FUNDAMENTA
MATHEMATICAE
228 (2015)

Non-abelian group structure on the Urysohn universal space

by

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Abstract. We prove that there exists a non-abelian group structure on the Urysohn universal metric space. More precisely, we introduce a variant of the Graev metric that enables us to construct a free group with countably many generators equipped with a two-sided invariant metric that is isometric to the rational Urysohn space. We list several related open problems.

Introduction. There has recently been a lot of research connected to the Urysohn universal metric space. The space was constructed by P. Urysohn [14] in the 1920's but was forgotten for quite a long time. Nowadays, the Urysohn space, as well as the group of all its isometries, are a popular topic of research. A very interesting result was found by P. Cameron and A. Vershik [1] who proved that there is an abelian (monothetic) group structure on the Urysohn space. Later, P. Niemiec [10] proved that there is an abelian Boolean metric group that is isometric to the Urysohn space. And recently, Niemiec [11] rediscovered Shkarin's universal abelian Polish group [12] and proved that it is isometric to the Urysohn space as well (it is open though whether it differs from the group structures found by Cameron and Vershik). Niemiec also proved several negative results concerning group structures on the Urysohn space, e.g. he proved there is no abelian metric group of exponent 3 that is isometric to the Urysohn space [11, Proposition 2.18]. Let us also mention our previous work [3] where we showed the existence of a metrically universal separable abelian metric group (answering an open question of Shkarin [12]) which turned out to be yet another different abelian group isometric to the Urysohn space. Vershik then asked (personal communication and [15]) whether there also exists a non-abelian group structure on the Urysohn space. We answer this question affirmatively here. Thus the following is the main result of this paper.

2010 *Mathematics Subject Classification*: Primary 22A05; Secondary 54E50, 03C98.
Key words and phrases: Urysohn space, Graev metric, free groups, Fraïssé theory.

DOI: 10.4064/fm228-3-3 © Instytut Matematyczny PAN, 2015

THEOREM 0.1. *There exists a free group G on countably many generators equipped with a two-sided invariant metric that is isometric to the rational Urysohn space. In particular, there is a non-abelian group structure on the Urysohn space: the metric completion of G.*

The technical tool used for proving the theorem is an extension of a two-sided invariant metric on a group to its free product with a free group. The referee pointed out the connection of this tool to the classical Graev metric introduced in [6] and suggested giving it a more central role in the paper. We therefore state this tool in a general form, as suggested by the referee, as it might be of independent interest.

THEOREM 0.2. *Let (G, d_G) be a group with a two-sided invariant metric and let (X, d_X) be a metric space. Suppose that d' is a metric on the disjoint union $G \amalg X$ which extends both d_G and d_X, and such that for every $x \in X$ we have $\inf\{d'(g,x) : g \in G\} > 0$ (equivalently, G is closed in $G \amalg X$). Then d' extends to a two-sided invariant metric δ on $G * F(X)$, where $F(X)$ is the free group with X as the set of generators.*

REMARK 0.3. If $G = \{1\}$, then the δ from the statement of Theorem 0.2 corresponds to the standard Graev metric [6] on $F(X)$, with 1 as a unit, constructed over the pointed space $X \amalg \{1\}$.

We also refer the reader to [13] where a variant of the Graev metric on free products of groups having a common closed subgroup was defined.

The subject of group structures on the Urysohn space is still far from being exhausted, and several open questions are provided at the end of the paper. Since most of the groups isometric to the Urysohn space are constructed via Fraïssé theory, we also pose a few questions related to Fraïssé classes of metric groups.

1. Preliminaries and definitions.

Recall that the *Urysohn universal metric space* is a Polish metric space that contains an isometric copy of every finite metric space, and every partial isometry between two finite subsets extends to an autoisometry of the whole space. These properties characterize the Urysohn space uniquely up to isometry and moreover imply that it contains an isometric copy of every separable metric space.

The *rational Urysohn space* is a countable metric space with all distances rational that contains an isometric copy of every finite rational metric space, and every partial isometry between two finite subsets extends to an autoisometry of the whole space. Again, it follows that such a space is unique up to isometry and contains an isometric copy of every countable rational metric space. Moreover, one can prove that the completion of the rational Urysohn space is the Urysohn space.

Recall that a function $f : X \to \mathbb{R}_0^+$ is called *Katětov*, where (X, d) is some metric space, if $|f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y)$ for all $x, y \in X$. One should think about the Katětov function f as a function that prescribes distances from some, potentially new, point. We refer the reader to [8] for more information about Katětov functions and for the construction of the Urysohn space using them.

The following well known fact characterizes the Urysohn and the rational Urysohn spaces.

Fact 1.1.

1. *Let (X, d) be a countable metric space with a rational metric. Then it is isometric to the rational Urysohn space iff for every finite subset $A \subseteq X$ and every rational Katětov function $f : A \to \mathbb{Q}^+$ there exists $x \in X$ such that $d(a, x) = f(a)$ for all $a \in A$.*
2. *Let (X, d) be a Polish metric space. Then it is isometric to the Urysohn space iff for every finite subset $A \subseteq X$ and every Katětov function $f : A \to \mathbb{R}^+$ there exists $x \in X$ such that $d(a, x) = f(a)$ for all $a \in A$.*

Later, when we construct the free group with a two-sided invariant rational metric, we will check that it is isometric to the rational Urysohn space using the above characterization.

We now define a special type of metric that is completely determined by its values on pairs from a finite set.

Definition 1.2. Let (G, d) be a metric group. We say that the metric d is *finitely generated* if there exists a finite set $A_G \subseteq G$ (called a *generating set* for d) such that $1 \in A_G$, $A_G = A_G^{-1}$ and for every $a, b \in G$ we have

$$\begin{aligned} d(a,b) = \min\{&d(a_1, b_1) + \cdots + d(a_n, b_n) : n \in \mathbb{N}, \\ &\forall i \leq n\ (a_i, b_i \in A_G \wedge a = a_1 \cdot \ldots \cdot a_n,\ b = b_1 \cdot \ldots \cdot b_n)\}. \end{aligned}$$

In particular, G is (algebraically) generated by A_G.

Fact 1.3. *If d is a finitely generated metric on a group G, then d is two-sided invariant.*

Proof. Recall that (as can be easily verified) a metric d on a group G is two-sided invariant iff $d(a \cdot b, c \cdot d) \leq d(a, c) + d(b, d)$ for all a, b, c, d in G. It follows from the definition that finitely generated metrics have this property. ■

Let us now present the referee's observation which connects groups with a finitely generated metric with free groups with the Graev metric. Regarding the Graev metric on a free group, we refer the reader to [2, Section 3].

Observation 1.4. *G is a group with a finitely generated metric iff G is a factor group, with the factor metric of a free group on finitely many generators with the Graev metric.*

To see this, suppose that the finitely generated metric on G is generated by a finite set $A_G \subseteq G$. Consider the free group $F(A_G \setminus \{1\})$, with $A_G \setminus \{1\}$ the set of free generators and $1 \in A_G$ the unit, with the Graev metric. Since A_G also (algebraically) generates G, there is a natural homomorphism from $F(A_G \setminus \{1\})$ onto G. It follows from the definitions of the respective metrics that this homomorphism is 1-Lipschitz and that the distance between two elements of G is equal to the infimum distance between the corresponding classes in $F(A_G \setminus \{1\})$.

Conversely, if $F(X)$ is a free group with the Graev metric constructed over a finite pointed metric space $X \amalg \{1\}$, and $H \leq F(X)$ is a closed subgroup, then the factor metric on $F(X)/H$ is finitely generated. One can check that the generating set for the factor metric is $\{[a]_H : a \in X \amalg X^{-1} \amalg \{1\}\}$.

2. Proofs of the main theorems.

Having defined a finitely generated metric, we restate Theorem 0.2 here adding a special case when d_G is finitely generated and X is finite.

Theorem 2.1. *Let (G, d_G) be a group with a two-sided invariant metric and let (X, d_X) be a metric space.*

1. *Suppose that d' is a metric on the disjoint union $G \amalg X$ which extends both d_G and d_X, and such that for every $x \in X$ we have $\inf\{d'(g,x) : g \in G\} > 0$ (equivalently, G is closed in $G \amalg X$). Then d' extends to a two-sided invariant metric δ on $G * F(X)$, where $F(X)$ is the free group with X as the set of generators.*
2. *If d_G is finitely generated by values on pairs from some finite $A_G \subseteq G$, X is finite and d' is a metric on $A_G \amalg X$ which extends both $d_G{\restriction}A_G$ and d_X, then d' extends to a finitely generated metric δ on $G * F(X)$ such that $\delta{\restriction}G = d_G$.*

First, we show how to deduce Theorem 0.1 from Theorem 2.1. For every $m \in \mathbb{N}$, we shall denote by F_m the free group on m generators.

Suppose that f is a rational Katětov function defined on some finite set $B \subseteq F_m$, where F_m is equipped with a finitely generated rational metric. Call such a Katětov function *relevant*. Since there are only countably many finitely generated rational metrics on free groups on finitely many generators, it follows that there are only countably many relevant Katětov functions. So let $(f_n)_{n \in \mathbb{N}}$ be an enumeration of all relevant Katětov functions with infinite repetition.

We construct the group G inductively as a direct limit of free groups on finitely many generators equipped with a (two-sided invariant) finitely generated rational metric which is a variant of the Graev metric (defined below). A construction using a direct limit of groups equipped with the Graev metric was first used in [4] to produce a Polish group in which Lie sums and Lie brackets do not exist.

As base step, we set $F_1 \cong \mathbb{Z}$ to be the integers with the standard Euclidean metric d_1, which is clearly finitely generated and rational.

Suppose we have constructed a free group F_n with a finitely generated rational metric d_n which is generated by values on pairs from some finite set $A_n \subseteq F_n$. Consider the relevant Katětov function f_n. Then $f_n : (F_m, p) \supseteq B \to \mathbb{Q}^+$ for some $m \in \mathbb{Z}$, some finitely generated rational metric p and some finite set B. Suppose that $m \leq n$ and (F_m, p) is isometrically isomorphic to $(F_m, d_n \restriction F_m)$, where F_m is naturally identified with the free subgroup of F_n generated by m free generators. Then we can actually view f_n as defined on some finite set $B \subseteq F_n$. Without loss of generality, we may suppose that A_n, the generating set for d_n, is equal to B. Indeed, we could extend f_n to $A_n \cup B$ and then $A_n \cup B$ would still be a generating set for d_n.

Using Theorem 2.1(2) with $X = \{x\}$ and $d'(x, g) = f(g)$ for every g in $B = A_n$, we extend the metric d_n to a (finitely generated rational) metric d_{n+1} on $F_n * F_1 \cong F_{n+1}$ such that the Katětov function f_n is realized by the newly added generator.

If, on the other hand, either $m > n$ or $m \leq n$ but (F_m, p) is *not* isometrically isomorphic to $(F_m, d_n \restriction F_m)$, then we extend (F_n, d_n) to (F_{n+1}, d_{n+1}) arbitrarily (just ensuring that d_{n+1} is still finitely generated and rational).

When the inductive construction is finished, we have a free group with countably many generators, denoted by G, equipped with some two-sided invariant rational metric d. Then the group operations on G are continuous with respect to the topology induced by the metric. To see this, just observe that by invariance for any $g, h \in G$ we have

$$d(g, h) = d(g^{-1}, h^{-1}), \tag{2.1}$$

so the inverse operation is continuous (being an isometry), and for any $g_1, g_2, h_1, h_2 \in G$ we have

$$d(g_1 \cdot h_1, g_2 \cdot h_2) \leq d(g_1, g_2) + d(h_1, h_2) \tag{2.2}$$

by invariance and the triangle inequality.

Consider now the metric completion, denoted $\mathbb{G}$, of G. It is a separable complete metric space and the group operations extend to the completion. Indeed, the inverse operation extends because it is an isometry by (2.1), and the group multiplication extends because if $(g_n)_n, (h_n)_n \subseteq G$ are two

Cauchy sequences then $(g_n \cdot h_n)_n$ is a Cauchy sequence as well by (2.2). It follows that $\mathbb{G}$ is a Polish group equipped with a two-sided invariant metric. We refer the reader to [5] for an exposition of Polish (metric) groups.

We claim that G is isometric to the rational Urysohn space. It suffices to check the condition from Fact 1.1(1). So let $f : G \supseteq A \to \mathbb{Q}^+$ be an arbitrary rational Katětov function defined on a finite subset A. Then there exists some $m \in \mathbb{Z}$ such that $A \subseteq F_m$ and there are infinitely many n's such that f corresponds to f_n. Choose one such n that is greater than m. Then, in the nth induction step, we have guaranteed that f_n, and thus f, is realized in F_{n+1}.

The rest of the section is devoted to proving Theorem 2.1. We first prove (1) and then show how (2) follows.

The reader is invited to compare the tools of the proof with those in [2, Section 3]. Let X^{-1} be a disjoint copy of X, considered as the set of formal inverses of elements of X. For every $x \in X^i$, $i \in \{-1, 1\}$, x^{-1} denotes the corresponding element in X^{-i}. We extend d' to a distance d on $G \amalg X \amalg X^{-1}$ so that:

- For all $a, b \in G \amalg X^{-1}$ we have $d(a, b) = d'(a^{-1}, b^{-1})$.
- For all $a \in G \amalg X$ and $b \in G \amalg X^{-1}$ we have

$$d(a, b) = \inf\{d'(a, c) + d'(c^{-1}, b^{-1}) : c \in G\}.$$

In other words, first we define the distances between elements of $G \amalg X^{-1}$ so that the bijection $a \mapsto a^{-1}$ between $G \amalg X$ and $G \amalg X^{-1}$ is an isometry. Then we take the (greatest) metric amalgamation of $G \amalg X$ and $G \amalg X^{-1}$ over G.

Denote now $G \amalg X \amalg X^{-1}$ by S, and let $W(S)$ be the set of all words over S considered as an alphabet.

Definition 2.2. A word $w = w_1 \dots w_n \in W(S)$, where $w_i \in S$ for $i \leq n$, is called *irreducible* if for no $i < n$ do we have $w_i, w_{i+1} \in G$ or $w_i = w_{i+1}^{-1}$.

For every $w \in W(S)$ we shall denote by w' the corresponding element in $G * F(X)$. Note that the mapping $w \mapsto w'$ is a bijection between irreducible words from $W(S)$ and elements of $G * F(X)$.

For any $w \in W(S)$, let $|w|$ denote its length. If $v, w \in W(S)$ are two words of the same length n, then we define the pre-distance between them as

$$\rho(v, w) = d(v_1, w_1) + \cdots + d(v_n, w_n).$$

Finally, we define the *Graev metric* δ on $G * F(X)$ as

$$\delta(u, v) = \inf\{\rho(u^*, v^*) : u^*, v^* \in W(S),\ |u^*| = |v^*|,\ (u^*)' = u,\ (v^*)' = v\}$$

for any $u, v \in G * F(X)$. It is easy to check that δ is symmetric and that $\delta(u \cdot v, w \cdot x) \leq \delta(u, w) + \delta(v, x)$ for any $u, v, w, x \in G * F(X)$. The latter property also implies two-sided invariance and the triangle inequality. We need to check that it is indeed a metric, i.e. $\delta(u,v) > 0$ if $u \neq v$, and that it extends d'.

We need the following definition (cf. [2, Definition 3.3]).

DEFINITION 2.3 (Match). Let $w \in W(S)$ be a word of length n. Let $P \subseteq \{1,\dots,n\}$ be the subset such that for every $i \leq n$ we have $i \in P$ iff $w_i \in X \amalg X^{-1}$. We call a function $\theta : P \to P$ a *match* for w if

- for every $i \in P$, we have $\theta(\theta(i)) = i$,
- for every $i \in P$, we have $w_i = w_{\theta(i)}^{-1}$,
- for every $i \in P$, assuming without loss of generality that $i < \theta(i)$, we have
$$\prod_{i \leq j \leq \theta(i)} w'_j = 1.$$

LEMMA 2.4. *Let $w \in W(S)$ be such that $w' \in G$. Then there exists a match for w.*

Proof. Let $n = |w|$. We suppose that $w_i \in X \amalg X^{-1}$ for some $i \leq n$; otherwise, there is nothing to prove.

We claim that it suffices to prove the lemma when $w_1 \in X \amalg X^{-1}$, $w_n = w_1^{-1}$, $w' = 1$ and $(w_1 \dots w_i)' = 1$ for no $i < n$. Call such a sequence $w_1,\dots,w_n$ a *cancelling X-sequence* of length $|P|$, where again $P = \{i \leq n : w_i \in X \amalg X^{-1}\}$. Indeed, suppose the lemma is proved for that case. Consider the first index $1 \leq i_S < n$ such that $w_{i_S} \in X \amalg X^{-1}$. Since $w' \in G$ there must exist $i_S < i \leq n$ such that $(w_{i_S} \dots w_i)' = 1$. Let i_F be the least such i. Clearly, $w_{i_S} = w_{i_F}^{-1}$. By assumption, we can find a match θ_1 for the subword $w_{i_S} \dots w_{i_F}$. Then we look for the least index $i_F < i'_S$, if any, such that $w_{i'_S} \in X \amalg X^{-1}$. Again, we can find an appropriate $i'_S < i'_F \leq n$ and then find a match θ_2 for $w_{i'_S} \dots w_{i'_F}$. At the end, we can take as θ the union $\theta_1 \cup \theta_2 \cup \cdots$ of all the matches obtained that way.

We now prove the lemma (with the assumption that $w_1,\dots,w_n$ is a cancelling X-sequence) by induction on $|P|$. If $|P| = 2$ then clearly we may set $\theta(1) = n$ and $\theta(n) = 1$ and we are done.

Suppose now that $|P| = m > 2$ and the lemma has been proved for all (even) $l < m$. Suppose that $P = \{k_1 = 1,\dots,k_m = n\} \subseteq \{1,\dots,n\}$. Since $(w_1 \dots w_n)' = 1$ and $w_1 = w_n^{-1}$, we have $(w_2 \dots w_{n-1})' = 1$. Thus there must exist $2 < i < n$ such that $(w_{k_2} \dots w_{k_i})' = 1$. Take the least such i. Then both $w_{k_2},\dots,w_{k_i}$ and $w_1,\dots,w_{k_2-1},w_{k_i+1},\dots,w_n$ are cancelling X-sequences of length less than m. By induction hypothesis, we can find corresponding $\theta_1 : \{k_2,\dots,k_i\} \to \{k_2,\dots,k_i\}$ and $\theta_2 : \{k_1,k_{i+1},\dots,k_m\} \to \{k_1,k_{i+1},\dots,k_m\}$. We then set $\theta = \theta_1 \cup \theta_2$ and we are done. ■

Lemma 2.5. *Let* $g \in G * F(X)$. *Then* $\delta(g,1) = \inf\{\rho(w,u) : w,u \in W(S),\ |w| = |u|,\ w' = g,\ u' = 1,\ w \text{ is irreducible}\}$.

Proof. Let $w,u \in W(S)$ be such that $|w| = |u| = n$ and $w' = g$, $u' = 1$. Suppose that w is not irreducible. We will show that we can then reduce the words w and u to $\bar{w},\bar{u}$ such that $|\bar{w}| = |\bar{u}|$, $\bar{w}' = g$, $\bar{u}' = 1$, and $\rho(\bar{w},\bar{u}) \leq \rho(w,u)$.

Let $\theta : P \to P$ be a match for u. Since w is not irreducible, according to Definition 2.2 there is $i < n$ such that either $w_i, w_{i+1} \in G$, or $w_i, w_{i+1} \in X \amalg X^{-1}$ and $w_i = w_{i+1}^{-1}$. We shall treat these cases separately.

Case 1. Suppose that $w_i, w_{i+1} \in G$.

Subcase 1a. If $u_i, u_{i+1} \in G$ as well, then we could reduce w and u to $\bar{w},\bar{u}$ so that for every $j < i$, $\bar{v}_j = v_j$, $\bar{v}_i = v_i \cdot v_{i+1}$, and for every $i < j < n$, $\bar{v}_j = v_{j+1}$, where v is either w or u. In that case, we have $\rho(\bar{w},\bar{u}) \leq \rho(w,u)$ since $d(w'_i \cdot w'_{i+1}, u'_i \cdot u'_{i+1}) \leq d(w'_i, u'_i) + d(w'_{i+1}, u'_{i+1})$ by two-sided invariance of d on G.

Subcase 1b. Suppose that either u_i or u_{i+1} belongs to $X \amalg X^{-1}$, say $u_i \in X \amalg X^{-1}$. We shall find $\tilde{u} \in W(S)$ such that $|\tilde{u}| = |u|$, $\tilde{u}' = 1$, $\rho(w,\tilde{u}) \leq \rho(w,u)$ and $\tilde{u}_i \in G$. Let $j = \theta(i)$ and suppose that $j > i$; the other case is analogous. Then since θ is a match for u, we have $u_j = u_i^{-1}$. Thus $d(w_j,u_j) + d(u_i,w_i) \geq d(w_j^{-1}, w_i)$. Since $\prod_{i<k<j} u'_k = 1$, we can modify u to $\tilde{u}$ so that $\tilde{u}_i = w_j^{-1}$, $\tilde{u}_j = w_j$, and $\tilde{u}_k = u_k$ for $k \in \{1,\ldots,n\} \setminus \{i,j\}$. Then $\tilde{u}$ is as required since $\rho(w,u) - \rho(w,\tilde{u}) = (d(w_j,u_j) + d(u_i,w_i)) - (d(w_j^{-1}, w_i) + d(w_j,w_j)) \geq 0$. If $\tilde{u}_{i+1} \in G$, then we are in Subcase 1a. Otherwise, apply the procedure above also for $\tilde{u}_{i+1}$. Then we will be in Subcase 1a.

Case 2. Suppose that $w_i, w_{i+1} \in X \amalg X^{-1}$ and $w_{i+1} = w_i^{-1}$.

Subcase 2a. Suppose that either u_i or u_{i+1} is in G, say $u_i \in G$. Then $d(u_i,w_i) + d(w_{i+1},u_{i+1}) \geq d(u_i, u_{i+1}^{-1}) = d(u_i^{-1}, u_{i+1})$. We can replace w_i by u_i and w_{i+1} by u_i^{-1} in w to obtain $\tilde{w}$. Again, clearly $\rho(\tilde{w},u) \leq \rho(w,u)$. Note that both $\tilde{w}_i$ and $\tilde{w}_{i+1}$ are then in G. Thus we are in Case 1.

Subcase 2b. Suppose that both u_i and u_{i+1} are in $X \amalg X^{-1}$. Using the match θ and arguing as in Subcase 1a, we can check that $d(w_i,u_i) + d(u_{\theta(i)}, w_{\theta(i)}) \geq d(w_i^{-1}, w_{\theta(i)})$ and $d(w_{i+1},u_{i+1}) + d(u_{\theta(i+1)}, w_{\theta(i+1)}) \geq d(w_{i+1}^{-1}, w_{\theta(i+1)})$. It follows that we may modify u to $\tilde{u}$ so that $\tilde{u}_i = w_i$, $\tilde{u}_{i+1} = w_{i+1}$ and $\tilde{u}_{\theta(i)} = w_i^{-1}$, $\tilde{u}_{\theta(i+1)} = w_{i+1}^{-1}$; at other positions, $\tilde{u}$ is equal to u. It follows that $\rho(w,\tilde{u}) \leq \rho(w,u)$ and we can erase $w_i = \tilde{u}_i = w_{i+1}^{-1} = \tilde{u}_{i+1}^{-1}$ from w and $\tilde{u}$ respectively. ■

We are now ready to finish the proof of Theorem 2.1(1). Recall that it remains to prove that $\delta(x,y) > 0$ if $x \neq y$, and that δ extends d.

For the positivity, since δ is two-sided invariant it suffices to check that $\delta(x,1)>0$ for any $x\in G*F(S)$ such that $x\neq 1$. Let $w\in W(S)$ be the irreducible word such that $w'=x$. Let $n=|w|$. By Lemma 2.5, we have $\delta(x,1)=\inf\{\rho(w,u):u\in W(S),\ |u|=n,\ u'=1\}$. By assumption

$$\varepsilon_0=\min_{w_i\in X\amalg X^{-1}}\inf\{d(g,w_i):g\in G\}>0.$$

Let $u\in W(S)$ be arbitrary such that $|u|=n$ and $u'=1$. If there exists $i\leq n$ such that $w_i\in X\amalg X^{-1}\wedge u_i\in G$, then $\rho(w,u)\geq\varepsilon_0>0$. Suppose there exists $i\leq n$ such that $w_i\in G\wedge u_i\in X\amalg X^{-1}$. Let $\theta:P\to P$ be a match for u, where again $P=\{i\leq n:u_i\in X\amalg X^{-1}\}$. Then we could replace u by u^* such that $u^*_j=u_j$ for $j\in\{1,\dots,n\}\setminus\{i,\theta(i)\}$, and $u^*_i=w_i$ and $u^*_{\theta(i)}=w_i^{-1}$. Indeed, since $d(w_{\theta(i)},w_i^{-1})\leq d(w_i,u_i)+d(u_i^{-1},w_{\theta(i)})$, it follows that $\rho(w,u^*)\leq\rho(w,u)$.

Consequently, we may suppose that for every $i\leq n$ we have $w_i\in G$ iff $u_i\in G$. Indeed, if for some $i\leq n$ we have $w_i\in X\amalg X^{-1}$ and $u_i\in G$, then we have argued above that $\rho(w,u)\geq\varepsilon_0>0$. If, on the other hand, for some $i\leq n$ we have $w_i\in G$ and $u_i\in X\amalg X^{-1}$, then we have argued that we can replace u by u^* such that $(u^*)'=1$, $|u^*|=|u|$, $u^*_i\in G$ and $\rho(w,u^*)\leq\rho(w,u)$.

If $w_i\in G$ for every $i\leq n$, then since w is irreducible we have $w=w_1$, $u=u_1=1$, and clearly $\rho(w,u)=d(w,1)>1$. Therefore we suppose that $P\neq\emptyset$. Let

$$\begin{aligned}\varepsilon_1&=\min\{d(w_i,w_j^{-1}):i,j\in P,\ w_i\neq w_j^{-1}\},\\ \varepsilon_2&=\min\{d(w_j,1):w_j\in G\}.\end{aligned}$$

If there exists $i\in P$ such that $w_i\neq u_i$ then $\rho(w,u)\geq d(w_i,u_i)+d(w_{\theta(i)},u_{\theta(i)})$ $=d(w_i,u_i)+d(w_{\theta(i)},u_i^{-1})\geq d(w_i,w_{\theta(i)}^{-1})\geq\varepsilon_1>0$, since $u_i=u_{\theta(i)}^{-1}$.

Otherwise, $w_i=u_i$ for every $i\in P$. However, since $w'\neq 1$, there exists $i\in P$ such that $\theta(i)>i$ and for every $i<j<\theta(i)$ we have $j\notin P$. We claim that either $\theta(i)=i+1$ or $\theta(i)=i+2$. Indeed, if $\theta(i)>i+1$, then $w_{i+1}\in G$, and since w is irreducible we must have $w_{i+2}\in X\amalg X^{-1}$, so $i+2\in P$ and the claim follows. If the first case holds, i.e. $\theta(i)=i+1$, we have $\rho(w,u)\geq d(w_i,u_i)+d(w_{i+1},u_i^{-1})\geq d(w_i,w_{i+1}^{-1})\geq\varepsilon_1>0$. If $\theta(i)=i+2$, then by the definition of match we must have $u_{i+1}=1$, and thus $\rho(w,u)\geq d(w_{i+1},1)\geq\varepsilon_2>0$.

To prove that δ extends d, let $x,y\in S$. Clearly, $\delta(x,y)\leq d(x,y)$. By two-sided invariance of δ and Lemma 2.5 we have $\delta(x,y)=\delta(x\cdot y^{-1},1)=\inf\{d(x,z)+d(y^{-1},z^{-1}):z\in S\}$. However, the infimum is attained for $z=x$ or $z=y$ since $d(x,z)+d(y^{-1},z^{-1})=d(x,z)+d(z,y)\geq d(x,y)$, and we are done.

It remains to prove item (2) of Theorem 2.1. First of all, we consider the greatest metric d'' on $G \amalg X$ that extends d' on $A_G \amalg X$ and d_G on G. More precisely, d'' is the amalgam metric of d' on $A_G \amalg X$ and d_G on G over A_G, i.e. $d''(x,g) = \inf\{d'(x,g_0) + d_G(g_0,g) : g_0 \in A_G\}$ for $x \in X$ and $g \in G$. Observe that the infimum is in fact attained since A_G is finite. Thus in particular, if d' is rational, so is d''. Next, we extend d'' to δ as in item (1). We need to verify that δ is finitely generated, and if d' is rational, then so is δ.

Let d be the extension of d'' to $G \amalg X \amalg X^{-1}$ as in the proof of item (1). Define a metric γ on $G * F(X)$ as follows: for any $x, y \in G * F(X)$, set

$$\begin{aligned}\gamma(x,y) = \inf\{&d(x_1,y_1) + \cdots + d(x_n,y_n) : n \in \mathbb{N},\\ &x_1,y_1,\ldots,x_n,y_n \in A_G \amalg X \amalg X^{-1},\ x = x_1 \cdot \ldots \cdot x_n,\ y = y_1 \cdot \ldots \cdot y_n\}.\end{aligned}$$

Then γ is a two-sided invariant metric which is finitely generated by the values on pairs from $A_G \amalg X \amalg X^{-1}$. Moreover, the infimum is actually attained since $A_G \amalg X \amalg X^{-1}$ is finite. Thus, if d' is rational, then so is γ. Comparing the definitions of γ and δ in this particular case, one can see that they are equal. Indeed, for any $x, y \in G * F(X)$ we have

$$\begin{aligned}\delta(x,y) &= \inf\{\rho(w_x,w_y) : w_x,w_y \in W(S),\ w'_x = x,\ w'_y = y,\ |w_x| = |w_y|\}\\ &= \inf\{d(x_1,y_1) + \cdots + d(x_m,y_m) : x_1,y_1,\ldots,x_m,y_m \in G \amalg X \amalg X^{-1},\\ &\qquad x = x_1 \cdot \ldots \cdot x_m,\ y = y_1 \cdot \ldots \cdot y_m\}.\end{aligned}$$

Observe that in the previous equivalent definition of δ, the elements $x_1, y_1, \ldots, x_m, y_m$ are allowed to be from $G \amalg X \amalg X^{-1}$, while in the definition of γ they have to be from $A_G \amalg X \amalg X^{-1}$. Thus, $\delta(x,y) \leq \gamma(x,y)$. However, for every $f, g \in G$ we have $d(f,g) = d(f_1,g_1) + \cdots + d(f_j,g_j)$ for some $f_1, g_1, \ldots, f_j, g_j \in A_G$ since $d \restriction G = d_G$ is finitely generated by A_G. Similarly, for every $z \in X \amalg X^{-1}$ and $g \in G$ we have $d(z,g) = d(z,h) + d(h,g)$ for some $h \in A_G$. Thus for some $h_1, g_1, \ldots, h_l, g_l \in A_G$ we have $d(h,g) = d(h_1,g_1) + \cdots + d(h_l,g_l)$, so $d(z,g) = d(z,h) + d(h^{-1},h^{-1}) + d(h_1,g_1) + \cdots + d(h_l,g_l)$. So actually

$$\begin{aligned}\delta(x,y) = \inf\{&d(x_1,y_1) + \cdots + d(x_m,y_m) : x_1,y_1,\ldots,x_m,y_m \in A_G \amalg X \amalg X^{-1},\\ &x = x_1 \cdot \ldots \cdot x_m,\ y = y_1 \cdot \ldots \cdot y_m\} = \gamma(x,y).\end{aligned}$$

3. Open questions and problems

3.1. Groups isometric to the Urysohn space. To summarize, there are now five known group structures on the Urysohn space (1), the groups

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(1) One should rather talk about classes of group structures since Cameron–Vershik's example is a class of continuum many different monothetic group structures on the Urysohn space.

from [1], [10], [11] (and [12]), [3] and the present paper. Four of them are known to be different, but it is open whether Shkarin/Niemiec's group belongs to Cameron–Vershik's class. We provide some open questions from this area.

Let us start with groups of finite exponent. We have already mentioned in the introduction that Niemiec [10] proved that there is an abelian metric group of exponent 2 isometric to the Urysohn space and that he proved in [11] that there is no abelian metric group of exponent 3 isometric to the Urysohn space. Moreover, consider the Fraïssé class of all finite abelian groups of exponent n, where $n > 3$, equipped with an invariant rational metric. Niemiec showed [11, Theorem 5.5] that, surprisingly, the corresponding Fraïssé limit is not isometric to the rational Urysohn space. However, the following problem is still open.

Question 3.1 (Niemiec). *Does there exist an abelian metric group of finite exponent other than* 2 *and* 3 *that is isometric to the Urysohn space?*

Since all known metric groups isometric to the Urysohn space have an invariant metric and a countable dense subgroup isometric to the rational Urysohn space, the following problem is probably worthy of investigation.

Problem 3.2. *Characterize those countable groups that admit a* <u>*two-sided*</u> *invariant metric with which they are isometric to the rational Urysohn space.*

The reason we stressed that the metric should be two-sided invariant is that in that case the group operations are automatically continuous and the operations extend to the metric completion. The following question is thus natural in this context.

Question 3.3. *Does there exist a metric group that is isometric to the* (*rational*) *Urysohn space, but the metric is not two-sided invariant?*

3.2. Fraïssé classes of metric groups. The natural class of all finite abelian groups equipped with an invariant rational metric is rather easily checked to be a Fraïssé class, and the metric completion of the corresponding Fraïssé limit is the universal Polish abelian group from [12] and [11]. However, the analogous problem for the non-abelian case is open.

Question 3.4. *Does the class of all finite groups equipped with a two-sided invariant rational metric have the amalgamation property?*

Note that the class of all finite groups does have the amalgamation property [9] and the Fraïssé limit is Hall's universal locally finite group [7]. It is not hard to check that if the class from Question 3.4 were Fraïssé, then the Fraïssé limit would be algebraically isomorphic to the Hall group. It is not clear though whether it would be isometric to the rational Urysohn space.

Acknowledgements. The author is grateful to the referee for the excellent report pointing out the connection between our original construction and the Graev metric and suggesting a generalization which is presented in this final form of the article.

The author is also grateful to Wiesław Kubiś for discussions on this topic.

Part of this work was done during the trimester program "Universality and Homogeneity" at the Hausdorff Research Institute for Mathematics in Bonn. The author would therefore like to thank for the support and great working conditions there.

The author was supported by IMPAN's international fellowship programme partially sponsored by PCOFUND-GA-2012-600415.

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Received 13 March 2014;
in revised form 23 June 2014