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Ordered group invariants for one-dimensional spaces

by

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Abstract. We show that the Bruschlinsky group with the winding order is a homeomorphism invariant for a class of one-dimensional inverse limit spaces. In particular we show that if a presentation of an inverse limit space satisfies the Simplicity Condition, then the Bruschlinsky group with the winding order of the inverse limit space is a dimension group and is a quotient of the dimension group with the standard order of the adjacency matrices associated with the presentation.

1. Introduction. Ordered groups have been useful invariants for the classification of many different categories. A class of ordered groups, *dimension groups*, was used in the study of C^*-algebras to classify AF-algebras ([6]), and Giordano, Herman, Putnam and Skau ([8, 9]) defined (simple) dimension groups in terms of dynamical concepts to give complete information about the orbit structure of zero-dimensional minimal dynamical systems. Swanson and Volkmer ([15]) showed that the dimension group of a primitive matrix is a complete invariant for *weak equivalence*, which is called *C^*-equivalence* by Bratteli, Jørgensen, Kim and Roush ([5]). And Barge, Jacklitch and Vago ([3]) showed that, for a certain class of one-dimensional inverse limit spaces, two spaces are homeomorphic if and only if their associated substitutions are weak equivalent, and that if two inverse limit spaces are homeomorphic and the squares of their connection maps are orientation preserving, then the dimension groups of the adjacency matrices associated with the substitutions are order isomorphic.

A recent development ([2, 3, 4, 7, 8, 15]) is the refinement of $\check{H}^1(X)$ as a topological invariant for certain one-dimensional spaces X, by making this group an ordered group. Here $\check{H}^1(X)$ is the direct limit of first cohomology groups on graphs approximating the space X. There is a natural order on the first cohomology of a graph (a coset is positive if it contains a nonnegative

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function), and the *standard order* on $\check{H}^1(X)$ is the direct limit order derived from the natural graph orders (see Definition 3.7). Except for parts of [4] and [7], the ordered cohomology results have involved the standard order.

A second order on $\check{H}^1(X)$, the *winding order*, is geometrically natural as its positive elements are the homotopy classes of continuous orientation preserving maps from X to S^1. Boyle and Handelman ([4]) defined the *winding order* for suspension spaces of zero-dimensional dynamical systems, and showed that in some (but not all) cases it agrees with the standard order. Forrest ([7]) defined the winding order for the first Čech cohomology groups of directed graphs (thus taking the step of removing dynamics), and used this to show that whenever two one-dimensional inverse limit spaces are *pro-homotopy equivalent*, then their first Čech cohomology groups with the standard order are order isomorphic.

In this paper, we extend the definition of the winding order to a large class of one-dimensional spaces, "compact branched matchbox manifolds". We show that, for a compact connected orientable branched matchbox manifold with an inverse limit presentation satisfying the Simplicity Condition, the Bruschlinsky group with the winding order is a simple dimension group, and the winding order equals the standard order. This is a natural extension of the relations between zero-dimensional minimal systems and simple dimension groups in Giordano, Herman, Putnam and Skau ([8, 9]) to an appropriate class of one-dimensional spaces. As a corollary we obtain an independent proof of some results of Forrest and Barge, Jacklitch and Vago ([7, 3]) computing dimension group invariants for the oriented generalized one-dimensional solenoids of Williams ([16, 17, 18]).

The outline of the paper is as follows. In Section 2, using work of Aarts and Oversteegen ([1]), Mardešić and Segal ([12]) and Rogers ([14]), we define compact connected orientable branched matchbox manifolds, and show that they all have presentations by orientation preserving maps of finite directed nondegenerate graphs. In Section 3, we show that the Bruschlinsky group with the winding order of a compact connected orientable branched matchbox manifold with the Simplicity Condition is order isomorphic to the direct limit of the graph groups with the standard order defined from the presentation (and therefore the winding and standard orders agree). And in Section 4, we recall the axioms for one-dimensional generalized solenoids and calculate the Bruschlinsky groups with the winding order of an example in which the Bruschlinsky group is not given by the obvious direct limit of presenting matrices.

2. Branched matchbox manifolds and ordered groups. Aarts and Oversteegen ([1]) defined a *matchbox manifold* to be a separable metric space Y such that each point $y \in Y$ has a neighborhood which is homeomorphic to $S_y \times I_y$, where S_y is a zero-dimensional space and I_y is an open interval. For a topological embedding $h : S_y \times I_y \to Y$, they called $h(S_y \times I_y)$ a *matchbox neighborhood* of $y \in Y$. A matchbox manifold Y is called *orientable* if each arc component C_α, $\alpha \in A$, of Y has a parameterized immersed arc $p_\alpha : \mathbb{R} \to C_\alpha$ such that each point $y \in Y$ has a matchbox neighborhood $h(S_y \times I_y)$ with the following property: for each $\alpha \in A$ and each $t \in \mathbb{R}$ with $p_\alpha(t) \in h(S_y \times I_y)$ there exists an open interval I containing t such that $\mathrm{pr}_2 \circ h^{-1} \circ p_\alpha$ is increasing on I, where pr_2 is the canonical projection from $S_{p_\alpha(t)} \times I_{p_\alpha(t)}$ to $I_{p_\alpha(t)}$.

THEOREM 2.1 ([1]). *For a one-dimensional space Y, the following are equivalent*:

(1) *Y is an orientable matchbox manifold.*

(2) *Y is the phase space of a flow without rest point.*

(3) *There exists a cross section K with return time map r_K such that Y is the standard suspension of (K, r_K).*

Branched matchbox manifold. We define a *branched matchbox* to be a topological space homeomorphic to $U = ((S_1 \times (-1, 0]) \cup (S_2 \times [0, 1)))/{\sim}$ such that S_1 and S_2 are zero-dimensional separable metrizable spaces and there is a (closed) equivalence relation $\approx$ on $S_1 \cup S_2$ such that

(1) for every $s_1 \in S_1$ ($\sigma_2 \in S_2$, respectively) there exists at least one $s_2 \in S_2$ ($\sigma_1 \in S_1$, respectively) such that $s_1 \approx s_2$ ($\sigma_1 \approx \sigma_2$, respectively),

(2) $(S_1 \cup S_2)/{\approx}$ is a zero-dimensional metrizable space with the quotient topology, and

(3) $(s_1, i) \sim (s_2, j)$ if and only if either $s_1 \approx s_2$ and $i = j = 0$ or $s_1 = s_2$ and $i = j$.

REMARK 2.2. In this paper, we will always be concerned with the case where S_1 and S_2 are compact.

For $s_1 \in S_1$ and $s_2 \in S_2$ such that $s_1 \approx s_2$, the set

$$((\{s_1\} \times (-1, 0]) \cup (\{s_2\} \times [0, 1)))/{\sim}$$

is called a *match.*

A *branched matchbox manifold* is a separable metrizable space Y together with a collection of maps called *charts* such that

(1) a chart is a homeomorphism $h : V \to U$ where V is an open set in Y and U is a branched matchbox,

(2) every point in Y is in the domain of some chart, and

(3) for charts $h_1 : V_1 \to U_1$ and $h_2 : V_2 \to U_2$ the change of coordinates map $h_2 \circ h_1^{-1} : h_1(V_1 \cap V_2) \to h_2(V_1 \cap V_2)$ is continuous.

Every branched matchbox U has the direction given by the second coordinate, with a continuous projection $p_U : U \to (-1, 1)$ defined by $[(z, j)] \mapsto j$.

Following the approach of Aarts and Oversteegen ([1, §3]), we call a branched matchbox manifold Y *orientable* if it can be covered by branched matchboxes with directions agreeing on overlaps, i.e., there are oriented branched matchboxes U_i with projections $p_i : U_i \to (-1, 1)$, open sets V_i covering Y, and homeomorphisms $h_i : V_i \to U_i$ such that for every i, j and every locally one-to-one curve $\gamma : [0, 1] \to V_i \cap V_j$, $p_i \circ h_i \circ \gamma$ is increasing if and only if $p_j \circ h_j \circ \gamma$ is increasing. The particular collection of charts, maximal with respect to this change of coordinate property, is called an *orientation* of the branched matchbox manifold Y.

Ordered group. A *preordered group* is a pair (G, G_+) where G is an Abelian group, and the *positive cone* G_+ is a submonoid of G which generates G. We write $g_1 \le g_2$ if $g_2 - g_1 \in G_+$ for $g_1, g_2 \in G$. If (G, G_+) satisfies the additional condition $G_+ \cap -G_+ = \{0\}$, then (G, G_+) is called an *ordered group*.

An *order unit* in a preordered group is an element $u \in G_+$ such that for every $g \in G$ there exists a positive integer $n = n(g)$ such that $g \le nu$. A preordered group (G, G_+) is *unperforated* if for every $g \in G$ and positive integer n, $ng \in G_+$ implies $g \in G_+$. We say that an ordered group (G, G_+) has the *Riesz Interpolation Property* if given $g_1, g_2, h_1, h_2 \in G$ with $g_i \le h_j$ $(i, j = 1, 2)$, there is a $k \in G$ such that $g_i \le k \le h_j$.

Bruschlinsky group with the winding order. For a compact metric space Y, let $C(Y, S^1)$ be the set of continuous functions from Y to S^1, and

$$R(Y) = \{\phi \in C(Y, S^1) \mid \phi(y) = \exp(2\pi i g(y)) \text{ for some } g \in C(Y, \mathbb{R})\}.$$

Then $R(Y)$ is the subgroup of functions homotopic to a constant map in $C(Y, S^1)$. The *Bruschlinsky group* of Y ([13, §4.3]) is given by

$$\mathrm{Br}(Y) = C(Y, S^1)/R(Y).$$

It is well known that $\check{H}^1(Y)$, the first Čech cohomology group of Y, is isomorphic to the Bruschlinsky group of Y ([4, 10]).

Now suppose that Y is an oriented compact branched matchbox manifold. Let $C_\oplus(Y, S^1)$ be the set of $\phi \in C(Y, S^1)$ such that there exists a map $\psi \in R(Y)$ for which $\phi \cdot \psi$ is non-orientation reversing, i.e., for every orientation preserving parameterized curve $\gamma : \mathbb{R} \to Y$, $(\phi \cdot \psi)(\gamma)(t)$ does not move in the clockwise direction as $t \in \mathbb{R}$ increases.

Define $\mathrm{Br}_\oplus(Y) = \{[\phi] \mid \phi \in C_\oplus(Y, S^1)\}$. Then $(\mathrm{Br}(Y), \mathrm{Br}_\oplus(Y))$ is a preordered group. We call this preorder the *winding order* ([4, §4]).

Remark 2.3 ([4, 4.7]). It is possible that the Bruschlinsky group with the winding order of a compact orientable space is not an ordered group.

Observation 2.4. *Homeomorphic orientable compact metric spaces have order-isomorphic Bruschlinsky groups with the winding order.*

Proposition 2.5 ([10]). *The Bruschlinsky group of a compact branched matchbox manifold is a torsion-free group.*

Recall that a *continuum* is a compact connected metric space.

Lemma 2.6 ([10]). *Let Y be a continuum, $\phi \in C(Y, S^1)$, and $p_n : S^1 \to S^1$ defined by $z \mapsto z^n$ for every positive integer n. Then $n \cdot [\phi] = [p_n \circ \phi]$.*

Proposition 2.7. *The Bruschlinsky group with the winding order of a compact connected oriented branched matchbox manifold Y is unperforated.*

Proof. Suppose that $\phi \in C(Y, S^1)$ and $n \in \mathbb{Z}_+$ are such that $n \cdot [\phi] = [p_n \circ \phi] \in \mathrm{Br}_\oplus(Y)$. Then there exists a map $\psi \in R(Y)$ given by $y \mapsto \exp(2\pi i g(y))$ with $g \in C(Y, \mathbb{R})$ such that $(p_n \circ \phi) \cdot \psi$ is non-orientation-reversing.

Define $\widetilde{\psi} : Y \to S^1$ by $y \mapsto \exp(2\pi i \cdot \frac{1}{n} g(y))$. Then we have $\widetilde{\psi} \in R(Y)$ and $(p_n \circ \phi) \cdot \psi = p_n \circ (\phi \cdot \widetilde{\psi})$. For every orientation preserving parameterized curve $\gamma : \mathbb{R} \to Y$,

$$((p_n \circ \phi) \cdot \psi) \circ \gamma(t) = p_n \circ (\phi \cdot \widetilde{\psi}) \circ \gamma(t) = p_n \circ ((\phi \cdot \widetilde{\psi}) \circ \gamma(t))$$

does not move clockwise on S^1 as $t \in \mathbb{R}$ increases. So $\phi \cdot \widetilde{\psi}$ is non-orientation reversing as n is a positive integer. Therefore $\phi \in C_\oplus(Y, S^1)$, and $(\mathrm{Br}(Y), \mathrm{Br}_\oplus(Y))$ is unperforated. ∎

Remark 2.8. If Y is a compact connected orientable matchbox manifold, then the above Propositions 2.5 and 2.7 follow from Propositions 4.5 and 3.4 of [4] and Theorem 2.1.

One-dimensional continua. In [14], Rogers introduced the following notions for one-dimensional continua.

Suppose that X_1 and X_2 are graphs and that $\mathcal{V}_i$ and $\mathcal{E}_i$ are the vertex set and the edge set of X_i, respectively, $i = 1, 2$. A continuous onto map $f : X_2 \to X_1$ is called *simplicial relative to* $(\mathcal{V}_1, \mathcal{V}_2)$ if $f(\mathcal{V}_2) \subseteq \mathcal{V}_1$ and for every edge $e_2 \in \mathcal{E}_2$ there is an edge $e_1 \in \mathcal{E}_1$ such that $f|_{e_2 \setminus \mathcal{V}_2}$ is a homeomorphism onto $e_1 \setminus \mathcal{V}_1$ or a constant map. The map $f : X_2 \to X_1$ is *simplicial* if it is simplicial relative to some vertex sets of X_1 and X_2. And f is called *light* if the preimage of each point is totally disconnected.

An inverse limit sequence $\{X_k, f_k\}$ on graphs is called *light simplicial* if each f_k is light simplicial, and is called *light uniformly simplicial* if each X_k is a graph with a vertex set $\mathcal{V}_k$ and each map $f_k : X_k \to X_{k-1}$ is light simplicial relative to $(\mathcal{V}_{k-1}, \mathcal{V}_k)$.

Theorem 2.9 ([12, 14]). *Suppose that $\overline{X}$ is a one-dimensional continuum.*

(1) *$\overline{X}$ is homeomorphic to an inverse limit of a light simplicial sequence $\{X_k, f_k\}$ on graphs.*

(2) $\overline{X}$ *is homeomorphic to a light uniformly simplicial inverse limit on graphs if and only if there exists a map* $\pi : \overline{X} \to [0,1]$ *such that* $\pi^{-1}(\{0,1\})$ *is totally disconnected and* $\pi|_e$ *is a homeomorphism for every* e *which is the closure of a component of* $\overline{X} \setminus \pi^{-1}(\{0,1\})$.

Suppose that $\{X_k, f_k\}$ is a light simplicial sequence on graphs. Let

$$\overline{X} = X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} \ldots = \left\{ (x_0, x_1, \ldots) \in \prod_{k=0}^{\infty} X_k \,\middle|\, f_{k+1}(x_{k+1}) = x_k \right\}.$$

For a one-dimensional continuum Y, we call the sequence $\{X_k, f_k\}$ a *presentation* of Y if $\overline{X}$ is homeomorphic to Y.

NOTATION 2.10. Suppose that G is a directed graph. We consider a directed edge e of G as the image of a local homeomorphism from $[0,1]$ to e such that $e(0)$ is the initial point of e and $e(1)$ is the terminal point. Then we can represent each point $x \in e$ as $e(t)$ (possibly $e(0) = e(1)$).

Recall that a continuous map $p : [0,1] \to G$, a directed graph, is *orientation preserving* if $e^{-1} \circ p : I \to [0,1]$ is increasing for every interval $I \subset [0,1]$ such that $p(I)$ is a subset of a directed edge e. A continuous map $f : G_1 \to G_2$ between two directed graphs is *orientation preserving* if, for every orientation preserving map $p : [0,1] \to G_1$, $f \circ p : [0,1] \to G_2$ is orientation preserving ([7]). A directed graph is called *nondegenerate* if every vertex has at least one incoming edge and at least one outgoing edge.

Suppose that Y is a compact connected oriented branched matchbox manifold. Since Y is a one-dimensional continuum, there is a light simplicial presentation $\{X_k, f_k\}$ of Y by Theorem 2.9. The following proposition shows that the orientation of Y decides the directions of edges in each coordinate space X_k so that every connection map $f_k : X_k \to X_{k-1}$ is orientation preserving.

PROPOSITION 2.11. *Suppose that* Y *is a compact connected oriented branched matchbox manifold. Then* Y *has a light simplicial presentation by orientation preserving maps of directed nondegenerate graphs.*

Proof. Suppose that $\{h_U : V \to U\}$ is an orientation of Y where U is a branched matchbox with the projections $p_U : U \to (-1,1)$. Let $\{X_k, f_k\}$ be a light uniformly simplicial presentation of Y given by Theorem 2.9, and $\pi_k : Y \to X_k$ the canonical projection to the kth coordinate space. If e is an edge of X_k with $\pi_k^{-1}(e \setminus \mathcal{V}_k) \cap h_U^{-1}(U) \neq \emptyset$, then give the direction to the set $(e \setminus \mathcal{V}_k) \cap \left(\pi_k \circ h_U^{-1}(U)\right) \subset e$ so that, for every curve $\gamma : [0,1] \to \pi_k^{-1}(e \setminus \mathcal{V}_k) \cap h_U^{-1}(U)$, $p_U \circ h_U \circ \gamma$ is increasing if and only if $e^{-1} \circ \pi_k \circ \gamma$ is increasing. Since $\{h_U\}$ is an orientation of Y, we can extend this direction on $(e \setminus \mathcal{V}_k) \cap \pi_k \circ h_U^{-1}(U)$ to e, and each edge X_k has a direction induced by the orientation of Y.

Suppose that $x = (x_0, x_1, \ldots)$ is a point in Y such that $x_k \in X_k$ is a vertex and that U is a branched matchbox such that the domain of h_U contains x. Then there is a match $M \subset U$ containing $h_U(x)$ such that $p_U|_M \circ h_U(x) = t$ for some $t \in (-1,1)$. Since $\pi_k \circ h_U^{-1} \circ (p_U|_M)^{-1}((-1,t))$ and $\pi_k \circ h_U^{-1} \circ (p_U|_M)^{-1}((t,1))$ are nonempty sets in X_k, there exist an edge e_- such that $(\pi_k \circ h_U^{-1} \circ (p_U|_M)^{-1}((-1,t))) \cap e_- \neq \emptyset$, which is incoming to x_k, and an edge e_+ such that $(\pi_k \circ h_U^{-1} \circ (p_U|_M)^{-1}((t,1))) \cap e_+ \neq \emptyset$, which is outgoing from x_k. Therefore X_k is nondegenerate.

Suppose that $e_k \in \mathcal{E}_k$ and $e_{k-1} \in \mathcal{E}_{k-1}$ are two edges such that $e_{k-1} = f_k(e_k)$, and $h_U : V \to U$ is a chart such that $W = \pi_k \circ h_U^{-1}(U) \cap (e_k \setminus \mathcal{V}_k) \neq \emptyset$. Then $f_k(W) \subset \pi_{k-1} \circ h_U^{-1}(U) \cap (e_{k-1} \setminus \mathcal{V}_{k-1})$, and for every curve $\gamma : [0,1] \to h_U^{-1}(U) \cap \pi_k^{-1}(e_k \setminus \mathcal{V}_k)$, $e_k^{-1} \circ \pi_k \circ \gamma$ is increasing $\Leftrightarrow p_U \circ h_u \circ \gamma$ is increasing $\Leftrightarrow e_{k-1}^{-1} \circ \pi_{k-1} \circ \gamma$ is increasing.

Let $\gamma : [a,b] \to h_U^{-1}(U) \cap \pi_k^{-1}(e_k \setminus \mathcal{V}_k)$ be given by $\pi_k \circ \gamma(t) = e_k(t)$. Then we have $\pi_{k-1} \circ \gamma(t) = f_k \circ e_k(t)$, and $e_{k-1}^{-1} \circ \pi_{k-1} \circ \gamma(t) = e_{k-1}^{-1} \circ f_k \circ e_k(t)$ is increasing as t is increasing. Therefore $f_k : X_k \to X_{k-1}$ is orientation preserving. ∎

COROLLARY 2.12. *Suppose that Y is a compact connected orientable branched matchbox manifold. Then there is a continuous map $\pi : Y \to S^1$ such that $\pi^{-1}(1)$ is totally disconnected and $\pi|_\ell$ is an orientation preserving homeomorphism for every ℓ which is an arc component of $Y \setminus \pi^{-1}(1)$.*

Proof. Define $\pi : Y \to S^1$ by $x = (x_0, x_1, \ldots) \mapsto \exp(2\pi i t)$, where $t \in [0,1]$ is given by $x_0 = e(t) \in e \in \mathcal{E}_0$. Then π is well defined and $\pi^{-1}(1) = \{x \in Y \mid x_0 \in \mathcal{V}_0\}$ is a zero-dimensional set. Since ℓ, an arc component of $Y \setminus \pi^{-1}(1)$, is given by $\ell = (e_0 \setminus \mathcal{V}_0, e_1 \setminus \mathcal{V}_1, \ldots)$ where $e_i \in \mathcal{E}_i$, $\pi : \ell \to S^1$ given by $x = (e_0(t), e_1(t), \ldots) \mapsto \exp(2\pi i t)$ is an orientation preserving homeomorphism. ∎

We have the following proposition from Theorem 2.9.

PROPOSITION 2.13. *Every compact connected orientable branched matchbox manifold has a light uniformly simplicial presentation.*

STANDING ASSUMPTION 2.14. *From now on, a graph means a finite directed nondegenerate graph.*

3. Orientable one-dimensional inverse limit spaces. In this section we suppose that $\overline{X}$ is a compact connected oriented branched matchbox manifold with a presentation $\{X_k, f_k\}$ such that each X_k is a graph with a fixed vertex set $\mathcal{V}_k$ and each map $f_k : X_k \to X_{k-1}$ is an orientation preserving map such that $f_k(\mathcal{V}_k) \subset \mathcal{V}_{k-1}$ and $f_k|_{X_k \setminus \mathcal{V}_k}$ is locally one-to-one. Let $\mathcal{E}_k$ be the set of directed edges in X_k defined by $\mathcal{V}_k$, $C(\mathcal{E}_k, \mathbb{Z})$ the set of integer-valued functions on $\mathcal{E}_k$, and $C_+(\mathcal{E}_k,\mathbb{Z})$ the subset of $C(\mathcal{E}_k,\mathbb{Z})$ with range in the nonnegative integers $\mathbb{Z}_+$. For each vertex p_i of X_k, define the *vertex function* $v_i \in C(\mathcal{E}_k,\mathbb{Z})$ such that for every edge $e \in \mathcal{E}_k$,

$$v_i(e) = \begin{cases} 1 & \text{if } e \text{ is an edge from } p_i \text{ to another vertex point,} \\ -1 & \text{if } e \text{ is an edge from another vertex point to } p_i, \\ 0 & \text{if } p_i \text{ is the initial and terminal point of } e, \text{ or } p_i \notin e. \end{cases}$$

Write V_k for the set of integral combinations of $\{v_i\} \subset C(\mathcal{E}_k,\mathbb{Z})$, and call an element of V_k a *vertex coboundary*. Define

$$\mathcal{G}^k = C(\mathcal{E}_k,\mathbb{Z})/V_k \quad \text{and} \quad \mathcal{G}^k_+ = C_+(\mathcal{E}_k,\mathbb{Z})/V_k.$$

Then $(\mathcal{G}^k, \mathcal{G}^k_+, \mathbf{1})$ is a unital preordered group.

NOTATION 3.1. By a *path* in a graph X we mean a finite sequence $e_1^{s(1)} \dots e_n^{s(n)}$ of edges such that, for $1 \le i < n$, $s(i) = \pm 1$ represents the direction of e_i and the terminal vertex of $e_i^{s(i)}$ is the initial vertex of $e_{i+1}^{s(i+1)}$. We write $e^s \in \wp$ if $\wp$ is a path and e is an edge such that e^s is a factor of $\wp$. A *cycle* is a path $e_1^{s(1)} \dots e_n^{s(n)}$ such that the terminal vertex of $e_n^{s(n)}$ is the initial vertex of $e_1^{s(1)}$.

We say that a function g in $C(\mathcal{E}_k,\mathbb{Z})$ is *zero* (*nonnegative*, respectively) *on cycles* if the sum of $g(e)$ over the edges e of every cycle in X_k is zero (nonnegative, respectively).

LEMMA 3.2 ([4, §3]). *Suppose that g is an element of $C(\mathcal{E}_k,\mathbb{Z})$. Then*

(1) *g is an element of V_k if and only if g is zero on cycles in X_k, and*

(2) *$[g]$ is an element of $C_+(\mathcal{E}_k,\mathbb{Z})/V_k = \mathcal{G}^k_+$ if and only if g is nonnegative on cycles.*

Given $g \in C(\mathcal{E}_k,\mathbb{Z})$, define a continuous map

$$\phi_g : X_k \to S^1, \qquad x \mapsto \exp(2\pi i t g(e)) \quad \text{for } x = e(t),\ t \in [0,1].$$

Then ϕ_g is well defined as every vertex point maps to $1 \in S^1$, and ϕ_g is an element of $C(X_k, S^1)$.

LEMMA 3.3. *Suppose that g is an element of $C(\mathcal{E}_k,\mathbb{Z})$. Then g is an element of V_k if and only if ϕ_g is homotopic to a constant function* 1 *in* $C(X_k, S^1)$.

Proof. Suppose that g is an element of V_k. For each vertex function v_i defined at the vertex p_i of X_k, define a map $h_{sv_i} : X_k \to S^1$ for $0 \le s \le 1$ by

$$h_{sv_i}(e(t)) = \begin{cases} e^{2\pi i s t} & \text{if } e \text{ is an edge from } p_i \text{ to another vertex point,} \\ e^{-2\pi i s t} & \text{if } e \text{ is an edge from another vertex point to } p_i, \\ e^{2\pi i s} & \text{if } p_i \text{ is the initial and terminal point of } e, \\ 1 & \text{otherwise.} \end{cases}$$

Then $s \mapsto h_{sv_i}$, $0 \le s \le 1$, is a homotopy between ϕ_{v_i} and 1.

Now suppose that ϕ_g and 1 are homotopic on X_k. Since the winding number of the restriction of ϕ_g to every cycle in X_k is a homotopy invariant and $\sum_{e\in\ell} g(e)$ is the winding number for every cycle ℓ in X_k, we conclude that g is zero on every cycle, and that g is an element of V_k by Lemma 3.2. ∎

Therefore we have a well defined map

$$\iota_k : \mathcal{G}^k \to \mathrm{Br}(X_k) \quad \text{given by} \quad [g] \mapsto [\phi_g].$$

PROPOSITION 3.4. *Let ι_k be defined as above. Then ι_k is an isomorphism of preordered groups $(\mathcal{G}^k, \mathcal{G}^k_+)$ and $(\mathrm{Br}(X_k), \mathrm{Br}_\oplus(X_k))$.*

Proof. Since $\phi_{g+h} = \phi_g \cdot \phi_h$, ι_k is a group homomorphism. By Lemma 3.3, ϕ_g is homotopic to a constant function 1 if and only if g is a vertex coboundary. So $\iota_k : \mathcal{G}^k \to \mathrm{Br}(X_k)$ is injective.

To obtain an inverse of ι_k, suppose that ϕ belongs to $C(X_k, S^1)$. Then we can choose a map $\varrho : \mathcal{V}_k \to \mathbb{R}$ where $\mathcal{V}_k$ is the vertex set of X_k such that $\phi(p) = \phi(2\pi i \varrho(p))$ for every vertex p of X_k. Define $S_\varrho \in C(X_k, S^1)$ by

$$e(t) \mapsto \exp(2\pi i((1-t)\varrho(e(0)) + t\varrho(e(1)))), \qquad 0 \le t \le 1.$$

Then S_ϱ is homotopic to the constant map 1 by $H_u = S_{u\varrho}$ for $0 \le u \le 1$, ϕ is homotopic to ϕ/S_ϱ, and for every vertex p of X_k, $(\phi/S_\varrho)(p) = 1 \in S^1$.

For each edge $e \in \mathcal{E}_k$, let $r_\phi(e)$ be the number of times the loop $(\phi/S_\varrho)(x)$ winds around S^1 as $x = e(t)$ moves on e. Since $(\phi/S_\varrho)(p) = 1 \in S^1$ for every vertex p of X_k, $r_\phi(e)$ is well defined for each edge e. Then $r_\phi : e \mapsto r_\phi(e)$ is an element of $C(\mathcal{E}_k, \mathbb{Z})$, and ϕ_{r_ϕ} wraps around S^1 the same number of times as ϕ/S_ϱ. Therefore ϕ_{r_ϕ} is homotopic to ϕ/S_ϱ, and $[\phi] \mapsto [r_\phi]$ gives the desired inverse to ι_k.

Clearly, if $g \in C(\mathcal{E}_k, \mathbb{Z}_+)$, then $[\iota_k(g)] = [\phi_g]$ is a positive element in the winding order. Conversely, if $[\phi_g] \in \mathrm{Br}(X_k)$ is positive in the winding order, then there exists a map $\psi \in R(X_k)$ such that $\phi_g \cdot \psi$ is non-orientation reversing. It follows that g has to be nonnegative on cycles, and we have $[g] \in \mathcal{G}^k_+$ by Lemma 3.2. Therefore ι_k is an isomorphism of preordered groups. ∎

Since $f_{k+1} : X_{k+1} \to X_k$ is an orientation preserving map, if e is an edge in $\mathcal{E}_{k+1}$, then $f_{k+1}(e)$ is a path $e_1 \ldots e_n$ in X_k. Hence f_{k+1} induces a map

$$f^*_{k+1} : C(\mathcal{E}_k, \mathbb{Z}) \to C(\mathcal{E}_{k+1}, \mathbb{Z}), \qquad g \mapsto g \circ f_{k+1},$$

where $(g \circ f_{k+1})(e) = \sum_{i=1}^n g(e_i)$ such that $f_{k+1}(e) = e_1 \ldots e_n$ in $\mathcal{E}_k$. And f_{k+1} induces another map

$$\widetilde{f}^*_{k+1} : C(X_k, S^1) \to C(X_{k+1}, S^1), \qquad \phi \mapsto \phi \circ f_{k+1}.$$

LEMMA 3.5. *Let f^*_{k+1} and $\widetilde{f}^*_{k+1}$ be given as above. Then there are well defined homomorphisms from $\mathcal{G}^k$ to $\mathcal{G}^{k+1}$ and from $\mathrm{Br}(X_k)$ to $\mathrm{Br}(X_{k+1})$ defined by f^*_{k+1} and $\widetilde{f}^*_{k+1}$, respectively.*

Proof. For every $v \in V_k$ and every cycle ℓ in X_{k+1}, $f_{k+1}(\ell)$ is a cycle in X_k and $f_{k+1}^*(v)(\ell) = v(f_{k+1}(\ell)) = 0$ by Lemma 3.2. Therefore $f_{k+1}^*(v)$ is an element of V_{k+1}, and the map $\mathcal{G}^k \to \mathcal{G}^{k+1}$ given by $[g] \mapsto [f_{k+1}^*(g)]$ is a well defined homomorphism. That $\widetilde{f}_{k+1}^*$ induces a homomorphism follows from the definition of the Bruschlinsky group. ∎

Let us denote these well defined homomorphisms as f_{k+1}^* and $\widetilde{f}_{k+1}^*$, respectively, if they do not give any confusion.

PROPOSITION 3.6. *Let $\iota_k : \mathcal{G}^k \to \mathrm{Br}(X_k)$, f_{k+1}^* and $\widetilde{f}_{k+1}^*$ be given as above. Then $\iota_{k+1} \circ f_{k+1}^* = \widetilde{f}_{k+1}^* \circ \iota_k$, and moreover, f_{k+1}^* and $\widetilde{f}_{k+1}^*$ are order preserving homomorphisms.*

Proof. It is not difficult to check, for every $[g] \in \mathcal{G}^k$,

$$(\iota_{k+1} \circ f_{k+1}^*)([g]) = (\widetilde{f}_{k+1}^* \circ \iota_k)([g]),$$

and we have $\iota_{k+1} \circ f_{k+1}^* = \widetilde{f}_{k+1}^* \circ \iota_k$.

To show that $\widetilde{f}_{k+1}^*$ is order preserving, suppose $[\phi] \in \mathrm{Br}_\oplus(X_k)$. Then there exists a $\psi \in R(X_k)$ such that $\phi \cdot \psi$ is non-orientation reversing. Since $\widetilde{f}_{k+1}^*(\psi) = \psi \circ f_{k+1}$ is an element of $R(X_{k+1})$ by Lemma 3.5 and $f_{k+1} : X_{k+1} \to X_k$ is orientation preserving, for every orientation preserving parameterized curve $\gamma : \mathbb{R} \to X_{k+1}$, $f_{k+1} \circ \gamma$ is an orientation preserving parameterized curve in X_k, and

$$((\phi \circ f_{k+1}) \cdot (\psi \circ f_{k+1}))(\gamma(t)) = ((\phi \cdot \psi) \circ f_{k+1})(\gamma(t)) = (\phi \cdot \psi) \circ (f_{k+1} \circ \gamma)(t)$$

does not move in the clockwise direction as $t \in \mathbb{R}$ increases. Therefore $[\phi \circ f_{k+1}] = \widetilde{f}_{k+1}^*([\phi])$ is an element of $\mathrm{Br}_\oplus(X_{k+1})$, and $\widetilde{f}_{k+1}^*$ is an order preserving homomorphism. Since ι_k is an order preserving isomorphism by Proposition 3.4, $f_{k+1}^* = \iota_{k+1}^{-1} \circ \widetilde{f}_{k+1}^* \circ \iota_k$ is also order preserving. ∎

Then $\{\mathcal{G}^k, f_{k+1}^*\}$ and $\{\mathrm{Br}(X_k), \widetilde{f}_{k+1}^*\}$ are directed systems. Let $\varinjlim \mathcal{G}^k$ and $\varinjlim \mathrm{Br}(X_k)$ be the direct limits of $\{\mathcal{G}^k, f_{k+1}^*\}$ and $\{\mathrm{Br}(X_k), \widetilde{f}_{k+1}^*\}$, respectively.

DEFINITION 3.7. Recall that $C_+(\mathcal{E}_k, \mathbb{Z})$ is the subset of $C(\mathcal{E}_k, \mathbb{Z})$ with range in $\mathbb{Z}_+$, and that $\mathcal{G}_+^k$ is given by $C_+(\mathcal{E}_k, \mathbb{Z})/V_k$. Since $f_{k+1}^* : C(\mathcal{E}_k, \mathbb{Z}) \to C(\mathcal{E}_{k+1}, \mathbb{Z})$ defined by $g \mapsto g \circ f_{k+1}$ is an order preserving homomorphism by Proposition 3.6, $(\varinjlim \mathcal{G}^k)_+ = \varinjlim \mathcal{G}_+^k$ is well defined. This set, as a positive set, defines the order which is the *direct limit order* or the *standard order* on $\varinjlim \mathcal{G}^k$.

The standard isomorphism $\varinjlim \mathcal{G}^k \to \mathrm{Br}(\overline{X})$. Suppose $\overline{X} = \varprojlim X_k$ and that $\pi_k : \overline{X} \to X_k$ is the projection map to the kth coordinate space. If ϕ is an element of $C(X_k, S^1)$, then ϕ induces an element $\phi \circ \pi_k \in C(\overline{X}, S^1)$. We will use the isomorphism $\iota_k : \mathcal{G}^k \to \mathrm{Br}(X_k)$ and the natural map $\mathrm{Br}(X_k) \to \mathrm{Br}(\overline{X})$ defined by $[\phi] \mapsto [\phi \circ \pi_k]$ to make an isomorphism $\iota : \varinjlim \mathcal{G}^k \to \mathrm{Br}(\overline{X})$.

Let $1_{X_k} : X_k \to S^1$ and $1_{\overline{X}} : \overline{X} \to S^1$ be given by $x_k \mapsto 1 \in S^1$ and $x \mapsto 1$ for all $x_k \in X_k$ and $x \in \overline{X}$, respectively. Suppose that ϕ is an element of $C(X_k, S^1)$ such that ϕ is homotopic to 1_{X_k} by $H : X_k \times [0,1] \to S^1$. Then $\phi \circ \pi_k$ is homotopic to $1_{\overline{X}} = 1_{X_k} \circ \pi_k$ by the map $\overline{H} : \overline{X} \times [0,1] \to S^1$ given by $\overline{H}(x,t) = H(\pi_k(x), t)$. Thus there is a well defined map

$$\pi_k^* : \mathrm{Br}(X_k) \to \mathrm{Br}(\overline{X}), \qquad [\phi] \mapsto [\phi \circ \pi_k].$$

Since $(\phi_1 \cdot \phi_2) \circ \pi_k = (\phi_1 \circ \pi_k) \cdot (\phi_2 \circ \pi_k)$ for all $\phi_1, \phi_2 \in C(X_k, S^1)$, π_k^* is a homomorphism. That $f_{k+1} \circ \pi_{k+1} = \pi_k : \overline{X} \to X_k$ implies the following lemma.

LEMMA 3.8. *Let π_k^* and $\widetilde{f}_{k+1}^*$ be defined as above. Then $\pi_{k+1}^* \circ \widetilde{f}_{k+1}^* = \pi_k^*$ for all k.*

Let $\varphi_k^* : \mathrm{Br}(X_k) \to \varinjlim \mathrm{Br}(X_k)$ be the natural map for each k. If $\varphi_k^*([\phi]) = \varphi_l^*([\psi])$ for $[\phi] \in \mathrm{Br}(X_k)$ and $[\psi] \in \mathrm{Br}(X_l)$, then there is a positive integer $m \geq k, l$ such that $\widetilde{f}_{m+1}^* \circ \ldots \circ \widetilde{f}_{k+1}^*([\phi]) = \widetilde{f}_{m+1}^* \circ \ldots \circ \widetilde{f}_{l+1}^*([\psi])$. Hence

$$\begin{aligned} \pi_k^*([\phi]) &= \pi_{m+1}^* \circ \widetilde{f}_{m+1}^* \circ \ldots \circ \widetilde{f}_{k+1}^*([\phi]) \\ &= \pi_{m+1}^* \circ \widetilde{f}_{m+1}^* \circ \ldots \circ \widetilde{f}_{l+1}^*([\psi]) = \pi_l^*([\psi]), \end{aligned}$$

and there is a well defined group homomorphism

$$\pi^* : \varinjlim \mathrm{Br}(X_k) \to \mathrm{Br}(\overline{X}), \qquad \varphi_k^*([\phi]) \mapsto \pi_k^*([\phi]) = [\phi \circ \pi_k]\,.$$

LEMMA 3.9. *Suppose that ξ is an element of $C(\overline{X}, S^1)$. Then there exist $\xi' \in C(\overline{X}, S^1)$ and $k \geq 0$ such that ξ is homotopic to ξ' and $\xi'(x) = \xi'(y)$ if $x_k = y_k$.*

Proof. Define a metric d on $\overline{X}$ by

$$d(x,y) = \sum_{k=0}^{\infty} \frac{1}{2^k} d_k(x_k, y_k)$$

where $x = (x_0, x_1, \ldots)$, $y = (y_0, y_1, \ldots) \in \overline{X}$ and d_k is a metric on X_k compatible with its topology. Since $\overline{X}$ is a compact Hausdorff space, every element in $C(\overline{X}, S^1)$ is uniformly continuous. So, for given ξ and $\varepsilon > 0$, there exists a nonnegative integer k such that for $x, y \in \overline{X}$, $x_k = y_k$ implies $d(\xi(x), \xi(y)) < \varepsilon$.

For $x = (x_0, \ldots, x_k, \ldots) \in \overline{X}$, set $x^k = \{y \in \overline{X} \mid y_k = x_k\}$. Then $d(\xi(a), \xi(b)) < \varepsilon$ for all $a, b \in x^k$, and we can choose a point $\widetilde{x} \in S^1$ such that $\widetilde{x}$ is the center of the smallest interval containing $\xi(x^k)$ in S^1. Define $\xi' : \overline{X} \to S^1$ by $\xi'|_{x^k} = \widetilde{x}$. Then it is clear that $\xi' \in C(\overline{X}, S^1)$ and $\xi'(x) = \xi'(y)$ if $x_k = y_k$. Since $d(\xi(x), \xi'(x)) < \varepsilon$ for all $x \in \overline{X}$, ξ is homotopic to ξ'. ∎

PROPOSITION 3.10. *Let π^* be defined as above. Then π^* is a group isomorphism.*

Proof. To show that π^* is surjective, suppose $\xi \in C(\overline{X}, S^1)$ and that ξ' and k are given in Lemma 3.9. Define $\phi_k : X_k \to S^1$ by $x_k \mapsto \xi'(x)$ for $x = (x_0, \dots, x_k, \dots) \in \overline{X}$. Then ϕ_k is well defined, and it is trivial that $\phi_k \circ \pi_k = \xi'$. Therefore $\xi \in C(\overline{X}, S^1)$ is homotopic to $\phi_k \circ \pi_k$, and $\pi^* : \varinjlim \mathrm{Br}(X_k) \to \mathrm{Br}(\overline{X})$ is surjective.

Suppose $\xi_1, \xi_2 \in C(\overline{X}, S^1)$ and that ξ_1 is homotopic to ξ_2. Then by the surjectivity of π^*, there exist nonnegative integers $k \le l$ and $\phi \in C(X_k, S^1)$, $\psi \in C(X_l, S^1)$ such that ξ_1 is homotopic to $\phi \circ \pi_k$ and ξ_2 is homotopic to $\psi \circ \pi_l$. Since $\phi \circ \pi_k = \phi \circ f_{k+1} \circ \dots \circ f_l \circ \pi_l$, we have

$$\varphi_l^*([\psi]) = \varphi_l^*([\phi \circ f_{k+1} \circ \dots \circ f_l]) = \varphi_l^* \circ \widetilde{f}_l^* \circ \dots \circ \widetilde{f}_{k+1}^*([\phi]) = \varphi_k^*([\phi]).$$

Hence π^* is injective. ∎

Therefore the isomorphisms $\iota_k : \mathcal{G}^k \to \mathrm{Br}(X_k)$ and $\pi^* : \varinjlim \mathrm{Br}(X_k) \to \mathrm{Br}(\overline{X})$ induce an isomorphism $\iota : \varinjlim \mathcal{G}^k \to \mathrm{Br}(\overline{X})$.

Order isomorphism. Assume now that the presentation $\{X_k, f_k\}$ satisfies the following

SIMPLICITY CONDITION. For each $k \ge 1$ there exists $\kappa(k) \ge k$ such that $f_{k+1} \circ \dots \circ f_l(e) = X_k$ for every $l \ge \kappa(k)$ and $e \in \mathcal{E}_l$, where $\mathcal{E}_l$ is the edge set of X_l.

Then the winding order on $\mathrm{Br}(X_k)$ and $\mathrm{Br}(\overline{X})$ is an order.

THEOREM 3.11. *Suppose that the presentation $\{X_k, f_k\}$ satisfies the above Simplicity Condition. Then $\iota : (\varinjlim \mathcal{G}^k, \varinjlim \mathcal{G}_+^k) \to (\mathrm{Br}(\overline{X}), \mathrm{Br}_\oplus(\overline{X}))$ is an isomorphism of ordered groups.*

Proof. Trivial case. Suppose that all but finitely many X_k have a unique edge, i.e., X_k is homeomorphic to the circle S^1 with a unique vertex by the Standing Assumption 2.14, and that the connection map $f_k : X_k \to X_{k-1}$ is the identity map if $X_k = X_{k-1} = S^1$. Then it is obvious that

$$(\varinjlim \mathcal{G}^k, \varinjlim \mathcal{G}_+^k) \cong (\mathrm{Br}(\overline{X}), \mathrm{Br}_\oplus(\overline{X})) = (\mathbb{Z}, \mathbb{Z}_+),$$

and ι is an isomorphism.

Nontrivial case. First, ι is a group isomorphism, and clearly $\iota(\varinjlim \mathcal{G}_+^k) \subseteq \mathrm{Br}_\oplus(\overline{X})$. It remains to show that ι maps $\varinjlim \mathcal{G}_+^k$ onto $\mathrm{Br}_\oplus(\overline{X})$. So we assume that $[\phi]$ is an element of $\mathrm{Br}_\oplus(\overline{X})$. Then there is an $[h]$ in $\mathcal{G}^k$ for some $k \ge 0$ such that $[\phi] = [\phi_h \circ \pi_k]$, and we need to show $[h] \in \mathcal{G}_+^k$.

That $[\phi]$ is an element of $\mathrm{Br}_\oplus(\overline{X})$ implies that there is a map $\gamma \in R(\overline{X})$ such that $(\phi_h \circ \pi_k) \cdot \gamma$ is non-orientation reversing. Since γ is an element of $R(\overline{X})$, there is a continuous map $g : \overline{X} \to \mathbb{R}$ such that $\gamma(x) = \exp(2\pi i g(x))$.

For $y = (y_0, \ldots, y_k, \ldots) \in \overline{X}$, if $y_k = e(t)$ for $e \in \mathcal{E}_k$ and $t \in [0,1]$, then $\phi_h \circ \pi_k \cdot \gamma$ is defined by $y \mapsto \exp(2\pi i(th(e) + g(y)))$.

Suppose that $(\phi_h \circ \pi_k) \cdot \gamma$ is a constant map to S^1. Then $[(\phi_h \circ \pi_k) \cdot \gamma] = [\phi_h \circ \pi_k] \cdot [\gamma] = [\phi_h \circ \pi_k] = [1]$ in $\mathrm{Br}(\overline{X})$ as γ is homotopic to the identity element in $\mathrm{Br}(\overline{X})$. Hence the equivalence class of h is the identity element in $\varinjlim \mathcal{G}^k$, for $\iota : \varinjlim \mathcal{G}^k \to \mathrm{Br}(\overline{X})$ is an isomorphism.

Next suppose that $(\phi_h \circ \pi_k) \cdot \gamma$ is not constant on S^1. Then there are a nonnegative integer m, a small interval I contained in some edge e' of X_{k+m}, and $\varepsilon > 0$ such that if Γ is any orientation preserving curve in $\overline{X}$ and $\pi_{k+m}(\Gamma|_{[a,b]}) = I$, then $\mathrm{length}\{((\phi_h \circ \pi_k) \cdot \gamma) \circ \Gamma|_{[a,b]}\} > \varepsilon$.

Given an arbitrary constant L, by the Simplicity Condition we can choose a sufficiently large integer M such that e' is covered under $f^{k+m+1} \circ \ldots \circ f^{k+m+M}$ at least L times by every edge in $\mathcal{E}_{k+m+M}$.

Define

$$H = f^*_{k+m+M} \circ \ldots \circ f^*_{k+1}(h) = h \circ f_{k+1} \circ \ldots \circ f_{k+m+M} \in C(\mathcal{E}_{k+m+M}, \mathbb{Z}).$$

Then by Lemma 3.8, $\phi_H \circ \pi_{k+m+M} \in C(\overline{X}, S^1)$ is homotopic to $\phi_h \circ \pi_k$. For $x = (x_0, \ldots, x_{k+m+M}, \ldots) \in \overline{X}$, as x_{k+m+M} moves forward through a directed edge e of $\mathcal{E}_{k+m+M}$, its image under $\phi_H \circ \pi_{k+m+M}$ moves $\sum h(\widehat{e}) \cdot n_e(\widehat{e})$ times around S^1, where $n_e(\widehat{e})$ is the number of times e covers $\widehat{e} \in \mathcal{E}_k$ under the map $f^{k+1} \circ \ldots \circ f^{k+m+M}$.

LEMMA 3.12. *For every edge $e \in \mathcal{E}_{k+m+M}$, $H(e) \geq 2\pi L\varepsilon - 2\max|g|$.*

Proof. Regard e as a curve $e(t)$, $0 \leq t \leq 1$, and pick a curve $\Gamma : [0,1] \to \overline{X}$ such that $\pi_{k+m+M} \circ \Gamma(t) = e(t)$. As t increases from 0 to 1, the point

$$((\phi_h \circ \pi_k) \cdot \gamma) \circ \Gamma(t) = (\phi_h \circ \pi_k \circ \Gamma(t)) \cdot (\gamma \circ \Gamma(t))$$

moves counterclockwise on S^1 from $e^{2\pi i g(\Gamma(0))}$ to $e^{2\pi i(g(\Gamma(1)) + H(e))}$, covering an arclength A in the plane such that

$$A \leq \frac{1}{2\pi}(H(e) + 2\max|g|).$$

Because $\phi_h \circ \pi_k \circ \Gamma = \phi_h \circ f_{k+1} \circ \ldots \circ f_{k+m+M} \circ \pi_{k+m+M} \circ \Gamma$, as t runs from 0 to 1 the curve $f_{k+m+1} \circ \ldots \circ f_{k+m+M} \circ \pi_{k+m+M} \circ \Gamma(t)$ wraps around e' at least L times, and therefore $A \geq L\varepsilon$. Consequently, $2\pi L\varepsilon - 2\max|g| \leq H(e)$ as required. ∎

Since we can choose M to make L as large as we wish, we can make the choice to guarantee $H(e) > 0$ for every edge. Therefore $[H] = [h]$ is an element of $\mathcal{G}^k_+$. ∎

Dimension group. Let M be an $r \times s$ nonnegative integer matrix. Then the matrix M determines a homomorphism $\mathbb{Z}^s \to \mathbb{Z}^r$ by the ordinary matrix multiplication. The *simplicial order* on $\mathbb{Z}^r$ is the usual ordering $\mathbb{Z}^r_+ = \{(n_1, \ldots, n_r) \mid n_i \geq 0\}$. Then the corresponding homomorphism $M : \mathbb{Z}^s \to$

$\mathbb{Z}^r$ is *positive* with respect to the simplicial order, that is, $a \geq 0$ implies $M(a) \geq 0$.

DEFINITION 3.13 ([6, §2]). Let M_i be an $r(i) \times r(i-1)$ nonnegative integer matrix. For a system of ordered groups and positive maps

$$\mathbb{Z}^{r(0)} \xrightarrow{M_1} \mathbb{Z}^{r(1)} \xrightarrow{M_2} \dots$$

the set-theoretic direct limit $\varinjlim(\mathbb{Z}^{r(i)}, M_i)$ is an ordered group under the usual limit addition operation with the positive cone $\varinjlim(\mathbb{Z}_+^{r(i)}, M_i) = \bigcup_{i=1}^{\infty} M_{i\infty}(\mathbb{Z}_+^{r(i-1)})$ where $M_{i\infty}$ is the induced map from $\mathbb{Z}^{r(i-1)}$ to the direct limit $\varinjlim(\mathbb{Z}^{r(i)}, M_i)$.

An ordered group (G, G_+) is called a *dimension group* if it is order isomorphic to the limit of a system of simplicially ordered groups with positive maps.

Let (G, G_+) be a dimension group. A subgroup H of G is called an *order ideal* if H is an ordered group with the positive cone $H_+ = H \cap G_+$ and $0 \leq a \leq b \in H$ implies $a \in H$. The dimension group (G, G_+) is called *simple* if it has no proper order ideal.

In a simple dimension group (G, G_+) with an element $g \in G$, if neither g nor $-g$ lies in G_+, then g is called an *infinitesimal* element. If u is an order unit and g is an infinitesimal element of G, then $g + u$ is also an order unit.

It is well known that a dimension group defined as above by matrices M_i is simple if for every i there exists j such that all entries of the matrix $M_j M_{j-1} \dots M_{i+1} M_i$ are strictly positive.

Suppose that $\{X_k, f_k\}$ is a presentation of an (orientable) branched matchbox manifold with the edge set $\mathcal{E}_k$ of X_k. Then for each edge $e_i \in \mathcal{E}_k$, $f_k(e_i)$ is a path $e_{i,1}^{s(1)} \dots e_{i,j(i)}^{s(j(i))}$ in X_{k-1} such that $s(j) = \pm 1$ denotes the direction and the terminal point of $e_{i,j}^{s(j)}$ is the initial point of $e_{i,j+1}^{s(j+1)}$ for $1 \leq j < j(i)$. Therefore we can define an induced map $\check{f}_k : \mathcal{E}_k \to \mathcal{E}_{k-1}^*$ by

$$\check{f}_k : e_i \mapsto e_{i,1}^{s(1)} \dots e_{i,j(i)}^{s(j(i))}.$$

DEFINITION 3.14. Suppose that X_k has n_k edges for all $k \geq 0$. Then the *adjacency matrix* M_k of $(\check{f}_k, \mathcal{E}_k, \mathcal{E}_{k-1})$ is an $n_k \times n_{k-1}$ matrix such that for any edges $e_i \in \mathcal{E}_k$ and $e_j \in \mathcal{E}_{k-1}$, $M_k(i,j)$ is the number of times $\check{f}_k(e_i)$ covers e_j ignoring the direction of the covering.

LEMMA 3.15 ([6, §3]). *A countable ordered group is a dimension group if and only if it is unperforated and has the Riesz Interpolation Property.*

PROPOSITION 3.16. *Suppose that $\{X_k, f_k\}$ is a presentation of a compact connected orientable branched matchbox manifold with the adjacency matrices M_k. Then*

(1) $(\varinjlim C(\mathcal{E}_k,\mathbb{Z}), \varinjlim C_+(\mathcal{E}_k,\mathbb{Z})) \cong (\varinjlim(\mathbb{Z}^{n_k}, M_k), \varinjlim(\mathbb{Z}_+^{n_k}, M_k))$.

If the presentation satisfies the Simplicity Condition, then

(2) $(\varinjlim C(\mathcal{E}_k,\mathbb{Z}), \varinjlim C_+(\mathcal{E}_k,\mathbb{Z}))$ *and* $(\varinjlim \mathcal{G}^k, \varinjlim \mathcal{G}_+^k)$ *are simple dimension groups.*

Proof. (1) For each $g \in C(\mathcal{E}_{k-1},\mathbb{Z})$ and $f_k^* : C(\mathcal{E}_{k-1},\mathbb{Z}) \to C(\mathcal{E}_k,\mathbb{Z})$ given by $g \mapsto g \circ f_k$, if we represent g as $(g(e_1),\ldots,g(e_{n_{k-1}})) \in \mathbb{Z}^{n_{k-1}}$, then $C(\mathcal{E}_{k-1},\mathbb{Z})$ is isomorphic to $\mathbb{Z}^{n_{k-1}}$ and $f_k^*(g) = g \circ f_k$ is given by $M_k \cdot (g(e_1),\ldots,g(e_{n_{k-1}}))^t$. Hence we have $\varinjlim C(\mathcal{E}_{k-1},\mathbb{Z}) \cong \varinjlim(\mathbb{Z}^{n_k}, M_k)$. Since $C_+(\mathcal{E}_{k-1},\mathbb{Z})$ is the set of elements in $C(\mathcal{E}_{k-1},\mathbb{Z})$ with range in $\mathbb{Z}_+$, $C(\mathcal{E}_{k-1},\mathbb{Z})$ is simplicially ordered, and so is $\varinjlim C(\mathcal{E}_k,\mathbb{Z})$. Therefore $(\varinjlim C(\mathcal{E}_k,\mathbb{Z}), \varinjlim C_+(\mathcal{E}_k,\mathbb{Z}))$ is order isomorphic to $(\varinjlim(\mathbb{Z}^{n_k}, M_k), \varinjlim(\mathbb{Z}_+^{n_k}, M_k))$.

(2) Suppose that H is a proper order ideal of $(\varinjlim C(\mathcal{E}_k,\mathbb{Z}), \varinjlim C_+(\mathcal{E}_k,\mathbb{Z}))$ and that $b \in H_+$. Then there exist a nonnegative integer k and $h \in C_+(\mathcal{E}_k,\mathbb{Z})$ such that $b = [h] \in \varinjlim C(\mathcal{E}_k,\mathbb{Z})$. By the Simplicity Condition, there is a nonnegative integer $\kappa(k) \geq k$ such that $f_{k+1} \circ \ldots \circ f_l(e) = X_k$ for every $l \geq \kappa(k)$ and every edge $e \in \mathcal{E}_l$. If $a \in \varinjlim C_+(\mathcal{E}_k,\mathbb{Z})$, then we can choose a positive integer $l \geq \kappa(k)$ and $g \in C_+(\mathcal{E}_l,\mathbb{Z})$ such that $a = [g]$. Let $n = \max_{e\in\mathcal{E}_l} g(e)$. Then $n \cdot b = [n \cdot f_l^* \circ \ldots \circ f_{k+1}^* \circ h] \in H_+$ and $n \cdot f_l^* \circ \ldots \circ f_{k+1}^* \circ h - g \in C_+(\mathcal{E}_l,\mathbb{Z})$. So we have $0 \leq a \leq n \cdot b$ and $a \in H_+$. Therefore $H_+ = \varinjlim C_+(\mathcal{E}_k,\mathbb{Z})$, and $(\varinjlim C(\mathcal{E}_k,\mathbb{Z}), \varinjlim C_+(\mathcal{E}_k,\mathbb{Z}))$ is a simple dimension group.

The group $(\varinjlim \mathcal{G}^k, \varinjlim \mathcal{G}_+^k) \cong (\mathrm{Br}(\overline{X}), \mathrm{Br}_\oplus(\overline{X}))$ is an unperforated ordered group by Proposition 2.7, and its positive set is the image of the positive set of $\varinjlim C(\mathcal{E}_k,\mathbb{Z})$ under the quotient map $\chi : \varinjlim C(\mathcal{E}_k,\mathbb{Z}) \to \varinjlim \mathcal{G}^k$. We claim that with this quotient order, $(\varinjlim \mathcal{G}^k, \varinjlim \mathcal{G}_+^k)$ satisfies the Riesz Interpolation Property (and therefore by Lemma 3.15 is a dimension group). (We learned this argument from unpublished remarks of David Handelman. The general line of argument is also implicit in remarks on pp. 58 and 66 of [8].)

Let $V = \ker\chi$. Note that if V contains a nonzero positive element u, then for every $g \in \varinjlim C_+(\mathcal{E}_k,\mathbb{Z})$ we have $0 \leq g \leq nu$ for some integer n, and therefore $0 \leq \chi(g) \leq 0$, which contradicts the image of χ being a nontrivial ordered group. Therefore all elements of V are infinitesimals.

To show the Riesz Interpolation Property, suppose that $[a_1], [a_2], [b_1], [b_2] \in \varinjlim \mathcal{G}^k$ satisfy $[a_i] < [b_j]$ $(i,j = 1,2)$. Let $a_i, b_j \in \varinjlim C(\mathcal{E}_k,\mathbb{Z})$ be preimages of $[a_i]$ and $[b_j]$, respectively. Since $-[a_i] + [b_j]$ is a nonzero positive element of $\varinjlim \mathcal{G}^k$, there exists a $v_{i,j} \in V$ such that $-a_i + v_{i,j} + b_j$ is a nonzero positive element of $\varinjlim C(\mathcal{E}_k,\mathbb{Z})$. Because $v_{i,j}$ is an infinitesimal element, it follows that $-a_i + b_j$ is a nonzero positive element of $\varinjlim C(\mathcal{E}_k,\mathbb{Z})$, and $a_i < b_j$ for $i,j = 1,2$. Hence by the Riesz Interpolation Property for $\varinjlim C(\mathcal{E}_k,\mathbb{Z})$ there exists an element $c \in \varinjlim C(\mathcal{E}_k,\mathbb{Z})$ such that $a_i \leq c \leq b_j$.

Then by the definition of the quotient order we have $[a_i] \le [c] \le [b_j]$ for all i, j, as required. Therefore $(\varinjlim \mathcal{G}^k, \varinjlim \mathcal{G}_+^k)$ is a dimension group by Lemma 3.15.

Suppose that (G, G_+) is a proper order ideal of $(\varinjlim \mathcal{G}^k, \varinjlim \mathcal{G}_+^k)$. Then it is not difficult to see that $(H, H_+) = (\chi^{-1}(G), \chi^{-1}(G_+))$ is a proper order ideal of $(\varinjlim C(\mathcal{E}_k, \mathbb{Z}), \varinjlim C_+(\mathcal{E}_k, \mathbb{Z}))$ which is a simple dimension group. Therefore $(\varinjlim \mathcal{G}^k, \varinjlim \mathcal{G}_+^k)$ is a simple dimension group. ∎

If each graph X_k is a wedge of circles, then $V_k = \{0\}$ as each edge in X_k is a cycle. So we have the following corollary:

Corollary 3.17. *Suppose that the presentation $\{X_k, f_k\}$ satisfies the Simplicity Condition and that each graph X_k is a wedge of circles. Then $(\varinjlim \mathcal{G}^k, \varinjlim \mathcal{G}_+^k)$ is order isomorphic to $(\varinjlim(\mathbb{Z}^{n_k}, M_k), \varinjlim(\mathbb{Z}_+^{n_k}, M_k))$.*

The following corollary follows from Observation 2.4 and Theorem 3.11.

Corollary 3.18. *Suppose that $(\overline{X}_i, \bar{f}_i)$ is a compact connected orientable branched matchbox manifold with the Simplicity Condition for $i = 1, 2$. If $\overline{X}_1$ is homeomorphic to $\overline{X}_2$, then $\varinjlim \mathcal{G}_1^k$ is order isomorphic to $\varinjlim \mathcal{G}_2^k$.*

Remark 3.19. (1) The dimension group of adjacency matrices is not a homeomorphism invariant. See Example 4.4.

(2) The isomorphism in Corollary 3.18 need not respect distinguished order units ([4, §1]).

4. One-dimensional generalized solenoid. One interesting class of branched matchbox manifolds is one-dimensional branched solenoids, including one-dimensional generalized solenoids of Williams ([16, 17, 18]). Let X be a directed graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, and $f : X \to X$ a continuous map. We define some axioms which might be satisfied by (X, f) ([18]).

Axiom 0 (Indecomposability). (X, f) is indecomposable.

Axiom 1 (Nonwandering). All points of X are nonwandering under f.

Axiom 2 (Flattening). There is $k \ge 1$ such that for all $x \in X$ there is an open neighborhood U of x such that $f^k(U)$ is homeomorphic to $(-\varepsilon, \varepsilon)$.

Axiom 3 (Expansion). There are a metric d compatible with the topology and positive constants C and λ with $\lambda > 1$ such that for all $n > 0$ and all points x, y on a common edge of X, if f^n maps the interval $[x, y]$ into an edge, then $d(f^n x, f^n y) \ge C\lambda^n d(x, y)$.

Axiom 4 (Nonfolding). $f^n|_{X-\mathcal{V}}$ is locally one-to-one for every positive integer n.

Axiom 5 (Markov). $f(\mathcal{V}) \subseteq \mathcal{V}$.

Let $\overline{X}$ be the inverse limit space

$$\overline{X} = X \xleftarrow{f} X \xleftarrow{f} \cdots = \Big\{(x_0, x_1, x_2, \ldots) \in \prod_{k=0}^{\infty} X \,\Big|\, f(x_{n+1}) = x_n\Big\},$$

and $\bar{f} : \overline{X} \to \overline{X}$ the induced homeomorphism defined by

$$(x_0, x_1, x_2, \ldots) \mapsto (f(x_0), f(x_1), f(x_2), \ldots) = (f(x_0), x_0, x_1, \ldots).$$

Let Y be a topological space and $g : Y \to Y$ a homeomorphism. We call Y a 1-*dimensional generalized solenoid* or 1-*solenoid* and g a *solenoid map* if there exist a directed graph X and a continuous map $f : X \to X$ such that (X, f) satisfies all six axioms and $(\overline{X}, \bar{f})$ is topologically conjugate to (Y, g). If (X, f) satisfies all axioms except possibly the Flattening Axiom, then we call Y a *branched solenoid*. If we can choose the direction of each edge in X so that the connection map $f : X \to X$ is orientation preserving, then we call (X, f) an *orientable presentation*, and Y an *orientable* (branched) solenoid. If (Y, g) is a branched solenoid with a presentation (X, f), then there exists an $n \times n$ adjacency matrix $M_{X,f}$ where n is the cardinal number of the set of edges in X. If X is a wedge of circles and f leaves the unique branch point of X fixed, then we say (X, f) is an *elementary presentation*.

We get the following proposition from Theorem 3.11 and Corollary 3.17.

PROPOSITION 4.1. *Suppose that* $(\overline{X}, \bar{f})$ *is an orientable branched solenoid with an adjacency matrix* M. *Then* $\iota : (\varinjlim(\mathbb{Z}^n, M), \varinjlim(\mathbb{Z}^n_+, M)) \to (\mathrm{Br}(\overline{X}), \mathrm{Br}_{\oplus}(\overline{X}))$ *is an epimorphism of ordered groups. If* (X, f) *is an elementary presentation*, *then* ι *is an isomorphism.*

REMARK 4.2. We need the elementary presentation condition for the injectivity of ι. See Example 4.4.

EXAMPLE 4.3 ([18, §2] and [11, §7.5]). Let X be the unit circle on the complex plane. Suppose that 1 and -1 are the vertices of X, and that the upper half-circle e_1 and the lower half-circle e_2 with counterclockwise

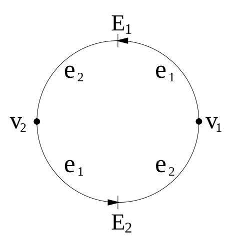

Fig. 1. (X, f) with the wrapping rule $\check{f}$

direction are the edges of X. Define $f : X \to X$ by $f : z \mapsto z^2$. The $\check{f} : \mathcal{E}_X \to \mathcal{E}_X^*$ is given by $\check{f} : e_1 \mapsto e_1e_2,\ e_2 \mapsto e_1e_2$, and the adjacency matrix is

$$M_{X,f} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Therefore we have

$$(\mathrm{Br}(\overline{X}), \mathrm{Br}_{\oplus}(\overline{X})) = (\mathbb{Z}[1/2], \mathbb{Z}[1/2] \cap \mathbb{R}_{+}).$$

Figure 1 represents the presentation (X, f) with the wrapping rule $\check{f}$.

Similarly, if (Y, g) is given by Figure 2, then (Y, g) does not satisfy

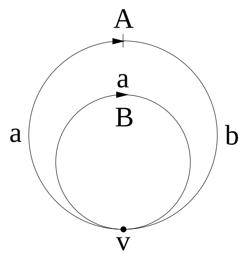

Fig. 2. (Y, g) with wrapping rule $\check{g}$

the Flattening Axiom and $(\overline{Y}, \overline{g})$ is a branched solenoid. The wrapping rule $\check{g} : \mathcal{E}_Y \to \mathcal{E}_Y^*$ is given by $a \mapsto ab$, $b \mapsto a$ and the adjacency matrix is

$$M_{Y,g} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus

$$\mathrm{Br}(\overline{Y}) = \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad \mathrm{Br}_{\oplus}(\overline{Y}) = \left\{ \mathbf{v} \in \mathbb{Z} \oplus \mathbb{Z} \,\middle|\, \mathbf{v} \cdot \left(\frac{1+\sqrt{5}}{2}, 1 \right) > 0 \right\} \cup \{\mathbf{0}\}.$$

The following example shows that the dimension group of adjacency matrices induced by a presentation is not a homeomorphism invariant.

Example 4.4 ([18, 4.8 and 5.1]). Let X be a wedge of two circles a, b with a unique vertex p, and $f : X \to X$ be defined by $a \mapsto aab$ and $b \mapsto ab$. So (X, f) is given by Figure 3. Suppose that Y is given by Figure 4 and

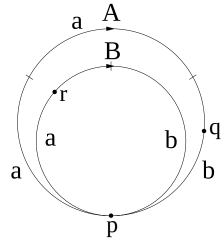

Fig. 3. (X, f) with a unique vertex $\{p\}$

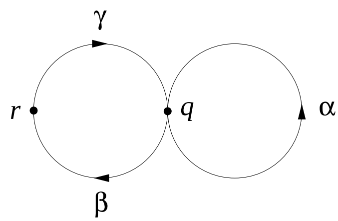

Fig. 4. The graph Y with two vertices $\{q, r\}$

that the wrapping rule $\check{g} : \mathcal{E}_Y \to \mathcal{E}_Y^*$ is given by

$$\alpha \mapsto \gamma\alpha\beta, \qquad \beta \mapsto \gamma, \qquad \gamma \mapsto \beta\gamma\alpha\beta.$$

Then it is shown in [18, 4.8] that $(\overline{X}, \bar{f})$ is topologically conjugate to $(\overline{Y}, \overline{g})$. Their adjacency matrices are given by the matrices

$$M_{(X,f)} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad M_{(Y,g)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Since the determinants of $M_{(X,f)}$ and $M_{(Y,g)}$ are 1 and -1, respectively, $M_{(X,f)}$ and $M_{(Y,g)}$ are invertible over $\mathbb{Z}$. Hence the dimension group of $M_{(X,f)}$ is $\mathbb{Z}^2$ and that of $M_{(Y,g)}$ is $\mathbb{Z}^3$. Therefore the dimension group of $M_{(X,f)}$ is not isomorphic to the dimension group of $M_{(Y,g)}$.

Since (X, f) is elementarily presented, the dimension group of $M_{(X,f)}$ is order isomorphic to the Bruschlinsky group of $(\overline{X}, \bar{f})$. And the Bruschlinsky group of $(\overline{Y}, \overline{g})$ is given by the dimension group of $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. Hence we have $\mathrm{Br}(\overline{X}) \cong \mathrm{Br}(\overline{Y}) \cong \mathbb{Z} \oplus \mathbb{Z}$ with

$$\mathrm{Br}_{\oplus}(\overline{X}) \cong \mathrm{Br}_{\oplus}(\overline{Y}) \cong \left\{ \mathbf{v} \in \mathbb{Z} \oplus \mathbb{Z} \;\middle|\; \mathbf{v} \cdot \left(\frac{3 + \sqrt{5}}{2}, 1 \right) > 0 \right\} \cup \{\mathbf{0}\}.$$

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References

[1] J. Aarts and L. Oversteegen, *Matchbox manifolds*, in: Continua (Cincinnati, OH, 1994), Lecture Notes in Pure and Appl. Math. 170, Marcel Dekker, 1995, 3–14.

[2] M. Barge and B. Diamond, *Homeomorphisms of inverse limit spaces of one-dimensional maps*, Fund. Math. 146 (1995), 509–537.

[3] M. Barge, J. Jacklitch and G. Vago, *Homeomorphisms of one-dimensional inverse limits with applications to substitution tilings*, *unstable manifolds*, *and tent maps*, in: Geometry and Topology in Dynamics, Contemp. Math. 246, Amer. Math. Soc., 1999, 1–15.

[4] M. Boyle and D. Handelman, *Orbit equivalence*, *flow equivalence and ordered cohomology*, Israel J. Math. 95 (1996), 169–210.

[5] O. Bratteli, P. E. T. Jørgensen, K. H. Kim and F. Roush, *Non-stationarity of isomorphism between AF algebras defined by stationary Bratteli diagrams*, Ergodic Theory Dynam. Systems 20 (2000), 1639–1656.

[6] E. Effros, *Dimensions and C^*-algebras*, CBMS Regional Conf. Ser. in Math. 46, Amer. Math. Soc., 1981.

[7] A. Forrest, *Cohomology of ordered Bratteli diagrams*, Pacific J. Math., to appear.

[8] T. Giordano, I. Putnam and C. Skau, *Topological orbit equivalence and C^*-crossed products*, J. Reine Angew. Math. 469 (1995), 41–111.

[9] R. Herman, I. Putnam and C. Skau, *Ordered Bratteli diagrams*, *dimension groups and topological dynamics*, Internat. J. Math. 3 (1992), 827–864.

[10] J. Krasinkiewicz, *Mappings onto circle-like continua*, Fund. Math. 91 (1976), 39–49.

[11] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge Univ. Press, 1995.

[12] S. Mardešić and J. Segal, *ε-Mappings onto polyhedra*, Trans. Amer. Math. Soc. 109 (1963), 146–164.

[13] W. Parry and S. Tuncel, *Classification Problems in Ergodic Theory*, London Math. Soc. Lecture Note Ser. 67, Cambridge Univ. Press, 1982.

[14] J. W. Rogers, Jr., *Inverse limits on graphs and monotone mappings*, Trans. Amer. Math. Soc. 176 (1973), 215–225.

[15] R. Swanson and H. Volkmer, *Invariants of weak equivalence in primitive matrices*, Ergodic Theory Dynam. Systems 20 (2000), 611–626.

[16] R. F. Williams, *One-dimensional non-wandering sets*, Topology 6 (1967), 473–487.

[17] —, *Classification of one-dimensional attractors*, in: Global Analysis, Proc. Sympos. Pure Math. 14, Amer. Math. Soc., 1970, 341–361.

[18] I. Yi, *Canonical symbolic dynamics for one dimensional generalized solenoids*, Trans. Amer. Math. Soc. 353 (2001), 3741–3767.

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