Topologically invariant σ-ideals on Euclidean spaces

by

T. Banakh (Lviv and Kielce), **M. Morayne** (Wrocław),
R. Rałowski (Wrocław) and **Sz. Żeberski** (Wrocław)

Abstract. We study and classify topologically invariant σ-ideals with an analytic base on Euclidean spaces, and evaluate the cardinal characteristics of such ideals.

1. Introduction. The σ-ideals of Lebesgue measure zero sets and meager sets have been the subject of extensive research devoted to revealing the fine structure of the real line and, more generally, the Euclidean spaces. This research resulted in finding the relations between the most important cardinal characteristics of these two σ-ideals. These relations are described by the Cichoń diagram (see e.g. [9], [3]). Both ideals have Borel base and differ by the property that the ideal $\mathcal{M}$ of meager sets is topologically invariant while the ideal $\mathcal{N}$ of Lebesgue null sets is not.

In this paper we examine the properties of non-trivial topologically invariant σ-ideals with Borel base on Euclidean spaces $\mathbb{R}^n$. In particular, we show that the σ-ideal of meager sets, $\mathcal{M}$, is the biggest topologically invariant σ-ideal with Borel base on $\mathbb{R}^n$, while the σ-ideal generated by the so-called tame Cantor sets, $\sigma\mathcal{C}_0$, is the smallest one. Our main results concern the four cardinal characteristics of these two σ-ideals: the additivity (add), the uniformity (non), the covering (cov), and the cofinality (cof). In fact, we show that the uniformity and the covering numbers are the same for all non-trivial topologically invariant σ-ideals with Borel base on Euclidean spaces, and the remaining two cardinals may be different from the corresponding characteristics of the ideal $\mathcal{M}$. Yet, the respective cardinal characteristics of the extremal ideals $\sigma\mathcal{C}_0$ and $\mathcal{M}$ coincide. The same concerns the σ-ideals $\sigma\mathcal{D}_k$ generated by the closed subsets of dimension $k < n$ in $\mathbb{R}^n$.

Properties of topologically invariant σ-ideals may be different in different topological spaces. There are other natural spaces where it would be interesting and useful to know these properties. The Hilbert cube case is examined in [2].

2. Definitions, notation and statement of principal results. The symbols $\mathbb{R}$, $\mathbb{Q}$, and ω will have the usual meaning: the real line, the set of rational numbers, and the set of finite ordinals (i.e. the set of non-negative integers), respectively. A *Euclidean space* is a topological space homeomorphic to $\mathbb{R}^n$ for some positive integer n. All topological spaces considered in this paper are assumed to be separable and metrizable.

Let us recall that a subset A of a topological space X is *analytic* if A is the image of a Polish space under a continuous map. A subset $A \subseteq X$ has the *Baire property* (briefly, A is a *BP-set*) if there is an open subset $U \subseteq X$ such that the symmetric difference $A \triangle U = (A \setminus U) \cup (U \setminus A)$ is meager in X.

A non-empty family $\mathcal{I}$ of subsets of a set X is called an *ideal* on X if $\mathcal{I}$ is *hereditary* (with respect to taking subsets) and *additive* in the sense that the union $A \cup B$ of any two sets $A, B \in \mathcal{I}$ belongs to $\mathcal{I}$. An ideal $\mathcal{I}$ is called a *σ-ideal* if the union of any countable subfamily $\mathcal{A} \subseteq \mathcal{I}$ belongs to $\mathcal{I}$. An ideal $\mathcal{I}$ on X is *non-trivial* if $\mathcal{I}$ contains an uncountable set and $\mathcal{I}$ is not equal to the ideal $\mathcal{P}(X)$ of all subsets of X.

A subfamily $\mathcal{B} \subseteq \mathcal{I}$ is a *base* of an ideal $\mathcal{I}$ if each element $A \in \mathcal{I}$ is a subset of some $B \in \mathcal{B}$. We say that an ideal $\mathcal{I}$ of subsets of a topological space X has *Borel* (resp. *analytic*, *BP-*) *base*, or that $\mathcal{I}$ is an *ideal with Borel* (resp. *analytic*, *BP-*) *base*, if there exists a base for $\mathcal{I}$ consisting of Borel (analytic, BP-) subsets of X.

It is well-known that each Borel subset of a Polish space is analytic and each analytic subset of a metrizable separable space X has the Baire property in X. This implies that for an ideal $\mathcal{I}$ on a Polish space we have the following implications:

$$\mathcal{I} \text{ has Borel base} \;\Rightarrow\; \mathcal{I} \text{ has analytic base} \;\Rightarrow\; \mathcal{I} \text{ has BP-base}.$$

A σ-ideal $\mathcal{I}$ on a topological space X is *topologically invariant* if $\mathcal{I}$ is transformed onto $\mathcal{I}$ by any homeomorphism h of X, i.e. $\mathcal{I} = \{h(A) : A \in \mathcal{I}\}$.

It is clear that for each topological space X the ideal $\mathcal{M}$ of meager subsets of X is topologically invariant. It turns out that this ideal is the largest one among non-trivial topologically invariant σ-ideals with BP-base on $X = \mathbb{R}^n$.

THEOREM 2.1. *Each non-trivial σ-ideal $\mathcal{I}$ with BP-base on $\mathbb{R}^n$ is contained in the ideal $\mathcal{M}$ of meager subsets.*

Proof. Let us assume that $\mathcal{I} \not\subseteq \mathcal{M}$. Fix any set $A \in \mathcal{I} \setminus \mathcal{M}$. Since $\mathcal{I}$ has BP-base, we can assume that the non-meager set A has the Baire property and hence contains a G_δ-subset $G_U \subseteq A$, dense in some open subset U of X. Since $\mathbb{R}^n$ is topologically homogeneous, we can choose a countable family H of homeomorphisms of $\mathbb{R}^n$ such that $\bigcup_{h \in H} h(U)$ is dense in $\mathbb{R}^n$. Then the $G_{\delta\sigma}$-set $D = \bigcup_{h \in H} h(G_U)$ is comeager in X and hence contains a subset $G \subseteq D$ which is dense G_δ in $\mathbb{R}^n$. By the topological invariance of $\mathcal{I}$, the set D and its G_δ-subset G belong to the σ-ideal $\mathcal{I}$.

By [7], for a dense G_δ-subset G of $\mathbb{R}^n$, there are homeomorphisms $h_0, \ldots, h_n : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathbb{R}^n = \bigcup_{k=0}^{n} h_k(G)$. Then $\mathbb{R}^n \in \mathcal{I}$ by the topological invariance of $\mathcal{I}$, which means that $\mathcal{I}$ is trivial. ∎

By Theorem 2.1, $\mathcal{M}$ is the largest non-trivial σ-ideal with Borel base on $\mathbb{R}^n$. Now we describe the smallest non-trivial σ-ideal with Borel base on $\mathbb{R}^n$. It is denoted by $\sigma\mathcal{C}_0$ and is generated by the tame Cantor sets in $\mathbb{R}^n$.

A subset C of a Polish space X is called a *Cantor set* if C is homeomorphic to the Cantor cube $\{0,1\}^\omega$. By Brouwer's characterization [11, 7.4], a subset $C \subseteq X$ is a Cantor set if and only if it is compact, zero-dimensional and has no isolated points.

Two subsets A, B of a topological space X are called *ambiently homeomorphic* if $h(A) = B$ for some homeomorphism $h : X \to X$ of X.

A subset C of $\mathbb{R}^n$ is called a *tame Cantor set* if it is ambiently homeomorphic to a Cantor set contained in the line $\mathbb{R} \times \{0\}^{n-1} \subseteq \mathbb{R}^n$. Since any two Cantor sets on the real line are ambiently homeomorphic, any two tame Cantor sets in $\mathbb{R}^n$ are ambiently homeomorphic.

By [15], a closed subset $C \subseteq \mathbb{R}^n$ is a tame Cantor set if and only if for each $\varepsilon > 0$ the set C is contained in the interior of the union $\bigcup \mathcal{F}$ of a finite family $\mathcal{F}$ of pairwise disjoint n-cells of diameter $< \varepsilon$. Replacing these n-cells by smaller cells we can additionally assume that the boundary of each n-cell $B \in \mathcal{F}$ is a bicollared $(n-1)$-spere in $\mathbb{R}^n$. Using this characterization, in Lemma 3.1 we shall prove that *each Cantor set in $\mathbb{R}^n$ contains a tame Cantor set.*

It is known [4] that for $n \leq 2$ each Cantor set in $\mathbb{R}^n$ is tame, while for $n \geq 3$ a Cantor subset $C \subseteq \mathbb{R}^n$ is tame if and only if C is a Z_2-*set* in $\mathbb{R}^n$. The latter means that each map $f : [0,1]^2 \to \mathbb{R}^n$ can be uniformly approximated by a map $f' : [0,1]^2 \to \mathbb{R}^n \setminus C$. Cantor sets which are not tame are called *wild* (see [1], [5], [16]).

We denote by $\sigma\mathcal{C}_0$ the σ-ideal generated by the tame Cantor sets in $\mathbb{R}^n$. It consists of all subsets of countable unions of tame Cantor sets in $\mathbb{R}^n$.

THEOREM 2.2. *The σ-ideal $\sigma\mathcal{C}_0$ is contained in each non-trivial σ-ideal $\mathcal{I}$ with analytic base on $\mathbb{R}^n$.*

Proof. The ideal $\mathcal{I}$, being non-trivial, contains an uncountable set A. Since $\mathcal{I}$ has analytic base, we can assume that A is analytic and hence contains a Cantor set C according to Suslin's Theorem [11, 29.1]. Since each Cantor set in $\mathbb{R}^n$ contains a tame Cantor set, we can assume that C is a tame Cantor set in $\mathbb{R}^n$. So, $\mathcal{I}$ contains a tame Cantor set. Since any two tame Cantor sets in $\mathbb{R}^n$ are ambiently homeomorphic, by topological invariance, the ideal $\mathcal{I}$ contains all tame Cantor sets, and being a σ-ideal, it contains the σ-ideal $\sigma\mathcal{C}_0$ generated by the tame Cantor sets in $\mathbb{R}^n$. ∎

COROLLARY 2.3. *If $\mathcal{I}$ is a non-trivial topologically invariant σ-ideal with analytic base on $\mathbb{R}^n$, then $\sigma\mathcal{C}_0 \subseteq \mathcal{I} \subseteq \mathcal{M}$.*

This corollary will be used to evaluate the cardinal characteristics of non-trivial topologically invariant σ-ideals with Borel base on Euclidean spaces.

Given an ideal $\mathcal{I}$ on a set $X = \bigcup \mathcal{I}$, we shall consider the following four cardinal characteristics of $\mathcal{I}$:

$$\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I}, \bigcup\mathcal{A} \notin \mathcal{I}\},$$
$$\mathrm{non}(\mathcal{I}) = \min\{|A| : A \subseteq X, A \notin \mathcal{I}\},$$
$$\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I}, \bigcup\mathcal{A} = X\},$$
$$\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I}, \forall B \in \mathcal{I} \,\exists A \in \mathcal{A}\, (B \subseteq A)\}.$$

In fact, they can be expressed using the following two cardinal characteristics defined for any pair $\mathcal{I} \subseteq \mathcal{J}$ of ideals:

$$\mathrm{add}(\mathcal{I}, \mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I}, \bigcup\mathcal{A} \notin \mathcal{J}\},$$
$$\mathrm{cof}(\mathcal{I}, \mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J}, \forall B \in \mathcal{I} \,\exists A \in \mathcal{A}\, (B \subseteq A)\}.$$

Namely,

$$\mathrm{add}(\mathcal{I}) = \mathrm{add}(\mathcal{I}, \mathcal{I}), \quad \mathrm{non}(\mathcal{I}) = \mathrm{add}(\mathcal{F}, \mathcal{I}),$$
$$\mathrm{cov}(\mathcal{I}) = \mathrm{cof}(\mathcal{F}, \mathcal{I}), \quad \mathrm{cof}(\mathcal{I}) = \mathrm{cof}(\mathcal{I}, \mathcal{I}),$$

where $\mathcal{F}$ stands for the ideal of finite subsets of X.

The cardinal characteristics of the largest σ-ideal $\mathcal{M}$ have been thoroughly studied (see [3]). The (relative) cardinal characteristics of the smallest σ-ideal $\sigma\mathcal{C}_0$ (in $\mathcal{M}$) are evaluated in the following theorem which will be proved in Section 5. Theorem 2.4 and the subsequent Corollary 2.5 are the principal results of this article.

THEOREM 2.4. *For the σ-ideal $\sigma\mathcal{C}_0$ on $\mathbb{R}^n$ the following equalities hold:*

(1) $\mathrm{cov}(\sigma\mathcal{C}_0) = \mathrm{cov}(\mathcal{M})$;
(2) $\mathrm{non}(\sigma\mathcal{C}_0) = \mathrm{non}(\mathcal{M})$;
(3) $\mathrm{add}(\sigma\mathcal{C}_0) = \mathrm{add}(\sigma\mathcal{C}_0, \mathcal{M}) = \mathrm{add}(\mathcal{M})$;
(4) $\mathrm{cof}(\sigma\mathcal{C}_0) = \mathrm{cof}(\sigma\mathcal{C}_0, \mathcal{M}) = \mathrm{cof}(\mathcal{M})$.

Corollary 2.3 and Theorem 2.4 imply:

Corollary 2.5. *For any non-trivial topologically invariant σ-ideal $\mathcal{I}$ with analytic base on $\mathbb{R}^n$ we get:*

(1) $\mathrm{cov}(\mathcal{I}) = \mathrm{cov}(\mathcal{M})$;
(2) $\mathrm{non}(\mathcal{I}) = \mathrm{non}(\mathcal{M})$;
(3) $\mathrm{add}(\mathcal{I}) \leq \mathrm{add}(\mathcal{M})$;
(4) $\mathrm{cof}(\mathcal{I}) \geq \mathrm{cof}(\mathcal{M})$.

Thus, on $\mathbb{R}^n$, the following variant of Cichoń's diagram describes relations between cardinal characteristics of the ideal $\mathcal{M}$ and any non-trivial topologically invariant σ-ideal $\mathcal{I}$ (here $a \to b$ stands for $a \leq b$):

$$\mathrm{non}(\mathcal{I}) = \mathrm{non}(\mathcal{M}) \longrightarrow \mathrm{cof}(\mathcal{M}) \longrightarrow \mathrm{cof}(\mathcal{I}) \longrightarrow \mathfrak{c}$$

$$\omega_1 \longrightarrow \mathrm{add}(\mathcal{I}) \longrightarrow \mathrm{add}(\mathcal{M}) \longrightarrow \mathrm{cov}(\mathcal{M}) = \mathrm{cov}(\mathcal{I})$$

The following example shows that the inequalities $\mathrm{add}(\mathcal{I}) \leq \mathrm{add}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{I})$ can be strict.

Below, for a subset A of a Polish space X we denote by $\mathcal{I}_A$ the smallest topologically invariant σ-ideal containing the set A. It consists of all subsets of countable unions $\bigcup_{n \in \omega} h_n(A)$, where $h_n : X \to X$, $n \in \omega$, are homeomorphisms.

Example 2.6. *The σ-ideal $\mathcal{I}_\mathbb{I} \subseteq \mathcal{P}(\mathbb{R}^2)$ generated by the interval $\mathbb{I} = [0,1] \times \{0\}$ in the plane $\mathbb{R}^2$ has cardinal characteristics*

$$\mathrm{add}(\mathcal{I}_\mathbb{I}) = \omega_1, \quad \mathrm{non}(\mathcal{I}_\mathbb{I}) = \mathrm{non}(\mathcal{M}), \quad \mathrm{cov}(\mathcal{I}_\mathbb{I}) = \mathrm{cov}(\mathcal{M}), \quad \mathrm{cof}(\mathcal{I}_\mathbb{I}) = \mathfrak{c}.$$

Proof. The equalities $\mathrm{cov}(\mathcal{I}_\mathbb{I}) = \mathrm{cov}(\mathcal{M})$ and $\mathrm{non}(\mathcal{I}_\mathbb{I}) = \mathrm{non}(\mathcal{M})$ follow from Corollary 2.5.

The equality $\mathrm{add}(\mathcal{I}_\mathbb{I}) = \omega_1$ will follow if we check that $\bigcup_{t \in T}[0,1] \times \{t\} \notin \mathcal{I}_\mathbb{I}$ for any uncountable subset $T \subseteq [0,1]$. Assuming the opposite, we can find a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that the set

$$\{t \in T : h(\mathbb{I}) \cap ([0,1] \times \{t\}) \text{ contains a line segment}\}$$

is uncountable. This yields an uncountable family of pairwise disjoint proper intervals in $[0,1]$, which is not possible.

To show $\mathrm{cof}(\mathcal{I}_\mathbb{I}) = \mathfrak{c}$, choose any subfamily $\mathcal{B} \subseteq \mathcal{I}_\mathbb{I}$ with $|\mathcal{B}| = \mathrm{cof}(\mathcal{I}_\mathbb{I})$ such that each $A \in \mathcal{I}_\mathbb{I}$ is contained in some $B \in \mathcal{B}$. Let $\mathcal{X} = \{[0,1] \times \{x\} : x \in \mathbb{R}\}$. Notice that every member of $\mathcal{B}$ contains at most countably many members of $\mathcal{X}$. This implies that $|\mathcal{B}| = \mathfrak{c}$. ∎

Corollary 2.5 will be applied to calculate the cardinal characteristics of the σ-ideal $\sigma\mathcal{D}_k$ generated by all closed subsets of dimension $\leq k$ in $\mathbb{R}^n$. By [8, 1.8.11], $\sigma\mathcal{D}_{n-1}$ coincides with the ideal $\mathcal{M}$ of meager subsets of $\mathbb{R}^n$. The following theorem will be proved in Section 6.

THEOREM 2.7. *For all $0 \le k < n$ the σ-ideal $\sigma\mathcal{D}_k$ generated by all closed at most k-dimensional subsets of $\mathbb{R}^n$ has cardinal characteristics*

$$\mathrm{add}(\sigma\mathcal{D}_k) = \mathrm{add}(\mathcal{M}), \quad \mathrm{cov}(\sigma\mathcal{D}_k) = \mathrm{cov}(\mathcal{M}),$$
$$\mathrm{non}(\sigma\mathcal{D}_k) = \mathrm{non}(\mathcal{M}), \quad \mathrm{cof}(\sigma\mathcal{D}_k) = \mathrm{cof}(\mathcal{M}).$$

We finish this section with two open problems. A topologically invariant σ-ideal $\mathcal{I}$ will be called 1-*generated* if $\mathcal{I} = \mathcal{I}_A$ for some subset $A \in \mathcal{I}$. Observe that the σ-ideals $\sigma\mathcal{C}_0$ and $\mathcal{M}$ on $\mathbb{R}^n$ are 1-generated: the σ-ideal $\sigma\mathcal{C}_0$ is generated by any tame Cantor set in $\mathbb{R}^n$, while $\mathcal{M}$ is generated by the generalized Menger cube M_{n-1}^n (see [14], [6, p. 128]).

PROBLEM 2.8. *What are the cardinal characteristics of a 1-generated topologically invariant σ-ideal $\mathcal{I}_A$ with Borel base on $\mathbb{R}^n$? Is it true that $\mathrm{add}(\mathcal{I}) \in \{\omega_1, \mathrm{add}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{I}) \in \{\mathrm{cof}(\mathcal{M}), \mathfrak{c}\}$ for any such ideal $\mathcal{I}$?*

Corollary 2.3 implies that $\mathcal{M} = \sigma\mathcal{C}_0$ is the unique topologically invariant σ-ideal with analytic base on the real line $\mathbb{R}^1$. For higher-dimensional Euclidean spaces the ideals $\sigma\mathcal{C}_0$ and $\mathcal{M}$ are distinct.

PROBLEM 2.9 (M. Sabok). *What is the cardinality of the family of all topologically invariant σ-ideals with Borel base on $\mathbb{R}^n$ for $n \ge 2$? Is this cardinality equal to $2^{\mathfrak{c}}$?*

3. Some properties of tame Cantor sets in Euclidean spaces. In this section we shall establish some auxiliary facts related to tame Cantor sets and homeomorphism groups of Euclidean spaces. These facts will be used in the proof of Theorem 2.4.

LEMMA 3.1. *Each Cantor set C in $\mathbb{R}^n$ contains a tame Cantor set $T \subset C$.*

Proof. Let $2^{<\omega} = \bigcup_{k\in\omega} 2^k$ be the set of finite binary sequences. For a binary sequence $s = (s_0, \ldots, s_{n-1}) \in 2^{<\omega}$ and a number $i \in 2 := \{0,1\}$ we denote by $s\hat{\ }i$ the sequence $(s_0, \ldots, s_{n-1}, i)$.

For a point $x \in \mathbb{R}^n$ and $\varepsilon > 0$ we denote by $B(x, \varepsilon) = \{y \in \mathbb{R}^n : \|x-y\| < \varepsilon\}$ and $\bar{B}(x, \varepsilon) = \{y \in \mathbb{R}^n : \|x - y\| \le \varepsilon\}$ the open and closed ε-balls in $\mathbb{R}^n$.

By induction we shall construct a sequence $(x_t)_{t\in 2^{<\omega}}$ of points of C and a sequence $(\varepsilon_t)_{t\in 2^{<\omega}}$ of positive real numbers such that for every binary sequence $t \in 2^{<\omega}$ the following conditions are satisfied:

(1_t) $x_{t\hat{\ }0}, x_{t\hat{\ }1}$ are distinct points of $C \cap B(x_t, \varepsilon_t)$;
(2_t) $\bar{B}(x_{t\hat{\ }0}, \varepsilon_{t\hat{\ }0}) \cap \bar{B}(x_{t\hat{\ }1}, \varepsilon_{t\hat{\ }1}) = \emptyset$ and $\bar{B}(x_{t\hat{\ }0}, \varepsilon_{t\hat{\ }0}) \cup \bar{B}(x_{t\hat{\ }1}, \varepsilon_{t\hat{\ }1}) \subset B(x_t, \varepsilon_t)$;
(3_t) $\max\{\varepsilon_{t\hat{\ }0}, \varepsilon_{t\hat{\ }1}\} \le \frac{1}{2}\varepsilon_t$.

We start the inductive construction selecting any point $x_\emptyset \in C$ and setting $\varepsilon_\emptyset = 1$. Assume that, for some binary sequence $t \in 2^{<\omega}$, a point $x_t \in C$ and a positive real number ε_t have been constructed. Since the non-empty open

subset $C \cap B(x_t, \varepsilon_t)$ of C has no isolated points, we can choose two distinct points $x_{t^\frown 0}, x_{t^\frown 1} \in C \cap B(x_t, \varepsilon_t)$. Next, choose two positive real numbers $\varepsilon_{t^\frown 0}, \varepsilon_{t^\frown 1}$ so that conditions (2_t) and (3_t) are satisfied. This completes the inductive step.

Now consider the closed subset $T = \bigcap_{k \in \omega} \bigcup_{t \in 2^k} (C \cap \bar{B}(x_t, \varepsilon_t))$ of C. We claim that T is a tame Cantor set in $\mathbb{R}^n$. To see that T is a Cantor set, observe that the map $f : 2^\omega \to T$ assigning to each $t \in 2^\omega$ the limit of the Cauchy sequence $(x_{t|k})_{k \in \omega}$ is a homeomorphism. To see that the Cantor set T is tame, for every $\varepsilon > 0$ we find $k \in \omega$ such that $\max\{\varepsilon_t : t \in 2^k\} < \varepsilon$ and observe that $T \subset \bigcup_{t \in 2^k} B(x_t, \varepsilon_t)$, which means that T is covered by the interiors of finitely many pairwise disjoint n-cells of diameter $< \varepsilon$. By Osborne's characterization [15], the Cantor set T is tame. ∎

Corollary 2 of [13] or the characterization [15] of tame Cantor sets imply:

LEMMA 3.2. *For any Cantor sets $C_1, \ldots, C_n$ on the real line, the product $\prod_{i=1}^n C_i$ is a tame Cantor set in $\mathbb{R}^n$.*

We denote by $\mathcal{H}(\mathbb{R}^n)$ the homeomorphism group of $\mathbb{R}^n$, endowed with the compact-open topology. It can be identified with the closed subgroup of the homeomorphism group of the one-point compactification $\alpha\mathbb{R}^n = \mathbb{R}^n \cup \{\infty\}$. This implies that $\mathcal{H}(\mathbb{R}^n)$ is a Polish group.

LEMMA 3.3. *For each tame Cantor set $C \subseteq \mathbb{R}^n$ and each open dense set $U \subseteq \mathbb{R}^n$ the set*
$$\mathcal{H}_C^U = \{h \in \mathcal{H}(\mathbb{R}^n) : h(C) \subseteq U\}$$
is open and dense in $\mathcal{H}(\mathbb{R}^n)$.

Proof. The openness of $\mathcal{H}_C^U$ follows from the openness of U and the definition of the compact-open topology on $\mathcal{H}(\mathbb{R}^n)$. It remains to prove that $\mathcal{H}_C^U$ is dense in $\mathcal{H}(\mathbb{R}^n)$. Given $h_0 \in \mathcal{H}(\mathbb{R}^n)$, a compact set $K \subseteq \mathbb{R}^n$, and $\varepsilon > 0$, we need to find $h \in \mathcal{H}_C^U$ such that $\sup_{x \in K} \|h(x) - h_0(x)\| < \varepsilon$.

Let $C_0 = h_0(C) \subseteq \mathbb{R}^n$. Being tame, the Cantor set C_0 admits a cover by the interiors of pairwise disjoint n-cells $B_1, \ldots, B_m$ of diameter $< \varepsilon$. For every n-cell B_i choose a homeomorphism $g_i : B_i \to B_i$ which is the identity on ∂B_i and maps the compact set $C_0 \cap B_i$ into $B_i \cap U$. Then $g_1, \ldots, g_m$ yield a homeomorphism $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $g|B_i = g_i$ and g is the identity on the complement of $B_1 \cup \cdots \cup B_m$. Hence the homeomorphism $h = g \circ h_0$ has the required property: $h(C) \subset U$ and h is ε-near to h_0. ∎

LEMMA 3.4. *For each $A \in \sigma\mathcal{C}_0$ and each dense G_δ-set $G \subseteq \mathbb{R}^n$ the set*
$$\mathcal{H}_A^G = \{h \in \mathcal{H}(\mathbb{R}^n) : h(A) \subseteq G\}$$
contains a dense G_δ-subset of $\mathcal{H}(\mathbb{R}^n)$. If A is σ-compact, then $\mathcal{H}_A^G$ is a G_δ-set in $\mathcal{H}(\mathbb{R}^n)$.

Proof. Let $A \in \sigma\mathcal{C}_0$. Then $A \subseteq \bigcup_{k \in \omega} C_k$ for some tame Cantor sets $C_k \subseteq \mathbb{R}^n$, $k \in \omega$. We have $G = \bigcap_{k \in \omega} U_k$ for a decreasing sequence of dense open sets $U_k \subset \mathbb{R}^n$, $k \in \omega$. It follows from Lemma 3.3 that for any $i, j \in \omega$ the set $\mathcal{H}_{C_j}^{U_i}$ is a dense open set in $\mathcal{H}(\mathbb{R}^n)$. Then $\bigcap_{i,j \in \omega} \mathcal{H}_{C_j}^{U_i} \subseteq \mathcal{H}_A^G$ is a dense G_δ-subset of $\mathcal{H}(\mathbb{R}^n)$ contained in $\mathcal{H}_A^G$. ∎

4. Some known facts about cardinal characteristics of ideals. In the proof of Theorem 2.4 we shall simultaneously work with σ-ideals on various topological spaces. To distinguish between such σ-ideals we shall use the following notation.

For a perfect Polish space X we denote by $\mathcal{M}(X)$ the σ-ideal of all meager subsets of X, i.e. subsets of countable unions of closed sets with empty interior. It is well-known that the cardinal characteristics of the σ-ideal $\mathcal{M}(X)$ do not depend on the space X.

PROPOSITION 4.1. *If X is a perfect Polish space, then*

$$\mathrm{add}(\mathcal{M}(X)) = \mathrm{add}(\mathcal{M}(\omega^\omega)), \quad \mathrm{cov}(\mathcal{M}(X)) = \mathrm{cov}(\mathcal{M}(\omega^\omega)),$$
$$\mathrm{non}(\mathcal{M}(X)) = \mathrm{non}(\mathcal{M}(\omega^\omega)), \quad \mathrm{cof}(\mathcal{M}(X)) = \mathrm{cof}(\mathcal{M}(\omega^\omega)).$$

Proof. There is an embedding $\theta : \omega^\omega \to X$ whose image $\theta(\omega^\omega)$ is a dense G_δ-subset of X (as can be proved elementarily by a direct construction or follows from [12, Theorem 2, Sec. 36, IV, and Theorem 3, Sec. 36, II]). This gives the desired equalities. ∎

The above proposition justifies why we often use the symbol $\mathcal{M}$ without mentioning a specific space X.

For a topological space X we denote by $\sigma\mathcal{K}(X)$ the σ-ideal generated by the compact subsets of X, and set $\sigma\mathcal{K} = \sigma\mathcal{K}(\omega^\omega)$. It is known that

$$\mathrm{add}(\sigma\mathcal{K}) = \mathrm{non}(\sigma\mathcal{K}) = \mathfrak{b}, \quad \mathrm{cov}(\sigma\mathcal{K}) = \mathrm{cof}(\sigma\mathcal{K}) = \mathfrak{d},$$

where $\mathfrak{b}$ (resp. $\mathfrak{d}$) is defined as the smallest cardinality of a subset $B \subset \omega^\omega$ which is *unbounded* (resp. *dominating*) in ω^ω in the sense that for each $f \in \omega^\omega$ there is $g \in B$ such that $g \not\leq^* f$ (resp. $f \leq^* g$). Here for $f, g \in \omega^\omega$ we write $f \leq^* g$ if $\{n \in \omega : f(n) > g(n)\}$ is finite.

We will use the following well-known equalities (see e.g. [3]).

LEMMA 4.2. *For the ideals $\sigma\mathcal{K}$ and $\mathcal{M}$ on the Baire space ω^ω the following equalities hold:*

(1) $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ (Truss, Miller);
(2) $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$ (Fremlin);
(3) $\mathrm{cof}(\sigma\mathcal{K}, \mathcal{M}) = \mathrm{cof}(\sigma\mathcal{K}) = \mathfrak{d}$ (Bartoszyński).

We shall denote by $\sigma\mathcal{C}_0(\mathbb{R}^n)$ the σ-ideal generated by the tame Cantor sets in $\mathbb{R}^n$. Since each Cantor set in $\mathbb{R}$ is tame, we get $\sigma\mathcal{C}_0(\mathbb{R}) = \mathcal{M}(\mathbb{R})$.

Lemma 3.2 and the fact that each meager set in $\mathbb{R}$ is contained in a union of countably many Cantor sets imply:

LEMMA 4.3. *For any meager subsets $A_1, \ldots, A_n \subseteq \mathbb{R}$ the product $\prod_{k=1}^{n} A_k$ belongs to the ideal $\sigma\mathcal{C}_0(\mathbb{R}^n)$.*

This yields another lemma.

LEMMA 4.4. *For any zero-dimensional subspace $Z \subseteq \mathbb{R}$,*
$$\sigma\mathcal{K}(Z^n) \subseteq \sigma\mathcal{C}_0(\mathbb{R}^n).$$

5. Proof of Theorem 2.4. Let $\mathcal{M}$ be the ideal of meager subsets of $\mathbb{R}^n$, and $\sigma\mathcal{C}_0$ be the σ-ideal generated by the tame Cantor sets in $\mathbb{R}^n$. The proof of Theorem 2.4 is divided into four parts corresponding to the equalities (1)–(4).

(1) $\mathrm{cov}(\sigma\mathcal{C}_0) = \mathrm{cov}(\mathcal{M})$. The inequality $\mathrm{cov}(\mathcal{M}) \leq \mathrm{cov}(\sigma\mathcal{C}_0)$ follows from the (trivial) inclusion $\sigma\mathcal{C}_0 \subseteq \mathcal{M}$.

We now prove $\mathrm{cov}(\mathcal{M}) \geq \mathrm{cov}(\sigma\mathcal{C}_0)$. By the definition of $\mathrm{cov}(\mathcal{M}(\mathbb{R})) = \mathrm{cov}(\mathcal{M})$ there exists a cover $\mathcal{U} \subseteq \mathcal{M}(\mathbb{R})$ of $\mathbb{R}$ such that $|\mathcal{U}| = \mathrm{cov}(\mathcal{M})$. The cover
$$\mathcal{U}^n = \Big\{ \prod_{k=1}^{n} C_k : C_1, \ldots, C_n \in \mathcal{U} \Big\}$$
of $\mathbb{R}^n$ has cardinality $|\mathcal{U}|^n = \mathrm{cov}(\mathcal{M})$ and by Lemma 4.3 is contained in $\sigma\mathcal{C}_0$, whence $\mathrm{cov}(\sigma\mathcal{C}_0) \leq |\mathcal{U}^n| = \mathrm{cov}(\mathcal{M})$.

(2) $\mathrm{non}(\sigma\mathcal{C}_0) = \mathrm{non}(\mathcal{M})$. The inequality $\mathrm{non}(\sigma\mathcal{C}_0) \leq \mathrm{non}(\mathcal{M})$ trivially follows from $\sigma\mathcal{C}_0 \subseteq \mathcal{M}$.

To prove $\mathrm{non}(\sigma\mathcal{C}_0) \geq \mathrm{non}(\mathcal{M})$, let $A \subseteq \mathbb{R}^n$ and $|A| < \mathrm{non}(\mathcal{M})$. It follows that $A \subseteq B^n$ for some $B \subseteq \mathbb{R}$ with $|B| \leq n \cdot |A| < \mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{M}(\mathbb{R}))$. Hence B is meager. By Lemma 4.3, $B^n \in \sigma\mathcal{C}_0$.

(3) $\mathrm{add}(\sigma\mathcal{C}_0) = \mathrm{add}(\sigma\mathcal{C}_0, \mathcal{M}) = \mathrm{add}(\mathcal{M})$. Since $\mathrm{add}(\sigma\mathcal{C}_0) \leq \mathrm{add}(\sigma\mathcal{C}_0, \mathcal{M})$, it suffices to prove the inequalities $\mathrm{add}(\sigma\mathcal{C}_0, \mathcal{M}) \leq \mathrm{add}(\mathcal{M})$ and $\mathrm{add}(\mathcal{M}) \leq \mathrm{add}(\sigma\mathcal{C}_0)$.

For the first, let $\mathcal{A} \subseteq \mathcal{M}(\mathbb{R})$ be a subfamily of cardinality $|\mathcal{A}| = \mathrm{add}(\mathcal{M}(\mathbb{R})) = \mathrm{add}(\mathcal{M})$ whose union $\bigcup \mathcal{A}$ is not meager in $\mathbb{R}$. It follows that the family $\mathcal{A}^n = \{\prod_{k=1}^{n} A_k : A_1, \ldots, A_n \in \mathcal{A}\}$ has cardinality $|\mathcal{A}|^n = \mathrm{add}(\mathcal{M})$ and, by Lemma 4.3, is contained in $\sigma\mathcal{C}_0(\mathbb{R}^n) = \sigma\mathcal{C}_0$. The Kuratowski–Ulam Theorem [11, 8.41] implies that the union $\bigcup \mathcal{A}^n = (\bigcup \mathcal{A})^n$ is not meager in $\mathbb{R}^n$. Hence $\mathrm{add}(\sigma\mathcal{C}_0, \mathcal{M}) \leq |\mathcal{A}^n| = \mathrm{add}(\mathcal{M})$.

The inequality $\mathrm{add}(\mathcal{M}) \leq \mathrm{add}(\sigma\mathcal{C}_0)$ will follow if we show that for each family $\mathcal{A}$ containing less than $\mathrm{add}(\mathcal{M})$ tame Cantor sets in $\mathbb{R}^n$, the union $\bigcup \mathcal{A}$ belongs to $\sigma\mathcal{C}_0$. Consider the dense G_δ-set $G = (\mathbb{R}\setminus\mathbb{Q})^n$ in $\mathbb{R}^n$. By Lemma 3.4, for each tame Cantor set $A \in \mathcal{A}$ the set $\mathcal{H}_A^G = \{h \in \mathcal{H}(\mathbb{R}^n) : h(A) \subseteq G\}$ is a

dense G_δ-set in $\mathcal{H}(\mathbb{R}^n)$. Since $|\mathcal{A}| < \mathrm{add}(\mathcal{M}) \leq \mathrm{cov}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}(\mathcal{H}(\mathbb{R}^n)))$, the intersection $\bigcap_{A\in\mathcal{A}} \mathcal{H}_A^G$ is not empty. Let $h \in \bigcap_{A\in\mathcal{A}} \mathcal{H}_A^G$. It follows that $h(\mathcal{A}) = \{h(A) : A \in \mathcal{A}\}$ is a family of less than $\mathrm{add}(\mathcal{M})$ compact subsets of $G = (\mathbb{R} \setminus \mathbb{Q})^n$, which is homeomorphic to ω^ω. Since $|h(\mathcal{A})| < \mathrm{add}(\mathcal{M}) \leq \mathfrak{b} = \mathrm{add}(\sigma\mathcal{K}(G))$, there is a σ-compact subset $K \subseteq G$ containing $\bigcup_{A\in\mathcal{A}} h(A)$. Lemma 4.4 guarantees that $K \in \sigma\mathcal{C}_0(\mathbb{R}^n)$. Thus $h^{-1}(K)$ belongs to $\sigma\mathcal{C}_0$ and $\bigcup \mathcal{A} \subseteq h^{-1}(K)$.

(4) $\mathrm{cof}(\sigma\mathcal{C}_0) = \mathrm{cof}(\sigma\mathcal{C}_0, \mathcal{M}) = \mathrm{cof}(\mathcal{M})$. Since $\mathrm{cof}(\sigma\mathcal{C}_0, \mathcal{M}) \leq \mathrm{cof}(\sigma\mathcal{C}_0)$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathrm{non}(\mathcal{M}), \mathfrak{d}\}$, it suffices to prove $\max\{\mathrm{non}(\mathcal{M}), \mathfrak{d}\} \leq \mathrm{cof}(\sigma\mathcal{C}_0, \mathcal{M})$ and $\mathrm{cof}(\sigma\mathcal{C}_0) \leq \max\{\mathrm{non}(\mathcal{M}), \mathfrak{d}\}$.

First we prove separately that (a) $\mathrm{non}(\mathcal{M}) \leq \mathrm{cof}(\sigma\mathcal{C}_0, \mathcal{M})$ and (b) $\mathfrak{d} \leq \mathrm{cof}(\sigma\mathcal{C}_0, \mathcal{M})$.

To prove (a), let $\mathcal{A} \subseteq \mathcal{M}$ with $|\mathcal{A}| = \mathrm{cof}(\sigma\mathcal{C}_0, \mathcal{M})$ be such that each $C \in \sigma\mathcal{C}_0$ is contained in some $A \in \mathcal{A}$. For each $A \in \mathcal{A}$ we choose $x_A \in X \setminus A$. It follows that $B = \{x_A : A \in \mathcal{A}\} \notin \sigma\mathcal{C}_0$. Hence $\mathrm{non}(\mathcal{M}) = \mathrm{non}(\sigma\mathcal{C}_0) \leq |B| \leq |\mathcal{A}| = \mathrm{cof}(\sigma\mathcal{C}_0, \mathcal{M})$.

Now let us prove prove (b). Let $G = (\mathbb{R} \setminus \mathbb{Q})^n$, which is homeomorphic to ω^ω. By Lemma 4.2(3), $\mathrm{cof}(\sigma\mathcal{K}(G), \mathcal{M}(G)) = \mathfrak{d}$. By Lemma 4.4, we have $\sigma\mathcal{K}(G) \subseteq \sigma\mathcal{C}_0(\mathbb{R}^n)$. Moreover, $\mathcal{M}(G) = \{G \cap M : M \in \mathcal{M}(\mathbb{R}^n)\}$. Now we see that $\mathfrak{d} = \mathrm{cof}(\sigma\mathcal{K}(G), \mathcal{M}(G)) \leq \mathrm{cof}(\sigma\mathcal{C}_0, \mathcal{M})$, and the proof of (b) is complete.

Finally, we will prove $\mathrm{cof}(\sigma\mathcal{C}_0) \leq \mathrm{cof}(\mathcal{M}) = \max\{\mathrm{non}(\mathcal{M}), \mathfrak{d}\}$. Since $\mathrm{cof}(\sigma\mathcal{K}(G)) = \mathfrak{d}$, the ideal $\sigma\mathcal{K}(G)$ has a base $\mathcal{D} \subseteq \sigma\mathcal{K}(G)$ with $|\mathcal{D}| = \mathfrak{d}$. By Lemma 4.4 we have $\mathcal{D} \subseteq \sigma\mathcal{C}_0(\mathbb{R}^n)$.

Fix any non-meager subset H in $\mathcal{H}(\mathbb{R}^n)$ with $|H| = \mathrm{non}(\mathcal{M}(\mathcal{H}(\mathbb{R}^n))) = \mathrm{non}(\mathcal{M})$. It is clear that the family $\mathcal{C} = \{h^{-1}(D) : h \in H, D \in \mathcal{D}\}$ has cardinality $|\mathcal{C}| \leq |H \times \mathcal{D}| \leq \max\{\mathrm{non}(\mathcal{M}), \mathfrak{d}\} = \mathrm{cof}(\mathcal{M})$, and $\mathcal{C} \subseteq \sigma\mathcal{C}_0$. We will complete the proof if we show that $\mathcal{C}$ is a base for $\sigma\mathcal{C}_0$. Let $A \in \sigma\mathcal{C}_0$. Without loss of generality we can assume that A is σ-compact. By Lemma 3.4, $\mathcal{H}_A^G = \{h \in \mathcal{H}(\mathbb{R}^n) : h(A) \subseteq G\}$ is a dense G_δ-set in $\mathcal{H}(\mathbb{R}^n)$, and hence it meets H. Consequently, there is an $h \in H$ such that $h(A) \subseteq G$. Because $\mathcal{D}$ is a base for $\sigma\mathcal{K}(G)$, the σ-compact set $h(A)$ is contained in some σ-compact set $D \in \mathcal{D}$. Then $A \subseteq h^{-1}(D) \in \sigma\mathcal{C}_0$, and the proof of Theorem 2.4 is complete. ∎

6. Proof of Theorem 2.7. Fix $n \in \mathbb{N}$ and $k < n$, and consider the σ-ideal $\sigma\mathcal{D}_k$ generated by the closed k-dimensional sets in the Euclidean space $\mathbb{R}^n$. By Corollary 2.5,

$$\mathrm{add}(\sigma\mathcal{D}_k) \leq \mathrm{add}(\mathcal{M}), \quad \mathrm{cov}(\sigma\mathcal{D}_k) = \mathrm{cov}(\mathcal{M}),$$
$$\mathrm{non}(\sigma\mathcal{D}_k) = \mathrm{non}(\mathcal{M}), \quad \mathrm{cof}(\sigma\mathcal{D}_k) \geq \mathrm{cof}(\mathcal{M}).$$

So, it remains to check that $\mathrm{add}(\sigma\mathcal{D}_k) \geq \mathrm{add}(\mathcal{M})$ and $\mathrm{cof}(\sigma\mathcal{D}_k) \leq \mathrm{cof}(\mathcal{M})$. Identify $\mathbb{R}^n$ with a linear subspace of $\mathbb{R}^m$ for some $m \geq 2n + 3 \geq 5$. Then $\mathbb{R}^n$ is a Z_2-set in $\mathbb{R}^m$.

By [10], $\mathbb{R}^m$ contains a k-dimensional σ-compact subset Σ such that for any k-dimensional Z_2-set $K \subset \mathbb{R}^m$ the set $\{h \in \mathcal{H}(\mathbb{R}^m) : h(K) \subset \Sigma\}$ is dense in $\mathcal{H}(\mathbb{R}^m)$. The set Σ, being k-dimensional, can be enlarged to a dense k-dimensional G_δ-subset $G \subset \mathbb{R}^m$ (see [8, Theorem 1.5.11]). Since $\dim(G) = k < m$, the Baire Theorem implies that G is not σ-compact, and hence it is the image of ω^ω under a perfect map. This yields $\mathrm{add}(\sigma\mathcal{K}(G)) = \mathrm{add}(\sigma\mathcal{K}) = \mathfrak{b}$ and $\mathrm{cof}(\sigma\mathcal{K}(G)) = \mathrm{cof}(\sigma\mathcal{K}) = \mathfrak{d}$.

To prove that $\mathrm{add}(\sigma\mathcal{D}_k) \geq \mathrm{add}(\mathcal{M})$, fix any $\mathcal{A} \subset \sigma\mathcal{D}_k$ with $|\mathcal{A}| < \mathrm{add}(\mathcal{M})$. Since each set $A \in \mathcal{A}$ is contained in a countable union of compact k-dimensional subsets, we lose no generality assuming that each $A \in \mathcal{A}$ is compact. Since $A \subset \mathbb{R}^n$ is a Z_2-set in $\mathbb{R}^m$, the choice of Σ guarantees that $\mathcal{H}_A^\Sigma = \{h \in \mathcal{H}(\mathbb{R}^m) : h(A) \subset \Sigma\}$ is dense in $\mathcal{H}(\mathbb{R}^m)$, and hence the G_δ-subset

$$\mathcal{H}_A^G = \{h \in \mathcal{H}(\mathbb{R}^m) : h(A) \subset G\} \supset \mathcal{H}_A^\Sigma$$

is also dense in $\mathcal{H}(\mathbb{R}^m)$. Since $|\mathcal{A}| < \mathrm{add}(\mathcal{M}) \leq \mathrm{cov}(\mathcal{M})$, the intersection $\bigcap_{A \in \mathcal{A}} \mathcal{H}_A^G$ contains some homeomorphism $h : \mathbb{R}^m \to \mathbb{R}^m$, which maps $\bigcup \mathcal{A}$ into G. Consequently, $h(\mathcal{A}) = \{h(A) : A \in \mathcal{A}\} \subset \sigma\mathcal{K}(G)$. Since $|h(\mathcal{A})| = |\mathcal{A}| < \mathrm{add}(\mathcal{M}) \leq \mathfrak{b} = \mathrm{add}(\sigma\mathcal{K}(G))$, we conclude that $h(\bigcup \mathcal{A})$ is contained in some σ-compact subset $K \subset G$. Then $A = \mathbb{R}^n \cap h^{-1}(K)$ is a σ-compact subset of $\mathbb{R}^n$ with $\dim(A) \leq \dim(h^{-1}(K)) = \dim(K) \leq \dim(G) = k$ containing $\bigcup \mathcal{A}$ and witnessing that $\mathrm{add}(\sigma\mathcal{D}_k) \geq \mathrm{add}(\mathcal{M})$.

Next, we prove that $\mathrm{cof}(\sigma\mathcal{D}_k) \leq \mathrm{cof}(\mathcal{M}) = \max\{\mathrm{non}(\mathcal{M}), \mathfrak{d}\}$. Since $\sigma\mathcal{K}(G) = \mathfrak{d}$, the ideal $\sigma\mathcal{K}(G)$ has a base $\mathcal{D} \subseteq \sigma\mathcal{K}(G)$ with $|\mathcal{D}| = \mathfrak{d}$. Fix any non-meager subset H in $\mathcal{H}(\mathbb{R}^m)$ with $|H| = \mathrm{non}(\mathcal{M}(\mathcal{H}(\mathbb{R}^m))) = \mathrm{non}(\mathcal{M})$. It is clear that the family $\mathcal{C} = \{\mathbb{R}^n \cap h^{-1}(D) : h \in H, D \in \mathcal{D}\}$ has cardinality $|\mathcal{C}| \leq |H \times \mathcal{D}| \leq \max\{\mathrm{non}(\mathcal{M}), \mathfrak{d}\} = \mathrm{cof}(\mathcal{M})$, and $\mathcal{C} \subseteq \sigma\mathcal{D}_k$. We will complete the proof if we show that $\mathcal{C}$ is a base for $\sigma\mathcal{D}_k$.

Let $A \in \sigma\mathcal{D}_k$. Without loss of generality we can assume that A is the countable union $A = \bigcup_{i \in \omega} A_i$ of compact subsets A_i, $i \in \omega$, of dimension $\leq k$. By the choice of Σ, the set

$$\mathcal{H}_A^G = \{h \in \mathcal{H}(\mathbb{R}^m) : h(A) \subseteq G\} = \bigcap_{i \in \omega} \mathcal{H}_{A_i}^G$$

is a dense G_δ-set in $\mathcal{H}(\mathbb{R}^m)$, and hence it meets H. Consequently, there is $h \in H$ such that $h(A) \subseteq G$. Because $\mathcal{D}$ is a base for $\sigma\mathcal{K}(G)$, the σ-compact set $h(A)$ is contained in some σ-compact set $D \in \mathcal{D}$. Then $A \subseteq \mathbb{R}^n \cap h^{-1}(D) \in \mathcal{C}$, and the proof of Theorem 2.7 is complete. ∎

References

[1] L. Antoine, *Sur l'homeomorphie de deux figures et de leurs voisinages*, J. Math. Pures Appl. 86 (1921), 221–325.

[2] T. Banakh, M. Morayne, R. Rałowski and Sz. Żeberski, *Topologically invariant σ-ideals on the Hilbert cube*, Israel J. Math, accepted; arXiv:1302.5658 (2013).

[3] T. Bartoszyński and H. Judah, *Set Theory. On the Structure of the Real Line*, A K Peters, Wellesley, MA, 1995.

[4] R. Bing, *Tame Cantor sets in E^3*, Pacific J. Math. 11 (1961), 435–446.

[5] W. Blankinship, *Generalization of a construction of Antoine*, Ann. of Math. (2) 53 (1951), 276–297.

[6] A. Chigogidze, *Inverse Spectra*, Elsevier, Amsterdam, 1996.

[7] K. Ciesielski and J. Wojciechowski, *Sums of connectivity functions on $\mathbb{R}^n$*, Proc. London. Math. Soc. 76 (1998), 406–426.

[8] R. Engelking, *Theory of Dimensions Finite and Infinite*, Sigma Ser. Pure Math. 10, Heldermann, Lemgo, 1995.

[9] D. Fremlin, *Measure-additive coverings and measurable selectors*, Dissertationes Math. 260 (1987), 116 pp.

[10] R. Geoghegan and R. Summerhill, *Pseudo-boundaries and pseudo-interiors in Euclidean spaces and topological manifolds*, Trans. Amer. Math. Soc. 194 (1974), 141–165.

[11] A. Kechris, *Classical Descriptive Set Theory*, Grad. Texts in Math. 156, Springer, New York, 1995.

[12] K. Kuratowski, *Topology. I*, PWN, Warszawa, and Academic Press, New York, 1966.

[13] D. McMillan, Jr., *Taming Cantor sets in E^n*, Bull. Amer. Math. Soc. 70 (1964), 706–708.

[14] K. Menger, *Allgemeine Räume und Cartesische Räume. II. Über umfassendste n-dimensionale Mengen*, Proc. Akad. Wet. Amsterdam 29 (1926), 1125–1128.

[15] R. Osborne, *Embedding Cantor sets in a manifold*, Michigan Math. J. 13 (1966), 57–63.

[16] R. Sher, *Concerning wild Cantor sets in E^3*, Proc. Amer. Math. Soc. 19 (1968), 1195–1200.

T. Banakh
Department of Mathematics
Ivan Franko National University of Lviv
Lviv, 79000, Ukraine
and
Institute of Mathematics
Jan Kochanowski University
25-406 Kielce, Poland
E-mail: t.o.banakh@gmail.com

M. Morayne, R. Rałowski, Sz. Żeberski
Faculty of Fundamental Problems
of Technology
Wrocław University of Technology
50-370 Wrocław, Poland
E-mail: michal.morayne@pwr.edu.pl
robert.ralowski@pwr.edu.pl
szymon.zeberski@pwr.edu.pl