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Keeping the covering number of the null ideal small

by

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Abstract. It is proved that ideal-based forcings with the side condition method of Todorcevic (1984) add no random reals. By applying Judah–Repický's preservation theorem, it is consistent with the covering number of the null ideal being $\aleph_1$ that there are no S-spaces, every poset of uniform density $\aleph_1$ adds $\aleph_1$ Cohen reals, there are only five cofinal types of directed posets of size $\aleph_1$, and so on. This extends the previous work of Zapletal (2004).

1. Introduction. In the early 1980s Saharon Shelah isolated the notion of properness for partial orders [8]. In the context of forcing in set theory, this property might be very useful since it is preserved under countable support iteration and because no proper poset collapses ω_1. In particular, proper posets enable us to introduce many consistency results which cannot be forced by ccc forcings. It is worth pointing out that Stevo Todorcevic has considerably extended this range of applications. In fact, he discovered the *side condition method* which is a general tool to add an uncountable set by means of a proper forcing (e.g. [12]). For example, one can add an uncountable discrete subset through any given right-separated hereditarily separable regular space using this technique [14, §8] (see Example 2.3).

Jindřich Zapletal proved that two kinds of forcings (and their iterations) do not increase the additivity of the null ideal [22]. One of them is of the form "the specializations of Aronszajn trees", and the other is a wide collection of posets equipped with side conditions, the so-called *ideal-based forcings* [22]. The forcing mentioned above for introducing an uncountable discrete subspace to a regular right-separated hereditarily separable space belongs to this class, and so it is consistent that the additivity of the null ideal is equal to $\aleph_1$ and there are no S-spaces (here, an *S-space* is a hereditarily separable regular space which has a non-Lindelöf subspace). Shelah and

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Zapletal proved that it is consistent that every complete Boolean algebra of uniform density $\aleph_1$ has a complete Boolean subalgebra which is isomorphic to $\mathbb{C}_{\omega_1}$ ([1]) [10]. Zapletal pointed out that an ideal-based forcing is used to prove this consistency.

In [6], Judah and Repický gave a general preservation theorem for countable support iterations of proper forcings. They defined the notion of a covering family (explained in §2.2). A poset which preserves any covering family adds no random reals, and so preservation of covering families guarantees that its forcing extension does not increase the covering number of the null ideal. It follows from their general preservation theorem that a countable support iteration of proper forcings which preserve any covering family adds no random reals, and they proved that a σ-centered forcing ([2]) preserves any covering family (see §2.2). As a corollary, they proved that any countable support iteration of σ-centered forcings adds no random reals.

In [18, §5.2], the author introduced a subclass of ccc forcings which includes the specializations of Aronszajn trees, and showed that forcings in this class do not increase the covering number of the null ideal. This extends one of the previous results due to Zapletal mentioned above. In this paper, it is proved that ideal-based forcings preserve any covering family. So, by combining results due to Judah and Repický mentioned above, any countable support iteration of ideal-based forcings adds no random reals. As an application, it is consistent with the covering number of the null ideal being $\aleph_1$ that there are no S-spaces, every poset of uniform density $\aleph_1$ adds $\aleph_1$ Cohen reals, etc.

2. Preliminaries

2.1. The ideal-based forcing with the side condition method

Definition 2.1 (Zapletal, [22, §3]). A triple $\langle A, \sqsubseteq, \mathfrak{J}\rangle$ is called an *ideal-based triple* if

(A) $A \subseteq [\omega_1]^{<\aleph_0}$ and $\sqsubseteq$ is a transitive relation on A which refines the set-inclusion such that

- for each $a \in A$ and $\beta \in \omega_1$, $a \cap \beta \in A$ and $a \cap \beta \sqsubseteq a$, and
- for each $a, b \in A$, if a and b are $\sqsubseteq$-compatible (i.e. there exists $c \in A$ such that $a \sqsubseteq c$ and $b \sqsubseteq c$), then $a \cup b$ is in A and is a $\sqsubseteq$-upper bound of a and b,

([1]) $\mathbb{C}_{\aleph_1}$ is the set of finite partial functions from ω_1 into $\{0, 1\}$, ordered by reverse inclusion, i.e. for $p, q \in \mathbb{C}_{\omega_1}$, $q \leq_{\mathbb{C}_{\omega_1}} p$ iff $q \supseteq p$.

([2]) A subset P of a poset $\mathbb{P}$ is called *centered* if every finite subset of P has a common extension in $\mathbb{P}$, and a poset is called *σ-centered* if it is a union of countably many centered subsets.

(B) $\mathfrak{J}$ is a non-principal ideal on ω_1 such that

- every $\mathfrak{J}$-positive set has a countable $\mathfrak{J}$-positive subset, and
- the σ-ideal $\sigma\mathfrak{J}$ generated by $\mathfrak{J}$ is a proper ideal,

(C) for each $a \in A$, there exists a $\sigma\mathfrak{J}$-positive set Z such that for every $\beta \in Z$, $a \cup \{\beta\}$ is in A and $a \sqsubseteq a \cup \{\beta\}$, and

(D) for each $a \in A$, there exists an $\mathfrak{J}$-large set Y such that for every $\beta \in Y$, if $(a \cap \beta) \cup \{\beta\}$ is in A and $a \cap \beta \sqsubseteq (a \cap \beta) \cup \{\beta\}$, then $a \cup \{\beta\}$ is in A and $a \sqsubseteq a \cup \{\beta\}$.

We will give a typical example of ideal-based triples below. For each ideal-based triple $\langle A, \sqsubseteq, \mathfrak{J}\rangle$, we will define the *ideal-based forcing derived from* $\langle A, \sqsubseteq, \mathfrak{J}\rangle$, denoted by $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$. For an ideal-based triple $\langle A, \sqsubseteq, \mathfrak{J}\rangle$, we identify $\mathfrak{J}$ with its (minimal) fixed base, and let $\kappa_{\mathfrak{J}}$ be the successor cardinal of the least cardinality of a basis of the ideal $\mathfrak{J}$. The cardinal $\kappa_{\mathfrak{J}}$ is not larger than $(2^{\aleph_1})^+$, and since $\mathfrak{J}$ is a non-principal ideal on ω_1, we have $\kappa_{\mathfrak{J}} \geq \aleph_2$. We apply $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$ to iterate by countable support when $\kappa_{\mathfrak{J}}$ is equal to $\aleph_2$. For an ideal-based triple $\langle A, \sqsubseteq, \mathfrak{J}\rangle$, we write $\mathcal{M}(A, \sqsubseteq, \mathfrak{J})$ for the set of countable elementary substructures of the structure $\langle H(\kappa_{\mathfrak{J}}), \in, \lhd_{\kappa_{\mathfrak{J}}}, A, \sqsubseteq, \mathfrak{J}\rangle$ ($\lhd_{\kappa_{\mathfrak{J}}}$ is a well-ordering of $H(\kappa_{\mathfrak{J}})$). For each $M \in \mathcal{M}(A, \sqsubseteq, \mathfrak{J})$, we denote by $\overline{M}$ the transitive collapse of M, and by Ψ_M the transitive collapsing map from M onto $\overline{M}$. This viewpoint is necessary to check that $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$ has the $\aleph_2$-properness isomorphism condition ($\aleph_2$-pic for short, which is defined by Shelah [9, Ch. VIII, §2], see also Definition 3.2 later, [13, §4] and [10, §3]). The $\aleph_2$-pic is used to guarantee that a countable support iteration of $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$ of length $\leq \omega_2$ has the $\aleph_2$-chain condition.

Definition 2.2 (Todorcevic, Zapletal, et al.). For an ideal-based triple $\langle A, \sqsubseteq, \mathfrak{J}\rangle$, $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$ consists of functions p such that

(i) $\mathrm{dom}(p)$ is a finite $\in$-chain of transitive collapses of members of $\mathcal{M}(A, \sqsubseteq, \mathfrak{J})$,

(ii) for each $\mathtt{t} \in \mathrm{dom}(p)$, $p(\mathtt{t})$ is a pair $\langle \xi^p_{\mathtt{t}}, \mathcal{N}^p_{\mathtt{t}}\rangle$ such that

- $\xi^p_{\mathtt{t}} \in \omega_1$,
- $\mathcal{N}^p_{\mathtt{t}}$ is a finite subset of $\mathcal{M}(A, \sqsubseteq, \mathfrak{J})$ such that the transitive collapse of each member of $\mathcal{N}^p_{\mathtt{t}}$ is equal to the structure $\mathtt{t}$,

(iii) for each $\mathtt{t} \in \mathrm{dom}(p)$ and $\mathtt{t}' \in \mathrm{dom}(p) \cap \mathtt{t}$, $\xi^p_{\mathtt{t}'} \in \mathtt{t}$,

(iv) $\{\xi^p_{\mathtt{t}}; \mathtt{t} \in \mathrm{dom}(p)\} \in A$,

(v) for each $\mathtt{t} \in \mathrm{dom}(p)$, $\mathtt{t}' \in \mathrm{dom}(p) \cap \mathtt{t}$ and $M' \in \mathcal{N}^p_{\mathtt{t}'}$, there exists $M \in \mathcal{N}^p_{\mathtt{t}}$ such that $M' \in M$, and

(vi) for each $\mathtt{t} \in \mathrm{dom}(p)$, $\xi^p_{\mathtt{t}} \notin \bigcup(\mathfrak{J} \cap (\bigcup \mathcal{N}^p_{\mathtt{t}}))$ (hence in particular $\xi^p_{\mathtt{t}} \notin M$ for each $M \in \mathcal{N}^p_{\mathtt{t}}$), and for each q and p in $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$,

$$q \leq_{\mathbb{P}(A,\sqsubseteq,\mathfrak{J})} p \;:\Leftrightarrow\; \begin{array}{l} \mathrm{dom}(p) \subseteq \mathrm{dom}(q) \text{ and for each } \mathtt{t} \in \mathrm{dom}(p), \\ \mathcal{N}^p_{\mathtt{t}} \subseteq \mathcal{N}^q_{\mathtt{t}} \text{ and } \{\xi^p_{\mathtt{t}};\, \mathtt{t} \in \mathrm{dom}(p)\} \sqsubseteq \{\xi^q_{\mathtt{t}};\, \mathtt{t} \in \mathrm{dom}(q)\}. \end{array}$$

This is the $\aleph_2$-pic version of forcings with the side condition method (Zapletal calls them *forcings amended ideal-based* in [22, §3]). Todorcevic and others proved that $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$ in each situation is a proper forcing which has the $\aleph_2$-pic (e.g. [13, §4] and [10, §3]). Shelah proved that a countable support iteration of $\aleph_2$-pic forcings of length $\leq \omega_2$ has the $\aleph_2$-chain condition [9, Ch. VIII, 2.3 Lemma] (see also [13, Lemma 7] and [10, Fact 32]). Therefore a countable support iteration of ideal-based forcings of length $\leq \omega_2$ does not collapse any cardinal, over the ground model which satisfies CH. Zapletal proved that every $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$ is friendly, which is a sufficient condition for keeping the additivity of the null ideal small [22]. By clause (C) in Definition 2.1, $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$ adds an uncountable $\sqsubseteq$-filter on A.

Example 2.3 (Shooting an uncountable discrete subspace into a right-separated hereditarily separable regular space). The following is in [14, 8.9. Theorem]. See also [11, Theorem 8 and Corollary 9].

Let (ω_1, τ) be a right-separated hereditarily separable regular space, that is,

- (*right-separated*) for each $\xi \in \omega_1$, ξ (which is considered as the set of ordinals less than ξ) is an open subset of this space (i.e. $\xi \in \tau$), and
- (*hereditarily separable*) every subspace is separable, that is, every subspace has a countable dense subset.

For each $\xi \in \omega_1$, we fix $U_\xi \in \tau$ such that $\xi \in U_\xi$ and $\mathrm{Cl}_\tau(U_\xi) \subseteq \xi + 1$. We define $A := [\omega_1]^{<\aleph_0}$, for each $a, b \in A$,

$$a \sqsubseteq b \;:\Leftrightarrow\; a \subseteq b \text{ and } \forall \xi \in b \setminus a\ \forall \eta \in a\ (\xi \notin U_\eta)$$

and

$$\mathfrak{J} := \{X \subseteq \omega_1;\, \mathrm{Cl}_\tau(X) \text{ is countable}\}.$$

Then A satisfies clause (A) in Definition 2.1. By hereditary separability, $\mathfrak{J}$ satisfies clause (B). For each $a \in A$, the set $[\max(a) + 1, \omega_1)$ is a $\sigma\mathfrak{J}$-positive set which witnesses clause (C) for this a. For each $a \in A$, the set $\omega_1 \setminus \bigcup_{\xi \in a} \mathrm{Cl}_\tau(U_\xi)$ is a $\mathfrak{J}$-large set which witnesses clause (D) for this a. Therefore $\langle A, \sqsubseteq, \mathfrak{J}\rangle$ is an ideal-based triple.

Let G be an uncountable filter on $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$. Then we can find an uncountable subset H of G such that the set

$$\big\{\{\xi^p_{\mathtt{t}};\, \mathtt{t} \in \mathrm{dom}(p)\};\, p \in H\big\}$$

forms a Δ-system with root Δ. Then by the definition of $\sqsubseteq$ and the fact that H is a subset of a $\sqsubseteq$-filter,

$$\bigcup_{p\in H}\big(\{\xi^p_{\mathtt{t}};\ \mathtt{t}\in\mathrm{dom}(p)\}\setminus\Delta\big)$$

is an uncountable discrete subspace.

EXAMPLE 2.4 (Forcing the P-ideal dichotomy by finite approximations). The P-ideal dichotomy was introduced by Todorcevic [16]. To force it, there are two options: One is forcing by countable approximations [16], and the other is forcing by finite approximations [13, p. 722] (see also [17, Theorem 20.6] and [7, §5.2]). The latter case used models as side conditions. This may not fit the form of ideal-based forcings, but it has very similar properties. In fact, the proof in §§3–4 can be applied to the following forcing $\mathbb{P}(\mathcal{I})$. This is an $\aleph_2$-pic version of the forcing described in [17, Theorem 20.6] and [7, §5.2]. The definition and proofs of the statements about (a non-$\aleph_2$-pic version of) $\mathbb{P}(\mathcal{I})$ are given in [17, Theorem 20.6].

Let S be an uncountable set and $\mathcal{I}$ a P-ideal on $[S]^{\le\aleph_0}$, that is, $\mathcal{I}$ is an ideal whose members are countable subsets of S with the property that for every $A\in[\mathcal{I}]^{\le\aleph_0}$, there exists $b\in\mathcal{I}$ such that $a\setminus b$ is finite (we say that a is almost contained in b, denoted by $a\subseteq^* b$) for every $a\in A$. The *P-ideal dichotomy* is the statement that for every uncountable set S and every P-ideal $\mathcal{I}$ on $[S]^{\le\aleph_0}$, there exists either an uncountable subset X of S such that $[X]^{\le\aleph_0}\subseteq\mathcal{I}$, or a countable decomposition $S=\bigcup_{n\in\omega}S_n$ such that each S_n is *orthogonal* to $\mathcal{I}$, that is, $S_n\cap a$ is finite for every $a\in\mathcal{I}$.

Suppose that S cannot be covered by countably many subsets of S orthogonal to $\mathcal{I}$. Now we assume that S has size $\aleph_1$. If not, the following definition does not make sense (3). To simplify notation, we assume that $S=\omega_1$. Here, we consider the structure $H((2^{\aleph_1})^+)$ equipped with its well-ordering. For a countable elementary substructure M of $H((2^{\aleph_1})^+)$, we write $\delta_{\overline{M}}:=\omega_1\cap M$. The value $\delta_{\mathtt{t}}$ for each transitive collapse $\mathtt{t}$ of countable elementary substructures of $H\big(\big(2^{\aleph_1}\big)^+\big)$ is well-defined. $\mathbb{P}(\mathcal{I})$ consists of functions p such that

- $\mathrm{dom}(p)$ is a finite $\in$-chain of transitive collapses of countable elementary substructures of $H((2^{\aleph_1})^+)$ which contains S and $\mathcal{I}$ as members,
- for each $\mathtt{t}\in\mathrm{dom}(p)$, $p(\mathtt{t})$ is a pair $\langle b^p_{\mathtt{t}},x^p_{\mathtt{t}},\mathcal{N}^p_{\mathtt{t}}\rangle$ such that
  - $b^p_{\mathtt{t}}\in\mathcal{I}$,
  - $x^p_{\mathtt{t}}\in S\setminus M$,
  - $\mathcal{N}^p_{\mathtt{t}}$ is a finite set of countable elementary substructures of $H((2^{\aleph_1})^+)$ which contains S and $\mathcal{I}$ as members such that the transitive collapse of each member of $\mathcal{N}^p_{\mathtt{t}}$ is equal to the structure $\mathtt{t}$,
  - $b^p_{\mathtt{t}}\subseteq S\cap\delta_{\mathtt{t}}$, and for every $M\in\mathcal{N}^p_{\mathtt{t}}$ and $a\in\mathcal{I}\cap M$, $a\subseteq^* b^p_{\mathtt{t}}$,

(3) For a general S, a non-amended version of $\mathbb{P}(\mathcal{I})$ is proper, but may not satisfy the $\aleph_2$-pic.

- for each $\mathtt{t} \in \mathrm{dom}(p)$ and $\mathtt{t}' \in \mathrm{dom}(p) \cap \mathtt{t}$, we have $\{b^p_{\mathtt{t}'}, x^p_{\mathtt{t}'}\} \in \mathtt{t}$,
- for each $\mathtt{t} \in \mathrm{dom}(p)$, $\mathtt{t}' \in \mathrm{dom}(p) \cap \mathtt{t}$ and $M' \in \mathcal{N}^p_{\mathtt{t}'}$, there exists $M \in \mathcal{N}^p_{\mathtt{t}}$ such that $M' \in M$, and
- for each $\mathtt{t} \in \mathrm{dom}(p)$, $M \in \mathcal{N}^p_{\mathtt{t}}$ and $Y \in \mathcal{P}(S) \cap M$ which is orthogonal to $\mathcal{I}$, we have $x^p_{\mathtt{t}} \notin Y$,

and for each p and q in $\mathbb{P}(\mathcal{I})$,

$$q \leq_{\mathbb{P}(\mathcal{I})} p \;:\Leftrightarrow\; \mathrm{dom}(p) \subseteq \mathrm{dom}(q) \text{ and for each } \mathtt{t} \in \mathrm{dom}(p), \\ b^p_{\mathtt{t}} = b^q_{\mathtt{t}},\ \mathcal{N}^p_{\mathtt{t}} \subseteq \mathcal{N}^q_{\mathtt{t}} \text{ and } (\{x^q_{\mathtt{s}} : \mathtt{s} \in \mathrm{dom}(q)\} \setminus \{x^p_{\mathtt{s}} : \mathtt{s} \in \mathrm{dom}(p)\}) \cap \delta_{\mathtt{t}} \subseteq b^p_{\mathtt{t}}.$$

For a filter G of $\mathbb{P}(\mathcal{I})$, letting

$$X_G := \{x^p_{\mathtt{t}};\ p \in G \text{ and } \mathtt{t} \in \mathrm{dom}(p)\},$$

we note that $[X_G]^{\leq\aleph_0} \subseteq \mathcal{I}$. By identifying a member $p \in \mathbb{P}(\mathcal{I})$ with the set $\{x^p_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(p)\}$ and letting $\mathfrak{J}$ be the ideal which consists of all subsets of S orthogonal to $\mathcal{I}$, the triple $\langle \mathbb{P}(\mathcal{I}), \leq_{\mathbb{P}(\mathcal{I})}, \mathfrak{J}\rangle$ looks like an ideal-based triple (but we cannot drop information about $b^p_{\mathtt{t}}$ for each $\mathtt{t} \in \mathrm{dom}(p)$ to force with the forcing $\mathbb{P}(\mathcal{I})$). In fact, $\langle \mathbb{P}(\mathcal{I}), \leq_{\mathbb{P}(\mathcal{I})}, \mathfrak{J}\rangle$ has the following properties:

(a) Every subset of any condition of $\mathbb{P}(\mathcal{I})$ is also a condition of $\mathbb{P}(\mathcal{I})$, and if conditions p and q of $\mathbb{P}(\mathcal{I})$ are compatible, then $p \cup q$ is their common extension in $\mathbb{P}(\mathcal{I})$.
(b) Clause (B) in Definition 2.1 holds for the ideal $\mathfrak{J}$.
(c) Every condition of $\mathbb{P}$ can be extended arbitrarily.
(d) For any condition $p \in \mathbb{P}(\mathcal{I})$, $\mathtt{t} \in \mathrm{dom}(p)$, $M \in \mathcal{N}^p_{\mathtt{t}}$ and a $\mathfrak{J}$-positive subset $Z \in M$ of S, the set $Z \cap \bigcap_{\mathtt{t}' \in \mathrm{dom}(p) \setminus \mathtt{t}} b^p_{\mathtt{t}'}$ is not empty.

It follows from these properties that $\mathbb{P}(\mathcal{I})$ is proper and satisfies the $\aleph_2$-pic and, by the similar proof below, one can show that $\mathbb{P}(\mathcal{I})$ adds no random reals.

Here, we should notice that it may be impossible to force the P-ideal dichotomy for P-ideals on ω_1 by an iteration of length ω_2, because a P-ideal is not necessarily an $\aleph_1$-structure, and hence the classical book-keeping arguments do not work. The P-ideal dichotomy is forced by a countable support iteration of length a supercompact cardinal with Laver function (as a book-keeping device), like a forcing of the Proper Forcing Axiom [16, Lemma 7]. However, it is possible to force the P-ideal dichotomy for $\aleph_1$-generated P-ideals on ω_1 with a book-keeping procedure like e.g. the one in [5, §4] ([4]).

For the other examples, Zapletal pointed out that the following statements can be forced by ideal-based forcings.

([4]) Iterations of forcings for ideals on ω_1 with $\aleph_1$ generators are studied in [2, §3] (also in [4, §1.5]). In [4] (and [1]), basics of iterations of $\aleph_2$-pic forcings are also detailed.

- Classification of transitive relations on $\aleph_1$ (Todorcevic [13, 15]).
- Shooting an uncountable set through a coherent sequence on ω_1 (Todorcevic [14, §8]).
- Making a poset of uniform density $\aleph_1$ add $\aleph_1$ Cohen reals (Shelah and Zapletal [10]).

The Open Coloring Axiom due to Todorcevic can be forced by the iteration of forcings with the side condition method [14, p. 85 and 8.0. Theorem] (see also [17, Theorem 11.2] and [7, §5.1]). But it is not known whether the standard forcing with the side condition method for the Open Coloring Axiom can be considered as an ideal-based forcing, or whether we have an ideal-based forcing for the Open Coloring Axiom.

2.2. Judah and Repický's preservation theorem for adding no random reals

Definition 2.5 (Judah and Repický [6], [3, §6.3.B]). Let

$$\mathcal{S} := \left\{ f \in {}^{\omega}\mathcal{P}(2^{<\omega});\ \text{for each } n \in \omega,\ f(n) \subseteq {}^{n}2 \text{ and } \sum_{n\in\omega} \frac{|f(n)|}{2^n} \le 1 \right\}.$$

For each $f \in \mathcal{S}$, write

$$O_f := \{x \in {}^{\omega}2;\ \forall n \in \omega\ \exists m \ge n \text{ such that } x{\restriction}m \in f(m)\}.$$

A subset $\mathcal{F}$ of $\mathcal{S}$ is called a *covering family* if

- $\mathcal{F}$ is σ-directed, i.e. for any $F \in [\mathcal{F}]^{\le\aleph_0}$, there exists $g \in \mathcal{F}$ such that for every $f \in F$,
$$\exists n \in \omega\ \forall m \ge n\ \big(f(n) \subseteq g(n)\big),$$
- for any $x \in {}^{\omega}2$, there exists $f \in \mathcal{F}$ such that $x \in O_f$.

$\mathcal{S}$ is a set of codes of G_δ-null sets, and a covering family is a set of codes whose decodes cover the set of the reals. The covering number of the null ideal is equal to the smallest size of a covering family. We note that a forcing $\mathbb{P}$ adds no random reals if and only if (letting V be the ground model)

$$\Vdash_{\mathbb{P}} \text{“the family } \mathcal{S} \cap V \text{ is still a covering family”.}$$

We say that a forcing $\mathbb{P}$ *preserves any covering family* if for any covering family $\mathcal{F} \subset \mathcal{S}$ (in the ground model),

$$\Vdash_{\mathbb{P}} \text{“}\mathcal{F} \text{ is still a covering family”.}$$

“Preserving any covering family” is somewhat stronger than “adding no random reals”. Actually, this notion is necessary because there is a two-step iteration of proper forcings such that neither step adds a random real but the iteration does ([5]). Judah and Repický proved that for any countable support iteration $\langle \mathbb{P}_n, \dot{\mathbb{Q}}_n;\ n \in \omega\rangle$ of proper forcings, if no $\mathbb{P}_n$ adds random reals, then $\mathbb{P}_\omega$ adds no random reals [6, Corollary 4(a)]. We note that for any two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ of proper forcings, if $\mathbb{P}$ adds no random reals and forces that $\dot{\mathbb{Q}}$ preserves any covering family, then $\mathbb{P} * \dot{\mathbb{Q}}$ adds no random reals. Thus a countable support iteration of proper forcings which preserve any covering family adds no random reals.

Judah and Repický also proved that σ-centered forcings preserve any covering family [6, Theorem 5]. Therefore, a countable support iteration of σ-centered forcings adds no random reals.

The author introduced the property $\mathsf{R}_{1,\aleph_1}$ [18, 20] and the rectangle refining property [19], both stronger than the countable chain condition. In [18, Theorem 5.4], the author proved that forcings with the property $\mathsf{R}_{1,\aleph_1}$ (or the rectangle refining property) add no random reals. It follows from this proof that forcings with the property $\mathsf{R}_{1,\aleph_1}$ preserve any covering family.

3. Proofs of properness and the $\aleph_2$-pic. In this section, we show that every ideal-based forcing is proper and satisfies the $\aleph_2$-pic. For the proof, we refer to papers about the side condition method listed in the references. Since the proof in the next section is fairly self-contained, this section can be skipped.

THEOREM 3.1 (Todorcevic, Zapletal, et al). *Every ideal-based forcing* $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$ *is proper.*

Proof. Let $\langle A, \sqsubseteq, \mathfrak{J}\rangle$ be an ideal-based triple, $\mathbb{P} := \mathbb{P}(A, \sqsubseteq, \mathfrak{J})$, θ a large enough regular cardinal, N a countable elementary submodel of $H(\theta)$ which contains $\langle A, \sqsubseteq, \mathfrak{J}\rangle$ and $H(\kappa_{\mathfrak{J}})$ as members, and $p \in \mathbb{P} \cap N$. By clause (C) in Definition 2.1, there exists η in $\omega_1 \setminus \bigcup(\mathfrak{J} \cap N)$ such that

$$\{\xi^p_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(p)\} \cup \{\eta\} \in A.$$

Define

$$p^+ := p \cup \bigl\{\bigl\langle \overline{H(\kappa_{\mathfrak{J}}) \cap N}, \langle \eta, \{H(\kappa_{\mathfrak{J}}) \cap N\}\rangle\bigr\rangle\bigr\}.$$

We will show that p^+ is an $(N, \mathbb{P})$-generic condition.

([5]) This is mentioned in [23, p. 114]. Actually, the iteration $\mathbb{C}_{\omega_1} * \mathbb{B}^V$ adds a random real, where $\mathbb{B}^V$ is the forcing notion in the extension with $\mathbb{C}_{\omega_1}$ such that conditions of $\mathbb{B}^V$ are ones of the random forcing in the ground model. This is because $\mathbb{C}_{\omega_1} * \mathbb{B}^V$ is forcing-equivalent to the product $\mathbb{C}_{\omega_1} \times \mathbb{B}$, where $\mathbb{B}$ is the random forcing. And we note that $\mathbb{C}_{\omega_1}$ adds no random reals, and $\mathbb{B}^V$ adds no random reals in the extension $V^{\mathbb{C}_{\omega_1}}$ with $\mathbb{C}_{\omega_1}$. To see this, let x be a real in the extension with $\mathbb{C}_{\omega_1} * \mathbb{B}^V$. Then there exists $\alpha < \omega_1$ such that x is a real in the extension with $\mathbb{C}_\alpha * \mathbb{B}^V$ which is forcing-equivalent to $\mathbb{C}_\alpha \times \mathbb{B}$. Let c be a Cohen real over $V^{\mathbb{C}_\alpha}$ in $V^{\mathbb{C}_{\omega_1}}$. Then c is Cohen over the extension with $\mathbb{C}_\alpha \times \mathbb{B}$. Since a Cohel real makes the set of ground model reals of measure zero, x belongs to the null set defined with c. So x is not random over $V^{\mathbb{C}_{\omega_1}}$.

Let $D \in N$ be an open dense subset of $\mathbb{P}$, and q an extension of p^+ in $\mathbb{P}$ such that $q \in D$. By extending each $\mathcal{N}^q_{\mathtt{t}}$ if necessary, we may assume that

($\blacktriangle$) for each $\mathtt{t} \in \mathrm{dom}(q) \cap N$,

$$\left\{\left((\Psi_{H(\kappa_{\mathfrak{J}})\cap N})^{-1} \circ \Psi_{M''}\right)(M');\ M'' \in \mathcal{N}^q_{\overline{H(\kappa_{\mathfrak{J}})\cap N}} \text{ and } M' \in \mathcal{N}^q_{\mathtt{t}} \cap M''\right\} \subseteq \mathcal{N}^q_{\mathtt{t}}.$$

Since $\omega_1 \cap M = \omega_1 \cap M'$ whenever M and M' in $\mathcal{M}(A, \sqsubseteq, \mathfrak{J})$ are isomorphic, taking these isomorphic copies as above does not interfere with clause (vi) in Definition 2.2. Let $l := |\mathrm{dom}(q) \setminus (\mathrm{dom}(q) \cap N)|$. By clause (v) in Definition 2.2, it is possible to take an increasing chain $\langle K_j;\ j < l\rangle$ of members of $\bigcup_{\mathtt{t}\in\mathrm{dom}(q)\setminus(\mathrm{dom}(q)\cap N)} \mathcal{N}^q_{\mathtt{t}}$ such that $K_0 := H(\kappa_{\mathfrak{J}}) \cap N$. Then for each $i < l$, $\langle \xi^q_{\overline{K_j}};\ j < i\rangle$ is in K_i. We define T_l to be the set of $\sigma \in {}^l\omega_1$ with the following property: There exists $s \in D$ such that

- $\mathrm{dom}(s)$ end-extends $\mathrm{dom}(q) \cap N$,
- ($\blacktriangledown$) for each $\mathtt{t} \in \mathrm{dom}(q) \cap N$, $\mathcal{N}^s_{\mathtt{t}}$ includes $\mathcal{N}^q_{\mathtt{t}} \cap N$ as a subset,
- $\{\xi^s_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(s)\}$ end-extends $\{\xi^q_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(q) \cap N\}$, and
- $\{\xi^s_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(s) \setminus (\mathrm{dom}(q) \cap N)\} = \mathrm{ran}(\sigma)$.

Then even if q is not in N, since N contains $\{\xi^q_{\mathtt{t}}, \mathcal{N}^q_{\mathtt{t}} \cap N;\ \mathtt{t} \in \mathrm{dom}(q) \cap N\}$ and D as members, T_l is in $N \cap H(\kappa_{\mathfrak{J}}) = K_0$. Also $\langle \xi^q_{\overline{K_j}};\ j < l\rangle$ is in T_l. By downward induction on $j < l$, we define T_j such that

$$T_j := T_{j+1} \setminus \left\{\sigma \in T_{j+1};\ \{\tau(j);\ \tau \in T_{j+1} \text{ and } \tau{\restriction}j = \sigma{\restriction}j\} \in \mathfrak{J}\right\}.$$

We note that the sequence $\langle T_j;\ j \le l\rangle$ is in K_0, hence in K_i for each $i < l$.

We will show that $\langle \xi^q_{\overline{K_j}};\ j < l\rangle$ is also in T_0. We have observed that $\langle \xi^q_{\overline{K_j}};\ j < l\rangle$ is in T_l. Suppose that $i < l$ and $\langle \xi^q_{\overline{K_j}};\ j < l\rangle \in T_{i+1}$. Then by clause (vi) in Definition 2.2, we have $\xi^q_{\overline{K_i}} \notin \bigcup(\mathfrak{J} \cap K_i)$, and also

$$\xi^q_{\overline{K_i}} \in \left\{\tau(i);\ \tau \in T'_{i+1} \text{ and } \tau{\restriction}i = \langle \xi^q_{\overline{K_j}};\ j < i\rangle\right\} \in K_i.$$

Thus the set $\{\tau(i);\ \tau \in T_{i+1} \text{ and } \tau{\restriction}i = \langle \xi^q_{\overline{K_j}};\ j < i\rangle\}$ is $\mathfrak{J}$-positive, and hence $\langle \xi^q_{\overline{K_j}};\ j < l\rangle \in T_i$.

Therefore, when we consider T_0 as a tree which consists of all initial segments of members of T_0, then T_0 has a cofinal branch (of length l), and each non-terminal node has $\mathfrak{J}$-positive many successors in T_0. By induction on $\nu < l$ we will take $\zeta_\nu \in \omega_1 \cap N = \omega_1 \cap K_0$ such that

- $\langle \zeta_\mu;\ \mu < \nu\rangle$ is an initial segment of some member of T_0,
- $\{\xi^r_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r)\} \cup \{\zeta_\mu;\ \mu < \nu\}$ is in A, and
- $\{\xi^r_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r)\} \sqsubseteq \{\xi^r_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r)\} \cup \{\zeta_\mu;\ \mu \le \nu\}$.

Given $\langle \zeta_\mu;\ \mu < \nu\rangle$, by clause (D) in Definition 2.1, there exists a $\mathfrak{J}$-large subset Y_ν of ω_1 such that for each $\beta \in Y_\nu$, if

$$\big((\{\xi^r_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r)\} \cup \{\zeta_\mu;\ \mu < \nu\}) \cap \beta\big) \cup \{\beta\} \in A$$

and

$$(\{\xi^r_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r)\} \cup \{\zeta_\mu;\ \mu < \nu\}) \cap \beta \\ \sqsubseteq \big((\{\xi^r_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r)\} \cup \{\zeta_\mu;\ \mu < \nu\}) \cap \beta\big) \cup \{\beta\},$$

then

$$\{\xi^r_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r)\} \cup \{\zeta_\mu;\ \mu < \nu\} \cup \{\beta\} \in A$$

and

$$\{\xi^r_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r)\} \cup \{\zeta_\mu;\ \mu < \nu\} \sqsubseteq \{\xi^r_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r)\} \cup \{\zeta_\mu;\ \mu < \nu\} \cup \{\beta\}.$$

By the properties of T_0, there exists $\zeta_\nu \in Y_\nu$ such that $\langle \zeta_\mu;\ \mu \le \nu\rangle$ is an initial segment of some member of T_0. Since $\zeta_\nu \in Y_\nu$ and

$$(\{\xi^r_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r)\} \cup \{\zeta_\mu;\ \mu < \nu\}) \cap \zeta_\nu = \{\xi^r_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r) \cap N\} \cup \{\zeta_\mu;\ \mu < \nu\},$$

by the properties of T_0 and Y_ν, and clause (A) in Definition 2.1, we conclude that

$$\{\xi^r_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r)\} \cup \{\zeta_\mu;\ \mu \le \nu\} \in A$$

and

$$\{\xi^r_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r)\} \cup \{\zeta_\mu;\ \mu < \nu\} \sqsubseteq \{\xi^r_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r)\} \cup \{\zeta_\mu;\ \mu \le \nu\},$$

which finishes the choice of ζ_ν.

We take $r \in \mathbb{P} \cap K_0$ which witnesses that $\langle \zeta_\nu;\ \nu < l\rangle \in T_0$. Then r is in D. We define r^+ such that

- $\mathrm{dom}(r^+) := \mathrm{dom}(r) \cup \mathrm{dom}(q)$,
- for each $\mathtt{t} \in \mathrm{dom}(r)$, $\xi^{r^+}_{\mathtt{t}} := \xi^r_{\mathtt{t}}$, and for each $\mathtt{t} \in \mathrm{dom}(q) \setminus \mathrm{dom}(r)$, $\xi^{r^+}_{\mathtt{t}} := \xi^q_{\mathtt{t}}$, and
- for each $\mathtt{t} \in \mathrm{dom}(q) \setminus \mathrm{dom}(r)$, we have $\mathcal{N}^{r^+}_{\mathtt{t}} := \mathcal{N}^q_{\mathtt{t}}$; for each $\mathtt{t} \in \mathrm{dom}(r) \setminus \mathrm{dom}(q)$,

$$\mathcal{N}^{r^+}_{\mathtt{t}} := \mathcal{N}^r_{\mathtt{t}} \cup \big\{((\Psi_{M''})^{-1} \circ \Psi_{H(\kappa_{\mathfrak{J}}) \cap N})(M');\\ M'' \in \mathcal{N}^q_{\overline{H(\kappa_{\mathfrak{J}}) \cap N}}\ \&\ M' \in \mathcal{N}^r_{\mathtt{t}} \cap N\big\};$$

  and for each $\mathtt{t} \in \mathrm{dom}(r) \cap \mathrm{dom}(q)$, $\mathcal{N}^{r^+}_{\mathtt{t}} := \mathcal{N}^r_{\mathtt{t}} \cup \mathcal{N}^q_{\mathtt{t}}$.

We now check that r^+ is a condition of $\mathbb{P}$. First, r^+ satisfies clauses (i)–(iii) in Definition 2.2 by the definition of r^+, and clause (iv) by the choice of $\langle \zeta_\nu;\ \nu < l\rangle$.

To check (v) we only consider the non-trivial case: Let $\mathtt{t} \in \mathrm{dom}(r) \setminus \mathrm{dom}(q)$, $\mathtt{t}' \in \mathrm{dom}(q) \cap \mathtt{t}$ (then $\mathtt{t}' \in \mathrm{dom}(q) \cap N$) and $K' \in \mathcal{N}^q_{\mathtt{t}'} \setminus \mathcal{N}^r_{\mathtt{t}'}$. By clause (v) for q, there exists $M'' \in \mathcal{N}^q_{\overline{H(\kappa_{\mathfrak{J}})\cap N}}$ such that $K' \in M''$. So by clause ($\blacktriangle$), the set

$$((\Psi_{H(\kappa_{\mathfrak{J}})\cap N})^{-1} \circ \Psi_{M''})(K')$$

is in $\mathcal{N}^q_{\mathtt{t}'} \cap N$. By clause ($\blacktriangledown$) and clause (v) for r, there exists $L \in \mathcal{N}^r_{\mathtt{t}}$ ($\subseteq N$) such that

$$((\Psi_{H(\kappa_{\mathfrak{J}})\cap N})^{-1} \circ \Psi_{M''})(K') \in L.$$

Then

$$K' \in ((\Psi_{M''})^{-1} \circ \Psi_{H(\kappa_{\mathfrak{J}})\cap N})(L) \in \mathcal{N}^{r^+}_{\mathtt{t}}.$$

To check clause (vi), we only consider the non-trivial case: Let $\mathtt{t} \in \mathrm{dom}(r)\backslash\mathrm{dom}(q)$, $M \in \mathcal{N}^{r^+}_{\mathtt{t}}\backslash N$ and $X \in \mathfrak{J}\cap M$; we will show that $\xi^{r^+}_{\mathtt{t}} \notin X$. By the definition of r^+, $\xi^{r^+}_{\mathtt{t}} = \xi^r_{\mathtt{t}}$ and there are $M'' \in \mathcal{N}^q_{H(\kappa_{\mathfrak{J}})\cap N}$, $M' \in \mathcal{N}^r_{\mathtt{t}} \cap N$ and $X' \in M'$ such that

$$M = ((\Psi_{M''})^{-1} \circ \Psi_{H(\kappa_{\mathfrak{J}})\cap N})(M'),$$
$$X = ((\Psi_{M''})^{-1} \circ \Psi_{H(\kappa_{\mathfrak{J}})\cap N})(X').$$

Since $(\Psi_{M''})^{-1} \circ \Psi_{H(\kappa_{\mathfrak{J}})\cap N}$ is an isomorphism from the structure $\langle H(\kappa_{\mathfrak{J}}) \cap N, A, \sqsubseteq, \mathfrak{J}\rangle$ onto $\langle M'', A, \sqsubseteq, \mathfrak{J}\rangle$, and $X \in \mathfrak{J} \cap M$ in M'', $X' \in \mathfrak{J} \cap M'$ in $H(\kappa_{\mathfrak{J}}) \cap N$, we have $\xi^r_{\mathtt{t}} \notin X'$ in $H(\kappa_{\mathfrak{J}})\cap N$. So since the isomorphism $(\Psi_{H(\kappa_{\mathfrak{J}})\cap N})^{-1} \circ \Psi_{M''}$ does not move $\xi^r_{\mathtt{t}}$, it follows that $\xi^r_{\mathtt{t}} \notin X$ in M''. $\dashv_{\text{Check } r^+ \in \mathbb{P}}$

By the choice of $\langle \zeta_\nu; \nu < l\rangle$, r^+ is a common extension of r and q. ■

Definition 3.2 (Shelah [9, Ch. VIII, 2.1 Definition]). A poset $\mathbb{P}$ satisfies the $\aleph_2$-*properness isomorphism condition* ($\aleph_2$-pic) if for any large enough regular cardinal θ, any $\alpha, \beta \in \omega_2$ with $\alpha < \beta$, any countable elementary submodels N_α and N_β of the structure $H(\theta)$ (equipped with its well-ordering) such that $\alpha \in N_\alpha$, $\beta \in N_\beta$, $\mathbb{P} \in N_\alpha \cap N_\beta$, $N_\alpha \cap \omega_2 \subseteq \beta$ and $N_\alpha \cap \alpha = N_\beta \cap \beta$, any $p \in \mathbb{P} \cap N_\alpha$ and any isomorphism $\pi : N_\alpha \to N_\beta$ such that $\pi(\alpha) = \beta$ and $\pi\restriction(N_\alpha \cap N_\beta) = \mathrm{id}$, there exists an $(N_\alpha, \mathbb{P})$-generic extension q of both p and $\pi(p)$ in $\mathbb{P}$ such that

$$q \Vdash_{\mathbb{P}} \text{“}\, \pi[\dot{G}_{\mathbb{P}} \cap N_\alpha] = \dot{G}_{\mathbb{P}} \cap N_\beta \,\text{”}.$$

Lemma 3.3 (Todorcevic, Zapletal, et al.). *Every ideal-based forcing* $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$ *satisfies the* $\aleph_2$*-pic.*

Proof. Suppose that θ is a large enough regular cardinal, α and β are in ω_2 with $\alpha < \beta$, N_α and N_β are countable elementary submodels of the structure $H(\theta)$ (equipped with its well-ordering) such that $\alpha \in N_\alpha$, $\beta \in N_\beta$, $\mathbb{P}(A, \sqsubseteq, \mathfrak{J}) \in N_\alpha \cap N_\beta$, $N_\alpha \cap \omega_2 \subseteq \beta$ and $N_\alpha \cap \alpha = N_\beta \cap \beta$, $p \in \mathbb{P}(A, \sqsubseteq, \mathfrak{J}) \cap N_\alpha$, and $\pi : N_\alpha \to N_\beta$ is an isomorphism such that $\pi(\alpha) = \beta$ and $\pi\restriction(N_\alpha \cap N_\beta) = \mathrm{id}$. Then $N_\alpha \cap \omega_1 = N_\beta \cap \omega_1$ and $N_\alpha \cap \mathcal{P}(\omega) = N_\beta \cap \mathcal{P}(\omega)$. Thus $\mathrm{dom}(p) = \mathrm{dom}(\pi(p))$ and $\xi^p_{\mathtt{t}} = \xi^{\pi(p)}_{\mathtt{t}}$ for each $\mathtt{t} \in \mathrm{dom}(p)$. Let $\mathtt{s}$ be the transitive collapse of $N_\alpha \cap H(\kappa_{\mathfrak{J}})$ and take $\eta \in A \setminus \bigcup(\mathfrak{J} \cap (N_\alpha \cup N_\beta))$. Define a function q with domain $\mathrm{dom}(p) \cup \{\mathtt{s}\}$ by setting $q(\mathtt{t}) := \langle \xi^p_{\mathtt{t}}, \mathcal{N}^p_{\mathtt{t}} \cup \mathcal{N}^{\pi(p)}_{\mathtt{t}} \rangle$ for each $\mathtt{t} \in \mathrm{dom}(p)$, and $q(\mathtt{s}) := \langle \eta, \{N_\alpha \cap H(\kappa_{\mathfrak{J}}), N_\beta \cap H(\kappa_{\mathfrak{J}})\}\rangle$. Then q is a common extension of p and $\pi(p)$. Using the previous argument that r and q are compatible in $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$, it follows that

$$q \Vdash_{\mathbb{P}} \text{“}\, \pi[\dot{G}_{\mathbb{P}} \cap N_\alpha] = \dot{G}_{\mathbb{P}} \cap N_\beta \,\text{”}. \ \blacksquare$$

Shelah proved that, under CH, for any iteration $\langle \dot{\mathbb{P}}_\alpha, \dot{\mathbb{Q}}_\beta;\ \alpha \leq \omega_2,\ \beta < \omega_2 \rangle$ with countable support, if for each $\alpha < \omega_2$, $\Vdash_{\mathbb{P}_\alpha}$ “$\dot{\mathbb{Q}}_\alpha$ satisfies the $\aleph_2$-pic”, then for each $\alpha < \omega_1$, $\mathbb{P}_\alpha$ also satisfies the $\aleph_2$-pic and forces CH, and $\mathbb{P}_{\omega_2}$ satisfies the $\aleph_2$-cc [9, Ch. VIII, 2.4 Lemma and 2.9 Claim] (see also [1, §5.4]). Therefore, by use of a countable support iteration of ideal-based forcings of length ω_2 with some book-keeping device under CH, it can be forced that the size of the continuum is $\aleph_2$, there are no S-spaces, there are only five cofinal types of directed sets of size $\aleph_1$, every poset of uniform density $\aleph_1$ adds $\aleph_1$ Cohen reals, etc.

4. The proof that no random reals are added. The outline of the following proof is similar to the one of [18, Theorem 5.4] and [21, §3].

THEOREM 4.1. *Every ideal-based forcing $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$ preserves any covering family.*

Proof. Let $\langle A, \sqsubseteq, \mathfrak{J}\rangle$ be an ideal-based triple, $\mathbb{P} := \mathbb{P}(A, \sqsubseteq, \mathfrak{J})$ and $\mathcal{F} \subseteq \mathcal{S}$ a covering family. We show that

$$\Vdash_{\mathbb{P}} \text{“}\mathcal{F} \text{ is still a covering family”}.$$

Suppose that $p \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name for a function in ${}^\omega 2$. Let λ be a regular cardinal such that $\langle A, \sqsubseteq, \mathfrak{J}\rangle$ and $H(\kappa_{\mathfrak{J}})$ are members of $H(\lambda)$, and define

$$\mathcal{M}(A, \sqsubseteq, \mathfrak{J})^+ := \big\{ M^+ \cap H(\kappa_{\mathfrak{J}});\ M^+ \prec H(\lambda) \text{ countable} \\ \& \ \{\langle A, \sqsubseteq, \mathfrak{J}\rangle, H(\kappa_{\mathfrak{J}})\} \in M^+ \big\}.$$

We consider members of $\mathcal{M}(A, \sqsubseteq, \mathfrak{J})^+$ as members of $\mathcal{M}(A, \sqsubseteq, \mathfrak{J})$. Let θ be a large enough regular cardinal, and N a countable elementary submodel of $H(\theta)$ which contains $\langle A, \sqsubseteq, \mathfrak{J}\rangle$, $H(\kappa_{\mathfrak{J}})$, $H(\lambda)$, $\mathcal{F}$ and $\dot{x}$ as members. We note that N contains $\mathbb{P}$ and $\mathcal{M}(A, \sqsubseteq, \mathfrak{J})^+$ as members. Since $\mathcal{F}$ is a covering family (in particular, is σ-directed) and ${}^\omega 2 \cap N$ is countable, there exists $f \in \mathcal{F}$ such that

$${}^\omega 2 \cap N \subseteq O_f.$$

By clause (C) in Definition 2.1, there exists ξ_0 in $\omega_1 \setminus \bigcup(\mathfrak{J} \cap N)$ such that

$$\{\xi^p_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(p)\} \cup \{\xi_0\} \in A.$$

Define

$$p^{+} := p \cup \left\langle \overline{H(\kappa_{\mathfrak{J}}) \cap N}, \langle \xi_0, \{H(\kappa_{\mathfrak{J}}) \cap N\}\rangle \right\rangle.$$

(As seen in the proof of Theorem 3.1, p^{+} is an $(N, \mathbb{P})$-generic condition. We do not use this fact below.) We will show that

$$p^{+} \Vdash_{\mathbb{P}} \text{“}\dot{x} \in O_f\text{”},$$

which finishes the proof.

Suppose that

$$p^{+} \not\Vdash_{\mathbb{P}} \text{“}\dot{x} \in O_f\text{”},$$

and take $q \leq_{\mathbb{P}} p^{+}$ and $n \in \omega$ such that

$$q \Vdash_{\mathbb{P}} \text{“}\forall m \geq n \ (\dot{x}{\restriction}m \notin f(m))\text{”}.$$

By extending each $\mathcal{N}^q_{\mathtt{t}'}$ if necessary, we may assume that for each $\mathtt{t}'$ in $\mathrm{dom}(q) \cap N$, we have

$$\{((\Psi_{H(\kappa_{\mathfrak{J}})\cap N})^{-1} \circ \Psi_M)(M');\ M \in \mathcal{N}^q_{\overline{H(\kappa_{\mathfrak{J}})\cap N}} \text{ and } M' \in \mathcal{N}^q_{\mathtt{t}'} \cap M\} \subseteq \mathcal{N}^q_{\mathtt{t}'}.$$

Since $\omega_1 \cap M = \omega_1 \cap M'$ whenever M and M' in $\mathcal{M}(A, \sqsubseteq, \mathfrak{J})$ are isomorphic, taking these isomorphic copies as above does not interfere with clause (vi) in Definition 2.2.

Here, for each finite $\in$-chain σ, let $\min(\sigma)$ denote the $\in$-minimal element of σ. For each $k \in \omega$, define S_k to be the set of $v \in {}^{k}2$ with the following property: There are $\gamma \in \omega_1$ and $M \in \mathcal{M}(A, \sqsubseteq, \mathfrak{J})^{+}$ containing $\dot{x}$ and $\{\xi^q_{\mathtt{t}'}, \mathcal{N}^q_{\mathtt{t}'} \cap N;\ \mathtt{t}' \in \mathrm{dom}(q) \cap N\}$ such that for every $r \in \mathbb{P}$, *if*

- $\mathrm{dom}(r)$ end-extends $\mathrm{dom}(q) \cap N$,
- $\{\xi^r_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r)\}$ end-extends $\{\xi^q_{\mathtt{t}'};\ \mathtt{t}' \in \mathrm{dom}(q) \cap N\}$,

(•) there exists $K \in \mathcal{N}^r_{\min(\mathrm{dom}(r)\setminus(\mathrm{dom}(q)\cap N))}$ with $M \in K$ such that for each $\mathtt{t}' \in \mathrm{dom}(q) \cap N$,

$$\{((\Psi_K)^{-1} \circ \Psi_{M''})(M');\ M'' \in \mathcal{N}^r_{\overline{K}} \text{ and } M' \in \mathcal{N}^r_{\mathtt{t}'} \cap M''\} \subseteq \mathcal{N}^r_{\mathtt{t}'}$$

and $\mathcal{N}^r_{\mathtt{t}'} \cap K = \mathcal{N}^q_{\mathtt{t}'} \cap N$, and

- $r \Vdash_{\mathbb{P}} \text{“}\dot{x}{\restriction}k \neq v\text{”}$,

then

$$\xi^r_{\min(\mathrm{dom}(r)\setminus(\mathrm{dom}(q)\cap N))} \leq \gamma.$$

In general, q may not be a member of N, but the sequence $\langle S_k;\ k \in \omega\rangle$ belongs to the model N. Since $\{v{\restriction}k;\ v \in S_{k+1}\}$ is a subset of S_k for each $k \in \omega$, the set $\bigcup_{k\in\omega} S_k$ forms a subtree of $(2^{<\omega}, \subseteq)$. The key point of the proof is the following claim.

Claim 4.2. *For each $k \in \omega$, S_k is not empty.*

We will show this claim later. From this claim, the set $\bigcup_{k\in\omega} S_k$ forms an infinite tree, therefore by the elementarity of N, there exists $u \in {}^{\omega}2 \cap N$ such that $u{\restriction}k \in S_k$ for every $k \in \omega$. Since $u \in {}^{\omega}2 \cap N \subseteq O_f$, there exists $m \geq n$ such that $u{\restriction}m \in f(m)$. Since $u{\restriction}m \in S_m$ in N, there are $\gamma \in \omega_1 \cap N$ and $M \in \mathcal{M}(A, \sqsubseteq, \mathfrak{J})^+ \cap N$ which witness that $u{\restriction}m \in S_m$. Since $\mathrm{dom}(q)$ end-extends $\mathrm{dom}(q) \cap N$, $\{\xi^q_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(q)\}$ end-extends $\{\xi^q_{\mathtt{t}'};\ \mathtt{t}' \in \mathrm{dom}(q) \cap N\}$,

$$\xi^q_{\min(\mathrm{dom}(q)\setminus(\mathrm{dom}(q)\cap N))} \geq \omega_1 \cap N > \gamma,$$

and q and $H(\kappa_{\mathfrak{J}}) \cap N$ satisfies condition $(\bullet)$, it follows that

$$q \not\Vdash_{\mathbb{P}} \text{“}\dot{x}{\restriction}m \neq u{\restriction}m\text{”}.$$

So there is $r \leq_{\mathbb{P}} q$ such that

$$r \Vdash_{\mathbb{P}} \text{“}\dot{x}{\restriction}m = u{\restriction}m\text{”}.$$

Then

$$r \Vdash_{\mathbb{P}} \text{“}\dot{x}{\restriction}m = u{\restriction}m \in f(m)\text{”},$$

which is a contradiction.

Therefore it suffices to prove Claim 4.2. The rest of the proof is devoted to doing that. The following argument is based on a proof of the properness of $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$. Let $k \in \omega$ and suppose that S_k is empty.

Subclaim 4.3. *For any $v \in {}^{k}2 \setminus S_k$, $\gamma \in \omega_1$ and $M \in \mathcal{M}(A, \sqsubseteq, \mathfrak{J})^+$ which contains $\dot{x}$ and $\{\xi^q_{\mathtt{t}'}, \mathcal{N}^q_{\mathtt{t}'} \cap N;\ \mathtt{t}' \in \mathrm{dom}(q) \cap N\}$, there are $l \in \omega$ and $T \subseteq {}^{l}\omega_1$ such that*

- *for each $\sigma \in T$, $\sigma(0) > \gamma$,*
- *for each $\sigma \in T$ and $j < l$, the set $\{\tau(j);\ \tau \in T$ and $\tau{\restriction}j = \sigma{\restriction}j\}$ is countable $\mathfrak{J}$-positive, and*
- *for each $\sigma \in T$, there exists $s \in \mathbb{P}$ such that*
  - $\mathrm{dom}(s)$ *end-extends* $\mathrm{dom}(q) \cap N$,
  - $\{\xi^s_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(s)\}$ *end-extends* $\{\xi^q_{\mathtt{t}'};\ \mathtt{t}' \in \mathrm{dom}(q) \cap N\}$,
  - $\{\xi^s_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(s) \setminus (\mathrm{dom}(q) \cap N)\} = \mathrm{ran}(\sigma)$,
  - *there exists $K \in \mathcal{N}^s_{\min(\mathrm{dom}(s)\setminus(\mathrm{dom}(q)\cap N))}$ with $M \in K$ such that for each $\mathtt{t}' \in \mathrm{dom}(q) \cap N$,*

    $$\{((\Psi_K)^{-1} \circ \Psi_{M''})(M');\ M'' \in \mathcal{N}^s_{\overline{K}} \text{ and } M' \in \mathcal{N}^s_{\mathtt{t}'} \cap M''\} \subseteq \mathcal{N}^s_{\mathtt{t}'}$$

    and $\mathcal{N}^s_{\mathtt{t}'} \cap K = \mathcal{N}^q_{\mathtt{t}'} \cap N$, and
  - *$s \Vdash_{\mathbb{P}}$ “$\dot{x}{\restriction}k \neq v$ ”.*

Proof of Subclaim 4.3. Let $v \in {}^{k}2 \setminus S_k$, $\gamma \in \omega_1$, and take $M \in \mathcal{M}(A, \sqsubseteq, \mathfrak{J})^+$ which contains $\dot{x}$ and $\{\xi^q_{\mathtt{t}'}, \mathcal{N}^q_{\mathtt{t}'} \cap N;\ \mathtt{t}' \in \mathrm{dom}(q) \cap N\}$ as members. Let M^+ be a countable elementary submodel of $H(\lambda)$ which contains $\langle A, \sqsubseteq, \mathfrak{J}\rangle$ and $H(\kappa_{\mathfrak{J}})$ as members such that $M = M^+ \cap H(\kappa_{\mathfrak{J}})$. We note that M^+ contains

$\mathbb{P}$ as a member. Then by the assumption that $v \notin S_k$, there are $r \in \mathbb{P}$ and K_0 such that

- $\mathrm{dom}(r)$ end-extends $\mathrm{dom}(q) \cap N$,
- $\{\xi^r_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r)\}$ end-extends $\{\xi^q_{\mathtt{t}'};\ \mathtt{t}' \in \mathrm{dom}(q) \cap N\}$,
- $M \in K_0 \in \mathcal{N}^r_{\min(\mathrm{dom}(r)\setminus(\mathrm{dom}(q)\cap N))}$, and for each $\mathtt{t}' \in \mathrm{dom}(q) \cap N$,

  $$\{((\Psi_{K_0})^{-1} \circ \Psi_{M''})(M');\ M'' \in \mathcal{N}^r_{\overline{K_0}} \text{ and } M' \in \mathcal{N}^r_{\mathtt{t}'} \cap M''\} \subseteq \mathcal{N}^r_{\mathtt{t}'}$$

  and $\mathcal{N}^r_{\mathtt{t}'} \cap K_0 = \mathcal{N}^q_{\mathtt{t}'} \cap N$,
- $r \Vdash_{\mathbb{P}}$ "$\dot{x}{\restriction}k \neq v$", and
- $\xi^r_{\min(\mathrm{dom}(r)\setminus(\mathrm{dom}(q)\cap N))} > \gamma$.

Let $l := |\mathrm{dom}(r) \setminus (\mathrm{dom}(q) \cap N)|$. By clause (v) in Definition 2.2, it is possible to take an increasing chain $\langle K_j;\ j < l\rangle$ of members of $\bigcup_{\mathtt{t}\in\mathrm{dom}(r)\setminus(\mathrm{dom}(q)\cap N)} \mathcal{N}^r_{\mathtt{t}}$, where K_0 is the one already chosen. Then $\langle \xi^r_{\overline{K_j}};\ j < i\rangle$ is in K_i for each $i < l$. We define T'_l to be the set of $\sigma \in {}^l\omega_1$ with the following property: There exists $s \in \mathbb{P}$ such that

- $\mathrm{dom}(s)$ end-extends $\mathrm{dom}(q) \cap N$,
- $\{\xi^s_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(s)\}$ end-extends $\{\xi^q_{\mathtt{t}'};\ \mathtt{t}' \in \mathrm{dom}(q) \cap N\}$,
- $\{\xi^s_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(s) \setminus (\mathrm{dom}(q) \cap N)\} = \mathrm{ran}(\sigma)$,
- there exists $K \in \mathcal{N}^s_{\min(\mathrm{dom}(s)\setminus(\mathrm{dom}(q)\cap N))}$ with $M \in K$ such that for each $\mathtt{t}' \in \mathrm{dom}(q) \cap N$,

  $$\{((\Psi_K)^{-1} \circ \Psi_{M''})(M');\ M'' \in \mathcal{N}^s_{\overline{K}} \text{ and } M' \in \mathcal{N}^s_{\mathtt{t}'} \cap M''\} \subseteq \mathcal{N}^s_{\mathtt{t}'}$$

  and $\mathcal{N}^s_{\mathtt{t}'} \cap K = \mathcal{N}^q_{\mathtt{t}'} \cap N$, and
- $s \Vdash_{\mathbb{P}}$ "$\dot{x}{\restriction}k \neq v$".

We note that T'_l is in $M^+ \cap H(\kappa_{\mathfrak{J}}) = M \subseteq K_0$ and $\langle \xi^r_{\overline{K_j}};\ j < l\rangle$ is in T'_l.

By downward induction on $j < l$, we define

$$T'_j := T'_{j+1} \setminus \bigl\{\sigma \in T'_{j+1};\ \{\tau(j);\ \tau \in T'_{j+1} \text{ and } \tau{\restriction}j = \sigma{\restriction}j\} \in \mathfrak{J}\bigr\}.$$

We note that the sequence $\langle T'_j;\ j \leq l\rangle$ is in K_0, hence in K_j for each $j < l$.

We will show that $\langle \xi^r_{\overline{K_j}};\ j < l\rangle$ is also in T'_0. We have observed that $\langle \xi^r_{\overline{K_j}};\ j < l\rangle \in T'_l$. Suppose that $i < l$ and $\langle \xi^r_{\overline{K_j}};\ j < l\rangle \in T'_{i+1}$. Then by clause (vi) in Definition 2.2, we have $\xi^r_{\overline{K_i}} \notin \bigcup(\mathfrak{J} \cap K_i)$, and also

$$\xi^r_{\overline{K_i}} \in \{\tau(i);\ \tau \in T'_{i+1} \text{ and } \tau{\restriction}i = \langle \xi^r_{\overline{K_j}};\ j < i\rangle\} \in K_i.$$

Therefore the set $\{\tau(i);\ \tau \in T'_{i+1} \text{ and } \tau{\restriction}i = \langle \xi^r_{\overline{K_j}};\ j < i\rangle\}$ is $\mathfrak{J}$-positive, and hence $\langle \xi^r_{\overline{K_j}};\ j < l\rangle \in T'_i$.

Next, since the set $\{\sigma(0);\ \sigma \in T_0'\}$ is $\mathfrak{J}$-positive, by clause (B) in Definition 2.1, we can take its countable $\mathfrak{J}$-positive subset $E_0^{\emptyset}$. We define

$$T_0'' := \{\sigma \in T_0';\ \sigma(0) \in E_0^{\emptyset}\}.$$

By induction on $j < l-1$, we build a set $\{E_{j+1}^{\sigma\restriction(j+1)};\ \sigma \in T_j''\}$ of countable $\mathfrak{J}$-positive sets and T_{j+1}'' such that for each $\sigma \in T_j''$,

$$E_{j+1}^{\sigma\restriction(j+1)} \subseteq \{\tau(j+1);\ \tau \in T_j'' \text{ and } \tau\restriction(j+1) = \sigma\restriction(j+1)\}$$

and

$$T_{j+1}'' := \{\sigma \in T_j';\ \sigma(j+1) \in E_{j+1}^{\sigma\restriction(j+1)}\}.$$

Then T_{l-1}'' is as desired. $\dashv_{\text{Subclaim 4.3}}$

Let $\{v_i;\ i < 2^k\}$ enumerate the set ${}^k2$, and denote $M_{-1} := H(\kappa_{\mathfrak{J}}) \cap N$. By the assumption that $S_k = \emptyset$, we have $v_i \notin S_k$ for every $i < 2^k$. Using the subclaim, by induction on $i < 2^k - 1$, we take $l_i \in \omega$, $T_i \subseteq {}^{l_i}\omega_1$ and $M_i \in \mathcal{M}(A, \sqsubseteq, \mathfrak{J})^+$ such that:

- for each $\sigma \in T_i$, $\sigma(0) > \omega_1 \cap M_{i-1}$,
- for each $\sigma \in T_i$ and $j < l_i$, the set $\{\tau(j);\ \tau \in T_i \text{ and } \tau\restriction j = \sigma\restriction j\}$ is countable $\mathfrak{J}$-positive (hence T_i is a countable subset of ${}^{l_i}\omega_1$),
- $\big\{T_i, M_{i-1}, \{\xi^q_{\mathtt{t}'}, \mathcal{N}^q_{\mathtt{t}'} \cap N;\ \mathtt{t}' \in \mathrm{dom}(q) \cap N\}\big\} \in M_i$, and
- for each $\sigma \in T_i$, there exists $s \in \mathbb{P} \cap M_i$ such that
  - $\mathrm{dom}(s)$ end-extends $\mathrm{dom}(q) \cap N$,
  - $\{\xi^s_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(s)\}$ end-extends $\{\xi^q_{\mathtt{t}'};\ \mathtt{t}' \in \mathrm{dom}(q) \cap N\}$,
  - $\{\xi^s_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(s) \setminus (\mathrm{dom}(q) \cap N)\} = \mathrm{ran}(\sigma)$,
  - there exists $K \in \mathcal{N}^s_{\min(\mathrm{dom}(s)\setminus(\mathrm{dom}(q)\cap N))}$ with $M_{i-1} \in K$ such that for each $\mathtt{t}' \in \mathrm{dom}(q) \cap N$,
  $$\{((\Psi_K)^{-1} \circ \Psi_{K''})(K');\ K'' \in \mathcal{N}^s_{\overline{K}} \text{ and } K' \in \mathcal{N}^s_{\mathtt{t}'} \cap K''\} \subseteq \mathcal{N}^s_{\mathtt{t}'}$$
  and $\mathcal{N}^s_{\mathtt{t}'} \cap K = \mathcal{N}^q_{\mathtt{t}'} \cap N$, and
  - $s \Vdash_{\mathbb{P}}$ "$\dot{x}\restriction k \neq v_i$".

By the subclaim again, we can take $r^+_{2^k-1} \in \mathbb{P}$ such that

- $\mathrm{dom}(r^+_{2^k-1})$ end-extends $\mathrm{dom}(q) \cap N$,
- $\{\xi^{(r^+_{2^k-1})}_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r^+_{2^k-1})\}$ end-extends $\{\xi^q_{\mathtt{t}'};\ \mathtt{t}' \in \mathrm{dom}(q) \cap N\}$,

(•) there exists $K \in \mathcal{N}^{(r^+_{2^k-1})}_{\min(\mathrm{dom}(r^+_{2^k-1})\setminus(\mathrm{dom}(q)\cap N))}$ with $M_{2^k-2} \in K$ such that for each $\mathtt{t}' \in \mathrm{dom}(q) \cap N$,

$$\{((\Psi_K)^{-1} \circ \Psi_{M''})(M');\ M'' \in \mathcal{N}^{(r^+_{2^k-1})}_{\overline{K}}\ \&\ M' \in \mathcal{N}^{(r^+_{2^k-1})}_{\mathtt{t}'} \cap M''\} \subseteq \mathcal{N}^{(r^+_{2^k-1})}_{\mathtt{t}'}$$

and $\mathcal{N}^{(r^+_{2^k-1})}_{\mathtt{t}'} \cap K = \mathcal{N}^q_{\mathtt{t}'} \cap N$,

- $r^+_{2^k-1} \Vdash_{\mathbb{P}}$ "$\dot{x}{\restriction}k \neq v_{2^k-1}$", and
- $\xi^{(r^+_{2^k-1})}_{\min(\mathrm{dom}(r^+_{2^k-1})\setminus(\mathrm{dom}(q)\cap N))} > \omega_1 \cap M_{2^k-2}$.

By downward induction on $i < 2^k - 1$, we will take $r_i \in \mathbb{P} \cap M_i$ and $r^+_i \in \mathbb{P}$ such that

- $\mathrm{dom}(r_i)$ end-extends $\mathrm{dom}(q) \cap N$,
- $\{\xi^{r_i}_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r_i)\}$ end-extends $\{\xi^q_{\mathtt{t}'};\ \mathtt{t}' \in \mathrm{dom}(q) \cap N\}$,
- $|\mathrm{dom}(r_i) \setminus (\mathrm{dom}(q) \cap N)| = l_i$,
- $\langle \xi^{r_i}_{K^i_j};\ j < l_i\rangle \in T_i$,

($\bullet$) there exists

$$K \in \mathcal{N}^{r_i}_{\min(\mathrm{dom}(r_i)\setminus(\mathrm{dom}(q)\cap N))} \quad \text{with} \quad M_{i-1} \in K$$

such that for each $\mathtt{t}' \in \mathrm{dom}(q) \cap N$,

$$\{((\Psi_K)^{-1} \circ \Psi_{M''})(M');\ M'' \in \mathcal{N}^{r_i}_{K} \text{ and } M' \in \mathcal{N}^{r_i}_{\mathtt{t}'} \cap M''\} \subseteq \mathcal{N}^{r_i}_{\mathtt{t}'}$$

and $\mathcal{N}^{r_i}_{\mathtt{t}'} \cap K = \mathcal{N}^q_{\mathtt{t}'} \cap N$,

- $r_i \Vdash_{\mathbb{P}}$ "$\dot{x}{\restriction}k \neq v_i$", and
- r^+_i is a common extension of r_i and r^+_{i+1} in $\mathbb{P}$ such that

$$\mathrm{dom}(r^+_i) = \mathrm{dom}(r_i) \cup \mathrm{dom}(r^+_{i+1})$$

(then it follows that $\{\xi^{(r^+_i)}_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r^+_i)\} = \{\xi^{r_i}_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r_i)\} \cup \{\xi^{(r^+_{i+1})}_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r^+_{i+1})\}$) and for each $\mathtt{t}' \in \mathrm{dom}(q) \cap N$,

$$\mathcal{N}^{(r^+_i)}_{\mathtt{t}'} = \bigcup_{i \le j < 2^k-1} \mathcal{N}^{r_j}_{\mathtt{t}'} \cup \mathcal{N}^{(r^+_{2^k-1})}_{\mathtt{t}'}.$$

This finishes the proof because then r^+_0 is a common extension of the conditions $r^+_{2^k-1}$ and r_i for every $i < 2^k - 1$, and hence

$$r^+_0 \Vdash_{\mathbb{P}} \text{“}\dot{x}{\restriction}k \notin {}^k2\text{”},$$

which is a contradiction.

Suppose that we have built r^+_j for each j with $i < j < 2^k$; we will find r_i and r^+_i. To do so, we will take $\zeta^i_\nu \in \omega_1$ by induction on $\nu < l_i$ so that

- $\langle \zeta^i_\mu;\ \mu \le \nu\rangle$ is an initial segment of some member of T_i,
- $\bigcup_{i<j<2^k}\{\xi^{(r^+_j)}_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r^+_j)\} \cup \{\zeta^i_\mu;\ \mu \le \nu\}$ is in A, and
- $\bigcup_{i<j<2^k}\{\xi^{(r^+_j)}_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r^+_j)\} \sqsubseteq \bigcup_{i<j<2^k}\{\xi^{(r^+_j)}_{\mathtt{t}};\ \mathtt{t} \in \mathrm{dom}(r^+_j)\} \cup \{\zeta^i_\mu;\ \mu \le \nu\}$.

Given $\langle \zeta^i_\mu ; \mu < \nu \rangle$, by clause (D) in Definition 2.1, there exists a $\mathfrak{J}$-large subset Y^i_ν of ω_1 such that for each $\beta \in Y^i_\nu$, if

$$\Big(\Big(\bigcup_{i<j<2^k} \{\xi^{(r^+_j)}_{\mathtt{t}} ; \mathtt{t} \in \operatorname{dom}(r^+_j)\} \cup \{\zeta^i_\mu ; \mu < \nu\}\Big) \cap \beta\Big) \cup \{\beta\} \in A$$

and

$$\Big(\bigcup_{i<j<2^k} \{\xi^{(r^+_j)}_{\mathtt{t}} ; \mathtt{t} \in \operatorname{dom}(r^+_j)\} \cup \{\zeta^i_\mu ; \mu < \nu\}\Big) \cap \beta \sqsubseteq \Big(\Big(\bigcup_{i<j<2^k} \{\xi^{(r^+_j)}_{\mathtt{t}} ; \mathtt{t} \in \operatorname{dom}(r^+_j)\} \cup \{\zeta^i_\mu ; \mu < \nu\}\Big) \cap \beta\Big) \cup \{\beta\},$$

then

$$\bigcup_{i<j<2^k} \{\xi^{(r^+_j)}_{\mathtt{t}} ; \mathtt{t} \in \operatorname{dom}(r_j)\} \cup \{\zeta^i_\mu ; \mu < \nu\} \cup \{\beta\} \in A$$

and

$$\bigcup_{i<j<2^k} \{\xi^{(r^+_j)}_{\mathtt{t}} ; \mathtt{t} \in \operatorname{dom}(r^+_j)\} \cup \{\zeta^i_\mu ; \mu < \nu\} \sqsubseteq \bigcup_{i<j<2^k} \{\xi^{(r^+_j)}_{\mathtt{t}} ; \mathtt{t} \in \operatorname{dom}(r^+_j)\} \cup \{\zeta^i_\mu ; \mu < \nu\} \cup \{\beta\}.$$

By the properties of T_i, there exists $\zeta^i_\nu \in Y^i_\nu$ such that $\langle \zeta^i_\mu ; \mu \leq \nu \rangle$ is an initial segment of some member of T_i. Since $\zeta^i_\nu \in Y^i_\nu$ and

$$\Big(\bigcup_{i<j<2^k} \{\xi^{(r^+_j)}_{\mathtt{t}} ; \mathtt{t} \in \operatorname{dom}(r^+_j)\} \cup \{\zeta^i_\mu ; \mu < \nu\}\Big) \cap \zeta^i_\nu = \{\xi^q_{\mathtt{t}'} ; \mathtt{t}' \in \operatorname{dom}(q \cap N)\} \cup \{\zeta^i_\mu ; \mu < \nu\},$$

by the properties of Y^i_ν and clause (A) in Definition 2.1, we conclude that the set

$$\bigcup_{i<j<2^k} \{\xi^{(r^+_j)}_{\mathtt{t}} ; \mathtt{t} \in \operatorname{dom}(r^+_j)\} \cup \{\zeta^i_\mu ; \mu \leq \nu\}$$

is in A and

$$\bigcup_{i<j<2^k} \{\xi^{(r^+_j)}_{\mathtt{t}} ; \mathtt{t} \in \operatorname{dom}(r^+_j)\} \cup \{\zeta^i_\mu ; \mu < \nu\} \sqsubseteq \bigcup_{i<j<2^k} \{\xi^{(r^+_j)}_{\mathtt{t}} ; \mathtt{t} \in \operatorname{dom}(r^+_j)\} \cup \{\zeta^i_\mu ; \mu \leq \nu\},$$

which finishes the choice of ζ^i_ν.

We take $r_i \in \mathbb{P} \cap M_i$ which witnesses that $\langle \xi^i_\nu;\, \nu < l_i\rangle \in T_i$. Let K_{i+1} in $\mathcal{N}^{(r^+_{i+1})}_{\min(\mathrm{dom}(r^+_{i+1})\setminus(\mathrm{dom}(q)\cap N))}$ witness condition ($\bullet$) for r^+_{i+1}. We define r^+_i such that

- $\mathrm{dom}(r^+_i) := \mathrm{dom}(r_i) \cup \mathrm{dom}(r^+_{i+1})$,
- – for each $\mathtt{t} \in \mathrm{dom}(r^+_{i+1}) \setminus \mathrm{dom}(r_i)$, $\xi^{(r^+_i)}_{\mathtt{t}} := \xi^{(r^+_{i+1})}_{\mathtt{t}}$, and
  – for each $\mathtt{t} \in \mathrm{dom}(r_i)$, $\xi^{(r^+_i)}_{\mathtt{t}} := \xi^{r_i}_{\mathtt{t}}$, and
- – for each $\mathtt{t} \in \mathrm{dom}(r^+_{i+1}) \setminus \mathrm{dom}(r_i)$, $\mathcal{N}^{(r^+_i)}_{\mathtt{t}} := \mathcal{N}^{(r^+_{i+1})}_{\mathtt{t}}$,
  – for each $\mathtt{t} \in \mathrm{dom}(r_i) \setminus \mathrm{dom}(r^+_{i+1})$,
$$\mathcal{N}^{(r^+_i)}_{\mathtt{t}} := \mathcal{N}^{r_i}_{\mathtt{t}} \cup \Big\{((\varPsi_{M''})^{-1} \circ \varPsi_{K_{i+1}})(M');\, M'' \in \mathcal{N}^{(r^+_{i+1})}_{K_{i+1}} \;\&\; M' \in \mathcal{N}^{r_i}_{\mathtt{t}} \cap K_{i+1}\Big\},$$
  – for each $\mathtt{t} \in \mathrm{dom}(r_i) \cap \mathrm{dom}(r^+_{i+1})$ (then $\mathtt{t} \in \mathrm{dom}(q) \cap N$),
$$\mathcal{N}^{(r^+_i)}_{\mathtt{t}} := \mathcal{N}^{r_i}_{\mathtt{t}} \cup \mathcal{N}^{(r^+_{i+1})}_{\mathtt{t}}.$$

By the choice of $\langle \zeta^i_\nu; \nu < l\rangle$, if r^+_i is a condition of $\mathbb{P}$, then r^+_i is a common extension of r_i and r^+_{i+1}. So it suffices to prove that r^+_i is a condition of $\mathbb{P}$. Now, r^+_i satisfies clauses (i)–(iii) in Definition 2.2 by the definition of r^+_i, and clause (iv) by the choice of $\langle \zeta^i_\nu;\, \nu < l\rangle$.

(The rest of the proof is similar to the one of Theorem 3.1.) To check clause (v) in Definition 2.2, we only consider the non-trivial case: We let $\mathtt{t} \in \mathrm{dom}(r_i) \setminus \mathrm{dom}(r^+_{i+1})$, $\mathtt{t}' \in \mathrm{dom}(r^+_{i+1}) \cap \mathtt{t}$ (then $\mathtt{t}' \in \mathrm{dom}(q) \cap N$) and $K' \in \mathcal{N}^{(r^+_{i+1})}_{\mathtt{t}'} \setminus \mathcal{N}^{r_i}_{\mathtt{t}'}$. Then $\mathtt{t} \in M_i \subseteq K_{i+1}$. By clause (v) for r^+_{i+1}, there exists $M'' \in \mathcal{N}^{(r^+_{i+1})}_{K_{i+1}}$ such that $K' \in M''$. By clause ($\bullet$), the set
$$((\varPsi_{K_{i+1}})^{-1} \circ \varPsi_{M''})(K')$$
is in $\mathcal{N}^{(r^+_{i+1})}_{\mathtt{t}'} \cap K_{i+1} = \mathcal{N}^{q}_{\mathtt{t}'} \cap N$ which is included in $\mathcal{N}^{r_i}_{\mathtt{t}'} \cap N$. By clause (v) for r_i, there exists $L \in \mathcal{N}^{r_i}_{\mathtt{t}}$ such that
$$((\varPsi_{K_{i+1}})^{-1} \circ \varPsi_{M''})(K') \in L.$$
Since $r_i \in M_i \subseteq K_{i+1}$, L is in $\mathcal{N}^{r_i}_{\mathtt{t}} \cap K_{i+1}$. Hence by the definition of r^+_i,
$$K' \in ((\varPsi_{M''})^{-1} \circ \varPsi_{K_{i+1}})(L) \in \mathcal{N}^{(r^+_i)}_{\mathtt{t}}.$$

To check clause (vi) in Definition 2.2, we only consider the non-trivial case: Let $\mathtt{t} \in \mathrm{dom}(r_i) \setminus \mathrm{dom}(r^+_{i+1})$, $M \in \mathcal{N}^{(r^+_i)}_{\mathtt{t}} \setminus N$ and $X \in \mathfrak{J} \cap M$; we will show that $\xi^{(r^+_i)}_{\mathtt{t}} \notin X$. Then by the definition of r^+_i, we have $\xi^{(r^+_i)}_{\mathtt{t}} = \xi^{r_i}_{\mathtt{t}}$ and there are $M'' \in \mathcal{N}^{(r^+_{i+1})}_{K_{i+1}}$, $M' \in \mathcal{N}^{r_i}_{\mathtt{t}} \cap K_{i+1}$ and $X' \in M'$ such that

$$M = ((\Psi_{M''})^{-1} \circ \Psi_{K_{i+1}})(M'), \qquad X = ((\Psi_{M''})^{-1} \circ \Psi_{K_{i+1}})(X').$$

Since $(\Psi_{M''})^{-1} \circ \Psi_{K_{i+1}}$ is an isomorphism from $\langle H(\kappa_{\mathfrak{J}}) \cap N, A, \sqsubseteq, \mathfrak{J}\rangle$ onto $\langle M'', A, \sqsubseteq, \mathfrak{J}\rangle$, and since $X \in \mathfrak{J} \cap M$ and $M'', X' \in \mathfrak{J} \cap M'$ in $H(\kappa_{\mathfrak{J}}) \cap N$, we have $\xi^{(r^+_i)}_{\mathtt{t}} \notin X'$ in $H(\kappa_{\mathfrak{J}}) \cap N$. So by the fact that $(\Psi_{K_{i+1}})^{-1} \circ \Psi_{M''}$ does not move $\xi^{(r^+_i)}_{\mathtt{t}}$, we have $\xi^{(r^+_i)}_{\mathtt{t}} \notin X$ in M''. ∎

Therefore, by the previous work in §§2–3, a countable support iteration of ideal-based forcings adds no random reals. Therefore we conclude that it is consistent with the covering number of the null ideal being $\aleph_1$ that there are no S-spaces, every poset of uniform density $\aleph_1$ adds $\aleph_1$ Cohen reals, etc.

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