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Topology and dynamics of unstable attractors

by

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Abstract. This article aims to explore the theory of unstable attractors with topological tools. A short topological analysis of the isolating blocks for unstable attractors with no external explosions leads quickly to sharp results about their shapes and some hints on how Conley's index is related to stability. Then the setting is specialized to the case of flows in $\mathbb{R}^n$, where unstable attractors are seen to be dynamically complex since they must have external explosions.

1. Introduction. In the realm of continuous dynamical systems the notion of *attractor* plays a very important role because it captures the long term evolution of the system in question, and therefore it seems important to study the structure, both dynamical and topological, of these objects. Very sharp results, mainly concerning the shape, have been obtained for *stable attractors* (see for example [6], [16], [18], [26], [27]), but when it comes to *unstable attractors* not much is known, and in fact the bibliography concerning the subject is quite scarce (essentially [1]–[4], [22], [29]). Let us remark that papers [1] and [29] use Milnor's notion of attractor which is slightly different from ours.

We add some contributions to this general picture, namely regarding the shape of unstable attractors (Example 1, Theorem 7) and the dynamics in their vicinity (Theorem 17). Other interesting facts are proved in passing, and relations with Conley's index are found (Theorem 9).

A general reference for dynamical systems, which we shall follow closely, is [5]. Conley's index theory can be found in his monograph [11]. On the topological side, [7] and [19] give complete information about ANR's, and shape theory is thoroughly exposed in [8], [9], [15], [21]. Finally, should a complement on algebraic topology be needed, [28] covers everything used in

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this article. In the paper singular homology and Čech cohomology are used throughout.

Our setting will be that of a continuous flow defined on a locally compact metric space M. A compact invariant set K is *stable* if it possesses a basis of positively invariant neighbourhoods (that is, every point which comes near K stays near K forever). Given any $x \in M$, the set

$$J^+(x) := \{y \in M : y = \lim x_n \cdot t_n \text{ for some } x_n \to x,\, t_n \to +\infty\}$$

is called the *positive prolongational limit set* of x and it is easy to check that K is stable if, and only if, $J^+(x) \subseteq K$ for all $x \in K$. The sets $J^+(x)$ are always closed and invariant and, when compact, also connected (at least in locally compact phase spaces).

A compact invariant set $K \subseteq M$ is an *attractor* if its region of attraction $\mathcal{A}(K) := \{x \in M : \emptyset \neq \omega(x) \subseteq K\}$ is a neighbourhood of K in M. If an attractor K is stable then $J^+(x) \subseteq K$ for all $x \in K$, but in fact much more is true since $J^+(x) \subseteq K$ for all $x \in \mathcal{A}(K)$. If we agree to call $x \in \mathcal{A}(K)$ an *explosion point* if $J^+(x) \not\subseteq K$ (this differs slightly from the convention used in [2]), then an attractor K is unstable if and only if there exists some explosion point in K. We shall be primarily interested in *unstable attractors* which have only *internal explosions*, that is, such that every explosion point is in K. In [3] a measure of the complexity of the flow in $\mathcal{A}(K)$ is introduced under the name of instability depth, which is an ordinal number. Unstable attractors having only internal explosions correspond to the first nontrivial case of instability depth 1.

Following Conley we shall deal only with *isolated* invariant sets. These are compact invariant sets K which possess a so-called *isolating neighbourhood*, that is, a compact neighbourhood N such that K is the maximal invariant set in N, or setting

$$N^+ := \{x \in N : x \cdot [0, +\infty) \subseteq N\}, \qquad N^- := \{x \in N : x \cdot (-\infty, 0] \subseteq N\},$$

such that $K = N^+ \cap N^-$. We shall make use of a special type of isolating neighbourhoods, the so-called isolating blocks, which have good topological properties. More precisely, an *isolating block* N is an isolating neighbourhood such that there are compact sets $N^{\mathrm{i}}, N^{\mathrm{o}} \subseteq \partial N$, called the *entrance* and *exit* sets, satisfying

(1) $\partial N = N^{\mathrm{i}} \cup N^{\mathrm{o}}$,
(2) for every $x \in N^{\mathrm{i}}$ there exists $\varepsilon > 0$ such that $x \cdot [-\varepsilon, 0) \subseteq M - N$ and for every $x \in N^{\mathrm{o}}$ there exists $\delta > 0$ such that $x \cdot (0, \delta] \subseteq M - N$,
(3) for every $x \in \partial N - N^{\mathrm{i}}$ there exists $\varepsilon > 0$ such that $x \cdot [-\varepsilon, 0) \subseteq \operatorname{int} N$ and for every $x \in \partial N - N^{\mathrm{o}}$ there exists $\delta > 0$ such that $x \cdot (0, \delta] \subseteq \operatorname{int} N$.

These blocks form a neighbourhood basis of K in M (see [10] and [12]).

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2. On the shape of unstable attractors. It was mentioned in the introduction that very sharp results are known on the topology of stable attractors. More precisely, a stable attractor in an ANR has the shape of a finite-dimensional polyhedron. This situation, which is the best one can expect, does not carry over to the case of unstable attractors.

EXAMPLE 1. Every finite-dimensional compact space K can be embedded as an unstable attractor in an ANR. Hence, unstable attractors in ANR's need not have polyhedral shape in general.

To prove this, observe first that since K is compact and finite-dimensional we can assume that K is embedded in some n-dimensional cube, say $K \subseteq P := [0,1]^n$ (see for example [20, Theorem V 2, p. 56]). Consider the $(n+1)$-dimensional cube $Q := P \times [-1,1]$ and the subset $R \subseteq Q$ which is the union of the copy of K which lies in $P \times \{0\}$ (we shall call it K again) and the upper and lower lids of Q, that is, $R := K \cup (P \times \{-1,1\})$.

Let $e_{n+1} := (0, \overset{(n)}{\ldots}, 0, 1)$ be the unit vector pointing upwards and integrate the Lipschitz vector field $X : \mathbb{R}^n \times \mathbb{R} \supseteq Q \to \mathbb{R}^{n+1}$ defined by $X(q) := \mathrm{d}(q,R)e_{n+1}$, where d is any metric for Q, to obtain a global flow in $\mathbb{R}^n \times \mathbb{R}$. This flow has as fixed points exactly those in R and otherwise moves points upwards. Moreover, Q is invariant under ψ and so we shall restrict ourselves to $\psi|_Q$. This flow has the following properties:

(1) K is a compact invariant isolated set,
(2) every $q \notin R$ is moved upwards until it approaches either K or $P \times \{1\}$,
(3) if $q = (p,s)$, where $-1 < s < 1$, we distinguish three cases:
  (a) if $p \notin K$ then $\alpha(q) = (p,-1)$ and $\omega(q) = (p,1)$,
  (b) if $p \in K,\ 0 < s < 1$ then $\alpha(q) = (p,0)$ and $\omega(q) = (p,1)$,
  (c) if $p \in K,\ -1 < s < 0$ then $\alpha(q) = (p,-1)$ and $\omega(q) = (p,0)$.

Choose a point p_0 in K and let M be the quotient space obtained from Q by collapsing the upper and lower lids, together with p_0, to a single point m_0. Then M is an adjunction space obtained by gluing $P \times \{-1,1\}$ and p_0 onto a single point; as all these spaces are ANR's, so is M ([19, Theorem 1.2, p. 178]). Moreover, it is clear that ψ induces a mapping $\varphi : M \times \mathbb{R} \to M$ because the points we have identified are fixed, and φ is continuous because every restriction $\varphi|_{M\times[k,k+1]}$ is continuous by virtue of [14, Theorem 4.1, p. 262]. Thus we have obtained a flow in a compact ANR M with the following properties:

(1) K is a global isolated attractor in M (this follows from property (3)(a) above),
(2) K is unstable because there are homoclinic orbits (apply (3)(b) above).

In the example above the attractor K has external explosions. Thus it is reasonable to focus our interest on unstable attractors which only have internal explosions.

Example 2. To get an idea of how these might look like, let us mention a procedure to build them in a fashion similar to that of the previous example.

Start from a compact manifold K (without boundary) endowed with a flow φ_1 and consider the unit interval $[0,1]$ together with a dynamical system φ_2 which has 0 and 1 as fixed points and otherwise moves points away from 0 and towards 1. The product flow $\varphi(x,s,t) := (\varphi_1(x,t), \varphi_2(s,t))$ in the phase space $K \times [0,1]$ restricts to φ_1 on $K \times \{0\}$ and $K \times \{1\}$, hence these can be identified to get a flow in the quotient space $K \times \mathbb{S}^1$ (with the obvious identifications). Observe that K is an isolated unstable attractor in $K \times \mathbb{S}^1$ whose explosion points are all internal.

Regarding the shape of isolated unstable attractors whose explosions are all internal, results much in the spirit of the ones cited in the introduction above hold, and we now aim to prove this. Let us begin with a lemma which describes the structure of their isolating blocks.

Lemma 3. *Let K be an isolated unstable attractor which has only internal explosions. Then K has a basis $(N_k)_{k\in\mathbb{N}}$ of isolating blocks such that $N_k = N_k^+ \cup N_k^-$. Moreover, if K is a continuum then every connected isolating block of K is of the form $N = N^+ \cup N^-$.*

Proof. Let U be a neighbourhood of K in $\mathcal{A}(K)$ and take an isolating block $(N, N^{\mathrm{i}}, N^{\mathrm{o}})$ of K with $N \subseteq U$. We claim that $\widehat{N} = N^+ \cup N^-$ is a (compact) neighbourhood of K. If not, there would exist a sequence $x_n \to K$ contained in $N - \widehat{N}$, and this last condition amounts to the existence of sequences $s_n \le 0 \le t_n$ with $y_n = x_n \cdot s_n \in N^{\mathrm{i}}$, $z_n = x_n \cdot t_n \in N^{\mathrm{o}}$ and $x_n \cdot [s_n, t_n] \subseteq N$. As N^{i} and N^{o} are compact we can, taking subsequences if necessary, assume that $y_n \to y \in N^{\mathrm{i}}$ and $z_n \to z \in N^{\mathrm{o}}$. If s_n were bounded then it would have a convergent subsequence $s_{n_k} \to s_0$ and $x_{n_k} = y_{n_k} \cdot (-s_{n_k}) \to y \cdot (-s_0) \notin K$, contradicting the fact that $x_n \to K$. Hence we can assume that $s_n \to -\infty$ and consequently $t_n - s_n \to \infty$, so $z \in J^+(y)$. It follows that $J^+(y) \nsubseteq K$ and the flow explodes at $y \notin K$, thus contradicting our hypothesis and proving the claim.

We shall prove that $\widehat{N}$ is open and closed in N, which shows that $\widehat{N}$ is also an isolating block of K. If $\widehat{N}$ were not open in N there would exist a sequence x_n contained in $N - \widehat{N}$ and converging to $x \in \widehat{N} = N^+ \cup N^-$. As

$\widehat{N}$ is a neighbourhood of K, in fact $x \in N^+ - K$ or $x \in N^- - K$. Thus, by the same argument as before, we can obtain sequences of times $s_n \le 0 \le t_n$ such that $y_n = x_n \cdot s_n \in N^{\mathrm{i}}$, $z_n = x_n \cdot t_n \in N^{\mathrm{o}}$ and $x_n \cdot [s_n, t_n] \subseteq N$. Furthermore we shall assume that $y_n \to y \in N^{\mathrm{i}}$ and $z_n \to z \in N^{\mathrm{o}}$. Suppose $x \in N^+ - K$. Then t_n must be bounded since otherwise we could argue as before to show that $z \in J^+(y)$, contrary to our hypothesis. Hence there exists a convergent subsequence $t_{n_k} \to t_0$ and $x_{n_k} \cdot t_{n_k} \to x \cdot t_0$, which implies that $z = x \cdot t_0 \in N^{\mathrm{o}}$, a contradiction with the fact that $x \in N^+$. The argument for the case $x \in N^- - K$ is similar.

Finally, if K is a continuum and N is a connected isolating block of K, then $N = N^+ \cup N^-$ since $N^+ \cup N^-$ is open and closed in N. ■

Remark 4. If $N = N^+ \cup N^- \subseteq \mathcal{A}(K)$ is an isolating block for K, every point in N^{o} must return to K, hence enter N again some time in the future. But this happens uniformly: since N^{o} is a compact set contained in $\mathcal{A}(K)$ and no point of N^{o} is an explosion point, there exists some $T > 0$ such that $N^{\mathrm{o}} \cdot [T, +\infty) \subseteq N^+ \cup N^-$ because this set is a neighbourhood of K. In particular, $N^- \cdot \mathbb{R} = N^- \cup N^{\mathrm{o}} \cdot [0, T] \cup N^{\mathrm{o}} \cdot [T, +\infty)$ is compact since the first two summands are compact and every limit point of the third either belongs to that set or to $K \subseteq N^-$.

Proposition 5. *Let K be an isolated unstable attractor with only internal explosions and let N be an isolating block of the form $N = N^+ \cup N^-$. Then the inclusion $i : K \hookrightarrow N$ is a shape equivalence.*

Proof. We first show that if N_1 is an isolating block contained in the interior of N and with the structure $N_1 = N_1^+ \cup N_1^-$ then N_1 is a retract of N. To prove this consider for every $x \in N - \operatorname{int} N_1$ the time t_x such that $x \cdot t_x \in \partial N_1^+$ if $x \in N^+$ and $x \cdot t_x \in \partial N_1^-$ if $x \in N^-$. Define the map $r : N \to N_1$ in the following way:

$$r(x) = \begin{cases} x \cdot t_x & \text{if } x \in N - \operatorname{int} N_1, \\ x & \text{otherwise.} \end{cases}$$

It is easy to see that r is continuous, hence a retraction, since K has no external explosions.

Consider now a sequence of conveniently chosen isolating blocks of K, $N_{k+1} \subset \operatorname{int} N_k$ (all of them with the structure $N_k = N_k^+ \cup N_k^-$) and a sequence of retractions $r_k : N \to N_k$ defined as before. We shall use t_x^k to denote the time such that $x \cdot t_x^k \in \partial N_k^+$ or ∂N_k^- for x outside $\operatorname{int} N_k$. We may assume that the isolating blocks N_k form a fundamental system of neighbourhoods of K in M. The map $\varphi_k : N \times [0,1] \to N_k$ given by

$$\varphi_k(x, s) = \begin{cases} r_k(x) \cdot s t^{k+1}_{r_k(x)} & \text{if } x \in N - \operatorname{int} N_{k+1}, \\ x & \text{otherwise,} \end{cases}$$

defines a homotopy $r_k \simeq r_{k+1}$ in N_k. Moreover, the map $\phi_k : N \times [0,1] \to N$ given by

$$\phi_k(x,s) = \begin{cases} x \cdot st_x^k & \text{if } x \in N - \operatorname{int} N_k, \\ x & \text{otherwise,} \end{cases}$$

defines a homotopy $r_k \simeq \mathrm{id}_N$ (the identity in N). Now it is easy to deduce that $i : K \hookrightarrow N$ is a shape equivalence. ∎

As a consequence we obtain a cohomological criterion to detect external explosions.

Corollary 6. *Let K be a continuum which is an isolated unstable attractor. If there is a connected isolating block N such that $H^*(K) \neq H^*(N)$ then K has external explosions.*

We can now state the following result.

Theorem 7. *Let M be an ANR and let K be an isolated unstable attractor in M which has only internal explosions. Then K has polyhedral shape.*

Proof. Let N be an isolating block of K. In Proposition 5 it is proved that if N_1 is an isolating block contained in the interior of N and with the structure $N_1 = N_1^+ \cup N_1^-$ then N_1 is a retract of N. Since M is an ANR, it follows that N_1 is also an ANR, which again by Proposition 5 and West's theorem [30] implies that K has polyhedral shape. ∎

For every unstable attractor K the smallest stable set $\widehat{K}$ which contains K (it always exists since an intersection of stable sets is again stable) turns out to be an attractor contained in $\mathcal{A}(K)$ and with the same region of attraction, that is, $\mathcal{A}(\widehat{K}) = \mathcal{A}(K)$. Specifically, $\widehat{K} := \{x \in \mathcal{A}(K) : J^-(x) \cap K \neq \emptyset\}$ (see [5, Theorem 1.25, p. 64]). We shall call $\widehat{K}$ the *stabilization* of K. In the particular case of interest to us this stabilization has a simple description.

Proposition 8. *Let K be an isolated unstable attractor which does not have external explosions and let $N = N^+ \cup N^- \subseteq \mathcal{A}(K)$ be an isolating block for K* (*exists by Proposition* 3). *Then $\widehat{K} = N^- \cdot \mathbb{R}$.*

Proof. We need to check that $J^-(x) \cap K \neq \emptyset$ if and only if $x \in N^- \cdot \mathbb{R}$. Assuming first that $J^-(x) \cap K \neq \emptyset$, there exist sequences $x_n \to x$ and $t_n \to -\infty$ such that $x_n \cdot t_n \to K$, so $(x_n \cdot t_n)_{n \geq n_0} \subseteq N$ for some $n_0 \in \mathbb{N}$ because N is a neighbourhood of K. Since $N = N^+ \cup N^-$, by taking an appropriate subsequence we can suppose that either $(x_n \cdot t_n)_{n \geq n_0} \subseteq N^+$ or $(x_n \cdot t_n)_{n \geq n_0} \subseteq N^-$. In the first case it is clear that $x \in K$ and we are finished. In the second case $(x_n)_{n \geq n_0} \subseteq N^- \cdot \mathbb{R}$, which is compact (Remark 4), hence $x \in N^- \cdot \mathbb{R}$.

The other implication is easier: if $x \in N^- \cdot \mathbb{R}$, then for some $t \in \mathbb{R}$ we have $x \cdot t \in N^-$ and $K \supseteq \alpha(x \cdot t) \subseteq J^-(x \cdot t) = J^-(x)$. Since $\alpha(x \cdot t) \neq \emptyset$, the proof is complete. ■

Recall that the *Conley index* of an isolated invariant set K with isolating block N is the homotopy type of the compact pair (N, N^o), N^o being the exit set. However, the cohomology module $CH^*(K) := H^*(N, N^o)$ is normally used for computational purposes, and is called the *cohomological Conley index*. In the case of interest to us this index gives some information about the unstability taking place in K.

Theorem 9. *Let K be a connected isolated unstable attractor which has only internal explosions. Then* $\#($*components of* $\widehat{K} - K) \leq \operatorname{rank} CH^1(K) + 1$. *Moreover, if $H^1(K)$ is trivial, then equality holds.*

Proof. Let N be an isolating block for K of the form $N = N^+ \cup N^-$ contained in $\mathcal{A}(K)$. The first half of Athanassopoulos' proof of [2, Proposition 3.2, p. 205] shows that $\varphi|_{N^o \times \mathbb{R}} : N^o \times \mathbb{R} \to N^o \cdot \mathbb{R}$ is a homeomorphism. Now observe that by Proposition 8, $\widehat{K} - K = (N^- \cdot \mathbb{R}) - K = (N^- - K) \cdot \mathbb{R} = N^o \cdot \mathbb{R}$, hence N^o is a deformation retract of $\widehat{K} - K$ and both have the same number of connected components. Now, it can be proved as in Proposition 5 that the inclusion $j : (N^-, N^o) \hookrightarrow (N, N^o)$ is a shape equivalence, which implies that $CH^*(K)$ is also given by the Čech cohomology of (N^-, N^o). The cohomology sequence of the pair (N^-, N^o) then reads

$$0 = \widetilde{H}^0(N^-) \to \widetilde{H}^0(N^o) \to CH^1(K) \to H^1(N^-) = H^1(K) \to \cdots,$$

where the equality $H^*(K) = H^*(N^-)$ (which is readily deduced from the proof of Proposition 5) and the hypothesis that K is connected have been used. Truncating this sequence to obtain

$$0 \to \widetilde{H}^0(N^o) \to CH^1(K) \to \operatorname{im} CH^1(K) \subseteq H^1(K) \to 0$$

and applying the rank formula we get $\operatorname{rank} \widetilde{H}^0(N^o) = \operatorname{rank} CH^1(K) - \operatorname{rank} \operatorname{im} CH^1(K) \leq \operatorname{rank} CH^1(K)$ (clearly, equality holds if $H^1(K) = 0$). It only remains to observe that $\operatorname{rank} \widetilde{H}^0(N^o) = \#(\text{components of } N_0) - 1 = \#(\text{components of } \widehat{K} - K) - 1$, and the inequality is proved. ■

Corollary 10. *Let K be a connected isolated unstable attractor whose Conley index satisfies $CH^1(K) = 0$. If K disconnects $\widehat{K}$ then external explosions occur in $\mathcal{A}(K)$.*

In [2, Proposition 3.6, p. 208] another bound for the number of connected components of $\widehat{K} - K$ is given, namely $\operatorname{rank} H^1(\mathcal{A}(K))$. It should be observed that our bound is of a genuinely different nature, because it depends only on the local dynamics near K. This can be checked in the following example.

Example 11. Consider the cylinder $C := \mathbb{S}^1 \times [0,1] \subseteq \mathbb{R}^3$ and the $\mathcal{C}^\infty$ vector field, defined in cylindrical coordinates,

$$X(\theta, h) := (-\sin(\theta), \cos(\theta), h(1-h)).$$

The flow obtained by integrating this field has the property that every point in $\mathbb{S}^1 \times (0,1)$ spirals forwards to $\mathbb{S}^1 \times \{1\}$ and backwards to $\mathbb{S}^1 \times \{0\}$, both of which are periodic orbits. Identify these two orbits to a single periodic orbit to get an unstable attractor K for a flow in the 2-dimensional torus. Its Conley index is easily calculated and turns out to be that of a pointed cone, therefore trivial. Hence Theorem 9 is sharp, because it asserts $\widehat{K} - K$ is connected, as it effectively is (here $\widehat{K}$ is the whole torus, which is not disconnected by K). On the other hand, the bound in [2] informs us that the number of components in $\widehat{K} - K$ is less than or equal to 2.

In [2, Theorem 3.7, p. 200] it is proven that the cohomology groups of K are finitely generated. This agrees with the fact just shown that K has the shape of a polyhedron.

3. Isolated unstable attractors in $\mathbb{R}^n$. This section is devoted to an analysis of isolated unstable attractors in $\mathbb{R}^n$, together with some particular results in the plane (Theorem 18). Our aim is to prove that every connected isolated unstable attractor in $\mathbb{R}^n$ must have external explosions, and this will be done by showing that no isolating blocks of the form prescribed in Lemma 3 can ever exist. The rest of the proof is purely topological in nature and is contained in the lemmata which precede Theorem 17.

Lemma 12. *Let $K \subseteq \mathbb{R}^n$ be a continuum. Then*

(1) *$\mathbb{R}^n - K$ is an open set with a finite or countable number of connected components*,
(2) *for every neighbourhood V of K in $\mathbb{R}^n$, almost all connected components of $\mathbb{R}^n - K$ (i.e., all but a finite number) lie in V.*

The proof of this lemma is easy and we omit it.

Lemma 13. *Let $K \subseteq \mathbb{R}^n$ be connected and compact. There exists a decreasing neighbourhood basis $(V_k)_{k\in\mathbb{N}}$ of K in $\mathbb{R}^n$ such that*

(1) *each V_k is a connected and compact ($\mathcal{C}^\infty$) n-manifold*,
(2) *for every component U of $\mathbb{R}^n - K$ and every $k \in \mathbb{N}$ the set $U - V_k$ is connected (possibly empty).*

Proof. First of all we show that for every $\varepsilon > 0$ there exists a neighbourhood W of K satisfying (1) and the condition $W \subseteq B_\varepsilon(K)$. Then it will be proven that every such W can be reduced so that it also satisfies (2).

Fix $\varepsilon > 0$ and approximate the continuous function $\mathrm{d}_K(x) := \mathrm{d}(x, K)$, where d denotes Euclidean distance, by a $\mathcal{C}^\infty$ function $\delta_K : \mathbb{R}^n \to [0, +\infty)$

such that $\mathrm{d}_K \leq \delta_K \leq \mathrm{d}_K + \varepsilon/3$ (see [25, Exercise 36, p. 152]). Observe that these inequalities and the fact that $\mathrm{d}_K|_K = 0$ imply that $\delta_K(\mathbb{R}^n) \supseteq [\varepsilon/3, +\infty)$ and, by Sard's theorem ([23, Corollary, p. 11]), a regular value $a \in (\varepsilon/3, 2\varepsilon/3)$ can be found. Then $\delta_K^{-1}((-\infty, a])$ is a compact n-manifold ([23, Lemma 3, p. 12]) in $\mathbb{R}^n$ which is a neighbourhood of K because for $x \in K$ we have $\delta_K(x) \leq \mathrm{d}_K(x) + \varepsilon/3 = \varepsilon/3 < a$, so $K \subseteq \delta_K^{-1}((-\infty, a))$, which is an open set contained in $\delta_K^{-1}((-\infty, a])$. Setting W equal to the component of $\delta_K^{-1}((-\infty, a])$ which contains K completes the first step of the proof.

Now we shall modify W in such a way that it also satisfies (2). Because of Lemma 12 almost every component of $\mathbb{R}^n - K$ is contained in W, so we only have to deal with a finite number of them. Call one of these U and let us show first that $U - W$ has a finite number of components. In fact, since U is a component of $\mathbb{R}^n - K$, we have $\partial U \subseteq K$, and as W is a neighbourhood of K, it follows that $\partial U \cap \partial W \subseteq K \cap \partial W = \emptyset$, hence $\overline{U} \cap \partial W = (U \cup \partial U) \cap \partial W = (U \cap \partial W) \cup (\partial U \cap \partial W) = U \cap \partial W$. We then see that $U \cap \partial W$ is an open and closed set in ∂W, so it must be a union of some of its (finite) components (remember that ∂W is a closed $(n-1)$-manifold, being the boundary of a compact n-manifold). In particular, $U \cap \partial W$ is again a closed $(n-1)$-manifold so it separates U into a finite number of components, by Lefschetz duality ([28, Theorem 19, p. 297]). Now, if two points $p, q \in U - W$ cannot be connected by a path in $U - W$ (path connectedness and connectedness are equivalent because we are dealing with open sets in $\mathbb{R}^n$), then every path in U joining them (U is connected) meets W, so it must also meet ∂W. Hence they cannot be connected in $U - (U \cap \partial W)$ either, which means they are in different components of $U - (U \cap \partial W)$. But these are finite in number, so the same can be said about $U - W$.

To complete the proof it will be enough to show that W can be reduced so that $U - W$ is connected. This can be done as follows: suppose as before that there exist $p, q \in U - W$ which cannot be connected by any path in $U - W$. Since U is connected, we can find a polygonal path in U connecting p and q, and this path can be easily chosen so that it crosses ∂W transversally. Deleting an appropriate tubular neighbourhood of this path from W such that the result is an n-manifold we get a new neighbourhood of K which we shall denote again by W and which still satisfies (1) (observe that this process may disconnect W—just in the 2-dimensional case, if the tubular neighbourhood is reasonably chosen—but then only the component containing K should be kept). But now p and q are connected in $U - W$, and any two points which were connected before are still connected. Hence this new W separates U into at least one component less than the previous one, and since the number of components of $U - W$ was finite, repeating this process finitely many times proves the claim. ∎

Lemma 14. *Let $\mathbb{R}^n$ be the disjoint union of two connected sets A and B. If ∂A (or ∂B) is compact, then it is connected.*

Proof. For every $k \in \mathbb{N}$ let A_k and B_k be the open balls of radius $1/k$ centred at A and B respectively. Observe that, since for every $x \in \partial A = \partial B$ we have $\mathrm{d}(x, A) = 0 = \mathrm{d}(x, B)$, the inclusion $\partial A \subseteq \bigcap_{k\in\mathbb{N}} A_k \cap B_k$ follows. Moreover, if $x \in A_k \cap B_k$ then $B_{1/k}(x) \cap A \neq \emptyset$ and $B_{1/k}(x) \cap B \neq \emptyset$ so $B_{1/k}(x) \cap \partial A \neq \emptyset$ because the balls in $\mathbb{R}^n$ are connected. Therefore $x \in B_{1/k}(\partial A)$ and we deduce that $(\overline{A_k \cap B_k})_{k\in\mathbb{N}}$ is a neighbourhood basis of ∂A in $\mathbb{R}^n$. If we prove that every $A_k \cap B_k$ is connected the proof will be finished, since then the compact set ∂A will be the intersection of a decreasing neighbourhood basis of closed connected sets, hence connected. But this is easy, because the Mayer–Vietoris exact sequence in reduced singular homology for the union $\mathbb{R}^n = A_k \cup B_k$,

$$\cdots \to \widetilde{H}_1(\mathbb{R}^n) \to \widetilde{H}_0(A_k \cap B_k) \to \widetilde{H}_0(A_k) \oplus \widetilde{H}_0(B_k) \to \cdots,$$

reads

$$\cdots \to 0 \to \widetilde{H}_0(A_k \cap B_k) \to 0 \to \cdots$$

and so $\widetilde{H}_0(A_k \cap B_k) = 0$ and $A_k \cap B_k$ is connected. ∎

Lemma 15. *Let $K \subseteq \mathbb{R}^n$ be connected and compact. There exists a decreasing neighbourhood basis $(V_k)_{k\in\mathbb{N}}$ of K such that*

(1) *each V_k is a connected and compact ($\mathcal{C}^\infty$) n-manifold,*
(2) *for every component U of $\mathbb{R}^n - K$ and every $k \in \mathbb{N}$ the set $U \cap \partial V_k$ is connected (possibly empty).*

Proof. We shall see that the neighbourhood basis satisfying (1) and (2) in Lemma 13 suffices. So choose a particular V belonging to this basis and let us show (using Lemma 14) that $U \cap \partial V$ is connected, if not empty, for every component U of $\mathbb{R}^n - K$. Set $A = U - V$ and $B = V \cup (\mathbb{R}^n - U)$. It is clear that $\mathbb{R}^n$ is the disjoint union of A and B, because $\mathbb{R}^n - B = (\mathbb{R}^n - V) \cap U = A$. Observe that A is connected because of the special properties of V, so $\overline{A} \subseteq \mathbb{R}^n - K$ is also connected and it must lie entirely in a component of $\mathbb{R}^n - K$, which can only be U. Hence $\overline{A} \subseteq U \cap \overline{\mathbb{R}^n - V}$ and the reverse inclusion is easily seen to hold because U is open, so $\overline{A} = U \cap \overline{\mathbb{R}^n - V}$. Therefore $\partial A = \overline{A} \cap B = \overline{\mathbb{R}^n - V} \cap V \cap U = \partial V \cap U$. This set is compact (see the proof of Lemma 13) so if B is connected Lemma 14 would apply and finish the proof. But B is indeed connected, because

$$\mathbb{R}^n - U = \bigcup_{U' \neq U} (K \cup U'),$$

where U' ranges over the components of $\mathbb{R}^n - K$. Hence $\mathbb{R}^n - U$ is a union of connected sets (because U' and K are connected and $\emptyset \neq \partial U' \subseteq K$) which have nonempty intersection, so it is connected. Then $B = V \cup (\mathbb{R}^n - U)$ is a union of two connected sets with $\emptyset \neq \partial(\mathbb{R}^n - U) = \partial U \subseteq K \subseteq V$, hence connected. ∎

Remark 16. The situation described above can be easily pictured when $n = 2$ because then for every bounded component U of $\mathbb{R}^2 - K$ the set $U - V_k$ is homeomorphic to a disk (if not empty). To see this, think of $\mathbb{R}^2 \subseteq \mathbb{S}^2$ and recall from the proof that $\mathbb{R}^2 - (U - V_k)$ is connected, hence so is $\mathbb{S}^2 - (U - V_k)$. Now $U - V_k$ is an open set in $\mathbb{S}^2$ which is connected and whose complement is connected, so [13, Theorem 2.2, p. 202] shows that $U - V_k$ is an open disk.

Theorem 17. *Let $K \subseteq \mathbb{R}^n$ be a connected isolated attractor. If K is unstable, then it must have external explosions.*

Proof. Assume that K has no external explosions and pick an isolating block $N = N^+ \cup N^-$ for K contained in its basin of attraction, which exists by Lemma 3. Let V be a neighbourhood of K contained in N with the properties given by Lemma 15. Then for every component U of $\mathbb{R}^n - K$ the set $U \cap \partial V$ is a connected subset of $N^+ \cup N^-$, both of which are closed. Since $N^+ \cap N^- = K$ is disjoint from ∂V, it follows that $U \cap \partial V \subseteq N^+$ or $U \cap \partial V \subseteq N^-$.

As K is assumed to be an isolated unstable attractor there must exist a homoclinic orbit, i.e. there exists some $x \notin K$ such that $\emptyset \neq \alpha(x), \omega(x) \subseteq K$ ([5, Theorem 1.1, p. 114 and Corollary 1.2, p. 116]). This homoclinic orbit is contained in some component U of $\mathbb{R}^n - K$ but cannot be wholly contained in N (otherwise N would not isolate K), so it must meet ∂N and $U \cap \partial V$ in at least two points. But then neither $U \cap \partial V \subseteq N^+$ nor $U \cap \partial V \subseteq N^-$, which is a contradiction. ∎

If we restrict ourselves to the case $n = 2$ (the plane), quite strong results are available.

Theorem 18. *Every connected isolated global attractor K in $\mathbb{R}^2$ is stable.*

Proof. If K were unstable there would exist a point $x_0 \in \mathbb{R}^2 - K$ such that $\emptyset \neq \omega(x_0), \alpha(x_0) \subseteq K$, and we can assume that x_0 lies in the unbounded component U of $\mathbb{R}^2 - K$ (if not, the argument is only slightly different). Collapse K to a single point p and consider the flow $\widehat{\varphi}$ induced in the quotient space $\mathbb{R}^2/K$. Then $\{p\}$ is an isolated global attractor of $\widehat{\varphi}$ and $\overline{U} = U \cup \{p\}$ is homeomorphic to $\mathbb{R}^2$ (where the closure of U is taken in $\mathbb{R}^2/K$). This last assertion can be proved as follows: the set $K^* := \mathbb{R}^2 - U \supseteq K$ (equal to K plus the bounded components of $\mathbb{R}^2 - K$) does not disconnect the plane. Then $\mathcal{D} := \{K^*\} \cup \{\{x\} : x \notin K^*\}$ is an upper semicontinous decomposition of $\mathbb{R}^2$ none of whose elements separates the plane, hence the quotient space $\mathbb{R}^2/K^* \cong \mathbb{R}^2/\mathcal{D}$ is homeomorphic to $\mathbb{R}^2$ by [24, Theorem 22]. But the closure of U in $\mathbb{R}^2/K$ is homeomorphic $\mathbb{R}^2/K^*$, so the assertion follows and we have reduced the proof to the case where K is a single point p.

In $\mathbb{R}^2/K$ the condition about the limit sets of x_0 says $\omega(x_0) = \alpha(x_0) = \{p\}$. This implies that $\gamma(x_0)$ is disjoint from its limit sets and homeomorphic to $\mathbb{R}$. But then $\overline{\gamma(x_0)} = \gamma(x_0) \cup \{p\}$ is a one-point compactification of $\gamma(x_0) \cong \mathbb{R}$, hence it must be homeomorphic to $\mathbb{S}^1$. It follows that $\overline{\gamma(x_0)}$ separates $\overline{U}$ into two connected components, exactly one of which is bounded (say U_{x_0}), and with common boundary $\overline{\gamma(x_0)} = \gamma(x_0) \cup \{p\}$. Observe that U_{x_0} and $\overline{U}_{x_0}$ are invariant and homeomorphic to an open disk and a closed disk, respectively.

Now $x_0 \in J^+(p)$, so $x_0 \in \widehat{\{p\}}$ (the stabilization of the attractor $\{p\}$), and since $\widehat{\{p\}}$ is compact and invariant, $\overline{\gamma(x_0)} \subseteq \widehat{\{p\}}$. But now $\widehat{\{p\}}$ is a global stable attractor in $\overline{U} \cong \mathbb{R}^2$, hence by [17] its shape must be trivial, so it does not disconnect $\overline{U}$ and it follows that $U_{x_0} \subseteq \widehat{\{p\}}$. By [2, Proposition 4.4, p. 211] this implies that $\alpha(x) = \{p\}$ for every $x \in U_{x_0}$ so the argument and notations introduced above for x_0 extend to all $x \in \overline{U}_{x_0}$. That is, if $x \in \overline{U}_{x_0}$ then $\overline{\gamma(x)} = \gamma(x) \cup \{p\}$ separates $\mathbb{R}^2$ into two connected components. If we denote by U_x the bounded one, it is an invariant set with boundary $\overline{\gamma(x)}$. Observe that if $y \in U_x$ then $\gamma(y) \subseteq U_x$, and since U_x is homeomorphic to a disk, $U_y \subseteq U_x$.

Let N be an isolating neighbourhood for p. It is clear that for every $p \neq x \in U_{x_0}$ the inclusion $\gamma(x) \subseteq N$ cannot hold since otherwise p would not be isolated by N, hence $\gamma(x) \cap \partial N \neq \emptyset$ and $\overline{U}_x \cap \partial N \neq \emptyset$. If $x, y \in U_{x_0}$ are not in the same trajectory, then $x \in U_y$ or $y \in U_x$, so $\overline{U}_x \subseteq \overline{U}_y$ or $\overline{U}_x \subseteq \overline{U}_y$. In any case the intersection $\overline{U}_x \cap \overline{U}_y \cap \partial N$ coincides with either $\overline{U}_x \cap \partial N$ or $\overline{U}_y \cap \partial N$ and therefore the family $\{\overline{U}_x \cap \partial N\}_{p \neq x \in U_{x_0}}$ has the finite intersection property. By the compactness of ∂N there exists $y \in \bigcap_{p \neq x \in U_{x_0}} \overline{U}_x \cap \partial N$, and in particular $y \neq p$. However, $y \in \overline{U}_{x_0}$, hence U_y is an open disk whose boundary contains p. Consequently, there must exist some $x \in U_y \cap \operatorname{int} N$, which implies $U_x \subseteq U_y$ and $y \in \overline{U}_x = U_x \cup \gamma(x) \cup \{p\} \subseteq U_y \cup \{p\}$; but this is a contradiction since $y \notin U_y \cup \{p\}$. ∎

Let us remark here that the conclusion of Theorem 18 is false if the attractor K is not global, as Mendelson's famous example of an isolated unstable attractor in the plane shows ([22]). However, *every* isolated invariant continuum $K \subseteq \mathbb{R}^2$ has polyhedral shape. To prove this note that by Alexander's duality $H^1(K) \cong \widetilde{H}_0(\mathbb{R}^2 - K)$, hence $H^1(K)$ is free and finitely generated. Now it follows from a theorem of Borsuk on plane continua [9, Theorem 7.1, p. 221] that K has the shape of a polyhedron (in fact, a finite bouquet of circles).

References

[1] P. Ashwin and M. Timme, *Unstable attractors: existence and robustness in networks of oscillators with delayed pulse coupling*, Nonlinearity 18 (2005), 2035–2060.

[2] K. Athanassopoulos, *Explosions near isolated unstable attractors*, Pacific J. Math. 210 (2003), 201–214.

[3] —, *Remarks on the region of attraction of an isolated invariant set*, Colloq. Math. 104 (2006), 157–167.

[4] J. Auslander, N. P. Bhatia, and P. Seibert, *Attractors in dynamical systems*, Bol. Soc. Mat. Mexicana (2) 9 (1964), 55–66.

[5] N. P. Bhatia and G. P. Szegö, *Stability Theory of Dynamical Systems*, Grundlehren Mat. Wiss. 161, Springer, 1970.

[6] S. A. Bogatyĭ and V. I. Gutsu, *On the structure of attracting compacta*, Differentsial'nye Uravneniya 25 (1989), 907–909, 920 (in Russian).

[7] K. Borsuk, *Theory of Retracts*, Monografie Mat. 44, Państwowe Wydawnictwo Naukowe, Warszawa, 1967.

[8] —, *Concerning homotopy properties of compacta*, Fund. Math. 62 (1968), 223–254.

[9] —, *Theory of Shape*, Monografie Mat. 59, Państwowe Wydawnictwo Naukowe, Warszawa, 1975.

[10] R. C. Churchill, *Isolated invariant sets in compact metric spaces*, J. Differential Equations 12 (1972), 330–352.

[11] C. Conley, *Isolated Invariant Sets and the Morse Index*, CBMS Reg. Conf. Ser. Math. 38, Amer. Math. Soc., 1978.

[12] C. Conley and R. Easton, *Isolated invariant sets and isolating blocks*, Trans. Amer. Math. Soc. 158 (1971), 35–61.

[13] J. B. Conway, *Functions of One Complex Variable*, 2nd ed., Grad. Texts in Math. 11, Springer, 1978.

[14] J. Dugundji, *Topology*, Allyn and Bacon, 1966.

[15] J. Dydak and J. Segal, *Shape Theory. An Introduction*, Lecture Notes in Math. 688, Springer, 1978.

[16] A. Giraldo, M. A. Morón, F. R. Ruiz del Portal, and J. M. R. Sanjurjo, *Shape of global attractors in topological spaces*, Nonlinear Anal. 60 (2005), 837–847.

[17] A. Giraldo and J. M. R. Sanjurjo, *On the global structure of invariant regions of flows with asymptotically stable attractors*, Math. Z. 232 (1999), 739–746.

[18] B. Günther and J. Segal, *Every attractor of a flow on a manifold has the shape of a finite polyhedron*, Proc. Amer. Math. Soc. 119 (1993), 321–329.

[19] S. Hu, *Theory of Retracts*, Wayne State Univ. Press, 1965.

[20] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton Math. Ser. 4, Princeton Univ. Press, 1948.

[21] S. Mardešić and J. Segal, *Shape Theory. The Inverse System Approach*, North-Holland Math. Library 26, North-Holland, 1982.

[22] P. Mendelson, *On unstable attractors*, Bol. Soc. Mat. Mexicana (2) 5 (1960), 270–276.

[23] J. W. Milnor, *Topology from the Differentiable Viewpoint*, The University Press of Virginia, 1965.

[24] R. L. Moore, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc. 27 (1925), no. 4, 416–428.

[25] E. Outerelo and J. M. Ruiz, *Topología diferencial*, Addison-Wesley Iberoamericana, 1998.

[26] J. M. R. Sanjurjo, *Multihomotopy, Čech spaces of loops and shape groups*, Proc. London Math. Soc. (3) 69 (1994), 330–344.
[27] —, *On the structure of uniform attractors*, J. Math. Anal. Appl. 192 (1995), 519–528.
[28] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, 1966.
[29] M. Timme, F. Wolf, and T. Geisel, *Unstable attractors induce perpetual synchronization and desynchronization*, Chaos 13 (2003), 377–387.
[30] J. E. West, *Mapping Hilbert cube manifolds to ANR's: a solution of a conjecture of Borsuk*, Ann. of Math. (2) 106 (1977), 1–18.

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