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Coloring grids

by

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Abstract. A structure $\mathcal{A} = (A; E_i)_{i\in n}$ where each E_i is an equivalence relation on A is called an *n-grid* if any two equivalence classes coming from distinct E_i's intersect in a finite set. A function $\chi : A \to n$ is an *acceptable coloring* if for all $i \in n$, the set $\chi^{-1}(i)$ intersects each E_i-equivalence class in a finite set. If B is a set, then the n-cube B^n may be seen as an n-grid, where the equivalence classes of E_i are the lines parallel to the ith coordinate axis. We use elementary submodels of the universe to characterize those n-grids which admit an acceptable coloring. As an application we show that if an n-grid $\mathcal{A}$ does not admit an acceptable coloring, then every finite n-cube is embeddable in $\mathcal{A}$.

1. Introduction. Following [3], for a natural number $n \geq 2$ we define an *n-grid* to be a structure of the form $\mathcal{A} = (A; E_i)_{i\in n}$ such that each E_i is an equivalence relation on the set A and $[a]_i \cap [a]_j$ is finite whenever $a \in A$ and $i < j < n$ (where $[a]_i$ denotes the equivalence class of a with respect to the relation E_i). An *n-cube* is a particular kind of n-grid where A is of the form $A = A_0 \times \cdots \times A_{n-1}$ and each E_i is the equivalence relation on A whose equivalence classes are the lines parallel to the ith coordinate axis (i.e. two n-tuples are E_i-related if and only if all of their coordinates coincide except perhaps for the ith one). An *acceptable coloring* for an n-grid $\mathcal{A}$ is a function $\chi : A \to n$ such that $[a]_i \cap \chi^{-1}(i)$ is finite for all $a \in A$ and $i \in n$.

In [3], J. H. Schmerl gives a nice characterization of those semialgebraic n-grids which admit an acceptable coloring:

THEOREM 1.1 (Schmerl). *Suppose that $2 \leq n < \omega$, $\mathcal{A}$ is a semialgebraic n-grid and $2^{\aleph_0} \geq \aleph_{n-1}$. Then the following are equivalent:*

(1) *Some finite n-cube is not embeddable in $\mathcal{A}$.*
(2) *$\mathbb{R}^n$ is not embeddable in $\mathcal{A}$.*
(3) *$\mathcal{A}$ has an acceptable n-coloring.*

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In this note, we present a characterization that works for any n-grid (see Definition 2.1 and Theorem 2.7). Then we use this characterization to show that (1)⇒(3) in the previous theorem holds for arbitrary n-grids (see Theorem 3.1). In fact, the size of the continuum turns out to be irrelevant for this implication. The implication (3)⇒(2) for arbitrary n-grids follows from a result of Kuratowski, as mentioned in [3]. None of these implications can be reversed for arbitrary n-grids, regardless of the size of the continuum.

2. Twisted n-grids. In this section we use elementary submodels of the universe to obtain a characterization of those n-grids which admit an acceptable coloring. At first sight this characterization seems rather cumbersome, but it is the key to our results in the next section. The case $n = 3$ was already obtained in [1] with a bit different terminology and later used in [2].

As has become customary, whenever we say that M is an *elementary submodel of the universe*, we in fact mean that $(M, \in)$ is an elementary submodel of $(H(\theta), \in)$, where $H(\theta)$ is the set of all sets of hereditary cardinality less than θ, and θ is a large enough regular cardinal (e.g. when we are studying a fixed n-grid $\mathcal{A}$ on a transitive set A, $\theta = (2^{|A|})^+$ is large enough).

Given an equivalence relation E on a set A, we say that $B \subseteq A$ is *E-small* if the E-equivalence classes restricted to B are all finite. Note that the E-small sets form an ideal in the power set of A. Using this terminology, an n-coloring $\chi : A \to n$ is acceptable for the n-grid $(A; E_i)_{i\in n}$ if and only if $\chi^{-1}(i)$ is E_i-small for each $i \in n$.

A *test set* for an n-grid $\mathcal{A}$ is a set $\mathcal{M}$ of elementary submodels of the universe such that $\mathcal{A} \in \bigcap \mathcal{M}$, $|\mathcal{M}| = n - 1$ and $\mathcal{M}$ is linearly ordered by $\in$.

Definition 2.1. We say that that an n-grid $\mathcal{A} = (A; E_i)_{i\in n}$ is *twisted* if for every test set $\mathcal{M}$ for $\mathcal{A}$ and every $k \in n$, the set

$$\{x \in A \setminus \bigcup \mathcal{M} : [x]_i \in \bigcup \mathcal{M} \text{ for all } i \neq k\}$$

is E_k-small.

The rest of this section is devoted to showing that twisted n-grids are exactly the ones that admit acceptable colorings. For this, let us fix an arbitrary n-grid $\mathcal{A} = (A; E_i)_{i\in n}$; our first task is to cover A with countable elementary submodels in a way that allows us to define a suitable rank function for elements of A and for E_i-equivalence classes of elements of A.

We fix M_Λ an elementary submodel such that $A \cup \{\mathcal{A}\} \subseteq M_\Lambda$ and we let $\kappa = |M_\Lambda|$. Thinking of κ as an initial ordinal, we let $T = \bigcup_{m\in\omega} \kappa^m$ be the set of finite sequences of ordinals in κ. We have two natural orders on T, the tree (partial) order $\subseteq$ and the lexicographic order $\leq$. In both orders we have the same minimum element Λ, the empty sequence. For $\sigma \in T$ and

$\alpha \in \kappa$ we write $\sigma^\frown\alpha = \sigma \cup \{\langle|\sigma|, \alpha\rangle\}$. Given $\sigma \in T \setminus \{\Lambda\}$ we write $\sigma + 1$ for the successor of σ in the lexicographic order of $\kappa^{|\sigma|}$, that is,

$$\sigma + 1 = (\sigma\restriction(|\sigma| - 1))^\frown(\sigma(|\sigma| - 1) + 1).$$

We shall write $\sigma \wedge \tau$ for the infimum of σ and τ with respect to the tree order; thus for $\sigma \neq \tau$ we have

$$\sigma \wedge \tau = \sigma\restriction|\sigma \wedge \tau| = \tau\restriction|\sigma \wedge \tau| \quad \text{and} \quad \sigma(|\sigma \wedge \tau|) \neq \tau(|\sigma \wedge \tau|).$$

Now we can find inductively (on the length of $\sigma \in T$) elementary submodels M_σ such that:

(i) the sequence $\langle M_{\sigma^\frown\alpha} : \alpha \in \mathrm{cof}(|M_\sigma|)\rangle$ is a continuous (increasing) elementary chain,
(ii) $M_\sigma \subseteq \bigcup\{M_{\sigma^\frown\alpha} : \alpha \in \mathrm{cof}(|M_\sigma|)\}$,
(iii) $\{\mathcal{A}\} \cup \{M_\tau : \tau + 1 \subseteq \sigma\} \subseteq M_{\sigma^\frown 0}$,
(iv) if $\tau \subsetneq \sigma$ and M_τ is uncountable then $|M_\tau| > |M_\sigma|$.

We actually do not need to (and will not) define $M_{\sigma^\frown\alpha}$ when M_σ is countable or if $\alpha \geq \mathrm{cof}(|M_\sigma|)$.

Although the lexicographic order on T is not a well-order, it is not hard to see that conditions (ii) and (iv) allow the following definition of rank to make sense:

Definition 2.2. For $x \in M_\Lambda$ we define $\mathrm{rk}(x)$ as the minimum $\sigma \in T$ (in the lexicographic order) such that M_σ is countable and $x \in M_\tau$ for all $\tau \subseteq \sigma$.

Note that by the continuity of the elementary chains in condition (i), we know that $\mathrm{rk}(x)$ is always a finite sequence of ordinals which are either successor ordinals or 0. In particular, if $\sigma_x = \mathrm{rk}(x)$, $\sigma_y = \mathrm{rk}(y)$, $\sigma_x < \sigma_y$ and $m = |\sigma_x \wedge \sigma_y|$, then $\sigma_y(m)$ is a successor ordinal, say $\alpha + 1$, and we can define

$$\Delta(x, y) = (\sigma_x \wedge \sigma_y)^\frown\alpha.$$

This last definition will only be used in the proof of Lemma 2.5. The following remark summarizes the basic properties of $\Delta(x, y)$ that we will be using; all of them follow rather easily from the definitions.

Remark 2.3. If $\mathrm{rk}(x) < \mathrm{rk}(y)$ then

- $x \in M_{\Delta(x,y)}$ and $y \notin M_{\Delta(x,y)}$,
- $\Delta(x, y) + 1 \subseteq \mathrm{rk}(y)$,
- if $\sigma \supsetneq \Delta(x, y) + 1$ then $M_{\Delta(x,y)} \in M_\sigma$ (by conditions (i) and (iii)).

After assigning a rank to each member of M_Λ, we need a way to order in type ω all the elements of M_Λ of the same rank. This is easily done by fixing an injective enumeration

$$M_\sigma = \{t^\sigma_m : m \in \omega\}$$

for each σ for which M_σ is countable, and defining the degree of an element of M_Λ as follows:

Definition 2.4. For $x \in M_\Lambda$ we define $\deg(x)$ as the unique natural number satisfying

$$x = t^{\mathrm{rk}(x)}_{\deg(x)}.$$

The following two lemmas will be used to construct an acceptable coloring for $\mathcal{A}$ in the case that $\mathcal{A}$ is twisted, although the second one does not make any assumptions on $\mathcal{A}$.

Lemma 2.5. *If $\mathcal{A}$ is twisted then there is a set $B \subseteq A$ and a partition $B = \bigcup_{k\in n} B_k$ such that:*

(a) *each B_k is E_k-small,*

(b) $|\{i \in n : \mathrm{rk}([x]_i) = \mathrm{rk}(x)\}| \geq 2$ *for any $x \in A \setminus B$.*

Proof. For each $k \in n$ we let B_k be the set of all $x \in A$ such that $\mathrm{rk}([x]_k) > \mathrm{rk}([x]_i)$ for all $i \neq k$. Let $B = \bigcup_{k\in n} B_k$.

Note that for any $x \in A$ and $i \in n$ we have $\mathrm{rk}([x]_i) \leq \mathrm{rk}(x)$. On the other hand, if $\sigma = \mathrm{rk}([x]_k) = \mathrm{rk}([x]_j)$ for some $k \neq j$, then by elementarity and the fact that $[x]_k \cap [x]_j$ is finite, it follows that $\mathrm{rk}(x) \leq \sigma$ and hence $\mathrm{rk}(x) = \sigma$. This observation easily implies that condition (b) is met. It also implies that if $x \in B_k$ then

$$\mathrm{rk}([x]_{k_0}) < \cdots < \mathrm{rk}([x]_{k_{n-2}}) < \mathrm{rk}([x]_k) \leq \mathrm{rk}(x)$$

for some numbers $k_0, \ldots, k_{n-2}$ such that $\{k_0, \ldots, k_{n-2}, k\} = n$.

Now we put $\mathcal{M} = \{M_{\Delta([x]_{k_i},x)} : i \in n-1\}$, and use $\mathcal{M}$ as a test set for $\mathcal{A}$ to conclude that, since $\mathcal{A}$ is twisted, B_k is E_k-small.

To see that $\mathcal{M}$ is indeed a test set, it is enough to show that $M_{\Delta([x]_{k_i},x)} \in M_{\Delta([x]_{k_j},x)}$ for $i < j$. So fix $i < j$ and note that since $[x]_{k_i} \cap [x]_{k_j}$ is finite we have $\Delta([x]_{k_i}, x) = \Delta([x]_{k_i}, [x]_{k_j})$, and therefore by Remark 2.3,

$$\Delta([x]_{k_i}, x) + 1 \subseteq \mathrm{rk}(x) \wedge \mathrm{rk}([x]_{k_j}).$$

But then $\Delta([x]_{k_i}, x) + 1 \subsetneq \Delta([x]_{k_j}, x)$, and again by Remark 2.3 we get $M_{\Delta([x]_{k_i},x)} \in M_{\Delta([x]_{k_j},x)}$. ■

Lemma 2.6. *For all $i, k \in n$ with $i \neq k$, the set*

$$C_{i,k} = \{x \in A : \mathrm{rk}([x]_i) = \mathrm{rk}([x]_k) \text{ and } \deg([x]_i) < \deg([x]_k)\}$$

is E_k-small.

Proof. Fix $a \in A$ and let $\sigma = \mathrm{rk}([a]_k)$ and $d = \deg([a]_k)$. Note that if $x \in C_{i,k} \cap [a]_k$ then there is an $m < d$ (namely $m = \deg([x]_i)$) such that $x \in t^\sigma_m \cap t^\sigma_d$ and $t^\sigma_m \cap t^\sigma_d$ is finite. Hence $C_{i,k} \cap [a]_k$ is contained in a finite union of finite sets. ■

We are finally ready to prove the main result of this section.

Theorem 2.7. *The following are equivalent:*

(1) $\mathcal{A}$ *is twisted.*
(2) $\mathcal{A}$ *admits an acceptable coloring.*

Proof. Suppose first that $\mathcal{A}$ is twisted. Let B and B_k for $k \in n$ be as in Lemma 2.5, and let $C_{i,k}$ for $i, k \in n$ be as in Lemma 2.6. For each $k \in n$ define C_k as the set of all $x \in A \setminus B$ such that:

(i) $\mathrm{rk}(x) = \mathrm{rk}([x]_k)$,
(ii) for all $i \in n \setminus \{k\}$, if $\mathrm{rk}([x]_i) = \mathrm{rk}([x]_k)$ then $\deg([x]_i) < \deg([x]_k)$.

By condition (b) in Lemma 2.5, we have $C_k \subseteq \bigcup_{i \in n} C_{i,k}$ and therefore each C_k is E_k-small. It also follows that the C_k's form a partition of $A \setminus B$ so that we can define an acceptable coloring for $\mathcal{A}$ by

$$\chi(x) = k \quad \text{if and only if} \quad x \in B_k \cup C_k.$$

Now suppose that $\mathcal{A}$ admits an acceptable coloring and fix a test set $\mathcal{M}$ and $k \in n$. We want to show that the set

$$X = \{x \in A \setminus \bigcup \mathcal{M} : [x]_i \in \bigcup \mathcal{M} \text{ for all } i \neq k\}$$

is E_k-small. For this let $\chi : A \to n$ be an acceptable coloring such that (using elementarity and the fact that $\mathcal{M}$ is linearly ordered by $\in$) χ belongs to each $M \in \mathcal{M}$. Now if $x \in X$ and $i \neq k$ then there is an $M \in \mathcal{M}$ such that $[x]_i \cap \chi^{-1}(i) \in M$ and hence $[x]_i \cap \chi^{-1}(i) \subset M$ (since χ is acceptable); this implies that $\chi(x) \neq i$. Consequently, $X \subseteq \chi^{-1}(k)$ so that X is E_k-small. ∎

3. Embedding cubes into n-grids. Given an n-grid $\mathcal{A} = (A; E_i)_{i \in n}$ it will be convenient in this section to have a name $\rho_i : A \to A/E_i$ for the quotient maps ($\rho_i(\cdot) = [\cdot]_i$). Note that if $i \neq k$, $C \subseteq A$ is infinite and $\rho_k \restriction C$ is constant, then there is an infinite $D \subseteq C$ such that $\rho_i \restriction D$ is injective. We will make repeated use of this fact, without explicit mention, in the proof of the following:

Theorem 3.1. *If $\mathcal{A}$ is a non-twisted n-grid then any finite n-cube l^n (with $l \in \omega$) can be embedded in $\mathcal{A}$.*

Proof. By definition, since $\mathcal{A}$ is not twisted, there is a test set $\mathcal{M}$ and a $k \in n$ such that for some $a \in A$, the set

$$B = \{x \in [a]_k \setminus \bigcup \mathcal{M} : [x]_i \in \bigcup \mathcal{M} \text{ for all } i \neq k\}$$

is infinite. For each $x \in B$ and each $i \in n \setminus \{k\}$ there is an $M_i^x \in \mathcal{M}$ such that $[x]_i \in M_i^x$. Since $\mathcal{M}$ is finite, there must be an infinite $C \subseteq B$ on which the map $x \mapsto \langle M_i^x : i \in n \setminus \{k\} \rangle$ is constant, say with value $\langle M_i : i \in n \setminus \{k\} \rangle$. Note that since C is disjoint from $\bigcup \mathcal{M}$, the map $i \mapsto M_i$ must be injective and hence $\mathcal{M} = \{M_i : i \in n \setminus \{k\}\}$, because $|\mathcal{M}| = n - 1$. Finally, we can find an infinite set $D \subseteq C$ such that $\rho_i \restriction C$ is injective for all $i \neq k$.

Now taking $k_1 = k$ and letting φ be any injection from l into D, we easily see that the following statement is true for $j = 1$:

$P(j)$ There are distinct $k_1, \dots, k_j \in n$ and an embedding $\varphi : l^j \to (A; E_{k_1}, \dots, E_{k_j})$ such that:

(a) for $i \in n \setminus \{k_1, \dots, k_j\}$, $\rho_i \circ \varphi$ is injective and belongs to M_i,
(b) φ takes values in $A \setminus \bigcup\{M_i : i \in n \setminus \{k_1, \dots, k_j\}\}$.

Note that when $j = n$, conditions (a) and (b) become trivially true, and $P(n)$ just says that there is an embedding (modulo an irrelevant permutation of coordinates) of the finite cube l^n into $\mathcal{A}$, which is exactly what we want to show. We already know that $P(1)$ is true, so we are done if we can show that $P(j)$ implies $P(j+1)$ for $1 \leq j < n$.

Assuming $P(j)$, let $\varphi : l^j \to (A; E_{k_1}, \dots, E_{k_j})$ be such an embedding, and let $k_{j+1} \in n \setminus \{k_1, \dots, k_j\}$ be such that $M_{k_{j+1}}$ is the $\in$-maximum element of $\{M_i : i \in n \setminus \{k_1, \dots, k_j\}\}$. Let us call

$$\delta := \rho_{k_{j+1}} \circ \varphi \in M_{k_{j+1}}.$$

Now note that $\varphi \notin M_{k_{j+1}}$ and at the same time φ satisfies the following conditions (on the free variable Φ), all of which can be expressed using parameters from $M_{k_{j+1}}$:

- $\Phi : l^j \to (A; E_{k_1}, \dots, E_{k_j})$ is an embedding,
- $\rho_{k_{j+1}} \circ \Phi = \delta$,
- for $i \in n \setminus \{k_1, \dots, k_j, k_{j+1}\}$, $\rho_i \circ \Phi$ is injective and belongs to M_i,
- Φ takes values in $A \setminus \bigcup\{M_i : i \in n \setminus \{k_1, \dots, k_j, k_{j+1}\}\}$.

This means that there must be an infinite set (in fact there must be an uncountable one, but we will not be using this) $\{\varphi_m : m \in \omega\}$ of distinct functions satisfying those properties. Going to a subsequence l^j times, we may assume without loss of generality that for each $t \in l^j$, the map $m \mapsto \varphi_m(t)$ is either constant or injective. Now since they cannot all be constant, it is not hard to see that in fact all these maps have to be injective: just note that if $t, t' \in l^j$ are in a line parallel to the $(r-1)$th coordinate axis then it cannot be the case that the map associated with t is constant while the one associated with t' is injective, since otherwise $\{\varphi_m(t') : m \in \omega\}$ would be an infinite set contained in $[\varphi_0(t)]_{k_r} \cap [\varphi_0(t')]_{k_{j+1}}$. To see this, just note that in that situation we would have $[\varphi_m(t')]_{k_r} = [\varphi_m(t)]_{k_r} = [\varphi_0(t)]_{k_r}$ and $[\varphi_m(t')]_{k_{j+1}} = (\rho_{k_{j+1}} \circ \varphi_m)(t') = \delta(t') = (\rho_{k_{j+1}} \circ \varphi_0)(t') = [\varphi_0(t')]_{k_{j+1}}$.

Next we can find an infinite $I \subseteq \omega$ such that for each $t \in l^j$ and each $i \in n \setminus \{k_{j+1}\}$ the map $m \mapsto [\varphi_m(t)]_i$ is injective when restricted to I. Consequently, here one can find (one at a time) l distinct elements $m_0, \dots, m_{l-1}$

of I such that for all $t, t' \in l^j$, for all $r, r' \in l$ with $r \neq r'$ and for all $i \in n \setminus \{k_{j+1}\}$, we have $[\varphi_{m_r}(t)]_i \neq [\varphi_{m_{r'}}(t')]_i$.

Finally we let $\psi : l^{j+1} \to (A; E_{k_1}, \ldots, E_{k_{j+1}})$ be the function defined by $\psi(t, r) = \varphi_{m_r}(t)$. From the way we constructed the m_r's and using the fact that all the φ_m's are embeddings and δ is injective, one can see that ψ is in fact an embedding. From the fact that ψ is essentially a finite union of some φ_m's and from the choice of those φ_m's, it follows that conditions (a) and (b) in $P(j+1)$ are satisfied. ∎

This last theorem only goes one way: for example, the n-cube ω^n is twisted for $n \geq 2$, but of course any finite n-cube can be embedded in it. I suspect that only for very "nice" classes of n-grids can one reverse this theorem. Schmerl's theorem does it for semialgebraic n-grids; perhaps some form of o-minimality is what is required.

The question of when an infinite cube can be embedded in an arbitrary n-grid seems more subtle. For instance, let us consider the case $n = 2$. Using the same idea as in the proof of 3.1, one can easily show:

THEOREM 3.2. *If $\mathcal{A}$ is a non-twisted 2-grid then either $l \times \omega_1$ can be embedded in $\mathcal{A}$ for all $l \in \omega$, or $\omega_1 \times l$ can be embedded in $\mathcal{A}$ for all $l \in \omega$.*

However, it is not true that $\omega \times \omega$ embeds in any non-twisted 2-grid. For example, fix an uncountable family $\{A_\alpha : \alpha \in \omega_1\}$ of almost disjoint subsets of ω and let $A = \{(n, \alpha) \in \omega \times \omega_1 : n \in A_\alpha\}$. Think of A as a subgrid of the 2-cube $\omega \times \omega_1$. It is easy to see that this is a non-twisted grid, but not even $\omega \times 2$ can be embedded in it.

References

[1] R. de la Vega, *Decompositions of the plane and the size of the continuum*, Fund. Math. 203 (2009), 65–74.

[2] J. H. Schmerl, *Covering the plane with sprays*, Fund. Math. 208 (2010), 263–272.

[3] J. H. Schmerl, *A generalization of Sierpiński's paradoxical decompositions: coloring semialgebraic grids*, J. Symbolic Logic 77 (2012), 1165–1183.

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