FUNDAMENTA
MATHEMATICAE
168 (2001)

Chain conditions in maximal models

by

Paul Larson (Toronto) and **Stevo Todorčević** (Paris)

Abstract. We present two $\mathbb{P}_{\max}$ varations which create maximal models relative to certain counterexamples to Martin's Axiom, in hope of separating certain classical statements which fall between MA and Suslin's Hypothesis. One of these models is taken from [19], in which we maximize relative to the existence of a certain type of Suslin tree, and then force with that tree. In the resulting model, all Aronszajn trees are special and Knaster's forcing axiom $\mathcal{K}_3$ fails. Of particular interest is the still open question whether $\mathcal{K}_2$ holds in this model.

1. Introduction. If P is a partial order, then P satisfies the *countable chain condition*, or is c.c.c., if every set of pairwise incompatible elements of P is countable. Similarly, if P is a partition on n-tuples or on finite subsets of some underlying set, we say that P is c.c.c. if the partial order to force an uncountable homogeneous subset of P by finite approximations is c.c.c. It was shown in [32] that $\mathrm{MA}_{\aleph_1}$ is equivalent to the statement that every c.c.c. partition on finite subsets of ω_1 has an uncountable homogeneous subset. In this paper, we continue the project from [32] of analyzing certain classical statements which fall between $\mathrm{MA}_{\aleph_1}$ and Suslin's Hypothesis.

To do this, we study the models derived from two $\mathbb{P}_{\max}$ variations, and the theories they satisfy for c.c.c. partial orders. These variations are called $\mathbb{S}^{\mathrm{T}}_{\max}$ and $\mathbb{P}^{\mathrm{st}}_{\max}$, the first of which first appeared in [19]. These forcings are meant to be applied to models of determinacy, and for notational ease, we will call the $\mathbb{S}^{\mathrm{T}}_{\max}$ and $\mathbb{P}^{\mathrm{st}}_{\max}$ extensions of $L(\mathbb{R})$, in the context of $\mathrm{AD}^{L(\mathbb{R})}$, $\mathcal{M}$ and $\mathcal{N}$ respectively. In $\mathcal{M}$, there is a particular Suslin tree, which we call S_G, and we will call the extension by $\mathcal{M}$ created by forcing with this

2000 *Mathematics Subject Classification*: Primary 03E40; Secondary 03E02, 03E35, 03E50.

The first author was partially supported by the North Atlantic Treaty Organization and the Japan Society for the Promotion of Science. The second author is partially supported by the Serbian Foundation for Science and the Natural Sciences and Engineering Research Council.

tree $\mathcal{M}[H]$. Our interest in $\mathcal{M}[H]$ lies in its relationship with the question from [32] of whether Knaster's forcing axioms $\mathcal{K}_2$ and $\mathcal{K}_3$ (see Section 4) are equivalent. $\mathcal{K}_3$ fails in this model, and several of the known consequences of $\mathcal{K}_2$ hold there. The model $\mathcal{N}$ is intended to maximize the fragment of $\mathrm{MA}_{\aleph_1}$ consistent with the existence of an unfilled tower, in the hope that that fragment will include one of the $\mathcal{K}_n$'s.

We work with $\mathbb{P}_{\max}$ and in the context of $\mathrm{AD}^{L(\mathbb{R})}$ not because the questions we are interested in have large cardinal strength, but rather because the models these variations create are canonical for the types of questions we are interested in. For instance, the theory of $\mathcal{M}[H]$ is the answer to the question of how much of $\mathrm{MA}_{\aleph_1}$ can hold after forcing with a coherent Suslin tree. In the end, the results about c.c.c. partitions which these models yield should translate into ZFC arguments. $\mathbb{P}_{\max}$, then, is being used to streamline our investigation, though the models created here are also of interest.

Sections 2 and 3 give a few introductory remarks about trees and towers. Section 4 sets the stage by giving a summary of the consequences of some of the statements we are considering, including $\mathcal{C}^2$, $\mathcal{K}_2$, $\mathcal{K}_3$ and $\mathcal{K}_4$. Section 5 gives the definitions of our forcings, and in Sections 6–8 we analyze $\mathcal{M}[H]$. In the final section we rule out an optimistic way of showing that $\mathcal{K}_2$ holds in $\mathcal{M}[H]$ by showing that it is possible for a partition on pairs to fill a tower.

The following chart gives a partial summary of results. The horizontal implications are presented in Section 4. Only one of them, that under $\mathcal{K}_3$ every tower of length ω_1 contained in an analytic filter is filled (Theorem 4.11(5)), is new. The facts listed about $\mathcal{M}[H]$ are shown in Sections 6–8.

	True in $\mathcal{M}[H]$	False in $\mathcal{M}[H]$
MA		$\mathfrak{t}_{\mathrm{s}} > \omega_1$
$\mathcal{K}_4$		All sets Q-sets
		Ladders uniformized
$\mathcal{K}_3$	$\mathfrak{t}_{\mathrm{a}} > \omega_1$	$(2^{\omega_1}, <_{\mathrm{lex}}) \hookrightarrow {}^{\omega}\omega/U$
$\mathcal{K}_2$	Aronszajn trees special	
	Locally countable sets special	
$\mathcal{C}^2$	$\mathfrak{b} > \omega_1$	

Here $\mathfrak{t}_{\mathrm{s}}$ and $\mathfrak{t}_{\mathrm{a}}$ are the lengths of the shortest unfilled towers which are slow and contained in an analytic filter respectively. The remaining definitions are given in Sections 2–4.

Under the right formulation, all Π_2 sentences for $H(\omega_2)$ which hold in $\mathcal{M}[H]$ should hold in $\mathcal{N}$, but in general much less is known about $\mathcal{N}$ than about $\mathcal{M}[H]$. The maximality of $\mathcal{M}[H]$ implies that for a Π_2 sentence ϕ for $H(\omega_2)$, if it is possible for ϕ to hold after forcing with a coherent Suslin tree, then ϕ holds in $\mathcal{M}[H]$. Whether the remaining consequences of $\mathcal{K}_2$ and $\mathcal{C}^2$ listed in Section 4 hold in $\mathcal{M}[H]$ are important test questions, though some of them look to be as hard as the main questions themselves, i.e., whether $\mathcal{C}^2$ and $\mathcal{K}_2$ hold in $\mathcal{M}[H]$.

2. Trees

DEFINITION 2.1. A sequence $\langle a_\alpha : \alpha \to \omega \mid \alpha < \omega_1 \rangle$ of functions is *coherent* if $\{\gamma \mid a_\alpha(\gamma) \neq a_\beta(\gamma)\}$ is finite for all $\alpha < \beta < \omega_1$.

DEFINITION 2.2. A *coherent Suslin tree* T is a Suslin tree induced by a coherent sequence $\langle a_\alpha \mid \alpha < \omega_1 \rangle$, in the sense that T is the tree of all functions $b : \beta \to \omega$ for some countable ordinal β such that $\{\gamma < \beta \mid b(\gamma) \neq a_\beta(\gamma)\}$ is finite, ordered by extension.

Coherent trees are strongly homogeneous, in the terminology of [19] and [25], and vice versa [15]. We recall some terminology and simple observations from [19]. In order to make the order on the tree consistent with the order of the corresponding forcing, we adopt the convention that trees grow downwards.

DEFINITION 2.3. For models $M \subset N$, if T is an κ-tree in M, and $A \in N \setminus M$ is a subset of T, then A is a *deep antichain* of T with respect to M if for all antichains $B \subset T$ such that $B \in M$, there is an element of A below some element of B in T.

LEMMA 2.4 (Antichain Lemma). *Suppose κ is a regular cardinal, T is a κ-tree and $\mathbb{Q}$ is a partial order. If forcing with $\mathbb{Q}$ can add a deep antichain A of T with respect to the ground model, then forcing with T can put a κ-antichain through $\mathbb{Q}$.*

DEFINITION 2.5. If S is a coherent tree, and p and q are elements of the same level α of S, then π^S_{pq} is the isomorphism from the trivial subtree of S below p to the trivial subtree of S below q such that for $\beta > \alpha$ and $p' <_S p$ with $\beta \in \operatorname{dom}(p')$, $\pi^S_{pq}(p')(\beta) = p'(\beta)$.

LEMMA 2.6. *If S is a coherent Suslin tree, τ and σ are S-names for paths through S, and $G \subset S$ is generic, then in the extension by G there is some finite $a \subset \omega_1$ such that for all $p \in \sigma_G$ and $q \in \tau_G$ and $\alpha \in \operatorname{dom}(p) \cap \operatorname{dom}(q) \setminus a$, $p(\alpha) = q(\alpha)$.*

The following fact is also useful.

LEMMA 2.7. *If S is a coherent Suslin tree, then after forcing with S every subset of S of size ω_1 contains a set $\langle p_\alpha \mid \alpha < \omega_1 \rangle$ such that there exists a strictly decreasing chain $\langle q_\alpha \mid \alpha < \omega_1 \rangle \subset S$ such that $p_\alpha \leq_S q_\alpha$ for all α.*

Proof. Let τ be an S-name for an uncountable subset of S, and consider the set of pairs $A = \{(r,p) \in S \times S \mid r \Vdash \check{p} \in \tau\}$. Given $r, p \in S$, let $q(r,p)$ be the least node in the tree that is above both p and r. Using the fact that every uncountable subset of a Suslin tree is somewhere dense, we will be done if we can find $s_0, s_1 \in S$ such that $\{q(\pi^S_{s_0 s_1}(r), p) \mid (r,p) \in A\}$ has cardinality ω_1, since then s_0 forces that there will be ω_1 many r's in the generic with the corresponding p's and $q(r,p)$'s as desired.

To find s_0 and s_1, pick a club $C \subset \omega_1$, and a sequence $\langle (r_\alpha, p_\alpha) \in A \mid \alpha \in C\rangle$ such that $\alpha < \mathrm{lev}_S(r_\alpha), \mathrm{lev}_S(p_\alpha) < \inf(C \setminus (\alpha+1))$ for all $\alpha \in C$. Now by a regressive function argument, we may pick a level $\overline{\alpha}$ and a stationary set $D \subset C$ such that $\{\gamma \mid r_\alpha(\gamma) \neq p_\alpha(\gamma)\} \cap (\overline{\alpha}, \alpha) = \emptyset$ for all $\alpha \in D$. Furthermore, we may refine D so that for some pair of nodes s_0, s_1, for all $\alpha \in D$ we have $r_\alpha \leq_S s_0$ and $p_\alpha \leq_S s_1$. Then s_0 and s_1 are as desired. ■

The following construction shows that it is possible that forcing with an Aronszajn tree can destroy a stationary subset of ω_1. This contrasts with (but does not contradict) Proposition 38 of [31] which says no Aronszajn tree can be embedded in the tree of attempts to shoot a club through a stationary, costationary subset of ω_1.

LEMMA 2.8. *If diamond holds then there is a stationary set $S \subset \omega_1$ and an ω_1-tree T such that forcing with T preserves ω_1 but destroys the stationarity of S.*

Proof. Let $S \subset \omega_1$ be stationary, costationary, and let $\langle \sigma_\alpha \mid \alpha \in \overline{S}\rangle$ be a $\diamondsuit(\overline{S})$-sequence. We construct an ω_1-tree T with a subset A of T as follows. For $p \in T$ and $q \leq_T p$, we say that q is a *clean* successor of p if there is no $p' \in A$ with $q \leq_T p' <_T p$. We construct T and A having the following properties.

(i) For α a limit ordinal in $\overline{S}$ and $\beta < \alpha$, if $p \in T$ is on level β of T, then p has a successor in A at level α.

(ii) If $\alpha \in S$, no $p \in T$ on level α is in A.

(iii) If $p \in T$ and cofinally many predecessors of p are in A, then p is in A.

(iv) Every $p \in T$ has at least two clean successors at each lower level.

(v) For $\alpha \in \overline{S}$, if σ_α codes ω-many maximal antichains in $T{\upharpoonright}\alpha$, then each node in $T{\upharpoonright}\alpha$ has a successor on the αth level of T such that the path between them hits each of the maximal antichains coded by σ_α.

Conditions (i)–(iii) imply that the set of levels at which the generic path through T hits a member of A will be a club subset disjoint from S. Condition (iv) serves as an induction hypothesis to help us build the tree. Condition (v) ensures that forcing with T will preserve ω_1.

Subject to these constraints, we build as follows. We let level 0 have one node, not in A, and at successor levels we build ω successors of each node, none in A. For limit λ, we build depending on whether λ is in S or $\overline{S}$.

If $\lambda \in S$, then using a cofinal sequence in λ of ordertype ω, we apply condition (iv) below to choose for each $p \in T{\restriction}\lambda$ two paths above p disjoint from A and put a node (not in A) on the λth level above each of these paths. The λth level of T will consist of only these choices.

For $\lambda \in \overline{S}$, we first pick two clean successors for each node as in the previous case. Then for each $p \in T$, we find a path above p meeting each maximal antichain coded by σ_λ (if σ_λ does indeed code an ω-sequence of maximal antichains in $T{\restriction}\lambda$) and different from all the clean paths, and put a node in A on the λth level above that path.

Given these steps, the construction is completed as desired. ■

One interesting consequence of the above lemma is that shooting a continuous increasing ω_1-sequence through a projective stationary set (see [10]) can be rewritten as a Levy collapse followed by forcing with an Aronszajn tree.

3. Towers. Following [7], we let Ω denote the set of limit ordinals less than ω_1. The following definition also comes from [7].

DEFINITION 3.1. A sequence $\langle e_\delta : \omega \to \delta \mid \delta \in \Omega \rangle$ of functions is a *ladder system* on ω_1 if each e_δ is strictly increasing with range cofinal in δ.

We use $a \subset^* b$ to mean that all but finitely many members of a are members of b.

DEFINITION 3.2. For any ordinal γ, a sequence $\langle t_\alpha \subset \omega \mid \alpha < \gamma \rangle$ is a *tower* if $t_\beta \subset^* t_\alpha$ for all $\alpha < \beta < \gamma$. A tower is *filled* if there exists an infinite $x \subset \omega$ such that $x \subset^* t_\alpha$ for all $\alpha < \gamma$, otherwise it is *unfilled*.

Following convention, we let $\mathfrak{t}$ denote the length of the shortest unfilled tower and $\mathfrak{c}$ denote the cardinality of the continuum.

DEFINITION 3.3. Let f be any function from ω to ω such that the preimage of every integer is infinite. Let $z \subset \omega$ be infinite, let $E = \langle e_\alpha \mid \alpha < \omega_1 \rangle$ be a ladder system on ω_1, and let A be a function from ω_1 to ω. Then $\mathrm{ST}(z, f, A, E)$ is the tower $\langle t_\alpha \mid \alpha < \omega_1 \rangle$ constructed in the following way.

- $t_0 = z$.
- At successor stages, $n \in t_{\alpha+1}$ if and only if for some $m \in \omega$, n is the mth member of t_α and $f(m) = A(\alpha)$.
- $t_\alpha = \{i \in \omega \mid \exists n \in \omega$ such that i is the nth member of $\bigcap\{t_{e_\alpha(m)} \mid m < n\}\}$ for limit α.

A tower is called *slow* if it is of the form $\mathrm{ST}(z, f, A, E)$ for suitable z, f, A, and E.

The idea behind this definition is the following.

LEMMA 3.4. *Suppose that T is an Aronszajn tree such that forcing with T preserves ω_1. Let $z \subset \omega$ be infinite, $f : \omega \to \omega$ be such that all preimages are infinite, and let E be a ladder system on ω_1. Fix some wellordering of the successors of each node of T in ordertype ω. Then after forcing a generic path H through T, if we let $A : \omega_1 \to \omega$ be such that for all α the member of H on the $(\alpha+1)$st level of T is the $A(\alpha)$th successor of the member of H on the αth level in the fixed wellordering, then in the extension by H, $\mathrm{ST}(z, f, A, E)$ is unfilled.*

Proof. Note that by the definition of slow towers, there is a function $g : T \to \mathcal{P}(\omega)$ such that in the extension by H, the αth member of $\mathrm{ST}(z, f, A, E)$ will be the value of g at the member of H on the αth level of T. Note that incompatible nodes in T define initial segments of $\mathrm{ST}(z, f, A, E)$ with disjoint elements, which means that any real filling $\mathrm{ST}(z, f, A, E)$ would define a path through T. Since forcing with T does not add reals, $\mathrm{ST}(z, f, A, E)$ is unfilled in the extension. ∎

We have not resolved whether there is always a slow unfilled tower of length $\mathfrak{t}$. Intuitively, slow towers should be more likely to be filled, and so conceivably one could have all slow towers filled and still $\mathfrak{t} = \omega_1$.

For $x \subset \omega$, the *lower asymptotic density* of x is $\liminf_{n\to\omega} |x \cap n|/n$. Upper asymptotic density is defined similarly, with limsup in place of liminf, and asymptotic density is the common value when the two agree. Under CH, it is possible to construct unfilled towers of length ω_1 consisting of reals of asymptotic density 1. We will see, however, that there are no such towers in $\mathcal{M}[H]$. We also note that the same remarks apply with any Borel filter instead of the set of reals of asymptotic density 1.

The following concept comes from [14], though the terminology is ours.

DEFINITION 3.5. A *Q-sequence* is a sequence $\langle x_\alpha \subset \omega \mid \alpha < \omega_1 \rangle$ of pairwise almost disjoint sets such that for all $A \subset \omega_1$ there is a $y \subset \omega$ such that for all $\alpha < \omega_1$, $\alpha \in A \Leftrightarrow y \cap x_\alpha$ is infinite.

It is a standard fact that there is a Q-sequence if and only if there is a Q-set, i.e., an uncountable set of reals all of whose subsets are relatively G_δ. A straightforward consequence of the existence of Q-sets is that $2^\omega = 2^{\omega_1}$.

4. Chain conditions. We let $\mathcal{C}^2$ denote the statement that the product of c.c.c. partial orders is c.c.c. The question of whether the countable chain condition is a productive property was asked long ago by Marczewski [23], who soon afterwards showed that the stronger chain condition $\mathcal{K}_2$ of

Knaster (see below) is productive. The fundamental importance of $\mathcal{C}^2$ was first realized ten years later by Kurepa [17] when he made the connection of this statement with Suslin's Hypothesis.

DEFINITION 4.1. $\mathfrak{b}$ is the least cardinal κ such that there exists $\langle f_\alpha \mid \alpha < \kappa\rangle \subset {}^\omega\omega$ such that no such function dominates each f_α mod finite.

THEOREM 4.2. *$\mathcal{C}^2$ implies the following statements.*

(i) ([17]) *Suslin's Hypothesis.*
(ii) ([16]) *Every Hausdorff gap in* $\mathcal{P}(\omega)/\mathrm{Fin}$ *is indestructible.*
(iii) ([29]) $\mathfrak{b} > \omega_1$.
(iv) ([27]) *The continuum has cofinality greater than* ω_1.
(v) ([11]) *The Lebesgue measure cannot be extended to a countably additive measure defined on all sets of reals.*

A natural counterexample to $\mathcal{C}^2$ is obtained using a so-called entangled set of reals (see [27], Theorem 6). Such a set can be obtained either assuming that the continuum has cofinality ω_1 (see [27], Theorem 1) or assuming that the Lebesgue measure extends to all sets of reals (see [11], 7F). This is how parts (iv) and (v) of Theorem 4.2 were proved. That Suslin's Hypothesis does not imply statements (iii)–(iv) of Theorem 4.2 follows from results of Jensen [6] and Laver [22].

A partial order P is *powerfully c.c.c.* if for every integer n, the partial order P^n is c.c.c., and *productively c.c.c.* if for every c.c.c. Q, $P \times Q$ is c.c.c. One approach to questions about $\mathcal{C}^2$ is to consider partial orders which are productively or powerfully c.c.c. for particular reasons.

DEFINITION 4.3. A partial order P on ω_1 is *split* if there exist $\langle a_\gamma \mid \gamma < \omega_1\rangle$, $\langle b_\gamma \mid \gamma < \omega_1\rangle$ contained in P such that for all finite $A \subset \omega_1$ there are uncountably many γ such that for all $\overline{\gamma} \in A$, $a_{\overline{\gamma}}$ and b_γ are incompatible. Otherwise it is *unsplit.*

The idea for the definition above is that we would like some property such that the product of any two c.c.c. partial orders with the property is c.c.c. Being unsplit is such a property, but perhaps not maximal in this regard.

As in [19], we use the following standard ultrafilter lemma, which appeared in [2].

LEMMA 4.4. *Let U be a uniform ultrafilter on ω_1. Let P be a set with $K \subset P \times P$, and let $\langle p_0^\alpha, \ldots, p_n^\alpha \mid \alpha < \omega_1\rangle$ for some fixed integer n be such that for U-many $\alpha < \omega_1$,*

$$\{\beta < \omega_1 \mid \exists i, j \le n \ \ (p_i^\alpha, p_j^\beta) \in K\} \in U.$$

There there is a set $A \in U$ and a pair $\bar{i}, \bar{j}$ such that $\{\beta \mid (p^\alpha_{\bar{i}}, p^\beta_{\bar{j}}) \in K\} \in U$ for all $\alpha \in A$.

A straightforward application of Lemma 4.4 gives us the following lemmas. They have essentially the same proof, with the second being slightly more general.

LEMMA 4.5. *A finite product of unsplit partial orders on ω_1 is unsplit.*

LEMMA 4.6. *If $P \subset [\omega_1]^2$ is a partition on pairs from ω_2 which is split, then there is a sequence $\langle(\alpha_\gamma, \beta_\gamma) \mid \gamma < \omega_1\rangle$ of pairs of ordinals such that for all finite $a \subset \omega_1$ there are uncountably many γ such that $\{\alpha_{\overline{\gamma}}, \beta_\gamma\} \notin P$ for all $\overline{\gamma} \in a$.*

Proof. If P is split, then for some integers n, m, for each $\gamma < \omega_1$ there are finite subsets $a_\gamma = \{\alpha^i_\gamma \mid i < n\}$, $b_\gamma = \{\beta^j_\gamma \mid j < m\}$ of ω_1 such that for all $A \in [\omega_1]^{<\omega}$ the set

$$X_A = \{\gamma \mid \forall \eta \in A\ \exists i < n,\ j < m\ \ \{\alpha^i_\eta, \beta^j_\gamma\} \notin P\}$$

is uncountable. Let U be an ultrafilter on ω_1 such that each such X_A is in U. Now Lemma 4.4 finishes the proof. ■

A typical example of an unsplit poset is the poset of all finite antichains of an Aronszajn tree, a fact that was first (implicitly) established in [2].

The idea behind the following definitions comes from [30].

DEFINITION 4.7. Given a partial order P, let $N(P) \subset [[P]^{<\omega}]^2$ be the partition on pairs of finite subsets of P such that $\{a, b\} \in N(P)$ if $a \cap b \neq \emptyset$ or if every member of a is compatible with every member of b.

DEFINITION 4.8. We say that a partial order P has *property N* if for all disjoint sequences $\langle a_\alpha \mid \alpha < \omega_1\rangle$ of finite subsets of P, there exist $\alpha < \beta < \omega_1$ such that every element of a_α is compatible with every element of a_β.

A partial order P having property N means that there is no uncountable 1-homogeneous set for $N(P)$.

To see an example of a property N poset, consider subsets of $\mathcal{P}(\omega)$ as partially ordered sets ordered by inclusion. If X is such a subset of $\mathcal{P}(\omega)$ then the natural forcing notion of all finite antichains of X has property N whenever X is *locally countable*, i.e., whenever

$$X(< y) = \{x \in X \mid x \subset y\}$$

is countable for all $y \in X$. This fact was essentially proved in [8], in the course of proving that under $\mathrm{MA}_{\aleph_1}$ every well founded subposet of $\mathcal{P}(\omega)$ contains an uncountable antichain.

Unsplit partial orders have property N, and partial orders with property N are powerfully c.c.c. Further, for any finite collection $P_0, \ldots, P_n$ of unsplit partial orders, there is no 1-homogeneous set for the product of their respective $N(P_i)$'s. The proof of Theorem 4.2(iii), however, shows that $\mathfrak{b} = \omega_1$ implies that there are two partial orders with property N whose product is not c.c.c. Note that the poset of all finite antichains of a locally countable subset of $\mathcal{P}(\omega)$ is productively c.c.c.

A family of weakenings of $\mathrm{MA}_{\aleph_1}$ was suggested by Knaster in the Scottish Book [23] as follows.

DEFINITION 4.9. Let $n \in \omega$. A subset X of partially ordered set P is *n-linked* if

$$\forall F \in [X]^n \; \exists p \in P \; \forall q \in F \;\; p \le q.$$

A poset P has *property K_n* if $\forall X \in [P]^{\omega_1} \; \exists Y \in [X]^{\omega_1} \;\; Y$ is n-linked. $\mathcal{K}_n$ is the statement that every c.c.c. poset has property K_n.

The following are some of the sharpest results for consequences of the $\mathcal{K}_n$'s.

THEOREM 4.10 ([30]). *$\mathcal{K}_2$ implies the following.*

(i) *Every property N poset of size $\aleph_1$ is σ-linked.*
(ii) *All Aronszajn trees are special.*
(iii) *Every locally countable subset of $\mathcal{P}(\omega)$ of size $\aleph_1$ is special.*

THEOREM 4.11. *$\mathcal{K}_3$ implies the following.*

(i) ([32]) *Every c.c.c. poset of size $\aleph_1$ is σ-linked.*
(ii) ([29]) $(2^{\omega_1}, <_{\mathrm{lex}})$ *is embeddable in every nontrivial ultrapower ω^ω/U.*
(iii) ([28]) *Every uncountable subset of ω^ω contains an uncountable* 2-*splitting subset.*
(iv) ([24]) *The Lebesgue measure and Baire category are $\aleph_1$-additive.*
(v) *Every tower of length ω_1 that can be embedded inside an analytic filter is filled.*

Proof. We give a proof for the fifth statement, which is the only new fact on the list. Let $\langle t_\alpha \mid \alpha < \omega_1\rangle$ be a tower contained in an analytic filter G, where G is the range of a continuous function f with domain ω^ω. For each $\alpha < \omega_1$, fix $s_\alpha \in f^{-1}(t_\alpha)$. By Theorem 4.2(iii), $\mathfrak{b} > \omega_1$, and so there is a function s which bounds each of the s_α mod finite. For each finite modification s' of s, the image under f of the functions totally bounded by s' is a compact set, and so the tower is contained in an F_σ filter F.

From [18], then, we have an increasing sequence of natural numbers $\langle n_k : k \in \omega\rangle$ and sets $\{a_i^k \subset [n_k, n_{k+1}) \mid k \in \omega,\; i < m_k\}$ such that

- $\forall x \in [m_k]^{\le 2^k} \;\; \bigcap_{i \in x} a_i^k \ne \emptyset$,
- $\forall X \in F \;\; \{k \in \omega \mid \forall i < m_k \;\; a_i^k \not\subset X\}$ is finite.

Now fix an enumeration $\langle a_n \mid n \in \omega\rangle$ of the finite subsets of ω, and for each $\alpha < \omega_1$ let $f_\alpha : \omega \to \omega$ be such that $(t_\alpha\restriction n_k = a_m) \Rightarrow f_\alpha(k) = m$. Apply (iii) to the f_α's, getting an uncountable 2-splitting set $A \subset \omega_1$. Then $B_k = |\{f_\alpha\restriction n_k \mid \alpha \in A\}| \le 2^k$ for each $k \in \omega$. For each $a \in B_k$, choose an $a_i^k \subset a$ if one exists, and define our filling real $t \subset \omega$ on the interval $[n_k, n_{k+1})$ by taking the intersection of the chosen a_i^k's. Then t will have at least one element in each interval $[g(n), g(n+1))$, and will be contained mod finite in each t_α. ∎

Theorem 4.12 ([32]). *$\mathcal{K}_4$ implies that every ladder system can be uniformized, and that every uncountable set of reals is a Q-set.*

The material in this paper is motivated by the following conjecture.

Conjecture 4.13 ([32]). *$\mathcal{K}_2$ does not imply $\mathcal{K}_3$.*

One key question we have yet to resolve is whether $\mathcal{C}^2$ holds in $\mathcal{N}$. It is possible that a $\mathbb{P}_{\max}$ variation could help resolve the following test question for whether $\mathcal{C}^2$ implies $\mathcal{K}_2$.

Question 4.14. *Does $\mathcal{C}^2$ imply that every Aronszajn tree is special?*

Definition 4.15. If $K \subset [\omega_1]^{<\omega}$, then let $P_K = \{b \in [\omega_1]^{<\omega} \mid \forall a \subset b\ a \in K\}$, ordered by inclusion.

Similarly, for $n \in \omega$, if $K \subset [\omega_1]^n$, then let $P_K = \{b \in [\omega_1]^{<\omega} \mid \forall a \in [b]^n\ a \in K\}$, ordered by inclusion.

In each case, we say that K is c.c.c. if the partial order P_K is c.c.c.; co-c.c.c. if $\overline{K}$ is c.c.c.; and powerfully c.c.c. if for all $n \in \omega$ the partial order $(P_K)^n$ is c.c.c.

For $n < \omega$, we denote by $\omega_1 \xrightarrow{\text{ccc}} (\omega_1, \omega_1)^n$ the statement that every c.c.c. partition on n-tuples from ω_1 into two colors has an uncountable homogeneous subset, and by $\omega_1 \xrightarrow{\text{ccc}} (\omega_1, \omega_1)^{<\omega}$ that every c.c.c. partition on finite subsets of ω has an uncountable homogeneous subset. $\omega_1 \xrightarrow[\text{ccc}]{\text{ccc}} (\omega_1, \omega_1)^n$ means simply that there are no c.c.c., co-c.c.c. partitions of n-tuples on ω_1.

A proof of the following statement for $n = 2$ appears in [30]. We include a proof of the general case for completeness.

Lemma 4.16 ([30]). *$\omega_1 \xrightarrow{\text{ccc}} (\omega_1, \omega_1)^n$ implies that every powerfully c.c.c. poset has property K_n.*

Proof. Let P be a powerfully c.c.c. poset, and let $\langle p_\alpha \mid \alpha < \omega_1\rangle \subset P$. Define $[\omega_1]^n = K_0 \cup K_1$ by letting $\{\alpha_0, \ldots, \alpha_{n-1}\} \in K_0$ if and only if there is a lower bound for $\{p_{\alpha_0}, \ldots, p_{\alpha_{n-1}}\}$ in P. Let F be a disjoint family of finite homogeneous sets all of the same fixed size m. To each $f \in F$ associate the element of P^k, where $k = m + \binom{m}{n}$, which first lists p_α $(\alpha \in f)$ in increasing order (call them $p_{\alpha_0}, \ldots, p_{\alpha_{m-1}}$) and which in the $\{i_0 \ldots i_{n-1}\}$th place has a lower bound of $\{p_{\alpha_{i_0}}, \ldots, p_{\alpha_{i_{n-1}}}\}$ in P. Since P^k is c.c.c. we can find f and g in F for which the corresponding elements of P^k are compatible, in which case $f \cup g$ is 0-homogeneous. Thus $[\omega_1]^n = K_0 \cup K_1$ is a c.c.c. partition. Let H be an uncountable 0-homogeneous set. Then $\langle p_\alpha \mid \alpha \in H\rangle$ is such that every n-element subset has a lower bound. ∎

We note that all the consequences of the $\mathcal{K}_n$'s mentioned in this section are actually consequences of the corresponding c.c.c. partition relations. Also, it is not known whether MA for powerfully c.c.c. partial orders implies all of MA.

The following table, in which the asterisks denote the corresponding statements restricted to powerfully c.c.c. partial orders, gives the relationships among certain weakenings of MA. Each statement implies those directly below or directly to the right of it, except for the middle two on the last row, where nothing is known. We note that aside from the first two statements in the first row, which were shown in [32] to be equivalent, none of the reverse implications are resolved, and that all of the statements below except the last two in the last row (for which it is open) are known to imply Suslin's Hypothesis.

$\mathrm{MA}_{\aleph_1}$	$\omega_1 \xrightarrow{\mathrm{ccc}} (\omega_1,\omega_1)^{<\omega}$	$\mathrm{MA}^*_{\aleph_1}$	$\omega_1 \xrightarrow{\mathrm{ccc}^*} (\omega_1,\omega_1)^{<\omega}$
.	.	.	.
.	.	.	.
.	.	.	.
$\mathcal{K}_4$	$\omega_1 \xrightarrow{\mathrm{ccc}} (\omega_1,\omega_1)^4$	$\mathcal{K}_4^*$	$\omega_1 \xrightarrow{\mathrm{ccc}^*} (\omega_1,\omega_1)^4$
$\mathcal{K}_3$	$\omega_1 \xrightarrow{\mathrm{ccc}} (\omega_1,\omega_1)^3$	$\mathcal{K}_3^*$	$\omega_1 \xrightarrow{\mathrm{ccc}^*} (\omega_1,\omega_1)^3$
$\mathcal{K}_2$	$\omega_1 \xrightarrow{\mathrm{ccc}} (\omega_1,\omega_1)^2$	$\mathcal{K}_2^*$	$\omega_1 \xrightarrow{\mathrm{ccc}^*} (\omega_1,\omega_1)^2$
$\mathcal{C}^2$	$\omega_1 \xrightarrow[\mathrm{ccc}]{\mathrm{ccc}} (\omega_1,\omega_1)^2$	$\mathcal{C}^{2*}$	$\omega_1 \xrightarrow[\mathrm{ccc}]{\mathrm{ccc}^*} (\omega_1,\omega_1)^2$

A diagonal argument [12] (also, the proof of Theorem 4.2(iii)) shows that CH implies the existence of a powerfully c.c.c., co-powerfully-c.c.c. partition on ω_1. Jensen's result that CH+SH is consistent [6] then gives us that SH does not imply any of the above statements.

5. $\mathbb{P}_{\max}$ variations.

In this section, we give the definitions of the forcings we are considering, and briefly present their basic analysis. The omitted definitions, such as those for ψ_{AC} and "iterable", can be found in [33] and [19].

5.1. *A variation for one coherent Suslin tree.* The following is the forcing $\mathbb{S}^{\mathrm{T}}_{\max}$ from [19]. For notational ease, we denote by $\mathcal{M}$ the $\mathbb{S}^{\mathrm{T}}_{\max}$ extension of $L(\mathbb{R})$, assuming $\mathrm{AD}^{L(\mathbb{R})}$. In $\mathcal{M}$, there is a Suslin tree which is the union of the

Suslin trees selected by the conditions in the generic. The further extension by forcing with this tree is called $\mathcal{M}[H]$. The model $\mathcal{M}[H]$ satisfies Suslin's Hypothesis but not $\mathcal{K}_3$. It is not known whether it satisfies $\mathcal{C}^2$.

DEFINITION 5.1. $\mathbb{S}^{\mathrm{T}}_{\max}$ is the set $\langle\langle M_k \mid k < \omega\rangle, S, a\rangle$ of sequences such that:

- $a \in M_0$, $a \subset \omega_1^{M_0}$, and $\omega_1^{M_0} = \omega_1^{L[a,x]}$ for some $x \in \mathbb{R} \cap M_0$.
- Each M_k is a countable transitive model of ZFC.
- $M_k \in M_{k+1}$, $\omega_1^{M_k} = \omega_1^{M_{k+1}}$.
- $(I_{\mathrm{NS}})^{M_{k+1}} \cap M_k = (I_{\mathrm{NS}})^{M_{k+2}} \cap M_k$.
- $\bigcup\{M_k \mid k < \omega\} \models \psi^*_{\mathrm{AC}}$.
- $\langle M_k \mid k < \omega\rangle$ is iterable.
- $S \in M_0$ and $\forall k < \omega\ \ M_k \models S$ is a coherent Suslin tree.
- There exists $X \in M_0$ such that $X \subset \mathcal{P}(\omega_1)^{M_0} \backslash I_{\mathrm{NS}}^{M_1}$, $M_0 \models$ "$|X| = \omega_1$," and for all $A, B \in X$, if $A \neq B$ then $A \cap B \in I_{\mathrm{NS}}^{M_0}$.

The order on $\mathbb{S}^{\mathrm{T}}_{\max}$ is as follows:

$$\langle\langle N_k \mid k < \omega\rangle, S, b\rangle < \langle\langle M_k \mid k < \omega\rangle, \overline{S}, a\rangle$$

if $\langle M_k \mid k < \omega\rangle \in N_0$, $\langle M_k \mid k < \omega\rangle$ is hereditarily countable in N_0 and there exists an iteration

$$j : \langle M_k \mid k < \omega\rangle \to \langle M^*_k \mid k < \omega\rangle$$

such that:

- $j(a) = b$,
- $\langle M^*_k \mid k < \omega\rangle \in N_0$ and $j \in N_0$,
- $(I_{\mathrm{NS}})^{M^*_{k+1}} \cap M^*_k = (I_{\mathrm{NS}})^{N_1} \cap M^*_k$ for all $k < \omega$.
- $j(\overline{S}) = S$.

The following theorems summarize the relevant facts about $\mathbb{S}^{\mathrm{T}}_{\max}$ which carry over from [19]. The proofs of Theorems 5.2, 5.4 and 5.7 are minor variations of the corresponding proofs in [33]. Theorems 5.6 and 5.5 use the proof of the corresponding fact in [33] plus the optimal iteration lemma for coherent Suslin trees, which is proved in [25]. In the right large cardinal context, Theorem 5.6 says that any Π_2-sentence for the structure $\langle H(\omega_2), I_{\mathrm{NS}}, \in\rangle$ (where I_{NS} is a predicate for the nonstationary ideal) consistent with the existence of a coherent Suslin tree holds in the $\mathbb{S}^{\mathrm{T}}_{\max}$ extension of $L(\mathbb{R})$. Similarly, the maximal fragment of $\mathrm{MA}_{\aleph_1}$ consistent with the existence of a coherent Suslin tree should hold there.

THEOREM 5.2. *Assume AD holds in $L(\mathbb{R})$. Let n be an integer. Suppose that ϕ is a sentence such that it is a theorem of ZFC + "there exist n Woodin cardinals" that ϕ + "there exists a coherent Suslin tree" can be forced. Then* *the set of* $\langle\langle M_k \mid k < \omega\rangle, S, a\rangle \in \mathbb{S}^{\mathrm{T}}_{\max}$ *such that each* $M_k \models \phi$ *is dense in* $\mathbb{S}^{\mathrm{T}}_{\max}$.

Since an iteration of an $\mathbb{S}^{\mathrm{T}}_{\max}$ condition $\langle\langle M_k \mid k < \omega\rangle, S, a\rangle$ is uniquely determined by the image of a, and since compatible conditions agree about their selected sets a to the extent that they agree about countable ordinals, for a generic filter $G \subset \mathbb{S}^{\mathrm{T}}_{\max}$ we can define the set A_G to be the union of the sets a selected by the elements of G. This in turn allows the following definition.

DEFINITION 5.3. For a filter $G \subset \mathbb{S}^{\mathrm{T}}_{\max}$,

$$\mathcal{P}(\omega_1)_G = \bigcup\{(\mathcal{P}(\omega_1)^{M_0})^* \mid \langle\langle M_k \mid k < \omega\rangle, S, a\rangle \in G\}$$

and

$$S_G = \bigcup\{S \mid \exists\langle\langle M_k \mid k < \omega\rangle, S, a\rangle \in G\},$$

where for $x \in M_0$, x^* is the image of x under the iteration of $\langle M_k \mid k < \omega\rangle$ sending a to A_G.

Using this definition we can state the theorem which is the main tool for showing Π_2 facts in $\mathbb{P}_{\max}$-style extensions.

THEOREM 5.4. *In* $\mathcal{M}$, $\mathcal{P}(\omega_1)_G = \mathcal{P}(\omega_1)$.

Theorems 5.2 and 5.4 give the following.

THEOREM 5.5. *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Let n be an integer and ψ a Π_2 statement for the structure $\langle H(\omega_2), \in, I_{\mathrm{NS}}\rangle$. If it is a theorem of ZFC + "there exist n Woodin cardinals" that ψ + "there exists a coherent Suslin tree" can be forced to hold, then ψ holds in $\mathcal{M}$.*

Note that in the notation used in this paper, $L(\mathbb{R})^{\mathbb{S}^{\mathrm{T}}_{\max}}$ is $\mathcal{M}$, but we write the following theorem in standard notation for clarity.

THEOREM 5.6. *Assume* $\mathrm{AD}^{L(\mathbb{R})}$ *and that there exists a Woodin cardinal with a measurable above it, and that ψ is a Π_2 statement such that*

$$\langle H(\omega_2), \in, I_{\mathrm{NS}}\rangle \models \psi \wedge \text{“there exists a coherent Suslin tree.”}$$

Then in $L(\mathbb{R})^{\mathbb{S}^{\mathrm{T}}_{\max}}$, $\langle H(\omega_2), \in, I_{\mathrm{NS}}\rangle \models \psi$.

We also mention the following facts which carry over from the analysis in [33].

THEOREM 5.7. *In* $\mathcal{M}$, $\mathfrak{c} = \omega_2 = \utilde{\delta}^1_2$ *and* I_{NS} *is saturated.*

The following theorem follows from putting together Theorem 5.4 with an analysis of posets preserving Suslin trees. Again, $L(\mathbb{R})[G]$ in the statement of the theorem is the model $\mathcal{M}$ in this paper.

Theorem 5.8 ([19]). *Assume* $\mathrm{AD}^{L(\mathbb{R})}$ *and that* $G \subset \mathbb{S}^{\mathrm{T}}_{\max}$ *is* $L(\mathbb{R})$*-generic, and that* H *is* $L(\mathbb{R})[G]$*-generic for forcing with* S_G. *Then* $L(\mathbb{R})[G][H]$ *models Suslin's Hypothesis. Further, if* $\mathbb{P}$ *is a partial order in* $L(\mathbb{R})[G]$ *which is c.c.c. in* $L(\mathbb{R})[G][H]$, *and* $\langle D_\alpha \mid \alpha < \omega_1\rangle$ *is a sequence of dense subsets of* $\mathbb{P}$ *in* $L(\mathbb{R})[G][H]$, *then there is a filter* $F \subset \mathbb{P}$ *in* $L(\mathbb{R})[G]$ *meeting each* D_α.

5.2. *A variation for one unfilled tower.* Another $\mathbb{P}_{\max}$ variation for separating consequences of MA is the following variation for slow unfilled towers. The idea behind this variation is that it should satisfy the maximal fragment of $\mathrm{MA}_{\aleph_1}$ consistent with the existence of an unfilled slow tower. The key question is whether $\mathcal{K}_2$ implies that all slow towers are filled, and if the answer is no, then $\mathcal{N}$ should be a witness to that fact.

The reader will notice that most of this paper concerns $\mathcal{M}$ and $\mathcal{M}[H]$, and that there is relatively little discussion of $\mathcal{N}$. Part of this has to do with the fact that relatively little is known about what forcings preserve unfilled towers, aside from a few specific forcings (see for example [3] and [4]). Further, we need to resolve the iteration problems raised in the next subsection, a solution to which should lead to a model $\mathcal{N}$, perhaps arising from a modified version of $\mathbb{P}^{\mathrm{st}}_{\max}$, satisfying all the Π_2 sentences for $\langle H(\omega_2), \in, I_{\mathrm{NS}}\rangle$ holding in $\mathcal{M}[H]$. In any case, more on $\mathcal{N}$ will appear in [21].

Definition 5.9. $\mathbb{P}^{\mathrm{st}}_{\max}$ is the set of sequences $\langle\langle M_k \mid k < \omega\rangle, T, a\rangle$ such that:

- $a \in M_0$, $a \subset \omega_1^{M_0}$, and $\omega_1^{M_0} = \omega_1^{L[a,x]}$ for some $x \in \mathbb{R} \cap M_0$.
- Each M_k is a countable transitive model of ZFC.
- $M_k \in M_{k+1}$, $\omega_1^{M_k} = \omega_1^{M_{k+1}}$.
- $(I_{\mathrm{NS}})^{M_{k+1}} \cap M_k = (I_{\mathrm{NS}})^{M_{k+2}} \cap M_k$
- $\bigcup\{M_k \mid k < \omega\} \models \psi^*_{\mathrm{AC}}$.
- $\langle M_k \mid k < \omega\rangle$ is iterable.
- For some suitable $z, f, A, E \in M_0$, $T = \mathrm{ST}(z, f, A, E)$ as computed in M_0, and T is unfilled in M_k for each $k < \omega$.
- There exists $X \in M_0$ such that $X \subset \mathcal{P}(\omega_1)^{M_0} \setminus I^{M_1}_{\mathrm{NS}}$, $M_0 \models$ "$|X| = \omega_1$," and for all $A, B \in X$, if $A \neq B$ then $A \cap B \in I^{M_0}_{\mathrm{NS}}$.

The order on $\mathbb{P}^{\mathrm{st}}_{\max}$ is as follows:

$$\langle\langle N_k \mid k < \omega\rangle, T, b\rangle < \langle\langle M_k \mid k < \omega\rangle, \overline{T}, a\rangle$$

if $\langle M_k \mid k < \omega\rangle \in N_0$, $\langle M_k \mid k < \omega\rangle$ is hereditarily countable in N_0 and there exists an iteration

$$j : \langle M_k \mid k < \omega\rangle \to \langle M^*_k \mid k < \omega\rangle$$

such that:

- $j(a) = b$,
- $\langle M_k^* \mid k < \omega \rangle \in N_0$ and $j \in N_0$,
- $(I_{\mathrm{NS}})^{M_{k+1}^*} \cap M_k^* = (I_{\mathrm{NS}})^{N_1} \cap M_k^*$ for all $k < \omega$.
- $j(\overline{T}) = T$.

5.3. *Iteration lemmas.* In the terminology of [25], the optimal iteration lemma for a Σ_2 sentence ψ says roughly that given a $\mathbb{P}_{\max}$ condition modeling ψ, and given that ψ holds, there is an iteration of the condition of length ω_1 such that the image of the witness for ψ in the condition is a witness for ψ. The optimal iteration lemma for coherent Suslin trees is proved in [25]. It is noted there, by an argument of Woodin, that there is no optimal iteration lemma for $\mathfrak{t} = \omega_1$. This also follows from the fact that there are no unfilled towers of reals of asymptotic density 1 in $\mathcal{M}[H]$. This in turn follows from the fact that these reals form a filter definable in $L(\mathbb{R})$. Therefore, given a forcing P which does not add reals and a P-name for such a tower, there is an indestructibly c.c.c. partial order giving a real which is mod finite contained in all possible members of the tower. More generally, we have the following, by the same argument.

THEOREM 5.10. *Suppose that S is a Suslin tree, $\langle M_k \mid k < \omega \rangle$ is an iterable sequence and $T \in M_0$ is a tower such that for all iterations $j_0, \ldots, j_n$ of $\langle M_k \mid k < \omega \rangle$ and all $t_0, \ldots, t_n$ in $j_0(T), \ldots, j_n(T)$ respectively, $\bigcap_{i \leq n} t_i$ is infinite. Suppose that τ is an S-name for an iteration of $\langle M_k \mid k < \omega \rangle$ of length ω_1. Then there is an indestructibly c.c.c. forcing adding a real t such that for any path H through S, t fills the image of T under τ_H.*

Every tower is contained in a filter on ω. One consequence of Theorem 5.10 is that every tower in $\mathcal{M}[H]$ which is contained in a filter in $\mathcal{M}$ is filled. We would like to characterize those towers for which the hypothesis of Theorem 5.10 fails, that is, we would like to know when, given an iterable sequence $\langle M_k \mid k < \omega \rangle$ and an unfilled tower T in M_0 there exist iterations $j_0, \ldots, j_n$ of $\langle M_k \mid k < \omega \rangle$ and an ordinal γ such that $\bigcap_{i \leq n} j_i(T)_\gamma$ is finite. For the case of iterations of single models, we can show (see [21]) from the saturation of the nonstationary ideal plus Chang's Conjecture that there are such iterations for any tower not contained in a $\utilde{\Sigma}^1_2$ filter. The proof is abstract, however, and does not use any information about the combinatorics of the tower.

Another key question is whether there is a type of unfilled tower for which there is an optimal iteration lemma. There is a relatively simple iteration lemma for any type of tower from CH, which we present below. It uses the following lemma.

LEMMA 5.11. *Let $\langle M_k \mid k < \omega \rangle$ be an iterable sequence, and let $T \in M_0$ be a tower which is unfilled in each M_k. Let $B \subset \omega_1^{M_0}$ be an element of M_k*

stationary in M_{k+1}, and let $x \subset \omega$ be infinite. Then there is an iteration

$$j : \langle M_k \mid k < \omega \rangle \to \langle M_k^* \mid k < \omega \rangle$$

of length one such that $j(T)_{\omega_1^{M_0}}$ does not contain x mod finite.

Proof. Let $T = \langle t_\alpha \mid \alpha < \omega_1^{M_0} \rangle$. The ultrafilter inducing the embedding is constructed in ω steps with the usual bookkeeping, with the key point as follows. For each step, we have some set $A \in \mathcal{P}(\omega_1)^{M_k} \setminus I_{\mathrm{NS}}^{M_{k+1}}$ for some integer k, and some regressive function $f \in M_k$ with domain A. We would like to find a set $B \in \mathcal{P}(A)^{M_k} \setminus I_{\mathrm{NS}}^{M_{k+1}}$ such that f is constant on B, and such that $\{\alpha \in B \mid k' \in t_\alpha\} \in I_{\mathrm{NS}}^{M_{k+1}}$ for some integer $k' \in x \setminus k$. Pick any $\overline{B} \in \mathcal{P}(A)^{M_k} \setminus I_{\mathrm{NS}}^{M_{k+1}}$ such that f is constant on $\overline{B}$. Since T is not filled in M_{k+1} and $j(T)_{\omega_1^{M_0}}$ will fill T, the set $\{n \in \omega \mid \{\alpha \in \overline{B} \mid n \notin t_\alpha\} \in I_{\mathrm{NS}}^{M_{k+1}}\}$ must be finite. Thus since x is infinite, we can refine $\overline{B}$ to B excluding some element of $x \setminus k$ from $j(T)_{\omega_1^{M_0}}$. ■

The proof of the lemma below is just like that of the basic iteration lemma for $\mathbb{P}_{\max}$ in [33], using the above lemma and CH to ensure that no real fills the image of the tower.

LEMMA 5.12. *Assume CH, and let $\langle \langle M_k \mid k < \omega \rangle, T, a \rangle$ be a $\mathbb{P}_{\max}^{\mathrm{st}}$ condition. Then there is an iteration*

$$j : \langle M_k \mid k < \omega \rangle \to \langle M_k^* \mid k < \omega \rangle$$

such that $j(T)$ is an unfilled tower and $I_{\mathrm{NS}} \cap M_k^* = I_{\mathrm{NS}}^{M_{k+1}^*} \cap M_k^*$ for each $k < \omega$.

6. Failures of MA in $\mathcal{M}[H]$. It is well known that forcing with a Suslin tree cannot recover MA. The easiest way to see this is to note that MA (in fact, $\mathcal{K}_4$ [32]) implies that every ω_1-sequence of almost disjoint subsets of ω_1 is a Q-sequence. No sequence from the ground model can be a Q-sequence after forcing with a Suslin tree, though, since forcing with the Suslin tree adds no reals, and so no real can code the generic path through the tree by a ground model sequence. In fact, something stronger holds.

THEOREM 6.1. *There are no Q-sequences after forcing with a Suslin tree.*

Proof. Let S be a Suslin tree, and let τ be an S-name for an almost disjoint ω_1-sequence of subsets of ω. We may assume that there is a club set C of levels of S such that for $\alpha \in C$, the αth member of τ is decided by the α^+th level, where α^+ is the least member of C above α. It is easy, then, to construct an S-name σ for a subset of ω_1 such that for all $\alpha \in C$, "$\check{\alpha} \in \sigma$" is never decided until level $\alpha^+ + 1$. Then no real x from the ground model can code the realization of σ, since for each $\alpha \in C$, the question of whether

x has infinite intersection with the αth member of τ is decided before the question of whether α is in σ. But forcing with S adds no reals. ∎

Given a ladder system $\langle e_\delta \mid \delta \in \Omega \rangle$, a *coloring* of the system is a function $f : \omega \times \Omega \to 2$. A *uniformization* of the coloring is a function

$$F : \{e_\delta(i) \mid i < \omega,\ \delta \in \Omega\} \to 2$$

such that for all $\delta \in \Omega$ there exists an $i_\delta \in \omega$ such that $F(e_\delta(j)) = f(j, \delta)$ for all $j > i_\delta$. It is shown in [32] that $\mathcal{K}_4$ implies that every coloring on every ladder system is uniformized. An argument very similar to that for Theorem 6.1 shows that in $\mathcal{M}[H]$, every ladder system has an un-uniformized coloring.

THEOREM 6.2. *Let S be a Suslin tree, and let σ be an S-name for a ladder system. After forcing a generic path H through S there is a coloring for which σ_H is not uniformized.*

Proof. Let $C \subset \Omega$ be a club such that if $\alpha \in C$ and p is on the αth level of S, then p decides the first α members of σ. Build a name τ for a coloring such that for all $\alpha \in C$, if p is on the αth level of S then there are nodes $p', p'' \leq_S p$ deciding the coloring for σ_α such that for all i, $p' \Vdash \tau(\check{i}, \check{\alpha}) = 0 \Leftrightarrow p'' \Vdash \tau(\check{i}, \check{\alpha}) = 1$. Then there can be no S-name for a uniformization of this coloring, since if ν were such a name, then for some $\alpha \in C$ and p on the αth level of C, p would decide ν up to α. But then one of the corresponding p', p'' as above would make ν disagree with the coloring on σ_α infinitely often. ∎

It is shown in Chapter 7 of [29] that $\mathcal{K}_3$ implies that for any nonprincipal ultrafilter U on ω_1, $(2^{\omega_1}, <_{\mathrm{lex}})$ can be embedded in ${}^\omega\omega/U$. The following then shows that $\mathcal{K}_3$ fails in $\mathcal{M}[H]$.

THEOREM 6.3. *Let U be an ultrafilter on ω and T an Aronszajn tree preserving ω_1. After forcing with T, there is no embedding of $(2^{\omega_1}, <_{\mathrm{lex}})$ into ${}^\omega\omega/U$.*

Proof. Fix U and T, and let τ be a T-name for such an embedding. For $t \in T$, let $X_t = \{f \in 2^\omega \mid t \text{ forces that } \tau(\check{\pi}[g]) = \check{a}/\check{U}\}$. Note that under the initial segment topology, the intersection of ω_1 dense open subsets of 2^{ω_1} is dense. Since the union of the X_t's is all of 2^{ω_1}, some X_t must be somewhere dense.

Let $t \in T$ and $w \in 2^{<\omega_1}$ be such that X_t is dense in the set of extensions of w. Let π be an embedding of the extensions of t in T into the extensions of w in $2^{<\omega_1}$. Find an extension s of t which decides that $\tau(\pi[g]) = a/U$, where g is the generic branch through T. Now consider

$$(X_t)^- = \{x \in X_t \mid t \Vdash \tau(\check{x}) < \check{a}/\check{U}\},$$
$$(X_t)^+ = \{x \in X_t \mid t \Vdash \check{a}/\check{U} < \tau(\check{x})\}.$$

This is a pregap in $(2^{\omega_1}, <_{\mathrm{lex}})$ which gives us an uncountable branch through T as follows:

$$B = \{v \in T \mid v \text{ extends } s \text{ and } \pi[v] \text{ has extensions in both } (X_t)^- \text{ and } (X_t)^+\}. \ \blacksquare$$

Let $\{a_x \mid x \in \mathfrak{c}\}$ be a fixed independent family of subsets of ω (see [13]). Then for every subset X of $\mathfrak{c}$ there is a nonprincipal ultrafilter U on ω such that $x \in X$ iff $a_x \in U$. It follows that a forcing notion adds a new subset of $\mathfrak{c}$ if and only if it introduces a new ultrafilter on ω. Further, it is a theorem of ZFC + $2^\omega = 2^{\omega_1}$ that there is an ultrafilter U on ω such that $(2^{\omega_1}, <_{\mathrm{lex}})$ can be embedded into ${}^\omega\omega/U$. This follows from the fact (see [5]) that there exists in ZFC a set $\{a_x \mid x \in \mathfrak{c}\}$ of elements in ${}^\omega\omega$ such that for all $x_0, \ldots, x_n \in \mathfrak{c}$ and all orders σ on $n+1$ the set $\{i \in \omega \mid \forall j,k \le n \ a_{x_j}(i) < a_{x_k}(i) \Leftrightarrow \sigma(j) < \sigma(k)\}$ is infinite. Then for any assignment of 2^{ω_1} to the a_x's there is an ultrafilter as desired.

7. Fragments of MA in $\mathcal{M}[H]$. It is shown in [19] that Suslin's Hypothesis holds in $\mathcal{M}[H]$. The argument proceeds by taking an S-name for an Aronszajn tree, where S is our coherent tree, and showing that one can force the existence of a name for an uncountable antichain in the Aronszajn tree without adding an uncountable antichain to S. In this section we abstract the key points in that proof to show that another fragment of MA holds in $\mathcal{M}[H]$ (1).

DEFINITION 7.1. A partial order $(\{p_\alpha \mid \alpha < \omega_1\}, <)$ is *stable* if there exists an increasing sequence $\langle \gamma_\alpha \mid \alpha < \omega_1 \rangle$ of ordinals such that for all $\alpha, \beta < \omega_1$ there exist $\overline{\beta} < \gamma_\alpha$ and $p_{\beta^*} \le p_\beta$ such that p_{β^*} and $p_{\overline{\beta}}$ are compatible with the same elements of $\{p_\gamma \mid \gamma < \alpha\}$.

This notion of stability was introduced in [1]. Stable partitions include those derived from the compatibility relation on Aronszajn trees, as well as partitions derived from taking a subset of an Aronszajn tree. Other examples can be formed by taking a partition on nodes from the same level of an Aronszajn tree, and saying that a pair of nodes on the tree are compatible if their greatest common point is in the set, or if the pair composed of their first two points of difference is in the partition. Also, a partition $K \subset [\omega_1]^2$ is stable if every uncountable subset of it can be split into countably many sets

(1) *Added in proof* (February 2001). The arguments in this section have been considerably improved. To begin with, $H_0, \ldots, H_n$ in the statement of Lemma 7.2 can be replaced by a single path H, and so in Corollary 7.4 being unsplit can be replaced with property N. The reader is referred to the sequel to this paper [21] as well as to the authors' *Katetov's problem*, in preparation.

A_i such that for some partition $B \subset [\omega]^2$, for each pair of distinct integers i, j and any $p \in A_i$ and $q \in A_j$, $\{p, q\} \in K \Leftrightarrow \{i, j\} \in B$.

LEMMA 7.2. *Let S be a coherent Suslin tree, and τ an S-name for a stable c.c.c. partial order on ω_1. Suppose also that S and τ satisfy the condition that if $H_0, \ldots, H_n$ are paths through S after forcing with S then there is no 1-homogeneous set for $\prod_{i \leq n} N(\tau_{H_i})$. Then there is a c.c.c. forcing preserving the Suslinity of S and adding an S-name for a subset of τ meeting any given ω_1 dense subsets.*

Proof. Let S be a coherent Suslin tree, let τ be an S-name for a stable partial order on ω_1, and let $\langle \sigma_\alpha \mid \alpha < \omega_1 \rangle$ be S-names for dense sets in τ. Let $\langle \nu_\alpha \mid \alpha < \omega_1 \rangle$ be S-names for a sequence of ordinals witnessing that τ is strongly stable. Let Q be the forcing whose conditions are finite $a \subset S \times \omega_1$, under the superset order, with the following added stipulations:

- If $(p, \alpha) \in a$, then p decides $\tau \cap (\alpha \times \alpha)$.
- If $(p, \alpha), (q, \beta) \in a$, and $p \leq_S q$, then there exists $(r, \gamma) \in a$ such that $r \leq_S p$ and $r \Vdash_S \check{\gamma} \leq_\tau \check{\alpha} \wedge \check{\gamma} \leq_\tau \check{\beta}$.

Suppose that $\beta < \omega_1$, $a \in Q$ and $p \in S$. We would like to be able to expand a to an a' such that there exists $(q, \alpha) \in a'$ such that $q \Vdash \check{\alpha} \in \sigma_\beta$ and $q \leq_S p$. By extending p if necessary, we may assume that there are no $(r, \gamma) \in a$ such that $r \leq_S p$. Let r be S-least such that $r \geq_S p$ and there exists γ with $(r, \gamma) \in a$. By the second condition above, there is some γ with $(r, \gamma) \in a$ such that $r \Vdash \check{\gamma} \leq_\tau \check{\delta}$ for all $(r', \delta) \in a$ with $r \leq_S r'$. Then we can pick a pair (q, α) with $q \leq_S p$ and $q \Vdash \check{\alpha} \leq_\tau \check{\gamma} \wedge \check{\alpha} \in \sigma_\beta$. Therefore, Q forces the existence of an S-name for a filter contained in the realization of τ meeting the realizations of all the σ_β's.

By Lemma 2.4, we will be done if we show that Q is c.c.c. after forcing with S, since then Q preserves that S is Suslin. Supposing otherwise, let $\{a_\gamma = \langle (p_i^\gamma, \alpha_i^\gamma) \mid i < n \rangle \mid \gamma < \omega_1\}$ be an antichain in Q after forcing with S. We note that the incompatibility of a_β and a_γ implies that there exist integers i, j such that p_i^β and p_j^γ are compatible but $p_i^\beta \wedge p_j^\gamma$ forces that α_i^β and α_j^γ are incompatible in τ. By the usual Δ-system argument, then, we may assume that the a_γ's are disjoint, and that $\gamma < \gamma' \Rightarrow \alpha_i^\gamma < \alpha_j^{\gamma'}$. By Lemma 2.7, and by thinning our sequence, we can fix $\langle q_i^\gamma \mid i < n,\ \gamma < \omega_1 \rangle$ such that $\gamma < \gamma' \Rightarrow q_i^{\gamma'} \leq_S q_i^\gamma$ for all $i < n$, and $p_i^\gamma \leq_S q_i^\gamma$ for all i, γ. Further, we may assume that for each γ, the q_i^γ's are on the same level of S. Let H_i be the path defined by the q_i^γ's. By Lemma 2.6 each $q_i^\gamma \in \pi^S_{q_0^0 q_i^0}[H_0]$. By thinning our sequence again, we may assume that for each i, γ there are $\xi_i^\gamma, \delta < \min\{\alpha_i^\gamma, \mathrm{lev}_S(q_i^\gamma)\}$ such that $q_i^\gamma \Vdash \nu_\delta = \check{\xi}_i^\gamma$ and $\delta > \alpha_{\overline{i}}^{\overline{\gamma}}$, for all $\overline{i}, \overline{\gamma}$ such that $p_{\overline{i}}^{\overline{\gamma}} \geq_S p_i^\gamma$. But then by the definition of stability we can pick

$\overline{\alpha}_i^\gamma, \overline{p}_i^\gamma \leq p_i^\gamma$ and $\lambda_i^\gamma < \xi_i^\gamma$ for each i, γ such that $\overline{p}_i^\gamma$ forces $\overline{\alpha}_i^\gamma \leq_\tau \alpha_i^\gamma$ and every element of δ incompatible with $\overline{\alpha}_i^\gamma$ is incompatible with λ_i^γ.

Define an equivalence relation on n by whether $q_i^0 = q_j^0$. Let m be the number of equivalence classes. Let $\langle i_l \mid l < m\rangle$ be a list of representatives for the equivalence classes, and let

$$b_\gamma = \{\{\lambda_j^\gamma \mid j < n \wedge q_j^0 = q_{i_l}^0\} \cup \{\overline{\alpha}_j^\gamma \mid p_j^\gamma \in H_{i_l}\} \mid l < m\}.$$

Then for each $\gamma < \gamma'$, since a_γ and $a_{\gamma'}$ are incompatible in Q, there are $i, j < n$ such that $\overline{p}_i^{\gamma'} \leq_S \overline{p}_j^\gamma$ and $\overline{p}_i^{\gamma'}$ forces that $\overline{\alpha}_i^{\gamma'}$ and $\overline{\alpha}_j^\gamma$ are incompatible. It follows that $q_i^{\gamma'}$ forces that $\lambda_i^{\gamma'}$ and $\overline{\alpha}_j^\gamma$ are incompatible. Then if l is such that $q_i^0 = q_j^0 = q_{i_l}^0$, the lth coordinates of b_γ and $b_{\gamma'}$ are incompatible in $N(\tau_{H_{i_l}})$. Thus the b_γ's form a 1-homogeneous set in $\prod_{l\leq m} N(\tau_{H_{i_l}})$, contradicting our hypothesis. ∎

The following is an immediate corollary of Theorems 5.4 and 7.2.

Corollary 7.3. *If τ is an S_G-name in $\mathcal{M}$ for a stable partial order, and τ has the property that for all paths $H_0, \ldots, H_n$ through S_G in $\mathcal{M}[H]$ there is no 1-homogeneous set in $\prod_{i\leq n} N(\tau_{H_i})$, then Martin's Axiom holds for τ_H in $\mathcal{M}[H]$.*

Since any product of unsplit partial orders has property N, we have the following. A simple variation of the proof in [19] shows that all Aronszajn trees are special in $\mathcal{M}[H]$. This fact does not appear to be subsumed by Corollary 7.4.

Corollary 7.4. *In $\mathcal{M}[H]$, Martin's Axiom holds for partial orders which are stable and unsplit.*

One consequence of Corollary 7.3 is that if $\mathcal{C}^2$ holds in $\mathcal{M}[H]$, then MA holds there for stable partial orders which have property N. Property N is fairly restrictive, however, as it rules out partial orders with disconnected rectangles.

A simpler version of the proof of Theorem 7.2, similar to the proof in [19] that SH holds in $\mathcal{M}[H]$, shows that in $\mathcal{M}[H]$ every nonspecial tree of cardinality ω_1 contains a path. Since these trees need not have countable levels, the corresponding partial order is not stable.

It is shown in [1] that ladder uniformization follows from MA(stable) and so MA(stable) fails in $\mathcal{M}[H]$. The forcing to uniformize a given ladder system does not have property N, however. It is also shown in [1] that MA(stable) is weaker than MA.

We note that the above strategy cannot work for S-names for c.c.c. partial orders with the property that every subset of size ω_1 contains a disconnected rectangle, since then the corresponding forcing Q would indeed destroy the Suslinity of S. The above argument will also fail if the product of the realizations of the name by two different paths through S is not c.c.c., since then there will be an antichain in the corresponding Q. We can, however, generalize the above proof in such a way as to get around these obstacles, which we do in the last section of this paper.

Lastly, we prove that for any $\alpha < \omega_1$ and $n < \omega$ the partition relation $\omega_1 \to (\omega_1, \alpha : n)^2$ holds in $\mathcal{M}[H]$, where $\alpha : n$ in the second coordinate means that there is a set of ordertype $\alpha + n$ such that the top n nodes are incompatible with each of the first α many. Partitions failing to meet this criterion are productively c.c.c. failures of $\mathcal{K}_2$. Recently [21], we have improved the theorem below to show that $\omega_1 \to (\omega_1, \alpha)^2$ holds in $\mathcal{M}[H]$ for all countable α.

THEOREM 7.5. *For all $\alpha < \omega_1$ and $n < \omega$, $\omega_1 \to (\omega_1, \alpha : n)^2$ holds in $\mathcal{M}[H]$.*

Proof. Note that there are uncountably many $\alpha < \omega_1$ such that for any finite collection of subsets of α of ordertype less than α, the ordertype of the union of the collection is less than α. We may assume that our α has this property. Fix n. Let S be a coherent Suslin tree, and let τ be an S-name for a partition on ω_1. The usual forcing to create an uncountable 0-homogeneous set by finite conditions preserves the Suslinity of S unless forcing with S makes it not c.c.c. Let σ be an S-name for an antichain in this forcing consisting of m-tuples for some fixed integer m, and fix an ultrafilter U on α all of whose members have ordertype α, by the above property of α. By Lemma 2.7, we can fix integers $i, j < n$ and assume that the first coordinates of the ith members of the antichain will form a chain in S, and that given the first α members of the antichain $\langle p_i^\beta \mid i < m,\ \beta < \alpha\rangle$, the jth member of each subsequent member of the antichain will be incompatible with U-many of the p_i^β. Fix $s \in S$ deciding the first α members of the antichain and let $s' \in S$ be below each of the first α-many p_i^β. Then densely below s' one can add elements to τ which are incompatible with U-many of the p_i^β. Any n such elements, and the intersection of their corresponding sets from U, complete the proof. ∎

8. More forcing axioms in $\mathcal{M}[H]$

DEFINITION 8.1. The *Open Coloring Axiom* (OCA) is the statement that if $O \subset \mathbb{R} \times \mathbb{R}$ is open and symmetric, and $A \subset \mathbb{R}$, then either there is an uncountable set $B \subset A$ such that $[B]^2 \subset O$, or A is the union of countably many sets $\langle C_n \mid n \in \omega\rangle$ such that $[C_n]^2 \cap O = \emptyset$ for each n.

The following facts about OCA appear in [9].

Theorem 8.2. *There is a partial order which forces OCA and preserves Suslin trees.*

Theorem 8.3. *OCA is preserved by forcing with Suslin trees.*

In [20], it is shown that OCA holds in the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$. The proof of the following uses essentially the same arguments, plus Theorem 8.3.

Theorem 8.4. *OCA holds in $\mathcal{M}$ and $\mathcal{M}[H]$.*

Since OCA is not a statement about ω_1, Theorem 8.4 cannot be proved simply by referring to Theorem 5.4; more $\mathbb{P}_{\max}$ machinery is required. Roughly, the proof is as follows. Since $\mathbb{S}^{\mathrm{T}}_{\max}$ does not add reals, we can assume that the open set $O \subset \mathbb{R} \times \mathbb{R}$ is defined by a real x. Let τ be an $\mathbb{S}^{\mathrm{T}}_{\max}$-name for a set of reals, and let $\langle\langle N_k \mid k < \omega\rangle, S, b\rangle \in \mathbb{S}^{\mathrm{T}}_{\max}$ with $x \in N_0$ force that the realization of τ is not contained in any countable union of sets 1-homogeneous for O. The key point is that one can find a countable transitive model M with the following properties, where A is a set of reals coding τ:

(i) $M \models \mathrm{ZFC}$ + "there exists a coherent Suslin tree."

(ii) $\{A \cap M, \langle\langle N_k \mid k < \omega\rangle, S, b\rangle\} \in M$.

(iii) There exists an ordinal δ which is Woodin in M, and such that the pair $(M, \mathbb{I}_{<\delta})$ is A-iterable (see [33]), i.e., all iterations of M by the nonstationary ideal corresponding to δ are well founded and compute A correctly.

(iv) $\langle H(\omega_1)^M, \in, A \cap M\rangle \prec \langle H(\omega_1), \in, A\rangle$.

(v) Properties (ii)–(iv) are true of any forcing extension of M preserving ω_1^M and the Woodiness of δ.

By property (iv) above, M is correct about its version of $\mathbb{S}^{\mathrm{T}}_{\max}$ and also which conditions force which statements about τ. By property (v), we may assume that CH holds in M. Working in M, build a decreasing sequence of $\mathbb{S}^{\mathrm{T}}_{\max}$ conditions of length ω_1^M, starting with $\langle\langle N_k \mid k < \omega\rangle, S, b\rangle$, such that the limit sequence $\langle N_k^* \mid k < \omega\rangle$ agrees with M about stationary subsets of ω_1, and the image of the selected coherent Suslin trees in the members of the sequence is in fact Suslin in M. Further, since M sees that no countable sequence of closed sets can cover τ, this sequence can be built so that for each sequence $\langle C_i \mid i < \omega\rangle$ of closed sets of reals in M, either there exist some $i < \omega$ and $x, y \in [C_i]^2 \cap O \cap M$ such that some member of the sequence forces $\{x, y\} \subset \tau$, or there exists a real x in $(M \cap O) \setminus \bigcup\{C_i \mid i < \omega\}$ such that some member of the sequence forces $x \in \tau$. By [29] (Chapter 4), then, there is a c.c.c. forcing in M preserving T which gives an uncountable (in the extension of M) 0-homogeneous set for O contained in X. Call this further extension $\overline{M}$. By standard techniques [33] one can build an $\mathbb{S}^{\mathrm{T}}_{\max}$ condition below the sequence constructed in M whose first member is an iterate of $\overline{M}$

by its version of the nonstationary tower up to δ. This condition will also be A-iterable, and so will force that the image of X as iterated through the generic will be an uncountable subset of τ from which all pairs are in O.

Theorems 5.4 and 8.2 can be used to see that the set of conditions satisfying OCA is dense in $\mathbb{S}^{\mathrm{T}}_{\max}$, and so one can see more simply that all the consequences of OCA for $H(\omega_2)$ hold in $\mathcal{M}$.

OCA implies that every locally countable subset of $\mathcal{P}(\omega)$ has an uncountable antichain, and so this fact holds in $\mathcal{M}[H]$. We also have the following direct proof that every locally countable set is special, verifying that the conclusion of Theorem 4.19(iii) holds in $\mathcal{M}[H]$.

THEOREM 8.5. *Given a coherent Suslin tree S and an S-name τ for a locally countable subset of $\mathcal{P}(\omega)$, the forcing to add an S-name for a specialization of τ by finite approximations preserves S.*

Proof. Conditions in the forcing are finite sets $\langle (p_i, r_i, k_i) \in S \times \mathcal{P}(\omega) \times \omega \mid i < n\rangle$ such that $p_i \Vdash r_i \in \tau$ for all $i < n$, and if p_i and p_j are compatible and k_i equals k_j, then neither of r_i and r_j is contained in the other.

By Lemma 2.4, we need to see only that this forcing is c.c.c. after forcing with S. By Lemma 2.7, any antichain in the forcing in the extension by S can be written as $\langle (p^j_{i,\alpha}, r^j_{i,\alpha}, k^j_{i,\alpha}) \mid \alpha < \omega_1,\ j < n,\ i < m_j\rangle$ for some $n \in \omega$ and $\{m_j \mid j < n\} \subset \omega$ such that there exist S-decreasing chains $\langle q^j_\alpha \mid \alpha < \omega_1\rangle$ for $j < n$ such that $p^j_{i,\alpha} <_S q^j_\alpha$ for all $i < m_j$. By the usual thinning procedures we assume that $\alpha < \beta$ implies that $\mathrm{lev}_S(p^j_{i,\alpha}) < \mathrm{lev}_S(q^j_\beta) < \mathrm{lev}_S(p^j_{i,\beta})$, each $k^j_{i,\alpha}$ depends only on i and j, and the question of whether $q^j_{\alpha+1} <_S p^j_{i,\alpha} <_S q^j_\alpha$ depends only on i and j.

We will be done after we prove the following claim, by applying it successively to each $j < n$, $k < m_j$ and $i < m_j$ such that $q^j_\alpha \geq p^j_{i,\alpha} \geq q^j_{\alpha+1}$: Let A, B be uncountable subsets of ω_1 and fix $j < n$. Then there exist finite $x \subset \omega$ and uncountable $A' \subset A$, $B' \subset B$ such that $x \subset r^j_{i,\alpha}$ for all $\alpha \in A'$, and $x \not\subset r^j_{k,\beta}$ for all $\beta \in B'$.

To see the claim, note that otherwise there are S-names for a pair A, B of uncountable subsets of ω_1 and (simplifyng notation slightly) sequences $\langle (p_{i,\alpha}, r_{i,\alpha}) \mid \alpha \in A\rangle$ and $\langle (p_{k,\alpha}, r_{k,\alpha}) \mid \alpha \in B\rangle$ with the property that $\alpha < \beta$ implies $p_{k,\beta} <_S p_{i,\alpha}$, and for which no x satisfies the claim. We may also assume that for each finite $x \subset \omega$, x is an initial segment of either uncountably many $r_{i,\alpha}$ or none, and that each x which is an initial segment of uncountably many $r_{i,\alpha}$ is contained in all $r_{k,\alpha}$. But since the set of potential $p_{k,\beta}$'s must be dense, the existence of such names implies that in the realization of τ there are uncountable sets $\{r_\alpha \mid \alpha \in A\}$ and $\{s_\beta \mid \beta \in B\}$ of reals such that every s_β contains every r_α, a contradiction since τ is a name for a locally countable subset. ∎

Similar arguments can be used to directly show that other consequences of OCA hold in $\mathcal{M}[H]$. For instance, one can show directly that $\mathcal{M}[H]$ satisfies the statement that if $\langle f_\alpha : \omega \to \omega \mid \alpha < \omega_1 \rangle$ is a mod-finite increasing sequence then there is an uncountable subset of ω_1 from which each corresponding pair of functions oscillates. It is shown in [29] that this statement implies $\mathfrak{b} > \omega_1$. Showing that $\mathfrak{b} > \omega_1$ in $\mathcal{M}[H]$ is even easier, however, since if S is a Suslin tree and τ is an S-name for an ω_1-sequence of functions from ω to ω then there is an indestructibly c.c.c., and thus Suslin tree preserving, partial order adding a function which dominates mod finite all those with some chance to be in τ.

9. A tower-filling partition. As we have mentioned, one key test question is whether $\mathcal{K}_2$ implies that $\mathfrak{t} > \omega_1$. In this section we show that it is possible to have an unfilled tower and a c.c.c. partition on pairs from ω_1 for which any uncountable homogeneous set would define a real filling the tower. We first prove a general lemma which shows roughly that if S is a coherent Suslin tree and μ is an S-name for a partition on finite subsets of ω_1 which has property K_2, then there is a way to force an S-name ν for a c.c.c., co-c.c.c. partition on pairs such that any 0-homogeneous set for the realization of ν is 0-homogeneous for the realization of μ.

LEMMA 9.1. *Let S be a coherent Suslin tree, and let μ be an S-name for a partition on finite subsets of ω_1. Suppose that μ is such that the partial order for adding an uncountable set which is 0-homogeneous for the realization of μ by finite approximations is forced to have property K_2, and such that every $p \in S$ decides μ on $[\mathrm{lev}_S(p)]^{<\omega}$. Let Q be the collection of finite $a \subset S \times [\omega_1]^2$ such that*:

(i) *for each $(p, \{\alpha, \beta\}) \in a$, $\mathrm{lev}_S(p) = \beta + 1$ and $p \Vdash \{\check{\alpha}, \check{\beta}\} \in \mu$,*

(ii) *for all $p \in S$, and for all finite $b \subset \mathrm{lev}_S(p)$, if for all $\alpha < \beta \in b$ there exists $q \leq_S p$ such that $(q, \{\alpha, \beta\}) \in a$, then $p \Vdash \check{b} \in \mu$.*

Then Q, ordered by inclusion, preserves the Suslinity of S. Further, if H is generic for Q and g is generic for S, then the partition

$$\nu = \{x \in [\omega_1]^2 \mid \exists p \in g \ \ (p, x) \in H\}$$

is uncountable and c.c.c., and all finite sets which are homogeneous for ν are in μ.

Proof. To see that Q preserves the Suslinity of T, it suffices to show that Q remains c.c.c. after forcing with T. Let $\langle a_\alpha \mid \alpha < \omega_1 \rangle$ be a sequence of conditions in Q after forcing with T. For each α, define $d_\alpha = \{\gamma \in \omega_1 \mid \exists (p, x) \in a_\alpha \ \ \gamma \in x\}$. By refining, we may assume that the d_α's form a Δ-system with root r, and also that the sets $l_\alpha = \{\gamma \mid \exists (p, x) \in a_\alpha \ \ \mathrm{lev}_T(p) = \gamma\}$

form a Δ-system with root s. Further, we may assume that the restriction of each a_α to the set of pairs (p, x) for which $\mathrm{lev}_T(p) \in s$ is the same. Then for all $\alpha < \beta$, if b is a finite set satisfying the premise of condition (ii) in the definition of Q for $a_\alpha \cup a_\beta$, then b satisfies the same premise for either a_α or a_β, and so a_α and a_β are compatible.

One can easily check that the partition on pairs induced by Q is indeed uncountable, i.e. that for every condition $a \in Q$, every $\alpha \in \omega_1$ and every $p \in S$, we can extend a to include a pair $(q, \{\beta, \gamma\})$ where $q \leq_S p$ and $\gamma > \alpha$.

To see that the induced partition ν is c.c.c., let (a, p) force that τ is a $Q * S$-name for an ω_1-sequence of 0-homogeneous n-tuples for ν, for some integer n. For each $\alpha < \omega_1$, let $(a_\alpha, p_\alpha) \leq (a, p)$ force that b_α is the αth member of this sequence. Applying the first paragraph, by refining we may assume that the a_α's are pairwise compatible. Note that $p_\alpha \Vdash b_\alpha \in \mu$. By extending p if necessary, and renumbering, we may assume that the p_α's are dense below p, in which case the sequence $\langle (p_\alpha, b_\alpha) : \alpha < \omega_1 \rangle$ defines an S-name below p for a subset of the realization of μ of size ω_1. Since μ is a name for a partition satisfying K_2, there is an S-name below p, χ, for an uncountable pairwise compatible subset of the set $\{b_\alpha \mid p_\alpha \in H\}$, where H is the generic path through S with $p \in H$. Let $\alpha < \beta < \omega_1$ and $q \leq_S p$ be such that $q \Vdash b_\alpha, b_\beta \in \chi$. Let $a_q = \{(q_{\gamma+1}, \{\delta, \gamma\}) \mid \delta \in b_\alpha,\ \gamma \in b_\beta\}$, where $q_{\gamma+1}$ is the predecessor of q on level $\gamma + 1$ of S. Then $a^* = a_q \cup a_\alpha \cup a_\beta \in Q$, and $(a^*, q) \leq_{Q*S} (a_\alpha, p_\alpha)$, (a_α, p_α) forces that b_α and b_β are μ-compatible, and thus ν-compatible, and so τ is not a $Q * S$-name for an uncountable antichain in ν. ■

A simpler argument shows that for any K_2 partition on finite subsets from ω_1 there is a c.c.c. forcing which gives a c.c.c. partition on pairs from ω_1 for which any homogeneous set is homogeneous for the original partition. In that case, however, we do not know if the forcing adds an uncountable homogeneous set for the original partition.

For any tower of length ω_1 there is a c.c.c. partition on finite subsets of ω_1 for which any uncountable homogeneous set defines a real filling the tower, as follows.

Definition 9.2 ([32]). Given a tower $t = \langle t_\alpha \mid \alpha < \omega_1 \rangle$, let P_t be the set of finite $a \subset \omega_1$ such that if we let $x = \bigcap \{t_\alpha \mid \alpha \in a\}$ and

$$y = \{n \in \omega \mid \exists \alpha, \beta \in a \text{ such that } t_\alpha \text{ and } t_\beta \text{ first differ at } n\},$$

then for all $i < |a|$, $|x \cap n_i| \geq i$, where n_i is the ith element of y. The order on P_t is inclusion.

It is easy to see that P_t satisfies K_2, and that any uncountable set of compatible conditions defines a real filling t. Now let S be a coherent Suslin tree, and let $\{a_p \mid p \in S\}$ be a set of subsets of ω such that for all $p, q \in S$,

$p \leq_S q \Leftrightarrow a_p \subset^* a_q$. Since forcing with S adds no reals, a generic path g through S gives an unfilled tower $t_g = \{a_\alpha \mid a_\alpha = a_p$ for $p \in g \wedge \mathrm{lev}_S(p) = \alpha\}$. Lemma 9.1 then says that there is a forcing preserving the Suslinity of S which gives an S-name for a c.c.c. partition on pairs from ω_1 such that any uncountable homogeneous set for this partition is also homogeneous for P_{t_g} and thus defines a real filling t_g. A similar argument shows, however, that the partition given by the ν corresponding to P_{t_g} is also co-c.c.c. If the partition could be made c.c.c. but not co-c.c.c., then one could not force $\mathcal{K}_2$ over $\mathcal{M}[H]$ in the usual manner without filling t_g. A key question related to the problem of $\mathcal{C}^2$ vs. $\mathrm{MA}_{\aleph_1}$ is whether there can exist an unfilled tower and a partition on pairs such that any uncountable 0-homogeneous set or antichain for the partition would fill the tower, in an absolute sense. We note that under $\mathrm{AD}^{L(\mathbb{R})}$ it is a simple consequence of the existence of $\mathbb{P}_{\max}$ conditions plus Lemma 5.12 that if the existence of such an unfilled tower and c.c.c. partition is always forceable, then CH implies that there is such a pair.

10. Questions. We collect here some of the key questions left unresolved. Several of them have been asked before.

1. Does $\mathcal{K}_2$ imply $\mathcal{K}_3$? Does it imply $\mathrm{MA}_{\aleph_1}$?

2. Does $\mathcal{C}^2$ imply $\mathcal{K}_2$? Does it imply $\mathrm{MA}_{\aleph_1}$?

3. Does MA hold in $\mathcal{M}[H]$ for all partial orders that do not add a real which along with some set from the ground model codes a path through S_G?

4. Can a forcing of size ω_1 ever recover MA? Can adding a dominating function ever restore MA? Can filling a tower ever restore MA?

5. Does $\mathfrak{t} = \omega_1$ imply the existence of a slow unfilled tower?

6. Is there an optimal iteration lemma for the existence of a slow unfilled tower? For any type of unfilled tower?

7. Does MA for powerfully c.c.c. partial orders imply MA?

References

[1] U. Avraham and S. Shelah, *Forcing with stable posets*, J. Symbolic Logic 47 (1982), 37–42.

[2] J. Baumgartner, *Results and independence proofs in combinatorial set theory*, Ph.D. thesis, Univ. of California at Berkeley, 1970.

[3] J. Baumgartner and P. Dordal, *Adjoining dominating functions*, J. Symbolic Logic 50 (1985), 94–101.

[4] J. Brendle, H. Judah and S. Shelah, *Combinatorial properties of Hechler forcing*, Ann. Pure Appl. Logic 58 (1992), 185–199.

[5] W. W. Comfort and S. Negrepontis, *The Theory of Ultrafilters*, Grundlehren Math. Wiss. 211, Springer, New York, 1974.

[6] K. Devlin and H. Johnsbråten, *The Souslin Problem*, Lecture Notes in Math. 405, Springer, 1974.

[7] K. Devlin and S. Shelah, *A weak version of* $\diamondsuit$ *which follows from* $2^{\aleph_0} < 2^{\aleph_1}$, Israel J. Math. 29 (1978), 239–247.

[8] E. K. van Douwen and K. Kunen, *L-spaces and S-spaces in* $\mathcal{P}(\omega)$, Topology Appl. 14 (1982), 143–149.

[9] I. Farah, *OCA and towers in* $\mathcal{P}(\mathbb{N})/Fin$, Comment. Math. Univ. Carolin. 37 (1996), 861–866.

[10] Q. Feng and T. Jech, *Projective stationary sets, and strong reflection principles*, J. London Math. Soc. (2) 58 (1998), 271–283.

[11] D. H. Fremlin, *Real-valued measurable cardinals*, in: Set Theory of the Reals (Ramat Gan, 1991), Israel Math. Conf. Proc., 6, Bar-Ilan Univ., Ramat Gan, 1993, 151–304.

[12] F. Galvin, *Chain conditions and products*, Fund. Math. 108 (1980), 33–48.

[13] T. Jech, *Set Theory*, Academic Press, 1978.

[14] R. Jensen and R. Solovay, *Some applications of almost disjoint sets*, in: Mathematical Logic and Foundations of Set Theory (Jerusalem, 1968), North-Holland, 1970, 84–104.

[15] B. König, *Strongly homogeneous = coherent*, preprint, 1999.

[16] K. Kunen, (κ, λ^*) *gaps under MA*, note of August 1976.

[17] G. Kurepa, *La condition de Souslin et une propriété caractéristique des nombres réels*, C. R. Acad. Sci. Paris 231 (1950), 1113–1114.

[18] C. Laflamme, *Zapping small filters*, Proc. Amer. Math. Soc. 114 (1992), 535–544.

[19] P. Larson, *An* $\mathbb{S}_{max}$ *variation for one Souslin tree*, J. Symbolic Logic 64 (1999), 81–98.

[20] —, *Showing OCA in* $\mathbb{P}_{\max}$*-style extensions*, in preparation.

[21] P. Larson and S. Todorčević, *Chain conditions in maximal models II*, in preparation.

[22] R. Laver, *Random reals and Souslin trees*, Proc. Amer. Math. Soc. 100 (1987), 531–534.

[23] R. D. Mauldin (ed.), *The Scottish Book. Mathematics from the Scottish Café. Including selected papers presented at the Scottish Book Conference held at North Texas State University in Denton, Texas, May 1979*, Birkhaüser, Boston, 1981.

[24] J. T. Moore, *Continuous colorings associated with certain characteristics of the continuum*, Discrete Math. 214 (2000), 263–273.

[25] S. Shelah and J. Zapletal, *Canonical models for* $\aleph_1$ *combinatorics*, Ann. Pure Appl. Logic 98 (1999), 217–259.

[26] S. Todorčević, *Trees and linearly ordered sets*, in: Handbook of Set-Theoretic Topology, North-Holland, 1984, 235–293.

[27] —, *Remarks on chain conditions in products*, Compositio Math. 55 (1985), 295–302.

[28] —, *Remarks on cellularity in products*, ibid. 57 (1986), 357–372.

[29] —, *Partition Problems in Topology*, Contemp. Math. 84, Amer. Math. Soc., 1989.

[30] —, *Remarks on Martin's Axiom and the continuum hypothesis*, Canad. J. Math. 43 (1991), 832–851.

[31] S. Todorčević and J. Väänänen, *Trees and Ehrenfeucht–Fraïssé games*, Ann. Pure Appl. Logic 100 (1999), 69–97.

[32] S. Todorčević and B. Veličković, *Martin's axiom and partitions*, Compositio Math. 63 (1987), 391–408.

[33] W. H. Woodin, *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal*, de Gruyter Ser. Logic Appl. 1, de Gruyter, 1999.

Department of Mathematics
University of Toronto
Toronto M5S 1A1, Canada
E-mail: larson@math.toronto.edu

C.N.R.S. (7056)
Université Paris VII
75251 Paris Cedex 05, France
E-mail: stevo@logique.jussieu.fr

Received 22 March 2000;
in revised form 8 September 2000