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Real C^k Koebe principle

by

Weixiao Shen (Hefei) and **Michael Todd** (Surrey)

Abstract. We prove a C^k version of the real Koebe principle for interval (or circle) maps with non-flat critical points.

1. Introduction. The real Koebe principle, providing estimates of the first derivative of iterates of a smooth interval map, plays a very important role in recent research of one-dimensional dynamics. See [MS]. Considering its complex counterpart, the (*complex*) Koebe distortion theorem, it is natural to look for a C^k, $k \geq 2$, version of this principle. This is the goal of this paper.

More precisely, let f be a C^k endomorphism of the compact interval $I = [0,1]$ (or the circle $\mathbb{R}/\mathbb{Z}$). We assume that f has only non-flat critical points, that is, for each critical point c, there exists $\alpha > 1$ such that near c,

$$f = \psi Q \phi, \tag{1}$$

where ϕ (resp. ψ) is a C^k diffeomorphism from a neighbourhood of c (resp. $f(c)$) onto a neighbourhood of 0, and $|Q(x)| = |x|^\alpha$. We use NF^k to denote the class of such maps.

As usual, we say that an interval T is a *κ-scaled neighbourhood* of an interval J if J is compactly contained in T, and both components of $T \setminus J$ have length at least $\kappa|J|$.

Theorem 1. *Let f be in the class NF^n, $n \geq 2$. Let T be an interval such that $f^s : T \to f^s(T)$ is a diffeomorphism. For each $S, \kappa > 0$ and each $1 \leq k \leq n$ there exist $\delta = \delta(S, \kappa, f) > 0$ and $K_k = K_k(\kappa) > 0$ satisfying the following. If $\sum_{j=0}^{s-1} |f^j(T)| \leq S$ and J is a subinterval of T such that*

- *$f^s(T)$ is a κ-scaled neighbourhood of $f^s(J)$;*
- *$|f^j(J)| < \delta$ for $0 \leq j < s$,*

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then, letting $\psi_0 : J \to I$ *and* $\psi_s : f^s(J) \to I$ *be affine diffeomorphisms, for each* $x \in I$, *we have*

$$|D^k(\psi_s f^s \psi_0^{-1})(x)| < K_k.$$

Furthermore, $K_1 \to 1$ *as* $\kappa \to \infty$ *and for each* $k > 1$, $K_k \to 0$ *as* $\kappa \to \infty$.

The well known real Koebe principle claims the existence of K_1. Our proof will show that $K_k(\kappa)$ is of order κ^{-k} when $\kappa \to 0$, and of order $\kappa^{-(k-1)}$ when $\kappa \to \infty$, for each $2 \le k \le n$.

1.1. *Proof of Theorem 1.* To prove this theorem, we shall approximate the map $\psi_s f^s \psi_0^{-1}$ by maps in the Epstein class, and then apply the (complex) Koebe distortion theorem. The main step is to prove the following theorem.

THEOREM 2. *Let* f *be a map in the class* NF^n, $n = 2, 3, \ldots$. *Let* T *be an interval such that* $f^s : T \to f^s(T)$ *is a diffeomorphism. For any* $S, \kappa, \varepsilon > 0$, *there exists* $\delta = \delta(S, \kappa, \varepsilon) > 0$ *satisfying the following. Suppose that* $\sum_{j=0}^{s-1} |f^j(T)| \le S$ *and* J *is a subinterval of* T *such that*

- $f^s(T)$ *is a* κ*-scaled neighbourhood of* $f^s(J)$;
- $|f^j(J)| < \delta$ *for* $0 \le j < s$.

Then, letting $\psi_0 : J \to I$ *and* $\psi_s : f^s(J) \to I$ *be affine diffeomorphisms, there exists a map* $G : I \to I$ *in the Epstein class* $\mathcal{E}_{\kappa/2}$ *such that*

$$\|\psi_s f^s \psi_0^{-1} - G\|_{C^n} < \varepsilon.$$

Here, we say that a diffeomorphism $G : I \to I$ is in the *Epstein class* $\mathcal{E}_\beta$ if G^{-1} extends to a (holomorphic) univalent map from $\mathbb{C}_{(-\beta,1+\beta)} := \mathbb{C} \setminus ((-\infty, -\beta] \cup [1+\beta, \infty))$ into $\mathbb{C}$.

This result, for $n = 2$, appears as part of the proof of the Yoccoz Lemma in [T].

Proof of Theorem 1 assuming Theorem 2. By the complex Koebe distortion theorem, the fact that $G \in \mathcal{E}_{\kappa/2}$ implies that the C^n distance between $G|[0,1]$ and the identity map is bounded by a constant $\varepsilon(\kappa)$, and $\varepsilon(\kappa) \to 0$ as $\kappa \to \infty$. Taking $\varepsilon = \varepsilon(\kappa)$ in Theorem 2, we see that the C^n distance between $\psi_s f^s \psi_0^{-1}|[0,1]$ and the identity map is at most $2\varepsilon(\kappa)$. ∎

Outline of proof of Theorem 2. By rescaling the map $f : f^j(J) \to f^{j+1}(J)$, we obtain a diffeomorphism $f_j : I \to I$. For each j, one can find a map $g_j : I \to I$ in the Epstein class such that the C^n distance between f_j and g_j is of order $o(|f^j(J)|)$. Using the classical real Koebe principle (the C^1 version of Theorem 1), we shall prove that $G = g_{s-1} \cdots g_0$ is in the Epstein class $\mathcal{E}_{\kappa/2}$ (Proposition 6). Finally, using a proposition concerning the composition operator (Proposition 8), we show that $f_{s-1} \cdots f_1$ is C^n close to the map G.

It should be mentioned that similar ideas have appeared in the proofs of Theorem A.6 of [FM] and Lemma 3 of [AMM], but our result applies in more general situations.

REMARK 3. For maps in the class NF^3, the C^1 version of Theorem 1 still holds if we replace the assumption $\sum_{j=0}^{s-1}|f^j(T)| \le S$ by "$f^s(T)$ is contained in a small neighbourhood of critical points which are not in the basin of periodic attractors". See [K, SV]. It would be interesting to know if the C^k versions of Theorems 1 and 2 remain true under this alternative assumption. See also the recent work [KS].

REMARK 4. In fact, the whole argument applies to more general maps. It is sufficient to assume that the function Q appearing in (1) is in the Epstein class on each side of 0.

2. Proof of Theorem 2. By means of a C^n coordinate change, we may assume that for each critical point c_i, there is a neighbourhood U_i of c_i such that $|f(x) - f(c)| = |x - c_i|^{\alpha_i}$ for $x \in U_i$. Let us also fix an open interval $U_i' \ni c_i$ such that $\overline{U_i'} \subset U_i$. Define $U := \bigcup_i U_i$ and $U' := \bigcup_i U_i'$. Let $\eta = d(\partial U, \partial U')$. Then any interval of length less than η is either contained in U or disjoint from U'.

We fix T, J, κ, S as in Theorem 2. Let $J_0 = J$ and $J_i = f^i(J)$. For every $0 \le i < s$ we have a diffeomorphism $f^{s-i} : f^i(T) \to f^s(T)$, where $f^s(T)$ is a κ-scaled neighbourhood of $f^s(J)$.

We will rescale our maps as follows. Let $\psi_i : J_i \to I$ be the affine homeomorphisms such that each $f_i = \psi_{i+1} f \psi_i^{-1}$ is increasing. Then the following diagram commutes:

$$\begin{array}{ccccccccc}
J_0 & \xrightarrow{f} & J_1 & \xrightarrow{f} & \cdots & \xrightarrow{f} & J_{s-1} & \xrightarrow{f} & J_s \\
{\scriptstyle\psi_0}\big\downarrow & & \big\downarrow{\scriptstyle\psi_1} & & \big\downarrow & & \big\downarrow{\scriptstyle\psi_{s-1}} & & \big\downarrow{\scriptstyle\psi_s} \\
[0,1] & \xrightarrow{f_0} & [0,1] & \xrightarrow{f_1} & \cdots & \xrightarrow{f_{s-2}} & [0,1] & \xrightarrow{f_{s-1}} & [0,1]
\end{array}$$

We then approximate f_i as follows. For $0 \le i \le s-1$, let

$$\xi_i = \int_0^1 D^2 f_i(t)\,dt, \qquad g_i(x) = \begin{cases} f_i(x) & \text{if } J_i \subset U, \\ (1 - \xi_i/2)x + (\xi_i/2)x^2 & \text{otherwise.} \end{cases}$$

We use $C^n(I)$ to denote the Banach space of C^n maps $\phi : I \to \mathbb{R}$ with the C^n-norm

$$\|h\|_n = \max\{|D^k\phi(x)| : 0 \le k \le n, x \in I\}.$$

Let $C^n(I; I)$ denote the closed subset of $C^n(I)$ consisting of all maps such that $\phi(I) \subset I$. Let $\mathrm{Diff}_+^n(I)$ denote the set of all orientation-preserving C^n automorphisms of I.

Lemma 5. *There exists a continuous increasing function* $w : (0,\infty) \to (0,\infty)$ (*depending on* f) *such that* $\lim_{t\to 0+} w(t) = 0$ *and such that for all* $0 \le i \le s-1$,

$$\|g_i - f_i\|_n \le w(|J_i|)|J_i|.$$

Proof. Assume J_i is not in U, otherwise $g_i = f_i$. We will first estimate $|D^2 g_i(x) - D^2 f_i(x)|$ for $x \in [0,1]$. Observe that

$$D^2 g_i(x) = \xi_i = \int_0^1 D^2 f_i(t)\,dt, \qquad D^2 f_i(x) = \frac{|J_i|^2}{|J_{i+1}|}\, D^2 f(\psi_i^{-1}(x)).$$

There is some $x_0 \in [0,1]$ such that $\int_0^1 D^2 f_i(t)\,dt = D^2 f_i(x_0)$, so $D^2 g_i(x) = D^2 f_i(x_0)$ and

$$\begin{aligned}|D^2 g_i(x) - D^2 f_i(x)| &= |D^2 f_i(x_0) - D^2 f_i(x)| \\ &= \frac{|J_i|^2}{|J_{i+1}|}\, |D^2 f(\psi_i^{-1}(x_0)) - D^2 f(\psi_i^{-1}(x))| \\ &\le \frac{|J_i|^2}{|J_{i+1}|}\, w_1(|J_i|) \le C|J_i| w_1(|J_i|),\end{aligned}$$

where $w_1(\varepsilon) = \sup_{|x-y|<\varepsilon} |D^2 f(x) - D^2 f(y)|$ is the modulus of continuity of $D^2 f$, and $C = \sup_{x\notin U'} |Df(x)|^{-1}$.

Note that there exists some $x_1 \in [0,1]$ such that $Df_i(x_1) = Dg_i(x_1)$. So for $x \in [0,1]$,

$$|Dg_i(x) - Df_i(x)| \le \int_{x_1}^{x} |D^2 g_i(t) - D^2 f_i(t)|\,dt \le C|J_i| w_1(|J_i|).$$

Similarly,

$$|g_i(x) - f_i(x)| \le \int_0^x |Dg_i(t) - Df_i(t)|\,dt \le C|J_i| w_1(|J_i|).$$

For any $2 < k \le n$, $D^k g_i = 0$. Hence, for $x \in I$,

$$|D^k(g_i - f_i)(x)| = |D^k f_i(x)| = \frac{|J_i|^k}{|J_{i+1}|}\, |D^k f(\psi_i^{-1}(x))| \le C|J_i|^{k-1}.$$

Setting $w(t) = C\max(w_1(t), t)$ completes the proof. ∎

The map $g_{s-1}\cdots g_0$ is our candidate for G. Let us first apply the classical real Koebe principle to prove that G is in the Epstein class.

Proposition 6. *Assume that* $\sup_{j=0}^{s-1} |f^j(J)|$ *is sufficiently small. Then for each* $0 \le j \le s-1$, $g_{s-1}\cdots g_j$ *belongs to the Epstein class* $\mathcal{E}_\beta$, *where* $\beta = \kappa/2$.

Proof. Let $1/2 < \lambda_1 < \lambda_2 < 1$ be arbitrarily chosen constants. Let T' be the open interval with $J \subset T' \subset T$ such that both components of

$f^s(T')\setminus f^s(J)$ have length $\kappa\lambda_2|f^s(J)|$. Let $\widehat{T}'_j=\psi_j(f^s(T'))$ for all $0\le j\le s$. Clearly f_j extends to a diffeomorphism from $\widehat{T}'_j$ onto $\widehat{T}'_{j+1}$. By the classical real Koebe principle, for all $x,y\in T'$, we have $|Df^s(x)|/|Df^s(y)|\le C$, where $C=C(S,\kappa)>1$ is a constant. Therefore, for each $0\le j\le s-1$, $f_{s-1}\cdots f_j$ is a well defined diffeomorphism from $\widehat{T}'_j$ onto $\widehat{T}'_s$ with derivative between $1/C$ and C. Clearly, for $\gamma=\lambda_2\kappa C$, we have $\widehat{T}'_j\subset[-\gamma,1+\gamma]$ for all j.

Note that for each $0\le j\le s-1$, g_j^{-1} extends to a univalent map from $\mathbb{C}_{\widehat{T}'_{j+1}}$ into $\mathbb{C}_{\widehat{T}'_j}$. Moreover, for a given γ, arguing as in the previous lemma, we see that for all $0\le j\le s-1$,

$$\sup_{y\in\widehat{T}'_j}|f_j(y)-g_j(y)|=o(|J_j|).$$

Claim. *There exists $\delta>0$ such that if $\sup_{j=0}^{s-1}|f^j(J)|<\delta$ then for any $x\in\widehat{T}'_0$ and any $0\le r\le s-1$, if $g_j\cdots g_0(x)\in\widehat{T}'_{j+1}$ for all $0\le j\le r-1$, then*

$$|f_{r-1}\cdots f_0(x)-g_{r-1}\cdots g_0(x)|<\min\biggl(\frac{(\lambda_2-\lambda_1)\kappa}{C},\Bigl(\lambda_1-\frac{1}{2}\Bigr)\kappa\biggr).$$

To prove this claim, let $A_r=B_{-1}=\mathrm{id}$ and for all $0\le i\le r-1$ let $A_i=f_{r-1}\cdots f_i$ and $B_i=g_i\cdots g_0$. Then

$$\begin{aligned}
|f_{r-1}\cdots f_0(x)-g_{r-1}\cdots g_0(x)|&=|A_0B_{-1}(x)-A_rB_{r-1}(x)|\\
&\le\sum_{i=0}^{r-1}|A_iB_{i-1}(x)-A_{i+1}B_i(x)|=\sum_{i=0}^{r-1}|A_{i+1}f_iB_{i-1}(x)-A_{i+1}g_iB_{i-1}(x)|\\
&\le\sum_{i=0}^{r-1}\sup_{z\in\widehat{T}'_{i+1}}|A_{i+1}(z)|\sup_{y\in\widehat{T}'_i}|f_i(y)-g_i(y)|\le C\sum_{i=0}^{r-1}o(1)|J_i|,
\end{aligned}$$

which is arbitrarily small provided that $\sup_{j=0}^{s-1}|f^j(J)|$ is small enough. This proves the claim.

Now let $\widehat{T}''_0$ be the subinterval of $\widehat{T}'_0$ such that

$$f_{s-1}\cdots f_0(\widehat{T}''_0)=[-\lambda_1\kappa,1+\lambda_1\kappa].$$

Then for any $x\in\widehat{T}''_0$ and $0\le r\le s-1$ we have

$$d(f_{r-1}\cdots f_0(x),\partial\widehat{T}'_r)\ge\kappa(\lambda_2-\lambda_1)/C.$$

Together with the claim, this implies (by induction on r) that for all $0\le r\le s-1$, $g_{r-1}\cdots g_0$ is well defined on $\widehat{T}''_0$ and maps $\widehat{T}''_0$ diffeomorphically onto a subinterval of $\widehat{T}'_r$. Moreover, the claim also gives us $G(\widehat{T}''_0)\supset[-\beta,1+\beta]$ for $\beta=\kappa/2$. This proves that for any $0\le j\le s-1$, $g_j^{-1}\cdots g_{s-1}^{-1}$ extends to a univalent map from $\mathbb{C}_{(-\beta,1+\beta)}$, so $g_{s-1}\cdots g_j$ is in the Epstein class $\mathcal{E}_\beta$. ∎

Together with the complex Koebe distortion theorem, this implies the following.

Corollary 7. *There exists a constant* $C = C(\kappa) > 0$ *such that for any* $0 \le j \le s-1$, *we have*

$$\|\log D(g_{s-1}\cdots g_j)\|_n \le C.$$

The proof of Theorem 2 is then completed by the following proposition and lemma.

Proposition 8. *Let* $n \in \mathbb{N}\cup\{0\}$, *and let* $g_j \in \mathrm{Diff}^{n+1}_+(I)$ *and* $f_j \in \mathrm{Diff}^n_+(I)$ *for* $0 \le j \le s-1$. *For any* $C > 1$ *there exists* $E = E(C,n) > 0$ *such that if the following hold*:

(1) *for each* $0 \le j < s$, $\|\log D(g_{s-1}\cdots g_j)\|_n \le C$;
(2) *if* $n \ge 1$, $\|\log Dg_j - \log Df_j\|_{n-1} \le C$ *for all* $0 \le j \le s-1$;
(3) $\sum_{j=0}^{s-1}\|g_j - f_j\|_n \le C$,

then

$$\|g_{s-1}\cdots g_0 - f_{s-1}\cdots f_0\|_n \le E\sum_{j=0}^{s-1}\|f_j - g_j\|_n.$$

The proof of this proposition will be given in the next section.

Lemma 9. *For any* $C > 1$ *and* $k \in \mathbb{N}$, *there exists* $C' = C'(C,k) > 1$ *with the following property. Let* $\phi, \widetilde{\phi}$ *be maps in* $C^k(I)$ *such that* $\|\phi\|_k, \|\widetilde{\phi}\|_k \le C$. *Then*

(1) $\|e^\phi\|_k \le C'$;
(2) $\frac{1}{C'}\|\phi - \widetilde{\phi}\|_k \le \|e^\phi - e^{\widetilde{\phi}}\|_k \le C'\|\phi - \widetilde{\phi}\|_k$.

Proof. Let $\psi = e^\phi$ and $\widetilde{\psi} = e^{\widetilde{\phi}}$. By induction it is easy to compute that for all $k \ge 1$, there exist polynomials P_k and Q_k such that

- $D^k(e^\phi) = e^\phi \cdot P_k(\phi, D\phi, \ldots, D^k\phi)$;
- $D^k(\phi) = Q_k(\psi, D\psi, \ldots, D^k\psi)/\psi^k$.

From these the lemma follows easily. ∎

Proof of Theorem 2 assuming Proposition 8. It suffices to check that the conditions in Proposition 8 are satisfied. The first condition was verified in Corollary 7. By Lemma 5, $\|f_j - g_j\|_n \le |J_j|w(|J_j|)$. Furthermore, from the proof of that lemma, we can show that $\|\log Df_j\|_{n-1}, \|\log Dg_j\|_{n-1}$ are bounded above. Hence by Lemma 9, provided that $\sup_{j=0}^{s-1}|f^j(J)|$ is small enough, the second condition is verified. For the third one, we use the assumption $\sum_{j=0}^{s-1}|f^j(J)| \le \sum_{j=0}^{s-1}|f^j(T)| \le S$ and the fact that $w(|J_j|)$ is small when $|J_j|$ is small. ∎

3. Proof of Proposition 8. The goal of this section is to prove Proposition 8. Let us begin with a small lemma.

LEMMA 10. *For any $k \in \mathbb{N} \cup \{0\}$ and $C > 0$ there exists $K = K(C, k)$ with the following property. Let $u, v, B \in C^k(I; I)$, and let $A \in C^{k+1}(I)$. Assume that $\|A\|_{k+1} \le C$ and $\|B\|_k \le C$. Then*

$$\|AuB - AvB\|_k \le K\|u - v\|_k.$$

Proof. This lemma is a straightforward consequence of the chain rule. ∎

Proof of Proposition 8. We first introduce some notation for our calculations. Let $A_s = B_{-1} = \mathrm{id}$ and for $0 \le j \le s-1$, let $A_j = g_{s-1} \cdots g_j$ and $B_j = f_j \cdots f_0$. Then

$$\begin{aligned} g_{s-1} \cdots g_0 - f_{s-1} \cdots f_0 &= A_0 B_{-1} - A_s B_{s-1} \\ &= \sum_{j=0}^{s-1} (A_j B_{j-1} - A_{j+1} B_j) = \sum_{j=0}^{s-1} (A_{j+1} g_j B_{j-1} - A_{j+1} f_j B_{j-1}). \end{aligned}$$

Writing $S_j := A_j B_{j-1} = A_{j+1} g_j B_{j-1} = g_{s-1} \cdots g_j f_{j-1} \cdots f_0$, we have

$$g_{s-1} \cdots g_0 - f_{s-1} \cdots f_0 = \sum_{j=0}^{s-1} (S_j - S_{j+1}).$$

The proof of the proposition will proceed by induction on n. First, by Lemmas 9 and 10, $\|S_j - S_{j+1}\|_0 \le K(C, 0)\|f_j - g_j\|_0$. Thus,

$$\|g_{s-1} \cdots g_0 - f_{s-1} \cdots f_0\|_0 \le \sum_{i=0}^{s-1} \|f_j - g_j\|_0.$$

This proves the lemma for the case $n = 0$.

Now let $m \ge 1$ and assume that the proposition holds for $n = m - 1$. Let us prove it for $n = m$.

First, for each $0 \le r \le s - 1$, applying the induction hypothesis to the mappings f_j, g_j, $0 \le j \le r$, we have

$$\|f_r \cdots f_0 - g_r \cdots g_0\|_{m-1} \le E_1 \sum_{i=0}^{j-1} \|f_i - g_i\|_{m-1}, \tag{2}$$

where E_1 is a constant (depending only on C and m). Also, it is easy to show that the first assumption of the proposition implies $\|\log D(g_r \cdots g_0)\|_n < 2C$. Therefore, by the first part of Lemma 9 we have $\|D(g_r \cdots g_0)\|_n < C'$. Hence,

$$\|g_r \cdots g_0\|_m = \max(1, \|D(g_r \cdots g_0)\|_{m-1}) \le C'.$$

Applying this to (2), we have

$$\|B_r\|_{m-1} \le C_1. \tag{3}$$

To complete the induction it suffices to prove that there exists a constant E_2 such that

$$\|D^m(S_j - S_{j+1})\|_0 \le E_2 \|f_j - g_j\|_m. \tag{4}$$

To this end let us first prove the following.

Claim. *There exists a constant C_2 depending only on C such that for all* $0 \le j \le s-1$, $\|\log DS_j - \log DS_{j+1}\|_{m-1} \le C_2 \|f_j - g_j\|_m$.

In fact, for each $0 \le j \le s-1$, by the chain rule,

$$\begin{aligned}
\log DS_j - \log DS_{j+1} \\
&= [\log(DA_{j+1} g_j B_{j-1}) + \log(Dg_j B_{j-1}) + \log DB_{j-1}] \\
&\quad - [\log(DA_{j+1} f_j B_{j-1}) + \log(Df_j B_{j-1}) + \log DB_{j-1}] \\
&= [\log(DA_{j+1} g_j B_{j-1}) - \log(DA_{j+1} f_j B_{j-1})] \\
&\quad + [\log(Dg_j B_{j-1}) - \log(Df_j B_{j-1})] \\
&=: P_j + Q_j.
\end{aligned}$$

From the assumption $\|\log DA_{j+1}\|_m \le C$ and from (3), by Lemma 10, we obtain

$$\|P_j\|_{m-1} \le K(C_1, m-1) \|f_j - g_j\|_{m-1},$$

and

$$\|Q_j\|_{m-1} \le K(C_1, m-1) \|\log Dg_j - \log Df_j\|_{m-1}.$$

Since $\|\log Dg_j\|_{m-1}$ and $\|\log Df_j\|_{m-1}$ are bounded from above, the second statement of Lemma 9 implies the claim.

Finally, let us deduce (4) from the claim. By the second part of Lemma 9, it suffices to show that $\|\log DS_j\|_{m-1}$ is bounded from above by a constant. Since $\|\log DS_0\|_{m-1} = \|\log DA_0\|_{m-1} \le C$, this follows from the third assumption by applying the claim. This completes the proof. ■

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Mathematics Department
University of Science and Technology of China
Hefei, 230026, China
E-mail: wxshen@ustc.edu

Mathematics Department
University of Surrey
Guildford, Surrey, GU2 7XH, UK
E-mail: m.todd@surrey.ac.uk

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