset f_{\mu_1|1}(G)$. Thus $K_m \subset f_{\mu_i|i}(G)$ for every $i \le m$. We find $\varepsilon > 0$ so small that if a compact $C \subset f_{\mu_m|m}(G)$ satisfies $C \subset B(K_m, \varepsilon)$, then $h_i(K_m, K_m \cup C) \le 2^{-m}$ for every $i \le m$.

Using Lemma 3.4 for this ε and $\alpha := \alpha_{m+1}$, $n := m$, $\mu := \mu_m$, $\xi := \xi^m$, $G := G$, $K := K_m$ we obtain $K_{m+1} \in \mathcal{K}^*(E)$ and $\mu_{m+1} \in \mathbb{N}^{\mathbb{N}}$ such that

- $\mu_{m+1}|m = \mu_m|m$,
- $K_{m+1} \overset{\mu_{m+1}}{\hookrightarrow} G$,
- $(\neg \mathbf{V}^{\xi^m})(K_m, K_{m+1})$,
- $\operatorname{rk}(K_{m+1}) \ge \alpha_{m+1}$,
- $K_m \subset K_{m+1} \subset B(K_m, \varepsilon)$.

We finish the construction of the desired sequences by choosing a complete metric h_{m+1} on $\mathcal{K}(f_{\mu_{m+1}|m+1}(G))$ which is equivalent to the Hausdorff metric on $\mathcal{K}(f_{\mu_{m+1}|m+1}(G))$. The desired K and μ are defined by

$$K = \overline{\bigcup_{q=1}^{\infty} K_q}, \qquad \mu(m) = \mu_m(m), \qquad m \in \mathbb{N}.$$

Let $q \in \mathbb{N}$. Using (5) we infer that $\{K_p\}_{p=q}^{\infty}$ is a Cauchy sequence in $(\mathcal{K}(f_{\mu|q}(G)), h_q)$. Thus $\{K_p\}_{p=q}^{\infty}$ converges to some $K^\star$ in $(\mathcal{K}(f_{\mu|q}(G)), h_q)$. Since the Hausdorff metric on $\mathcal{K}(f_{\mu|q}(G))$ is equivalent to h_q, it follows that $\{K_p\}_{p=q}^{\infty}$ converges to $K^\star$ with respect to the Hausdorff metric. Lemma 2.3(i) shows that $K^\star = K$. Thus $K \subset f_{\mu|q}(G)$ for every $q \in \mathbb{N}$ and therefore $K \overset{\mu}{\hookrightarrow} G$.

Finally, we have $\operatorname{rk}(K) \ge \alpha$ by Lemma 2.22(ii). ∎

LEMMA 3.5. *Let $G \subset E$ be a G_δ non-σ-$\mathbf{P}$-porous set. Then there exists a non-σ-$\mathbf{P}$-porous compact subset of G.*

Proof. Put $H = \ker_{\mathbf{P}}(G)$. Then H is a nonempty G_δ set with $\ker_{\mathbf{P}}(H) = H$ (Lemma 2.6(i)–(iii)). According to Lemma 3.2 we have

$$\sup\{\operatorname{rk}(K);\ K \in \mathcal{K}^\star(H)\} = \omega_1.$$

The set $\mathcal{K}^\star(H)$ is a G_δ subset of $\mathcal{K}^*(E)$ and so Lemma 2.23 implies that H contains a non-σ-$\mathbf{P}$-porous compact set. ∎

4. Inscribing into an analytic set. Let (E, ϱ), $\mathbf{P}$, Ξ, $\mathbf{R}_n^{\xi}$, $\mathbf{V}^{\xi}$, and ξ^k be as in Setting 2.11. Let $C = 2^{\mathbb{N}}$ be the Cantor set and let ϱ_0 be a metric on C giving the product topology on C.

Let $\mathcal{O}_n$, $n \in \mathbb{N}$, be finite systems of pairwise disjoint clopen sets in C such that for every $n \in \mathbb{N}$ we have

- $\bigcup \mathcal{O}_n = C$,
- $\forall I \in \mathcal{O}_n : \operatorname{diam} I < 1/n$,
- $\forall I \in \mathcal{O}_{n+1}\, \exists Z \in \mathcal{O}_n : I \subset Z$.

Let π be the projection of $E\times C$ onto E. We will work with the maximum metric $\varrho^\star$ on $E \times C$, i.e.

$$\varrho^\star((x_1,c_1),(x_2,c_2)) = \max\{\varrho(x_1,x_2), \varrho_0(c_1,c_2)\}.$$

Thus $(E \times C, \varrho^\star)$ is a compact metric space. Let $\mathcal{I}$ be the σ-ideal of all σ-$\mathbf{P}$-porous subsets of E. We define a σ-ideal $\mathcal{I}^\star$ of subsets of $E \times C$ by

$$A \in \mathcal{I}^\star \ \Leftrightarrow\ \pi(A) \in \mathcal{I}.$$

We put $\Xi^\star = \Xi \times \mathbb{N}$ and for $\xi^\star = (\xi, q) \in \Xi^\star$ and $n \in \mathbb{N}$ we define point-set relations on $E \times C$ by

$$\widetilde{\mathbf{R}}^{\xi^\star}_n(x,A) \ \Leftrightarrow\ \exists I \in \mathcal{O}_q : x \in E \times I \,\&\, \mathbf{R}^\xi_n(\pi(x), \pi(A \cap (E \times I))),$$

$$\widetilde{\mathbf{V}}^{\xi^\star} = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty \widetilde{\mathbf{R}}^{\xi^\star}_k, \qquad \mathbf{P}^\star = \bigcup_{\xi^\star \in \Xi^\star} \widetilde{\mathbf{V}}^{\xi^\star}.$$

Let $\varepsilon > 0$, $\xi^\star_1, \xi^\star_2 \in \Xi^\star$, $n \in \mathbb{N}$. The symbol $\mathbf{C}^\star(\varepsilon, \xi^\star_1, \xi^\star_2, n)$ means that the condition $\mathbf{C}(\varepsilon, (\widetilde{\mathbf{R}}^{\xi^\star_1}_k), (\widetilde{\mathbf{R}}^{\xi^\star_2}_k), n)$ holds (cf. Definition 2.9).

Our goal is to prove the next lemma.

Lemma 4.1. *The σ-ideal $\mathcal{I}^\star$ is good.*

To this end we need the following lemmas.

Lemma 4.2. *Let $\xi^\star = (\xi, q) \in \Xi^\star$, $n \in \mathbb{N}$. Then $\widetilde{\mathbf{R}}^{\xi^\star}_n$ satisfies* **(A1)**, **(A3)**, *and is stable.*

Proof. **(A1)** is obviously satisfied for $\widetilde{\mathbf{R}}^{\xi^\star}_n$.

(A3) Suppose that $x \in E \times C$, $A \subset E \times C$, and $\widetilde{\mathbf{R}}^{\xi^\star}_n(x,A)$. This implies $x \in E \times I$ and $\mathbf{R}^\xi_n(\pi(x), \pi(A \cap (E \times I))$ for some $I \in \mathcal{O}_q$. Since $\mathbf{R}^\xi_n$ satisfies **(A3)**, we have

$$\mathbf{R}^\xi_n(\pi(x), \overline{\pi(A \cap (E \times I)}).$$

Since $E \times I$ is clopen and π is continuous, we have

$$\pi(\bar{A} \cap (E \times I)) = \pi(\overline{A \cap (E \times I)}) \subset \overline{\pi(A \cap (E \times I)}).$$

This implies $\mathbf{R}^\xi_n(\pi(x), \pi(\bar{A} \cap (E \times I))$ and, consequently, $\widetilde{\mathbf{R}}^{\xi^\star}_n(x, \bar{A})$.

To prove stability of $\widetilde{\mathbf{R}}^{\xi^\star}_n$ we define

$$\mathcal{A} = \{(x,K) \in (E \times C) \times \mathcal{K}^*(E \times C);\ \widetilde{\mathbf{R}}^{\xi^\star}_n(x,K)\},$$

$$\mathcal{J} = \{(z,L) \in E \times \mathcal{K}(E);\ \mathbf{R}^\xi_n(z,L)\},$$

$$f_I : \mathcal{K}(E \times C) \to \mathcal{K}(E), \qquad f_I(K) = \pi(K \cap (E \times I)), \qquad I \in \mathcal{O}_q.$$

The mapping $K \mapsto K \cap (E \times I)$ is continuous, since $E \times I$ is clopen (Lemma 2.3(iv)) and $K \mapsto \pi(K)$ is a continuous mapping of $\mathcal{K}(E \times C)$ to $\mathcal{K}(E)$ (Lemma 2.3(v)). Thus f_I is continuous. It is easy to see that

$$\mathcal{A} = \bigcup_{I \in \mathcal{O}_q} ((\pi \times f_I)^{-1}(\mathcal{J}) \cap ((E \times I) \times \mathcal{K}^*(E \times C))).$$

The set $\mathcal{J}$ is open since $\mathbf{R}_n^\xi$ is stable. Thus $\mathcal{A}$ is open and we are done. ∎

LEMMA 4.3. *Let* $\mathbf{C}(\varepsilon, \xi_1, \xi_2, n)$ *and* $q \in \mathbb{N}$. *Then there exists* $p \in \mathbb{N}$ *such that the condition* $\mathbf{C}^\star(\varepsilon, (\xi_1, q), (\xi_2, p), n)$ *is satisfied.*

Proof. Choose $p \in \mathbb{N}$ such that $p \geq q$ and $1/p < \varepsilon$. Set $\xi_1^\star = (\xi_1, q)$ and $\xi_2^\star = (\xi_2, p)$. Let $H^\star \subset E \times C$, $k \in \mathbb{N}$, $k \geq n$, and let $K^\star \subset E \times C$ be a compact set with $(\neg\widetilde{\mathbf{R}}_k^{\xi_2^\star})(K^\star, H^\star)$. Fix $I \in \mathcal{O}_p$. Put $K_I = \pi(K^\star \cap (E \times I))$ and $H_I = \pi(H^\star \cap (E \times I))$. We have $(\neg\mathbf{R}_k^{\xi_2})(K_I, H_I)$ and therefore there exists a finite system $\mathcal{S}_I$ of open sets in E such that

(i) each element of $\mathcal{S}_I$ intersects H_I,
(ii) $\bigcup \mathcal{S}_I \subset B(K_I, \varepsilon)$,
(iii) if $J \subset E$ intersects each element of $\mathcal{S}_I$, then $(\neg\mathbf{R}_k^{\xi_1})(K_I, J)$.

We put

$$\mathcal{S}_I^\star = \{S \times I;\, S \in \mathcal{S}_I\}, \qquad \mathcal{S}^\star = \bigcup\{\mathcal{S}_I^\star;\, I \in \mathcal{O}_p\}.$$

The system $\mathcal{S}^\star$ witnesses that the condition $\mathbf{C}^\star(\varepsilon, (\xi_1, q), (\xi_2, p), n)$ holds:

(1) Each element of $\mathcal{S}^\star$ clearly intersects $H^\star$.
(2) Since $\operatorname{diam}_\varrho I < \varepsilon$ for each $I \in \mathcal{O}_p$, (ii) implies $\bigcup \mathcal{S}^\star \subset B(K^\star, \varepsilon)$.
(3) Suppose that $J^\star \subset E \times C$ intersects each element of $\mathcal{S}^\star$. Take $x \in K^\star$. There exists $I \in \mathcal{O}_p$ with $x \in E \times I$. Then $(\neg\mathbf{R}_k^{\xi_1})(\pi(x), \pi(J^\star))$ since $\pi(J^\star)$ intersects each element of $\mathcal{S}_I$. This gives $(\neg\widetilde{\mathbf{R}}_k^{\xi_1^\star})(x, J^\star)$. ∎

Proof of Lemma 4.1. First we show that $\mathbf{P}^\star$, $\widetilde{\mathbf{V}}^{\xi^\star}$, and $\widetilde{\mathbf{R}}_n^{\xi^\star}$ have the properties **(G1)–(G3)**.

(G1) is satisfied for $\mathbf{P}^\star$ by the definition of $\mathbf{P}^\star$.

(G2) holds for $\widetilde{\mathbf{R}}_n^{\xi^\star}$ by Lemma 4.2.

(G3) Take $\varepsilon > 0$ and $\xi_1^\star = (\xi_1, q) \in \Xi^\star$. There exist $\xi_2 \in \Xi$ and $n \in \mathbb{N}$ such that $\mathbf{C}(\varepsilon, \xi_1, \xi_2, n)$. Using Lemma 4.3 we find $p \in \mathbb{N}$ such that $\mathbf{C}^\star(\varepsilon, (\xi_1, q), (\xi_2, p), n)$ is satisfied. Put $\xi_2^\star = (\xi_2, q)$ and we are done.

Using Lemma 2.13 we infer that $\mathbf{P}^\star$ is a porosity-like relation. It remains to show that $\mathcal{I}^\star$ is the σ-ideal of all σ-$\mathbf{P}^\star$-porous sets.

Take $A \subset E \times C$ with $\widetilde{\mathbf{V}}^{\xi^\star}(A, A)$ for some $\xi^\star = (\xi, q) \in \Xi^\star$. For every $I \in \mathcal{O}_q$, every $x \in A \cap (E \times I)$ and every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$, $k \geq n$, such that $\mathbf{R}_k^\xi(\pi(x), \pi(A \cap (E \times I)))$. Thus $\pi(A \cap (E \times I))$ is $\mathbf{P}$-porous. Therefore

$\pi(A) \in \mathcal{I}$ and $A \in \mathcal{I}^\star$. Using Lemma 2.2 we conclude that each σ-$\mathbf{P}^\star$-porous set is in $\mathcal{I}^\star$.

If $B \subset E$ satisfies $\mathbf{V}^\xi(B,B)$ for some $\xi \in \Xi$, then obviously $\mathbf{V}^{(\xi,p)}(B\times C, B\times C)$ for every $p \in \mathbb{N}$ and so $B \times C$ is $\mathbf{P}^\star$-porous. Further, if $A \in \mathcal{I}^\star$, then $\pi(A) \in \mathcal{I}$ and $\pi(A) = \bigcup_{n=1}^\infty B_n$, where each B_n satisfies $\mathbf{V}^\xi(B_n, B_n)$ for some $\xi \in \Xi$ (Lemma 2.2). Thus A is covered by countably many $\mathbf{P}^\star$-porous sets, since $A \subset \bigcup_{n=1}^\infty (B_n \times C)$. We conclude that A is σ-$\mathbf{P}^\star$-porous. ∎

Lemma 4.4. *Let A be an analytic set with $A \notin \mathcal{I}$. Then there exists a compact set $K \subset A$ with $K \notin \mathcal{I}$.*

Proof. We find a G_δ set $G^\star \subset E \times C$ with $\pi(G^\star) = A$. Since $A \notin \mathcal{I}$ we have $G^\star \notin \mathcal{I}^\star$. By Lemma 4.1 the σ-ideal $\mathcal{I}^\star$ is good. Thus there exists a compact set $K^\star \subset G^\star$ not in $\mathcal{I}^\star$ (Lemma 3.5). Then $K := \pi(K^\star)$ is as desired. ∎

Theorem 4.5. *Let X be a locally compact metric space, $\mathcal{J}$ be a good σ-ideal of subsets of X, and $A \subset X$ be an analytic subset with $A \notin \mathcal{J}$. Then there exists a compact set $K \subset A$ with $K \notin \mathcal{J}$.*

Proof. According to Lemma 2.6(ii) there exists an open set $G \subset X$ such that $A \cap G \notin \mathcal{I}$ and $\overline{G}$ is compact. Let $\mathbf{P}$, $\mathbf{V}^\xi$, $\mathbf{R}_n^\xi$ be relations witnessing that $\mathcal{J}$ is good. Let $\overline{\mathbf{P}}$, $\overline{\mathbf{V}^\xi}$, $\overline{\mathbf{R}_n^\xi}$ denote the restrictions of $\mathbf{P}$, $\mathbf{V}^\xi$, $\mathbf{R}_n^\xi$, respectively, to the set $\overline{G}$, i.e.

$$\overline{\mathbf{P}}(x,D) \quad \text{if and only if} \quad x \in \overline{G},\ D \subset \overline{G},\ \mathbf{P}(x,D),$$

and similarly for the other relations. Let $\bar{\mathcal{J}}$ be the set of all σ-$\overline{\mathbf{P}}$-porous sets. We have $A \cap G \notin \bar{\mathcal{J}}$. Having Lemma 4.4 and since $A \cap G$ is an analytic subset of $\overline{G}$ it is sufficient to show that $\bar{\mathcal{J}}$ is good (cf. Setting 2.11). We show that the relations $\overline{\mathbf{P}}$, $\overline{\mathbf{V}^\xi}$ and $\overline{\mathbf{R}_n^\xi}$ satisfy **(G1)–(G3)** of Definition 2.10. Conditions **(G1)** and **(G2)** are clearly satisfied.

(G3) Let $\varepsilon > 0$ and $\xi_1 \in \Xi$. Then there exist $\xi_2 \in \Xi$ and $n \in \mathbb{N}$ such that $\mathbf{C}(\varepsilon, \xi_1, \xi_2, n)$ is satisfied for the original relations on X. Let $H \subset \overline{G}$, $k \in \mathbb{N}$, $k \geq n$, and $K \subset \overline{G}$ be a compact set with $(\neg\overline{\mathbf{R}_k^{\xi_2}})(K,H)$. Then $(\neg\mathbf{R}_k^{\xi_2})(K,H)$ and there exists a finite system $\mathcal{S}$ of open sets in X such that

(i) each element of $\mathcal{S}$ intersects H,
(ii) $\bigcup \mathcal{S} \subset B(K,\varepsilon)$,
(iii) if $J \subset X$ intersects each element of $\mathcal{S}$, then $(\neg\mathbf{R}_k^{\xi_1})(K,J)$.

Using the system $\bar{\mathcal{S}} = \{W \cap \overline{G};\ W \in \mathcal{S}\}$ one can easily finish the proof of the validity of **(G3)** for our modified relations. ∎

Remark 4.6. Theorem 4.5 can be easily strengthened by replacing analyticity of A by the assumption that A is Suslin. Indeed, if A is Suslin then $A \cap G$ in the above proof is analytic.

5. Applications to concrete porosities. Now we will apply our abstract Theorem 4.5 to $\langle g\rangle$-porosity, ordinary porosity, and symmetrical porosity.

5.1. *Definitions.* First of all we recall the definitions of the above-mentioned porosities. We set

$$G := \{g : [0,\infty) \to [0,\infty);\ g(0) = 0,\ g(x) > x \text{ for every } x > 0,\ g \text{ is nondecreasing and continuous}\}.$$

The symbol g_α, $\alpha \in \mathbb{R}$, will stand for the function $x \mapsto \alpha x$, $x \in [0,\infty)$.

Let X be a metric space, $A \subset X$, $x \in X$, and $g \in G$. We say that

- A is *$\langle g\rangle$-porous at x* if there exists a sequence $\{B(x_n, r_n)\}_{n=1}^\infty$ of balls such that $x \in B(x_n, g(r_n))$, $\lim x_n = x$, $B(x_n, r_n) \cap A = \emptyset$,
- A is (ordinary, i.e. "upper") *porous at x* if A is $\langle g_\alpha\rangle$-porous at x for some $\alpha > 1$.

The point-set relations which correspond to $\langle g\rangle$-porosity and ordinary porosity are denoted by $\mathbf{P}_g$ and $\mathbf{P}_{\mathrm{or}}$, respectively.

Let $A \subset \mathbb{R}$, $x \in \mathbb{R}$, and $c > 0$. We say that

- A is *symmetrically porous at x* if there exists $c > 0$ and a sequence $\{B(x_n, r_n)\}_{n=1}^\infty$ of balls in $\mathbb{R}$ such that $\lim x_n = x$, $x \in B(x_n, cr_n)$ and $(B(x_n, r_n) \cup B(x + (x - x_n), r_n)) \cap A = \emptyset$.

The point-set relation which corresponds to symmetrical porosity is denoted by $\mathbf{P}_{\mathrm{sy}}$.

It is easy to see that $\mathbf{P}_g$, $\mathbf{P}_{\mathrm{or}}$, and $\mathbf{P}_{\mathrm{sy}}$ are porosity-like relations.

5.2. *The case of $\langle g\rangle$-porosity and ordinary porosity.* For $g \in G$ and $k \in \mathbb{N}$, define point-set relations $\mathbf{R}_k^g$ on E by

$$\mathbf{R}_k^g(x, A) \overset{\mathrm{def}}{\iff} \exists y \in E\ \exists r > 0 : \quad 1/(k+2) < \varrho(x,y) < 1/k,\ \varrho(x,y) < g(r),\ B(y,r) \cap A = \emptyset.$$

Clearly $\mathbf{P}_g = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty \mathbf{R}_k^g$. We will need the following lemma.

LEMMA 5.1. *Let $h, h^* \in G$ and let E be a compact metric space. Then the following assertions hold.*

(a) *For every $k \in \mathbb{N}$, the relation $\mathbf{R}_k^h$ satisfies* **(A1)**, **(A3)**, *and is stable.*

(b) *Let $h < h^*$, $H \subset E$, $k \in \mathbb{N}$, $K \subset E$ be a compact set, and suppose $(\neg\mathbf{R}_k^{h^*})(K, H)$ holds. Then there exists a finite system $\mathcal{S}$ of open sets in E such that*

  (i) *each element of $\mathcal{S}$ intersects H,*

  (ii) $\bigcup \mathcal{S} \subset B(K, 2/k)$,

  (iii) *if $J \subset E$ intersects each element of $\mathcal{S}$, then $(\neg\mathbf{R}_k^h)(K, J)$.*

Proof. (a) Let $k \in \mathbb{N}$. It is obvious that $\mathbf{R}_k^h$ satisfies **(A1)** and **(A3)**. Since h is continuous, we easily see that $\mathbf{R}_k^h(x,A)$ if and only if

$$(1)\qquad \exists y \in E\ \exists r > 0:\ 1/(k+2) < \varrho(x,y) < 1/k,\quad \varrho(x,y) < h(r),\ \operatorname{dist}(B(y,r),A) > 0.$$

It is easy to see that for each $y \in E$ and $r > 0$ the set

$$\{(x,F) \in E \times \mathcal{C}_{\mathrm{b}}(E);\ 1/(k+2) < \varrho(x,y) < 1/k,\quad \varrho(x,y) < h(r),\ \operatorname{dist}(B(y,r),F) > 0\}$$

is open in $E \times \mathcal{C}_{\mathrm{b}}(E)$. Consequently, (1) implies that $\mathbf{R}_k^h$ is stable.

(b) Choose $\delta > 0$ such that $h(\delta) < 1/(k+2)$. Clearly $\delta < 1/k$. Since $\min\{h^*(t) - h(t);\ t \in [\delta, 1/k]\} > 0$, by uniform continuity of h^* on $[\delta/2, 1/k]$ we can find $0 < \eta < \delta/2$ such that

$$(2)\qquad h^*(s - 2\eta) > h(s) \quad \text{for each } s \in [\delta, 1/k].$$

Now find a finite set $Z \subset H \cap B(K, 2/k)$ such that $H \cap B(K, 2/k) \subset B(Z,\eta)$ and put $S := \{B(z,\eta) \cap B(K, 2/k);\ z \in Z\}$. Clearly (i) and (ii) are satisfied.

To prove (iii), suppose to the contrary that $J \subset E$ intersects each element of $\mathcal{S}$, but $(\neg\mathbf{R}_k^h)(K,J)$ does not hold. Consequently, there exist $x \in K$, $y \in E$, and $r > 0$ such that $1/(k+2) < \varrho(x,y) < 1/k$, $\varrho(x,y) < h(r)$, and $B(y,r) \cap J = \emptyset$. Put $s := \min\{1/k, r\}$. Since $h(r) > \varrho(x,y) > 1/(k+2)$ and $h(\delta) < 1/(k+2)$, we have $r > \delta$ and thus $s \in [\delta, 1/k]$. Since $h(1/k) > 1/k > \varrho(x,y)$, we have $\varrho(x,y) < h(s)$ and (2) implies $h^*(s-2\eta) > \varrho(x,y)$. Therefore $(\neg\mathbf{R}_k^{h^*})(K,H)$ yields a point $w \in B(y, s-2\eta) \cap H$. Since clearly $w \in B(K, 2/k)$, there exists $z \in Z$ with $\varrho(z,w) < \eta$. Since J intersects $B(z,\eta)$, we obtain $J \cap B(y,s) \neq \emptyset$, which contradicts $B(y,r) \cap J = \emptyset$. ∎

THEOREM 5.2. *Let E be a locally compact metric space, $g \in G$, and $A \subset E$ be an analytic set which is not σ-$\langle g\rangle$-porous. Then there exists a compact set $K \subset A$ which is not σ-$\langle g\rangle$-porous.*

Proof. First suppose that E is a compact space. By Lemma 4.5 it is sufficient to prove that the σ-ideal $\mathcal{I}$ of all σ-$\langle g\rangle$-porous subsets of E is a good σ-ideal. For $n \in \mathbb{N}$, put $g^{(n)} := g \circ \cdots \circ g$ (n-fold composition).

Let $\Xi := \mathbb{N}$ and for $\xi \in \Xi$ put $\mathbf{V}^\xi := \mathbf{P}_{g^{(\xi)}}$, $\mathbf{R}_k^\xi := \mathbf{R}_k^{g^{(\xi)}}$; further put $\mathbf{P} := \bigcup_{\xi \in \Xi} \mathbf{V}^\xi$.

We will show that the σ-ideal $\mathcal{I}^*$ of all σ-$\mathbf{P}$-porous sets is good and then we will observe that $\mathcal{I} = \mathcal{I}^*$.

Condition **(G1)** from Definition 2.10 is clearly satisfied and **(G2)** holds by Lemma 5.1(a). To prove **(G3)**, let $\varepsilon > 0$ and $\xi_1 \in \Xi = \mathbb{N}$. Put $\xi_2 := \xi_1 + 1$ and choose $n \in \mathbb{N}$ for which $2/n < \varepsilon$. Since $g^{(\xi_1)}(t) < g^{(\xi_2)}(t)$ for $t > 0$, Lemma 5.1(b) immediately implies $\mathbf{C}(\varepsilon, (\mathbf{R}_k^{\xi_1}), (\mathbf{R}_k^{\xi_2}), n)$; consequently, **(G3)** holds. Thus $\mathcal{I}^*$ is a good σ-ideal.

The equality $\mathcal{I} = \mathcal{I}^*$ is an easy consequence of the fact (see [Za$_1$, Proposition 4.1]) that, for each $n \in \mathbb{N}$, a set $M \subset E$ is σ-$\langle g^{(n)} \rangle$-porous if and only if it is σ-$\langle g \rangle$-porous. Indeed, suppose that $A \subset E$ is $\mathbf{P}$-porous. For $\xi \in \Xi$, put $A_\xi := \{x \in A;\ \mathbf{V}^\xi(x, A)\}$. Then clearly $A = \bigcup_{\xi \in \Xi} A_\xi$ and each A_ξ is $\mathbf{V}^\xi$-porous (i.e., $\langle g^{(\xi)} \rangle$-porous). By the above mentioned result of [Za$_1$] each A_ξ is σ-$\langle g \rangle$-porous and therefore $A \in \mathcal{I}$. Thus we have proved $\mathcal{I}^* \subset \mathcal{I}$. Since $\mathcal{I} \subset \mathcal{I}^*$ is obvious, the first part of the proof is complete.

Now suppose that E is an arbitrary locally compact metric space. Then we can choose $x \in \ker_{\mathbf{P}_g}(A)$ and $r > 0$ such that $B(x, r)$ is relatively compact. The set $A^* := A \cap B(x, r)$ is clearly non-σ-$\langle g \rangle$-porous in the compact space $E^* := \overline{B(x, r)}$. By the first part of the proof we can find a compact set $K \subset A^* \subset A$ which is not σ-$\langle g \rangle$-porous in E^*. Since clearly K is not σ-$\langle g \rangle$-porous in E, the proof is complete. ∎

Corollary 5.3. *Let E be a locally compact metric space and $A \subset E$ be an analytic set which is not σ-porous. Then there exists a compact set $K \subset A$ which is not σ-porous.*

Proof. This is a special case of Theorem 5.2 for $g(x) := 3x$. Indeed, in this case $A \subset E$ is σ-porous if and only if it is σ-$\langle g \rangle$-porous; this follows easily from [Za$_3$, Section 4]. ∎

5.3. *The case of symmetrical porosity*

Theorem 5.4. *Let $A \subset \mathbb{R}$ be an analytic set which is not σ-symmetrically porous. Then there exists a compact set $K \subset A$ which is not σ-symmetrically porous.*

Proof. By Theorem 4.5 it is sufficient to prove that the σ-ideal $\mathcal{I}$ of all σ-symmetrically porous subsets of $\mathbb{R}$ is good.

To prove this, define for $\alpha > 1$ and $k \in \mathbb{N}$ a point-set relation $\mathbf{R}_k^\alpha$ on $\mathbb{R}$ by

$$(3)\qquad \mathbf{R}_k^\alpha(x, A) \overset{\text{def}}{\Longleftrightarrow} \exists y \in \mathbb{R}\ \exists r > 0 : 1/(k+2) < |x - y| < 1/k,$$
$$|x - y| < \alpha r,\ ((y - r, y + r) \cup (2x - y - r, 2x - y + r)) \cap A = \emptyset.$$

Put $\Xi = \mathbb{N} \setminus \{1\}$ and

$$\mathbf{V}^\xi := \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty \mathbf{R}_k^\xi$$

for $\xi \in \Xi$. Clearly

$$\mathbf{P}_{\mathrm{sy}} = \bigcup_{\xi \in \Xi} \mathbf{V}^\xi.$$

Each $\mathbf{R}_k^\xi$ clearly satisfies **(A1)** and **(A3)** and it is easy to prove that the relations $\mathbf{R}_k^\xi$ are stable. Indeed, this follows (cf. the proof above that $\mathbf{R}_k^h$ are stable) almost immediately from the obvious fact that $\mathbf{R}_k^\xi(x, A)$ if and only if

$$\exists y \in \mathbb{R}\ \exists r > 0:\ 1/(k+2) < |x-y| < 1/k, |x-y| < \alpha r,$$
$$\operatorname{dist}((y-r, y+r) \cup (2x-y-r, 2x-y+r), A) > 0.$$

Thus conditions **(G1)** and **(G2)** from Definition 2.10 are satisfied. To prove **(G3)**, let $\varepsilon > 0$ and $\xi_1 \in \varXi = \mathbb{N} \setminus \{1\}$. Then $\mathbf{C}(\varepsilon, (\mathbf{R}_k^{\xi_1}), (\mathbf{R}_k^{\xi_2}), n)$ holds for $\xi_2 := \xi_1 + 1$ and any $n \in \mathbb{N}$ such that $n > 3$ and $2/n < \varepsilon$.

To prove this, we can proceed quite similarly to Subsection 5.2. Suppose that $H \subset \mathbb{R}$, $k \in \mathbb{N}$, $k \geq n$ and a compact set $K \subset \mathbb{R}$ with $(\neg \mathbf{R}_k^{\xi_2})(K, H)$ are given. Choose $\delta > 0$ such that $\xi_1 \delta < 1/(k+2)$ and $0 < \eta < \delta/2$ so small that $\xi_1 s < \xi_2(s - 2\eta)$ for each $s \in [\delta, 1/k]$. Then find a finite set $Z \subset H \cap B(K, 2/k)$ such that $H \cap B(K, 2/k) \subset B(Z, \eta)$. Now it is easy to verify that the choice $S := \{B(z, \eta) \cap B(K, 2/k);\ z \in Z\}$ works. Thus $\mathcal{I}$ is a good σ-ideal and the proof is complete. ∎

Remark 5.5. Proceeding similarly, we can easily prove that Question (Q) also has an affirmative answer for several other types of porosity. First let us mention right (or left) porosity on $\mathbb{R}$, which is defined in the obvious way (see [Za_2, p. 316]). (We can also deal with $\langle g \rangle$-right porosity or with several generalizations of right porosity in $\mathbb{R}^n$.) In 5.3, we could work with shell porosity in $\mathbb{R}^n$ (see [V] for the definition) as well, which is a generalization of symmetrical porosity.

Finally, let us mention another application concerning symmetrical porosity, which can be useful. Let $z > 1$, $A \subset \mathbb{R}$, and $x \in \mathbb{R}$. We will say that $\mathbf{P}_{\mathrm{sy}}^z(x, A)$ holds if there exist $c \in (1, z)$ and a sequence $(B(x_n, r_n))$ of open balls in $\mathbb{R}$ such that $\lim x_n = x$, $x \in B(x_n, c r_n)$ and $(B(x_n, r_n) \cup B(x + (x - x_n), r_n)) \cap A = \emptyset$. The system $\mathcal{I}_z$ of all σ-$\mathbf{P}_{\mathrm{sy}}^z$-porous sets is strictly smaller than the system of all σ-symmetrically porous sets. (This follows from [EH_1].) However, $\mathcal{I}_z$ is a good σ-ideal. To prove this, define $\mathbf{R}_k^\alpha$ as in (3) and put $\varXi = \mathbb{Q} \cap (1, z)$. Then clearly $\mathbf{P}_{\mathrm{sy}}^z = \bigcup_{\xi \in \varXi} \mathbf{V}^\xi$, where $\mathbf{V}^\xi = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty \mathbf{R}_k^\xi$. We have proved that all $\mathbf{R}_k^\xi$ satisfy **(A1)**, **(A3)** and are stable; **(G3)** can be proved by a simple modification of the proof of Theorem 5.4. Note that the σ-ideal $\mathcal{I} = \bigcap_{z>1} \mathcal{I}_z$ naturally appears in several papers (e.g., [EH_2] and [Za_6]). Since each $\mathcal{I}_z$ is a good σ-ideal, we easily obtain the following interesting result:

If $A \subset \mathbb{R}$ is analytic and $A \notin \mathcal{I}$, then there exists a compact $K \subset A$ such that $K \notin \mathcal{I}$.

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