FUNDAMENTA
MATHEMATICAE
226 (2014)

Open subgroups of free topological groups

by

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Abstract. The theory of covering spaces is often used to prove the Nielsen–Schreier theorem, which states that every subgroup of a free group is free. We apply the more general theory of semicovering spaces to obtain analogous subgroup theorems for topological groups: Every open subgroup of a free Graev topological group is a free Graev topological group. An open subgroup of a free Markov topological group is a free Markov topological group if and only if it is disconnected.

1. Introduction. A well-known application of covering space theory is the Nielsen–Schreier theorem [21], which states that every subgroup of a free group is free [6, 13]. The corresponding situation for topological groups is more complicated since it is not true that every closed subgroup of a free topological group is free topological [7, 9, 11, 15]. The purpose of this paper is to use the theory of semicovering spaces developed in [3] to prove the following theorem.

THEOREM 1.1. *Every open subgroup of a free Graev topological group is a free Graev topological group.*

Free topological groups are important objects in the general theory of topological groups and have an extensive literature dating back to their introduction by A. A. Markov [17] in the 1940s. Markov first defined the free topological group $F_M(X)$ on a space X and Graev [11] later introduced the free topological group $F_G(X,*)$ on a space X with basepoint $* \in X$. For any space X, the existence of the groups $F_G(X,*)$ and $F_M(X)$ follows abstractly from categorical considerations [20]. Moreover, $F_M(X)$ is isomorphic to the free Markov topological group on the completely regular image of the canonical injection $\sigma : X \to F_M(X)$ (and similarly for Graev topological groups). Thus one may assume without loss of generality that X is

2010 *Mathematics Subject Classification*: Primary 22A05; Secondary 55R65, 55Q52.
Key words and phrases: free topological group, topological fundamental group, semicovering map.

DOI: 10.4064/fm226-1-2

completely regular. Since the methods of the current paper are categorical in nature, neither the spaces nor the groups in question are required to satisfy any separation axioms. For more on the theory of free topological groups, we refer the reader to [1, 22, 23].

Topological versions of the Nielsen–Schreier theorem have appeared for both abelian and non-abelian free topological groups. Morris and Pestov prove in [18] that every open subgroup of a free abelian topological group is a free abelian topological group. In the non-abelian case, Brown and Hardy [8] (see also [19]) consider free topological groups on Hausdorff k_ω-spaces, i.e. spaces which are the inductive limit of a sequence of compact subspaces. In particular, it is shown that a subgroup of a free Graev topological group which admits a continuous Schreier transversal is free Graev topological. As a special case, it follows that every open subgroup of a free Graev topological group on a Hausdorff k_ω-space is free Graev topological. In the current paper, we extend this open subgroup theorem to all free topological groups.

Covering-theoretic proofs of the algebraic Nielsen–Schreier theorem typically require the use of covering spaces and fundamental group(oid)s of graphs. Our proof of Theorem 1.1 generalizes this approach by replacing covering theory with the theory of semicoverings [3], graphs with **Top**-graphs (i.e. topological graphs with discrete vertex spaces), and the fundamental groupoid (fundamental group) with the fundamental **Top**-groupoid [3] (topological fundamental group [4]). Our application of the classification of semicoverings uses the fact that the theory applies naturally to certain non-locally path connected spaces, called locally wep-connected spaces, which are not included in the traditional classification of covering spaces.

This paper is structured as follows. In Section 2, we recall the basic theory of free topological groups and include a general comparison of the two notions of free topological groups (those in the sense of Graev and those in the sense of Markov). Using Theorem 1.1, we obtain a structure theorem for open subgroups of free Markov topological groups (Theorem 2.5): *An open subgroup of a free Markov topological group is a free Markov topological group if and only if it is disconnected.*

In Section 3, we extend the usual notion of an algebraic graph by allowing edge spaces to have non-discrete topologies; the resulting objects are called **Top**-*graphs.* We then present some universal constructions of topologically enriched categories and groupoids to be used in the computations of the next section. In Section 4, we show that the fundamental **Top**-groupoid (respectively, topological fundamental group) of a **Top**-graph is a free **Top**-groupoid (respectively, free Graev topological group). Finally, in Section 5, semicovering theory is applied to **Top**-graphs. In analogy to the fact that a covering of a graph is a graph, we find that a semicovering of a **Top**-graph is a **Top**-graph. The paper concludes with a proof of Theorem 1.1.

2. Free topological groups

Definition 2.1. Let X be a topological space. The *free Markov topological group* on X is the unique (up to isomorphism) topological group $F_M(X)$ equipped with a map $\sigma : X \to F_M(X)$, universal in the sense that every map $f : X \to G$ to a topological group G induces a unique, continuous homomorphism $\hat{f} : F_M(X) \to G$ such that $\hat{f}\sigma = f$.

The existence of free Markov topological groups is guaranteed by the General Adjoint Theorem [20]. In particular, if **Top** is the category of topological spaces and **TopGrp** is the category of topological groups, then F_M : **Top** $\to$ **TopGrp** is left adjoint to the forgetful functor **TopGrp** $\to$ **Top**. Algebraically, $F_M(X)$ is the free group on the underlying set of X and $\sigma : X \to F_M(X)$ is the canonical injection of generators.

Definition 2.2. Let X be a space with a basepoint $* \in X$. The *free Graev topological group* on $(X, *)$ is the unique (up to isomorphism) topological group $F_G(X, *)$ equipped with a map $\sigma_* : X \to F_G(X, *)$ such that $\sigma_*(*)$ is the identity element of $F_G(X, *)$ and universal in the sense that every map $f : X \to G$ to a topological group G which takes $*$ to the identity element of G induces a unique, continuous homomorphism $\widetilde{f} : F_G(X, *) \to G$ such that $\widetilde{f}\sigma_* = f$.

Similarly to the unbased case, if $\mathbf{Top}_*$ is the category of based topological spaces, then $F_G : \mathbf{Top}_* \to$ **TopGrp** is left adjoint to the forgetful functor **TopGrp** $\to \mathbf{Top}_*$. Here, the basepoint of a topological group is the identity element. Algebraically, $F_G(X, *)$ is the free group on the set $X \setminus \{*\}$, however, it is not necessarily isomorphic to $F_M(X \setminus \{*\})$ as a topological group. On the other hand, $F_M(X)$ is isomorphic to the free Graev topological group $F_G(X_+, *)$ where $X_+ = X \sqcup \{*\}$ has an isolated basepoint, and $F_G(X, *)$ is isomorphic to the quotient topological group $F_M(X)/N$ where N is the conjugate closure of $\{*\}$.

Graev shows in [11, Theorem 2] that the isomorphism class of $F_G(X, *)$ as a topological group does not depend on the choice of basepoint, i.e. given any other point $*' \in X$ there is an isomorphism $F_G(X, *) \to F_G(X, *')$ of topological groups. The following lemma, also due to Graev [11], identifies when $F_G(X, *)$ is isomorphic to the free Markov topological group $F_M(Y)$ on some space Y. For the sake of completion, we prove the lemma without any assumptions on the spaces or groups in question.

Lemma 2.3. *If X is the disjoint union $X = A_1 \sqcup A_2$ of non-empty open sets $A_i \subset X$ and $e_i \in A_i$, then $F_G(X, e_1)$ is isomorphic to the free Markov topological group $F_M(A_1 \vee A_2)$ on the wedge sum $A_1 \vee A_2 = X/\{e_1, e_2\}$.*

Proof. Let $q : X \to A_1 \vee A_2$ be the quotient map making the identification $q(e_1) = z = q(e_2)$. Define a map $f : X \to F_M(X)$ by

$$f(a)=\begin{cases} ae_2^{-1} & \text{if } a\in A_1,\\ a & \text{if } a\in A_2,\end{cases}$$

where ae_2^{-1} is the product in $F_M(X)$. Let $F_M(q):F_M(X)\to F_M(A_1\vee A_2)$ be the continuous homomorphism induced by q. Since $F_M(q)(e_1)=F_M(q)(e_2)$, the composition $\psi=F_M(q)f:X\to F_M(A_1\vee A_2)$ takes e_1 to the identity of $F_M(A_1\vee A_2)$ and induces a continuous homomorphism $\widetilde{\psi}:F_G(X,e_1)\to F_M(A_1\vee A_2)$. Note that $\widetilde{\psi}(e_2)=z$.

Now consider the map $g:X\to F_G(X,e_1)$ given by

$$g(a)=\begin{cases} ae_2 & \text{if } a\in A_1,\\ a & \text{if } a\in A_2,\end{cases}$$

where ae_2 is the product in $F_G(X,e_1)$. Since e_1 is the identity in $F_G(X,e_1)$, we have $g(e_1)=e_1e_2=e_2=g(e_2)$. Therefore, we obtain a continuous map $\phi:A_1\vee A_2\to F_G(X,e_1)$ on the quotient such that $\phi(z)=e_2$ and which induces a continuous homomorphism $\hat{\phi}:F_M(A_1\vee A_2)\to F_G(X,e_1)$.

A direct check shows that $\hat{\phi}\widetilde{\psi}$ is the identity homomorphism of $F_G(X,e_1)$ and $\widetilde{\psi}\hat{\phi}$ is the identity of $F_M(A_1\vee A_2)$. In particular, if $a\in A_1\setminus\{e_1\}$, then $\hat{\phi}\widetilde{\psi}(a)=\hat{\phi}(ae_2^{-1})=\hat{\phi}(a)\hat{\phi}(e_2)^{-1}=(ae_2)e_2^{-1}=a$ and $\widetilde{\psi}\hat{\phi}(a)=\widetilde{\psi}(ae_2)=\widetilde{\psi}(a)\widetilde{\psi}(e_2)=ae_2^{-1}e_2=a$. The other cases are straightforward and left to the reader. ∎

Theorem 2.4. *For any space X, the following are equivalent:*

(1) *X is connected,*
(2) *$F_G(X,*)$ is connected,*
(3) *$F_G(X,*)$ is not isomorphic to a free Markov topological group.*

Proof. (1)⇒(2) Suppose X is connected and let C be the connected component of the identity in $F_G(X,*)$. Since $\sigma_*:X\to F_G(X,*)$ is continuous, the generating set $\sigma_*(X)$ is a connected subspace of $F_G(X,*)$ containing $*$, and is therefore contained in C. The connected component of the identity element in a general topological group is a subgroup [1, 1.4.26]. Therefore $C=F_G(X,*)$.

(2)⇒(3) Every free Markov topological group is disconnected since the canonical map $X\to *$ collapsing X to a point induces a continuous homomorphism $F_M(X)\to F_M(*)=\mathbb{Z}$ onto the discrete group of integers. Therefore, if $F_G(X,*)$ is connected, $F_G(X,*)$ cannot be isomorphic to a free Markov topological group.

(3)⇒(1) This follows directly from Lemma 2.3. ∎

Combining Theorems 1.1 and 2.4 and the fact that every free Markov topological group is a free Graev topological group, we obtain a structure theorem for open subgroups of free Markov topological groups. This result generalizes that in [8] for free Markov topological groups on Hausdorff k_ω-spaces.

THEOREM 2.5. *An open subgroup of a free Markov topological group is a free Markov topological group if and only if it is disconnected.*

3. Topologically enriched graphs and categories. The rest of this paper is devoted to a proof of Theorem 1.1.

3.1. Top-graphs. A **Top**-*graph* Γ consists of a discrete space of vertices Γ_0, an edge space Γ, and continuous structure maps $\partial_0, \partial_1 : \Gamma \to \Gamma_0$. For convenience, we often let Γ denote the **Top**-graph itself. The set of composable edges in Γ is the pullback $\Gamma \times_{\Gamma_0} \Gamma = \{(e, e') \mid \partial_1(e) = \partial_0(e')\}$.

For each pair of vertices $x, y \in \Gamma_0$, let $\Gamma_x = \partial_0^{-1}(x)$, $\Gamma^y = \partial_1^{-1}(y)$, and $\Gamma(x, y) = \Gamma_x \cap \Gamma^y$. Since we require the vertex space of a **Top**-graph to be discrete, the edge space decomposes as the topological sum $\Gamma = \coprod_{(x,y)\in\Gamma_0\times\Gamma_0} \Gamma(x, y)$ over ordered pairs of vertices.

Since it is possible that both $\Gamma(x, y)$ and $\Gamma(y, x)$ are non-empty, we are motivated to make the following construction. Let $\Gamma(x, y)^{-1}$ denote a homeomorphic copy of $\Gamma(x, y)$ for each pair $(x, y) \in \Gamma_0 \times \Gamma_0$. Here $e \in \Gamma(x, y)$ corresponds to $e^{-1} \in \Gamma(x, y)^{-1}$. Define a new **Top**-graph $\Gamma^{\pm}$ to have vertex space Γ_0 and $\Gamma^{\pm}(x, y) = \Gamma(x, y) \sqcup \Gamma(y, x)^{-1}$. In particular, note that $\Gamma^{\pm}(x, x) = \Gamma(x, x) \sqcup \Gamma(x, x)^{-1}$.

A morphism $f : \Gamma \to \Gamma'$ of **Top**-graphs consists of a pair of continuous functions $(f_0, f) : (\Gamma_0, \Gamma) \to (\Gamma_0', \Gamma')$ such that $\partial_i' f = f_0 \partial_i$, $i = 1, 2$. Such a morphism is said to be *quotient* if f_0 and f are quotient maps of spaces (note f_0 only needs to be surjective to be quotient). There is also an obvious notion of sub-**Top**-graph $S \subseteq \Gamma$. We say such a sub-**Top**-graph is *wide* if $S_0 = \Gamma_0$. The category of **Top**-graphs is denoted **TopGraph**.

DEFINITION 3.1. The *geometric realization* of a **Top**-graph Γ is the topological space

$$|\Gamma| = \Gamma_0 \sqcup (\Gamma \times [0, 1])/\sim \quad \text{where} \quad \partial_i(\alpha) \sim (\alpha, i) \text{ for } i = 0, 1.$$

A **Top**-graph Γ is *connected* if $|\Gamma|$ is path connected, or equivalently, if for each $x, y \in \Gamma_0$, there is a sequence of vertices $x = a_1, a_2, \dots, a_n = y$ such that $\Gamma^{\pm}(a_j, a_{j+1}) \neq \emptyset$ for $j = 1, \dots, n - 1$.

In many instances, we use the term "**Top**-graph" to refer to the geometric realization $|\Gamma|$, and say a space X is a "**Top**-graph" if $X \cong |\Gamma|$ for some **Top**-graph Γ. We typically assume that **Top**-graphs are connected.

REMARK 3.2. For any $0 < r < 1$, the image of $\Gamma_x \times [0, r]$ in the quotient $|\Gamma|$ is homeomorphic to the cone $C\Gamma_x$ on Γ_x. Similarly, if $Z = \Gamma(x, y) \sqcup \Gamma(y, x)$, then the image of $Z \times [0, 1]$ in $|\Gamma|$ is the unreduced suspension SZ.

Finally, note that

$$|\Gamma| \setminus \Gamma_0 = \coprod_{(x,y)} (\Gamma(x,y) \times (0,1)).$$

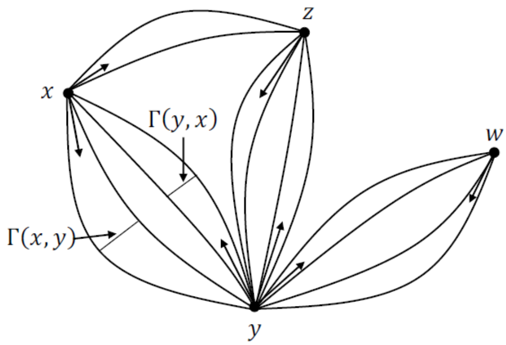

Fig. 1. The realization of a **Top**-graph Γ with four vertices. Here $\Gamma(z,x)$, $\Gamma(w,z)$, and $\Gamma(z,w)$ are all empty.

REMARK 3.3. It is unfortunate that $|\Gamma|$ need not be first countable at its vertices, however, it is possible to change the topology on $|\Gamma|$ without changing its homotopy type and so that each vertex has a countable neighborhood base. The *vertex neighborhood* of $x \in \Gamma_0$ of radius $r \in (0,1)$ is the image of

$$(\Gamma_x \times [0,r)) \cup (\Gamma^x \times (1-r,1])$$

in $|\Gamma|$ and is denoted $B(x,r)$. An *edge neighborhood* of a point $(e,t) \in \Gamma(x,y) \times (0,1)$ is the homeomorphic image of a set $U \times (a,b)$ where U is an open neighborhood of e in $\Gamma(x,y)$ and $0 < a < t < b < 1$. The basis consisting of vertex and edge neighborhoods is closed under finite intersection and generates a topology which can be strictly coarser than the quotient topology (but only at vertices). Note that each vertex neighborhood $B(x,r)$ is contractible onto x and the set $\{B(x,1/n) \mid n \geq 1\}$ is a countable neighborhood base at x.

From now on, we assume $|\Gamma|$ has the coarser topology generated by vertex and edge neighborhoods.

EXAMPLE 3.4. If Γ is a **Top**-graph with a single vertex, then $|\Gamma|$ is the *generalized wedge of circles* $\Sigma(\Gamma_+)$ (where Σ denotes reduced suspension) studied in detail in [2]. When $\Gamma_0 = \{x_0, x_1\}$ and the structure maps are the two constant maps $\partial_i : \Gamma \to \Gamma_0$, $\partial_i(\alpha) = x_i$ (equivalently, $\Gamma = \Gamma(x_0,x_1)$), then $|\Gamma|$ is the unreduced suspension $S\Gamma$ of the edge space. Thus, unlike a discrete graph, a **Top**-graph may be simply connected but not contractible, e.g. if $\Gamma_0 = \{x_0, x_1\}$ and $\Gamma = S^1$, then $|\Gamma| = S\Gamma = S^2$ is the 2-sphere.

Remark 3.5. Later on we will make use of the following construction on **Top**-graphs: The *path component space* of a topological space X is the quotient space $\pi_0(X)$ where each path component is identified to a point. If $\mathit{\Gamma}$ is a **Top**-graph, then $\pi_0(\mathit{\Gamma})$ is the **Top**-graph with vertex space $\mathit{\Gamma}_0$ and $\pi_0(\mathit{\Gamma})(x,y) = \pi_0(\mathit{\Gamma}(x,y))$. The canonical quotient morphism $q : \mathit{\Gamma} \to \pi_0(\mathit{\Gamma})$ of **Top**-graphs is the identity on vertices and takes an edge e to its path component $[e]$.

Remark 3.6. Another useful construction is a section to the path component functor $\pi_0 : \mathbf{TopGraph} \to \mathbf{TopGraph}$. Given any space X, there is a (paracompact Hausdorff) space $h(X)$ and a natural homeomorphism $\pi_0(h(X)) \cong X$ ([12]). Thus for any **Top**-graph $\mathit{\Gamma}$, we define $h(\mathit{\Gamma})$ to have object space $\mathit{\Gamma}_0$ and $h(\mathit{\Gamma})(x,y) = h(\mathit{\Gamma}(x,y))$, so that $\pi_0(h(\mathit{\Gamma})) \cong \mathit{\Gamma}$.

3.2. Top-categories and qTop-categories. Our use of enriched categories aligns with that in [16] but is restricted to small categories. If a **Top**-graph $\mathcal{C}$ comes equipped with continuous composition map $\mathcal{C} \times_{\mathcal{C}_0} \mathcal{C} \to \mathcal{C}$ making $\mathcal{C}$ a category in the usual way, then $\mathcal{C}$ is a **Top**-*category* (or a category enriched over **Top**). Since $\mathrm{Ob}(\mathcal{C}) = \mathcal{C}_0$ is discrete, $\mathcal{C} \times_{\mathcal{C}_0} \mathcal{C}$ decomposes as a topological sum of products $\mathcal{C}(x,y) \times \mathcal{C}(y,z)$. Thus to specify a **Top**-category one only need specify the hom-spaces $\mathcal{C}(x,y)$ and continuous composition maps $\mathcal{C}(x,y) \times \mathcal{C}(y,z) \to \mathcal{C}(x,z)$. If the composition maps are only continuous in each variable, then $\mathcal{C}$ is an **sTop**-category (the "s" is for "semitopological" as in [1]). A **Top**-functor $F : \mathcal{C} \to \mathcal{D}$ of **Top**-categories is a functor such that each function $F_{x,y} : \mathcal{C}(x,y) \to \mathcal{D}(F(x),F(y))$ is continuous. The category of **Top**-categories and **Top**-functors is denoted **TopCat**.

An *involution* on a small category $\mathcal{C}$ is a function $\mathcal{C} \to \mathcal{C}$ defined by functions $\mathcal{C}(x,y) \to \mathcal{C}(y,x)$, $f \mapsto f^*$, such that $(f^*)^* = f$, $(fg)^* = g^* f^*$, and $(\mathrm{id}_x)^* = \mathrm{id}_x$. A **Top**-category (respectively, an **sTop**-category) equipped with a continuous involution is an **Top**-*category with continuous involution* (respectively, a **qTop**-*category*). If $\mathcal{G}$ is a **Top**-category (respectively, **sTop**-category) whose underlying category is a groupoid and the involution determined by the inversion functions $\mathcal{G}(x,y) \to \mathcal{G}(y,x)$ is continuous, then $\mathcal{G}$ is a **Top**-*groupoid* (respectively, **qTop**-*groupoid*).

A functor $F : \mathcal{C} \to \mathcal{D}$ of categories with involution *preserves involution* if $F(f^*) = F(f)^*$. In particular, a **qTop**-functor $F : \mathcal{C} \to \mathcal{D}$ of **qTop**-categories is an involution preserving functor which is continuous on hom-spaces.

The notion of **qTop**-groupoid is particularly relevant and is studied in [3, Section 4]. The following lemma is a useful fact asserting that the category **TopGrpd** of **Top**-groupoids is a full reflective subcategory of the category **qTopGrpd** of **qTop**-groupoids.

Lemma 3.7 ([3, Lemma 4.5]). *The forgetful functor* **TopGrpd** $\to$ **qTopGrpd** *has a left adjoint* τ : **qTopGrpd** $\to$ **TopGrpd** *which is the identity on the underlying groupoids and functors.*

Remark 3.8. The *free* **Top**-*category* generated by a **Top**-graph Γ is the **Top**-category $\mathscr{C}(\Gamma)$ with object space Γ_0 and in which morphisms are finite sequences $e_1 \dots e_n$ of composable edges $e_i \in \Gamma$. In particular, the hom-space $\mathscr{C}(\Gamma)(x,y)$ is topologized as the topological sum $\coprod \Gamma(x,a_1) \times \Gamma(a_1,a_2) \times \dots \times \Gamma(a_n,y)$ where the sum ranges over all finite sequences $a_1,\dots,a_n$ in Γ_0. Composition is given by concatenation of sequences and, in order to obtain a true category, we add an isolated identity morphism $\{\mathrm{id}_x\}$ to each space $\mathscr{C}(\Gamma)(x,x)$. Note this construction yields a functor $\mathscr{C}$: **TopGraph** $\to$ **TopCat** left adjoint to the forgetful functor **TopCat** $\to$ **TopGraph**.

The construction of $\mathscr{C}(\Gamma)$ is easily modified to include a continuous involution. Recall the **Top**-graph $\Gamma^{\pm}$ described in the previous section. The free **Top**-category with (continuous) involution on Γ is the category $\mathscr{C}^{\pm}(\Gamma) = \mathscr{C}(\Gamma^{\pm})$ where non-identity morphisms are sequences $e_1^{\delta_1}\dots e_n^{\delta_n}$, $\delta_i \in \{\pm 1\}$, of composable edges, and which comes equipped with the continuous involution $e_1^{\delta_1}\dots e_n^{\delta_n} \mapsto e_n^{-\delta_n}\dots e_1^{-\delta_1}$.

Remark 3.9. The construction of the path component **Top**-graph $\pi_0(\Gamma)$ also applies to **Top**-categories. Recall that for spaces X, Y, there is a canonical, continuous bijection $\psi : \pi_0(X\times Y) \to \pi_0(X)\times\pi_0(Y)$ which is not necessarily a homeomorphism [2]. Consequently, if Γ is a **Top**-category (with continuous involution), then $\pi_0(\Gamma)$ naturally inherits the structure of an **sTop**-category (**qTop**-category) but is not always a **Top**-category (with continuous involution). While it is possible to avoid this difficulty by restricting to a Cartesian closed category of spaces, we remain in the usual topological category in order to prove Theorem 1.1 in full generality.

The observation on products in the previous remark immediately extends to the following lemma.

Lemma 3.10. *Given a* **Top**-*graph* Γ, *there is a canonical morphism* $\psi : \pi_0(\mathscr{C}(\Gamma)) \to \mathscr{C}(\pi_0(\Gamma))$ *of* **sTop**-*categories given by* $\psi([e_1\dots e_n]) = [e_1]\dots[e_n]$, *which is a natural isomorphism of the underlying categories.*

As an important case of Lemma 3.10, note that for the **Top**-graph $\Gamma^{\pm}$ we obtain the natural **qTop**-functor (and isomorphism of underlying categories)

$$\psi : \pi_0(\mathscr{C}^{\pm}(\Gamma)) = \pi_0(\mathscr{C}(\Gamma^{\pm})) \to \mathscr{C}(\pi_0(\Gamma^{\pm})) = \mathscr{C}(\pi_0(\Gamma)^{\pm}) = \mathscr{C}^{\pm}(\pi_0(\Gamma))$$

given by $\psi([e_1^{\delta_1}\dots e_n^{\delta_n}]) = [e_1]^{\delta_1}\dots[e_n]^{\delta_n}$.

Remark 3.11. Given a **Top**-graph Γ, the free **Top**-groupoid generated by Γ is denoted $\mathscr{F}(\Gamma)$ and is characterized by the following universal property: If $\mathcal{G}$ is a **Top**-groupoid, then any **Top**-graph morphism

$f : \Gamma \to \mathcal{G}$ extends uniquely to a **Top**-functor $\hat{f} : \mathscr{F}(\Gamma) \to \mathcal{G}$. In other words, $\mathscr{F}$: **TopGraph** $\to$ **TopGrpd** is left adjoint to the forgetful functor **TopGrpd** $\to$ **TopGraph**.

The underlying groupoid of $\mathscr{F}(\Gamma)$ is simply the free groupoid generated by the underlying algebraic graph of Γ, i.e. $\mathrm{Ob}(\mathscr{F}(\Gamma)) = \Gamma_0$ and a morphism is a reduced word $e_1^{\delta_1} \dots e_n^{\delta_n} \in \mathscr{C}^{\pm}(\Gamma)$ of composable edges. (See [6, 14] for more on free groupoids.) The topological structure of $\mathscr{F}(\Gamma)$ is characterized as follows: Let $\mathscr{F}_R(\Gamma)$ be the free groupoid on the underlying algebraic graph of Γ which is the quotient of $\mathscr{C}^{\pm}(\Gamma)$ with respect to the word reduction functor $R : \mathscr{C}^{\pm}(\Gamma) \to \mathscr{F}_R(\Gamma)$. It is straightforward to check that $\mathscr{F}_R(\Gamma)$ is a **qTop**-groupoid. The free **Top**-groupoid is the τ-reflection (recall Lemma 3.7)

$$\mathscr{F}(\Gamma) = \tau(\mathscr{F}_R(\Gamma)).$$

One can verify the universal property as follows: If $\mathcal{G}$ is a **Top**-groupoid and $f : \Gamma \to \mathcal{G}$ is a morphism of **Top**-graphs, it suffices to check that the unique functor $F : \mathscr{F}(\Gamma) \to \mathcal{G}$ extending f to the algebraic free groupoid is a **Top**-functor. The induced **Top**-functor $g : \mathscr{C}^{\pm}(\Gamma) \to \mathcal{G}$ (since $\mathcal{G}$ is a **Top**-category) factors as $g = F \circ R$, where R is the word reduction function. Since $R : \mathscr{C}^{\pm}(\Gamma) \to \mathscr{F}_R(\Gamma)$ is quotient, $F : \mathscr{F}_R(\Gamma) \to \mathcal{G}$ is a **qTop**-functor. It follows that the adjoint $F : \tau(\mathscr{F}_R(\Gamma)) \to \mathcal{G}$ is a morphism of **Top**-groupoids.

In the case that Γ has a single vertex [2], $\mathscr{C}^{\pm}(\Gamma)$ is the free topological monoid with continuous involution on Γ and $\mathscr{F}(\Gamma)$ is the free topological group $F_M(\Gamma) = F_G(\Gamma_+, *)$.

3.3. Vertex groups and free topological groups. We now show each vertex group of a free **Top**-groupoid is a free Graev topological group.

Definition 3.12. If Γ is a **Top**-graph, we say a sequence $e_1^{\delta_1} \dots e_n^{\delta_n}$ in $\Gamma^{\pm}(a_0, a_1) \times \Gamma^{\pm}(a_1, a_2) \times \cdots \times \Gamma^{\pm}(a_{n-1}, a_n) \subset \mathcal{C}^{\pm}(\Gamma)(a_0, a_n)$ is a *simple path* if the vertices $a_0, a_1, \dots, a_n$ are all distinct. A **Top**-graph T is a *tree* if for any pair of distinct vertices $x, y \in \Gamma_0$ there is a unique simple path from x to y.

If Γ is a **Top**-graph, a tree $T \subseteq \Gamma$ is *maximal* in Γ if $T_0 = \Gamma_0$. A *tree groupoid* is a groupoid $\mathcal{G}$ such that each set $\mathcal{G}(x, y)$ has exactly one element.

The above definition of "tree" is the classical definition, but we note that a tree T is equivalently a discrete **Top**-graph such that $|T|$ is simply connected. Moreover, note that if T is a tree, then $\mathscr{F}(T)$ is a discrete tree groupoid.

The standard argument that every connected graph contains a maximal tree is the same for **Top**-graphs.

Lemma 3.13. *Every connected* **Top**-*graph contains a maximal tree.*

Fix a **Top**-graph Γ, a maximal tree $T \subseteq \Gamma$, and a vertex $v \in \Gamma_0$. Let $\mathscr{F}(\Gamma)(v) = \mathscr{F}(\Gamma)(v,v)$ be the vertex topological group at v. Recall that if Γ has a single vertex v, then $\mathscr{F}(\Gamma) = \mathscr{F}(\Gamma)(v) \cong F_M(\Gamma) \cong F_G(\Gamma_+, *)$. Therefore, we restrict to the case when Γ has more than one vertex. In this case, T has non-empty edge space.

For vertex $x \in \Gamma_0$, let $\gamma_{v,x}$ be the unique element of $\mathscr{F}(T)(v,x)$. Define a retraction $r_T : \mathscr{F}(\Gamma) \to \mathscr{F}(\Gamma)(v)$ of groupoids so that if $\alpha \in \mathscr{F}(\Gamma)(x,y)$, then $r_T(\alpha) = \gamma_{v,x}\alpha\gamma_{y,v}$. By definition, if $i : \mathscr{F}(\Gamma)(v) \to \mathscr{F}(\Gamma)$ is the inclusion of the vertex group, then $r_T i$ is the identity of $\mathscr{F}(\Gamma)(v)$. Moreover, since composition in $\mathscr{F}(\Gamma)$ is continuous, r_T is a **Top**-functor.

LEMMA 3.14. *$r_T : \mathscr{F}(\Gamma) \to \mathscr{F}(\Gamma)(v)$ is a retraction of* **Top**-*groupoids.*

Let $\sigma : \Gamma \to \mathscr{F}(\Gamma)$ be the canonical **Top**-graph morphism. It is known that the underlying group of $\mathscr{F}(\Gamma)(v)$ is freely generated by the set $r_T\sigma(\Gamma \setminus T)$ (see, for instance, [6, 8.2.3]).

Note that if $\gamma \in \Gamma$, then $r_T\sigma(\gamma)$ is the identity element of the group $\mathscr{F}(\Gamma)(v)$ if and only if $\gamma \in T$. Let Γ/T be the quotient of the edge space Γ where the subspace T is collapsed to a point. Let $q : \Gamma \to \Gamma/T$ be the resulting quotient map, and choose the basepoint $* \in \Gamma/T$ to be the image of T under q. The function $r_T\sigma : \Gamma \to \mathscr{F}(\Gamma)(v)$ induces a continuous injection $s : \Gamma/T \to \mathscr{F}(\Gamma)(v)$ such that $sq = r_T\sigma$ and where $s(*)$ is the identity element of $\mathscr{F}(\Gamma)$. Since $r_T\sigma(\Gamma \setminus T)$ freely generates $\mathscr{F}(\Gamma)(v)$, the continuous group homomorphism $\tilde{s} : F_G(\Gamma/T, *) \to \mathscr{F}(\Gamma)(v)$ induced by s is an isomorphism of groups.

THEOREM 3.15. *If Γ has more than one vertex and $T \subseteq \Gamma$ is a maximal tree, then the vertex group $\mathscr{F}(\Gamma)(v)$ is isomorphic to the free Graev topological group $F_G(\Gamma/T, *)$.*

Proof. It suffices to show that the inverse of $\tilde{s} : F_G(\Gamma/T, *) \to \mathscr{F}(\Gamma)$ is continuous. Let $\sigma_* : \Gamma/T \to F_G(\Gamma/T, *)$ be the inclusion of generators. The composition $g = \sigma_* q : \Gamma \to F_G(\Gamma/T, *)$ may be viewed as a morphism of **Top**-graphs taking all vertices of Γ to the unique vertex of $F_G(\Gamma/T, *)$. Since $F_G(\Gamma/T, *)$ is a **Top**-groupoid, there is a unique **Top**-functor $\hat{g} : \mathscr{F}(\Gamma) \to F_G(\Gamma/T, *)$ such that $\hat{g}\sigma = g$. If $i : \mathscr{F}(\Gamma)(v) \to \mathscr{F}(\Gamma)$ is the inclusion of the vertex group, the composition $\hat{g}i : \mathscr{F}(\Gamma)(v) \to F_G(\Gamma/T, *)$ is continuous. We check that $\hat{g}i$ is the inverse of $\tilde{s}$.

Recall that $sq = r_T\sigma$. Therefore

$$\tilde{s}\hat{g}\sigma = \tilde{s}g = \tilde{s}\sigma_* q = sq = r_T\sigma.$$

Uniqueness of extensions then gives $\tilde{s}\hat{g} = r_T$.

$$\begin{tikzcd}
 & \Gamma \arrow[dl, "\sigma"'] \arrow[d, "g"] \arrow[dr, "q"] & \\
\mathscr{F}(\Gamma) \arrow[r, "\hat{g}"] \arrow[dr, shift left, "r_T"] & F_G(\Gamma/T, *) \arrow[d, "\tilde{s}"] & \Gamma/T \arrow[l, "\sigma_*"'] \arrow[dl, "s"] \\
 & \mathscr{F}(\Gamma)(v) \arrow[ul, shift left, "i"] &
\end{tikzcd}$$

It is now clear that $\tilde{s}\hat{g}i = r_T i = \mathrm{id}$. Finally, since $\tilde{s}\hat{g}i\tilde{s} = r_T i \tilde{s} = \tilde{s}$ and $\tilde{s}$ is injective, we have $\hat{g}i\tilde{s} = \mathrm{id}$. ∎

4. The fundamental Top-groupoid of a Top-graph

4.1. Path spaces and the fundamental Top-groupoid. For a given space X, let $\mathcal{P}X$ be the space of paths $[0,1] \to X$ with the compact-open topology generated by subbasis sets $\langle C, W\rangle = \{\alpha \mid \alpha(C) \subseteq W\}$ where $C \subseteq [0,1]$ is compact and $W \subseteq X$ is open. For a closed subinterval $K \subseteq [0,1]$, let $L_K\colon [0,1] \to K$ be the unique, increasing, linear homeomorphism. If $\alpha : [0,1] \to X$ is a path, let $\alpha_K = \alpha|_K \circ L_K$ be the restricted path of α to K. If $K = \{t\} \subseteq [0,1]$, take α_K to be the constant path $c_{\alpha(t)}$ at $\alpha(t)$. The concatenation $\alpha = \alpha_1 \cdot \dots \cdot \alpha_n$ of paths α_j satisfying $\alpha_{j+1}(0) = \alpha_j(1)$ is given by letting $\alpha_{[(j-1)/n, j/n]} = \alpha_j$. Let $\overline{\alpha}(t) = \alpha(1-t)$ denote the reverse path of a given path $\alpha \in \mathcal{P}X$.

Consider a basic open neighborhood $\mathcal{U} = \bigcap_{j=1}^{n} \langle C_j, U_j\rangle$ of a path α, and any closed interval $K \subseteq [0,1]$. Then $\mathcal{U}_K = \bigcap_{K \cap C_j \neq \emptyset} \langle L_K^{-1}(K \cap C_j), U_j\rangle$ is an open neighborhood of α_K. If $K = \{t\}$ is a singleton, let $\mathcal{U}_K = \langle [0,1], \bigcap_{t \in C_j} U_j\rangle$. On the other hand, if $\beta_K = \alpha$, then $\mathcal{U}^K = \bigcap_{j=1}^{n} \langle L_K(C_j), U_j\rangle$ is an open neighborhood of β. If $K = \{t\}$ so that $\alpha = c_{\alpha(t)}$, let $\mathcal{U}^K = \bigcap_{j=1}^{n} \langle \{t\}, U_j\rangle$.

Lemma 4.1. *Let $\mathcal{U} = \bigcap_{j=1}^{n} \langle C_j, U_j\rangle$ be an open neighborhood in $\mathcal{P}X$ such that $\bigcup_{j=1}^{n} C_j = [0,1]$. Then*

(1) *For any closed interval $K \subseteq [0,1]$, we have $(\mathcal{U}^K)_K = \mathcal{U} \subseteq (\mathcal{U}_K)^K$.*
(2) *If $0 = t_0 \leq t_1 \leq \dots \leq t_n = 1$, then $\mathcal{U} = \bigcap_{j=1}^{n} (\mathcal{U}_{[t_{j-1}, t_j]})^{[t_{j-1}, t_j]}$.*

In the case that X is a **Top**-graph, i.e. $X = |\Gamma|$ for some **Top**-graph Γ, recall that vertex neighborhoods $B(x, r)$ and edge neighborhoods $U \times (a, b)$ (as in Remark 3.3) form a basis $\mathscr{B}_\Gamma$ for the topology of $|\Gamma|$ which is closed under finite intersection. Thus sets of the form $\bigcap_{j=1}^{n} \langle [(j-1)/n, j/n], U_j\rangle$, where $U_j \in \mathscr{B}_\Gamma$, give a basis for the topology of $\mathcal{P}|\Gamma|$ which is convenient for our purposes. A basic open set of this form is said to be *standard*.

We now recall the topologically enriched version of the usual fundamental groupoid used in [3]. For a topological space X, let $\mathscr{P}X$ denote the

Top-graph whose vertex space is X with the discrete topology and such that $\mathscr{P}X(x,y)$ is the subspace of $\mathcal{P}X$ consisting of all paths from x to y. Though concatenation of paths gives a continuous operation $\mathscr{P}X(x,y) \times \mathscr{P}X(y,z) \to \mathscr{P}X(x,z)$, $\mathscr{P}X$ is not a **Top**-category because concatenation is not strictly associative. Note also that the compact-open topology of the path space $\mathcal{P}X$ is coarser than the topology of the edge space $\mathscr{P}X = \coprod_{(x,y)\in X\times X} \mathscr{P}X(x,y)$.

Definition 4.2 ([3]). The *fundamental* **qTop**-*groupoid* of a space X is the **qTop**-groupoid $\pi^{\mathrm{qtop}}X = \pi_0(\mathscr{P}X)$ with discrete object space X and where $\pi^{\mathrm{qtop}}X(x,y)$ is the quotient space $\pi_0(\mathscr{P}X(x,y))$ of path components of paths from x to y. The canonical quotient morphism is denoted $\pi : \mathscr{P}X \to \pi^{\mathrm{qtop}}X$. The *fundamental* **Top**-*groupoid* of X is the τ-reflection $\pi^{\tau}X = \tau(\pi^{\mathrm{qtop}}X)$.

Note that two paths $\alpha, \beta \in \mathscr{P}X(x,y)$ lie in the same path component if and only if they are homotopic relative to their endpoints. Thus the underlying groupoid of both $\pi^{\mathrm{qtop}}X$ and $\pi^{\tau}X$ is the usual fundamental groupoid πX.

In the case that $X = |\Gamma|$ is a **Top**-graph, we only wish to consider paths between vertices. Therefore we provide a separate definition of the fundamental **Top**-groupoid of a **Top**-graph. If Γ is a **Top**-graph, let $\mathscr{P}\Gamma \subset \mathscr{P}|\Gamma|$ be the **Top**-graph with object space Γ_0 and where $\mathscr{P}\Gamma(x,y) = \mathscr{P}|\Gamma|(x,y)$ is the space of paths from vertex x to vertex y.

Definition 4.3. The *fundamental* **qTop**-*groupoid* of a **Top**-graph Γ is the **qTop**-groupoid $\pi^{\mathrm{qtop}}(\Gamma,\Gamma_0) = \pi_0(\mathscr{P}\Gamma)$ with object space Γ_0 and where $\pi^{\mathrm{qtop}}(\Gamma,\Gamma_0)(x,y)$ is the quotient space $\pi_0(\mathscr{P}\Gamma(x,y))$ of homotopy classes of paths from x to y. The canonical quotient morphism is also denoted by $\pi : \mathscr{P}\Gamma \to \pi^{\mathrm{qtop}}(\Gamma,\Gamma_0)$. The *fundamental* **Top**-*groupoid* of Γ is the τ-reflection $\pi^{\tau}(\Gamma,\Gamma_0) = \tau(\pi^{\mathrm{qtop}}(\Gamma,\Gamma_0))$.

The underlying groupoid of both $\pi^{\mathrm{qtop}}(\Gamma,\Gamma_0)$ and $\pi^{\tau}(\Gamma,\Gamma_0)$ is the familiar fundamental groupoid $\pi(|\Gamma|,\Gamma_0)$ with set of basepoints Γ_0 ([6]).

4.2. $\pi^{\tau}(\Gamma,\Gamma_0)$ is a free Top-groupoid. Throughout this section, let Γ be a **Top**-graph.

Definition 4.4. A path $\alpha : [0,1] \to |\Gamma|$ is an *edge path* if $\alpha^{-1}(\Gamma_0) = \{0,1\}$. An edge path α is *trivial* if it is a null-homotopic loop. Equivalently, an edge path is *non-trivial* if its endpoints are distinct or if it traverses a generalized wedge of circles $\Sigma(\Gamma(x,x)_+) \subset |\Gamma|$ at some vertex $x \in \Gamma_0$. Let $\mathscr{E}\Gamma$ denote the wide sub-**Top**-graph of $\mathscr{P}\Gamma$ consisting of non-trivial edge paths.

Note that for each edge $e \in \Gamma(x,y)$, there is a canonical non-trivial edge path $\alpha_e : [0,1] \to |\Gamma|$ where $\alpha_e(t)$ is the image of $(e,t) \in \Gamma(x,y) \times [0,1]$

in $|\Gamma|$. The following lemma is a straightforward application of the existence of Lebesgue numbers.

LEMMA 4.5. *If $\mathcal{V}$ is an open neighborhood of an edge path $\alpha \in \mathscr{E}\Gamma(x,y)$, then there is a standard neighborhood $\mathcal{A} = \bigcap_{j=1}^{n} \langle [(j-1)/n, j/n], U_j \rangle$, $n > 2$, of α in $\mathcal{P}|\Gamma|$ such that $\mathcal{A} \cap \mathscr{E}\Gamma(x,y) \subseteq \mathcal{V}$, U_1 and U_n are vertex neighborhoods, and $U_2, \ldots, U_{n-1}$ are edge neighborhoods.*

For the following lemma, observe that on the free **Top**-category $\mathscr{C}(\mathscr{E}\Gamma)$, there is a continuous involution $\mathscr{C}(\mathscr{E}\Gamma) \to \mathscr{C}(\mathscr{E}\Gamma)$ given by $\alpha_1 \ldots \alpha_n \mapsto \overline{\alpha_n} \ldots \overline{\alpha_1}$. Consequently, $\pi_0(\mathscr{C}(\mathscr{E}\Gamma))$ naturally inherits the structure of a **qTop**-category.

LEMMA 4.6. *For any* **Top**-*graph Γ, there is a canonical embedding $\Gamma^{\pm} \to \mathscr{E}\Gamma$ of* **Top**-*graphs which induces a natural isomorphism $\pi_0(\mathscr{C}^{\pm}(\Gamma)) \to \pi_0(\mathscr{C}(\mathscr{E}\Gamma))$ of* **qTop**-*categories.*

Proof. The embedding $j : \Gamma^{\pm} \to \mathscr{E}\Gamma$ is the identity on objects and defined on edges as $j(e) = \alpha_e$ for $e \in \Gamma(x,y)$ and $j(e^{-1}) = \overline{\alpha_e}$ for $e^{-1} \in \Gamma(y,x)^{-1}$. We claim the **qTop**-functor $\mathscr{J} : \pi_0(\mathscr{C}^{\pm}(\Gamma)) = \pi_0(\mathscr{C}(\Gamma^{\pm})) \to \pi_0(\mathscr{C}(\mathscr{E}\Gamma))$ induced by j is the desired isomorphism of **qTop**-categories.

To construct an inverse, first note that if $\alpha \in \mathscr{E}\Gamma(x,y)$ is any edge path, then $\alpha(1/2)$ is the image of a unique point $(d,s) \in (\Gamma(x,y) \sqcup \Gamma(y,x)) \times (0,1)$ in $|\Gamma|$. If $d \in \Gamma(x,y)$, then α is homotopic to α_d, and if $d \in \Gamma(y,x)$, then α is homotopic to $\overline{\alpha_d}$. Define a **Top**-graph morphism $k : \mathscr{E}\Gamma \to \Gamma^{\pm}$, which is the identity on objects, as follows:

$$k(\alpha) = \begin{cases} d & \text{if } d \in \Gamma(x,y), \\ d^{-1} & \text{if } d \in \Gamma(y,x). \end{cases}$$

In the case $x = y$ and $\alpha(1/2) = (d,s) \in \Gamma(x,x) \times (0,1)$, we set $k(\alpha) = d$ if α is homotopic to α_d, and $k(\alpha) = d^{-1}$ if α is homotopic to $\overline{\alpha_d}$. We now show that the **qTop**-functor $\mathscr{K} : \pi_0(\mathscr{C}(\mathscr{E}\Gamma)) \to \pi_0(\mathscr{C}^{\pm}(\Gamma))$ induced by k is the inverse of $\mathscr{J}$.

Consider the **Top**-graph morphisms $j' : \pi_0(\Gamma^{\pm}) \to \pi_0(\mathscr{E}\Gamma)$ and $k' : \pi_0(\mathscr{E}\Gamma) \to \pi_0(\Gamma^{\pm})$ induced by j and k respectively. It is straightforward to check that these are inverse isomorphisms of **Top**-graphs. Consequently, j' induces an isomorphism

$$j'' : \mathscr{C}^{\pm}(\pi_0(\Gamma)) = \mathscr{C}(\pi_0(\Gamma)^{\pm}) = \mathscr{C}(\pi_0(\Gamma^{\pm})) \to \mathscr{C}(\pi_0(\mathscr{E}\Gamma))$$

of **Top**-categories. Moreover, it is clear that j'' preserves involution. Note that the inverse $k'' : \mathscr{C}(\pi_0(\mathscr{E}\Gamma)) \to \mathscr{C}^{\pm}(\pi_0(\Gamma))$ is induced by k'. Recall the natural morphism ψ of **sTop**-categories from Lemma 3.10, which is an isomorphism of underlying categories. The naturality of ψ guarantees the commutativity of the following squares:

$$\begin{tikzcd}
\pi_0(\mathscr{C}(\mathscr{E}\Gamma)) \arrow[r, "\mathscr{K}"] \arrow[d, "\psi"'] & \pi_0(\mathscr{C}^{\pm}(\Gamma)) \arrow[r, "\mathscr{J}"] \arrow[d, "\psi"'] & \pi_0(\mathscr{C}(\mathscr{E}\Gamma)) \arrow[r, "\mathscr{K}"] \arrow[d, "\psi"'] & \pi_0(\mathscr{C}^{\pm}(\Gamma)) \arrow[d, "\psi"'] \\
\mathscr{C}(\pi_0(\mathscr{E}\Gamma)) \arrow[r, "k''"] & \mathscr{C}^{\pm}(\pi_0(\Gamma)) \arrow[r, "j''"] & \mathscr{C}(\pi_0(\mathscr{E}\Gamma)) \arrow[r, "k''"] & \mathscr{C}^{\pm}(\pi_0(\Gamma))
\end{tikzcd}$$

Each functor in the diagram preserves involution and thus each arrow is a **qTop**-functor. On the level of underlying categories with involution, j'' and k'' are inverse isomorphisms. It follows that $\mathscr{J}$ and $\mathscr{K}$ are inverse isomorphisms of the underlying categories with involution. Moreover, since $\mathscr{J}$ and $\mathscr{K}$ are **qTop**-functors, we conclude that $\mathscr{J}$ is an isomorphism of **qTop**-categories. ∎

Since concatenation $(\alpha,\beta)\mapsto\alpha\cdot\beta$ of paths is continuous, the inclusion $\mathscr{E}\Gamma\to\mathscr{P}\Gamma$ gives rise to a **Top**-graph morphism $\mathscr{C}(\mathscr{E}\Gamma)\to\mathscr{P}\Gamma$, $\alpha_1\ldots\alpha_n\mapsto\alpha_1\cdot\ldots\cdot\alpha_n$, on the free **Top**-category. Application of the path component space functor gives a **qTop**-functor $\phi:\pi_0(\mathscr{C}(\mathscr{E}\Gamma))\to\pi_0(\mathscr{P}\Gamma)=\pi^{\mathrm{qtop}}(\Gamma,\Gamma_0)$, $\phi([\alpha_1\ldots\alpha_n])=[\alpha_1\cdot\ldots\cdot\alpha_n]$, to the fundamental **qTop**-groupoid.

Lemma 4.7. *The* **qTop**-*functor* $\phi:\pi_0(\mathscr{C}(\mathscr{E}\Gamma))\to\pi^{\mathrm{qtop}}(\Gamma,\Gamma_0)$ *is quotient.*

Proof. Consider any path $\alpha\in\mathscr{P}\Gamma(x,y)$ in $|\Gamma|$ connecting vertices x and y. There exists a unique $n\ge 0$ and $a_1,b_1,\ldots,a_n,b_n$ satisfying $0\le a_1<b_1\le\cdots\le a_n<b_n\le 1$, and such that the restriction $\alpha_i=\alpha_{[a_i,b_i]}$ is a non-trivial edge path. If $n=0$, then no restriction of α is a non-trivial edge path and α must be a null-homotopic loop based at $x=y$.

Define a morphism $\mathscr{D}:\mathscr{P}\Gamma\to\mathscr{C}(\mathscr{E}\Gamma)$ of underlying algebraic graphs (which is the identity on vertices) using the above decomposition of α: If $n=0$, let $\mathscr{D}(\alpha)$ be the identity id_x of the vertex at which α is based. If $n>0$, let $\mathscr{D}(\alpha)$ be the word $\alpha_1\ldots\alpha_n$ in $\mathscr{C}(\mathscr{E}\Gamma)$ whose letters are the restricted non-trivial edge paths. The "decomposition" morphism $\mathscr{D}$ is a direct generalization of the decomposition function in [2, p. 793]; it is important to note that $\mathscr{D}$ is only a morphism of underlying algebraic graphs since it is not continuous on edge spaces.

With $\mathscr{D}$ defined, we have the following factorization of π:

$$\begin{tikzcd}
\mathscr{C}(\mathscr{E}\Gamma) \arrow[d, "q"'] & \mathscr{P}\Gamma \arrow[l, "\mathscr{D}"'] \arrow[d, "\pi"] \\
\pi_0(\mathscr{C}(\mathscr{E}\Gamma)) \arrow[r, "\phi"] & \pi^{\mathrm{qtop}}(|\Gamma|,\Gamma_0)
\end{tikzcd}$$

where q is the canonical quotient **qTop**-function. Since α and $\alpha_1\cdot\ldots\cdot\alpha_n$ are homotopic paths (when $n>0$), the diagram commutes.

Given $x, y \in \Gamma_0$, suppose $U \subseteq \pi_1^{\mathrm{qtop}}(\Gamma, \Gamma_0)(x, y)$ is such that $\phi^{-1}(U)$ is open in $\pi_0(\mathscr{C}(\mathscr{E}\Gamma))(x, y)$. Since π is quotient, it suffices to show that $\pi^{-1}(U) = \mathscr{D}^{-1}(q^{-1}(\phi^{-1}(U)))$ is open in $\mathscr{P}\Gamma(x, y)$. Suppose $\alpha \in \pi^{-1}(U)$ is a fixed path with decomposition (n and $a_1, b_1, \ldots, a_n, b_n$) described above. Note that $q^{-1}(\phi^{-1}(U))$ is an open neighborhood of $\mathscr{D}(\alpha)$ in $\mathscr{C}(\mathscr{E}\Gamma)(x, y)$.

If $n = 0$, then $\mathscr{D}(\alpha) = \mathrm{id}_x$ for some x and the image of α lies in the contractible vertex neighborhood $B(x, 1)$. The neighborhood

$$\{\beta \in \mathscr{P}\Gamma(x, x) \mid \mathrm{Im}(\beta) \subseteq B(x, 1)\}$$

of α contains only null-homotopic loops and is therefore contained in $\pi^{-1}(U)$. Thus we may assume that $n > 0$.

If $n > 0$, then $\mathscr{D}(\alpha) = \alpha_1 \ldots \alpha_n \in q^{-1}(\phi^{-1}(U))$ where $\alpha_i \in \mathscr{E}\Gamma(x_i, x_{i+1})$. We construct a neighborhood of α contained in $\pi^{-1}(U)$. For convenience, rename the sets $[0, a_1], [b_1, a_2], \ldots, [b_{n-1}, a_n], [b_n, 1]$ (some of which may be singletons) as $K_1, K_2, \ldots, K_n, K_{n+1}$. Note that each restricted path α_{K_i} is a trivial loop based at a vertex x_i with image in some vertex neighborhood $B(x_i, r_i)$. In particular, $x = x_1$ and $x_{n+1} = y$.

Recall that $\mathcal{B}_i = \langle [0, 1], B(x_i, r_i) \rangle$ is a neighborhood of α_{K_i}, and thus $\mathcal{B}_i^{K_i}$ is a neighborhood of α, for each $i = 1, \ldots, n + 1$. Since $\mathscr{C}(\mathscr{E}\Gamma)$ is a **Top**-category, there are open neighborhoods V_i of α_i in $\mathscr{E}\Gamma(x_i, x_{i+1})$ such that the product $V_1 \ldots V_n$ is contained in $q^{-1}(\phi^{-1}(U))$. By Lemma 4.5, there is a standard neighborhood $\mathcal{A}_i = \bigcap_{j=1}^{m_i} \langle [(j-1)/m_i, j/m_i], U_j^i \rangle$, $m_i > 2$, of α_i in $\mathcal{P}|\Gamma|$ such that

(1) $\mathcal{U}_i = \mathcal{A}_i \cap \mathscr{E}\Gamma(x_i, x_{i+1}) \subseteq V_i$,
(2) U_1^i and $U_{m_i}^i$ are vertex neighborhoods, and
(3) $U_2^i, \ldots, U_{m_i-1}^i$ are edge neighborhoods.

We may also choose the $\mathcal{A}_i$ so that $U_1^1 \subset B(x_1, r_1)$, $U_{m_i}^i \cup U_1^{i+1} \subseteq B(x_{i+1}, r_{i+1})$ for $i = 1, \ldots, n-1$, and $U_{m_n}^n \subset B(x_{n+1}, r_{n+1})$.

Now

$$\mathcal{W} = \bigcap_{i=1}^{n} \mathcal{A}_i^{[a_i, b_i]} \cap \bigcap_{i=1}^{n+1} \mathcal{B}_i^{K_i} \cap \mathscr{P}\Gamma(x, y)$$

is an open neighborhood of α in $\mathscr{P}\Gamma(x, y)$.

Suppose that $\beta \in \mathcal{W}$. We clearly have $\beta(K_i) \subset B(x_i, r_i)$ for each $i = 1, \ldots, n+1$, however if $x_i = x_{i+2}$, it is possible that $\beta_{[a_i, b_{i+1}]}$ does not hit the vertex x_{i+1}. To deal with this possibility, we replace "small" portions of β without changing the homotopy class. For each $i = 2, \ldots, n$, let $s_i = L_{[a_{i-1}, b_{i-1}]}(1 - 1/m_{i-1})$ and $t_i = L_{[a_i, b_i]}(1/m_i)$ so that $K_i = [b_{i-1}, a_i] \subset [s_i, t_i]$. Now define a path γ to equal the path β with the following exceptions: replace the portion of β from s_i to b_{i-1} with the canonical arc from $\beta(s_i)$ to x_i, take γ to be the constant path at x_i on $[b_{i-1}, a_i]$, and replace the portion of β from a_i to t_i with the canonical arc from x_i to $\beta(t_i)$. Since γ is given by changing β only in the contractible neighborhoods $B(x_i, r_i)$, γ and β are homotopic paths, i.e. $\pi(\beta) = \pi(\gamma)$. Moreover, $\gamma_i = \gamma_{[a_i,b_i]}$ is an edge path for each i contained in $\mathcal{U}_i$. Thus

$$\mathscr{D}(\gamma) = \gamma_1 \dots \gamma_n \in \mathcal{U}_1 \dots \mathcal{U}_n \subseteq V_1 \dots V_n \subseteq q^{-1}(\phi^{-1}(U)).$$

Finally, we see that

$$\pi(\beta) = \pi(\gamma) = \phi(q(\mathscr{D}(\gamma))) \in U,$$

giving the inclusion $\mathcal{W} \subseteq \pi^{-1}(U)$. ∎

THEOREM 4.8. *The fundamental* **Top**-*groupoid* $\pi^\tau(\Gamma, \Gamma_0)$ *is naturally isomorphic to the free* **Top**-*groupoid* $\mathscr{F}(\pi_0(\Gamma))$.

Proof. The embedding $\Gamma \to \mathscr{P}\Gamma$ given by $e \mapsto \alpha_e$ induces a **Top**-graph morphism $\pi_0(\Gamma) \to \pi_0(\mathscr{P}\Gamma) = \pi^{\mathrm{qtop}}(\Gamma, \Gamma_0)$. Additionally, the identity functor $\pi^{\mathrm{qtop}}(\Gamma, \Gamma_0) \to \pi^\tau(\Gamma, \Gamma_0)$ is a morphism of **qTop**-groupoids. The composition $\sigma : \pi_0(\Gamma) \to \pi^\tau(\Gamma, \Gamma_0)$ of these two is a morphism of **Top**-graphs which induces a **Top**-functor $\hat{\sigma} : \mathscr{F}(\pi_0(\Gamma)) \to \pi^\tau(\Gamma, \Gamma_0)$ on the free **Top**-groupoid. A straightforward generalization of [4, 3.14] to **Top**-graphs with more than one vertex shows that $\hat{\sigma}$ is an isomorphism of the underlying groupoids. Therefore, it suffices to check that the inverse $\hat{\sigma}^{-1} : \pi^\tau(\Gamma, \Gamma_0) \to \mathscr{F}(\pi_0(\Gamma))$ is a **Top**-functor.

Consider the following commutative diagram. The upper horizontal functors are the **qTop**-isomorphism from Lemma 4.6 and the canonical **qTop**-functor $\psi : \pi_0(\mathscr{C}^\pm(\Gamma)) \to \mathscr{C}^\pm(\pi_0(\Gamma))$ from Lemma 3.10. The vertical functor R is the quotient **qTop**-functor given by word reduction (see Remark 3.11) and ϕ is the quotient **qTop**-functor of Lemma 4.7.

$$\begin{tikzcd}
\pi_0(\mathscr{C}(\mathscr{E}\Gamma)) \arrow[r, "\cong"] \arrow[d, "\phi"'] & \pi_0(\mathscr{C}^\pm(\Gamma)) \arrow[r, "\psi"] & \mathscr{C}^\pm(\pi_0(\Gamma)) \arrow[d, "R"] \\
\pi_1^{\mathrm{qtop}}(\Gamma, \Gamma_0) \arrow[rr, "\hat{\sigma}^{-1}"] & & \mathscr{F}_R(\pi_0(\Gamma))
\end{tikzcd}$$

Since the top composition is a **qTop**-functor and ϕ is quotient, $\hat{\sigma}^{-1} : \pi^{\mathrm{qtop}}(\Gamma, \Gamma_0) \to \mathscr{F}_R(\pi_0(\Gamma))$ is continuous on hom-spaces (by the universal property of quotient spaces), and is therefore a **qTop**-functor. Applying the τ-reflection implies that

$$\hat{\sigma}^{-1} : \pi^\tau(\Gamma, \Gamma_0) = \tau(\pi^{\mathrm{qtop}}(\Gamma, \Gamma_0)) \to \tau(\mathscr{F}_R(\pi_0(\Gamma))) = \mathscr{F}(\pi_0(\Gamma))$$

is a **Top**-functor. ∎

The topological group $\pi^\tau(\Gamma, \Gamma_0)(v)$ at a vertex $v \in \Gamma_0$ is, by definition, the topological fundamental group $\pi_1^\tau(|\Gamma|, v)$, as defined in [3, 4]. In light of Theorem 3.15, we have the following corollary.

Corollary 4.9. *The topological fundamental group* $\pi_1^\tau(|\Gamma|, v)$ *of a* **Top**-*graph* Γ *is a free Graev topological group. In particular, if* Γ *has more than one vertex and* $T \subset \pi_0(\Gamma)$ *is a maximal tree, then* $\pi_1^\tau(|\Gamma|, v) \cong F_G(\pi_0(\Gamma)/T, *)$. *If* Γ *has a single vertex, then* $\pi_1^\tau(|\Gamma|, v) \cong F_G(\pi_0(\Gamma)_+, *) \cong F_M(\pi_0(\Gamma))$.

Corollary 4.10. *Every free* **Top**-*groupoid is the fundamental* **Top**-*groupoid of some* **Top**-*graph.*

Proof. According to Remark 3.6, a given **Top**-graph Γ is isomorphic to $\pi_0(h(\Gamma))$ for some **Top**-graph $h(\Gamma)$. By Theorem 4.8,

$$\pi^\tau(h(\Gamma), h(\Gamma)_0) \cong \mathscr{F}(\pi_0(h(\Gamma))) \cong \mathscr{F}(\Gamma). \ \blacksquare$$

5. Semicoverings and a proof of Theorem 1.1. We recall the theory of semicovering spaces introduced in [3], and apply it to **Top**-graphs. Given a space X and a point $x \in X$, let $(\mathcal{P}X)_x$ and $(\Phi X)_x$ be respectively spaces of paths and homotopies (rel endpoints) of paths starting at x with the compact-open topology. Recall that $\mathscr{P}X(x, x')$ is the subspace of $(\mathcal{P}X)_x$ consisting of paths ending at x'. In particular, $\mathscr{P}X(x, x) = \Omega(X, x)$ is the space of based loops.

Definition 5.1. A *semicovering map* is a local homeomorphism $p : Y \to X$ such that for each $y \in Y$, the induced map $\mathcal{P}p : (\mathcal{P}Y)_y \to (\mathcal{P}X)_{p(y)}$ is a homeomorphism. The space Y is called a *semicovering* (*space*) of X. If α is a path starting at $p(y)$, then $\tilde{\alpha}_y$ denotes the unique lift of α starting at $y \in Y$.

Remark 5.2. The above definition of a "semicovering map" $p : Y \to X$ is slightly simpler than, but equivalent to, the original definition given in [3]. The exponential law for mapping spaces and the condition that $\mathcal{P}p : (\mathcal{P}Y)_y \to (\mathcal{P}X)_{p(y)}$ is a homeomorphism imply the induced map $\Phi p : (\Phi Y)_y \to (\Phi X)_{p(y)}$ of homotopies of paths is a homeomorphism. In particular, if p is a semicovering, then every map $f : (D^2, (1, 0)) \to (X, p(y))$ from the unit disk D^2 has a unique lift $\tilde{f}_y : (D^2, (1, 0)) \to (Y, y)$ such that $p \circ \tilde{f}_y = f$. This observation first appeared in [5, Remark 2.5].

Every covering (in the classical sense) is a semicovering, however, if X does not have a simply connected covering (e.g. the Hawaiian earring), a semicovering of X need not be a covering [3, 10]. Of particular importance to the current paper is the fact that a semicovering $p : Y \to X$ induces an open covering morphism $\pi^\tau p : \pi^\tau Y \to \pi^\tau X$ of fundamental **Top**-groupoids. That is, for each $y_1, y_2 \in Y$, $p(y_i) = x_i$, the induced map $p_* : \pi^\tau Y(y_1, y_2) \to \pi^\tau X(x_1, x_2)$ is an open embedding of spaces (or topological groups when $y_1 = y_2$). Just as in classical covering theory, if β is a path from x_1 to x_2, then $\tilde{\beta}_{y_1}(1) = y_2$ if and only if $[\beta]$ lies in the image of the embedding p_*.

Thus, since the image of p_* is open and $\pi : \mathscr{P}X(x_1, x_2) \to \pi^\tau X(x_1, x_2)$ is continuous, $\{\beta \mid \tilde{\beta}_{y_1}(1) = y_2\}$ is an open subspace of $\mathscr{P}X(x_1, x_2)$.

Since a **Top**-graph $|\Gamma|$ can fail to be locally path connected, the usual classification of covering spaces does not apply. We use the fact that semicovering theory applies to certain non-locally path connected spaces [3, Definition 6.4].

Definition 5.3. Let X be a space.

(1) A path $\alpha : [0,1] \to X$ is *well-targeted* if for every open neighborhood $\mathcal{U}$ of α in $(\mathcal{P}X)_{\alpha(0)}$ there is an open neighborhood V_1 of $\alpha(1)$ such that for each $b \in V_1$, there is a path $\beta \in \mathcal{U}$ with $\beta(1) = b$.
(2) A path $\alpha : [0,1] \to X$ is *locally well-targeted* if for every open neighborhood $\mathcal{U}$ of α in $(\mathcal{P}X)_{\alpha(0)}$ there is an open neighborhood V_1 of $\alpha(1)$ such that for each $b \in V_1$, there is a well-targeted path $\beta \in \mathcal{U}$ with $\beta(1) = b$.

A space X is *locally wep-connected* if for every pair of points $x, y \in X$, there is a locally well-targeted path from x to y.

We refer the reader to [5, Remark 2.20] for an explanation of the use of the letters "wep" in the definition of "locally wep-connected space".

Clearly, every locally path connected space is locally wep-connected. There are also many locally wep-connected spaces which are not locally path connected. For example, if Γ is a **Top**-graph with a single vertex, the generalized wedge of circles $|\Gamma| = \Sigma(\Gamma_+)$ is locally wep-connected [3, Proposition 6.7], but is only locally path connected if the edge space Γ is locally path connected. The following lemma generalizes this special case to arbitrary **Top**-graphs.

Lemma 5.4. *The space $|\Gamma|$ is locally wep-connected for any* **Top**-*graph Γ.*

Proof. Since $|\Gamma|$ is locally path connected at each vertex, it suffices to find a locally well-targeted path from a vertex to each point $z \in |\Gamma| \setminus \Gamma_0$. Suppose z is the image of (e, t) in $\Gamma(x, y) \times (0, 1)$. Any path $\alpha : [0,1] \to |\Gamma|$ such that $\alpha(0) = x$, $\alpha(1) = z$, and having image on the edge $\{e\} \times [0, 1) \subset |\Gamma|$ is locally well-targeted. The argument that α is locally well-targeted is identical to that in [3, Proposition 6.7]. ∎

Since **Top**-graphs are locally wep-connected, we call upon the classification of semicoverings to obtain the following lemma, which will be used in the proof of Theorem 1.1.

Lemma 5.5 ([3, Corollary 7.20]). *If Γ is a* **Top**-*graph, $x \in \Gamma_0$ is a vertex, and H is an open subgroup of the topological fundamental group $\pi_1^\tau(|\Gamma|, x)$, then there is a semicovering $p : Y \to |\Gamma|$, $p(y) = x$, such that the induced homomorphism $p_* : \pi_1^\tau(Y, y) \to \pi_1^\tau(|\Gamma|, x)$ is a topological embedding onto H.*

5.1. A semicovering of a Top-graph is a Top-graph. The following theorem generalizes the fact that a covering (in the classical sense) of a graph is a graph, and provides the last ingredient for our proof of Theorem 1.1.

THEOREM 5.6. *If $\mathit{\Gamma}$ is a* **Top***-graph and $p : Y \to |\mathit{\Gamma}|$ is a semicovering map, then there is a* **Top***-graph $\widetilde{\mathit{\Gamma}}$ such that $Y \cong |\widetilde{\mathit{\Gamma}}|$.*

Proof. It suffices to assume that the **Top**-graph and semicovering in question are connected. Since $\mathit{\Gamma}_0$ is a discrete subspace of $|\mathit{\Gamma}|$ and p is a local homeomorphism, $p^{-1}(\mathit{\Gamma}_0)$ is a discrete subspace of Y. Define the vertex space $\widetilde{\mathit{\Gamma}}_0 = p^{-1}(\mathit{\Gamma}_0)$. For $y_1, y_2 \in \widetilde{\mathit{\Gamma}}_0$ such that $x_i = p(y_i)$, define

$$\widetilde{\mathit{\Gamma}}(y_1, y_2) = \{e \in \mathit{\Gamma}(x_1, x_2) \mid \widetilde{(\alpha_e)}_{y_1}(1) = y_2\}$$

with the subspace topology of $\mathit{\Gamma}(x_1, x_2)$.

Define a map $h : |\widetilde{\mathit{\Gamma}}| \to Y$ as follows: The restriction of h to $\widetilde{\mathit{\Gamma}}_0$ is the identity. The map $h_{y_1,y_2} : \widetilde{\mathit{\Gamma}}(y_1, y_2) \times [0, 1] \to Y$ given by $h_{y_1,y_2}(e, t) = \widetilde{(\alpha_e)}_{y_1}(t)$ is continuous since $\mathit{\Gamma}(x_1, x_2) \to \mathscr{P}\mathit{\Gamma}(x_1, x_2)$, $e \mapsto \alpha_e$, is continuous, $\mathcal{P}p : (\mathcal{P}Y)_{y_1} \to (\mathcal{P}X)_{x_1}$ is a homeomorphism, and evaluation $(\mathcal{P}Y)_{y_1} \times [0, 1] \to Y$, $(\beta, t) \mapsto \beta(t)$, is continuous. The maps h_{y_1,y_2} induce the function h on the image of $\widetilde{\mathit{\Gamma}}(y_1, y_2) \times [0, 1]$ in $|\widetilde{\mathit{\Gamma}}|$. It follows from Lemma 5.7 below that h is bijective, and from Lemma 5.10 that h is continuous and open. ∎

To prove the following lemmas, we first make some observations about the edge spaces $\widetilde{\mathit{\Gamma}}(y_1, y_2)$. Let $A = \{\alpha_e \in \mathscr{P}\mathit{\Gamma}(x_1, x_2) \mid e \in \mathit{\Gamma}(x_1, x_2)\}$, and recall that $B = \{\beta \in \mathscr{P}\mathit{\Gamma}(x_1, x_2) \mid \widetilde{\beta}_{y_1}(1) = y_2\}$ is open in $\mathscr{P}\mathit{\Gamma}(x_1, x_2)$. It is now clear that $A \cap B$ is open in A and is the image of $\widetilde{\mathit{\Gamma}}(y_1, y_2)$ under the homeomorphism $\mathit{\Gamma}(x_1, x_2) \to A$, $e \mapsto \alpha_e$. Therefore, $\widetilde{\mathit{\Gamma}}(y_1, y_2)$ is an open subspace of $\mathit{\Gamma}(x_1, x_2)$. It follows that whenever $p(y_1) = x_1$,

- $\widetilde{\mathit{\Gamma}}_{y_1} = \mathit{\Gamma}_{x_1}$ and $\widetilde{\mathit{\Gamma}}^{y_1} = \mathit{\Gamma}^{x_1}$, and
- $\mathit{\Gamma}(x_1, x_2)$ decomposes as the topological sum $\coprod_{p(y_2)=x_2} \widetilde{\mathit{\Gamma}}(y_1, y_2)$.

LEMMA 5.7. *$h : |\widetilde{\mathit{\Gamma}}| \to Y$ is a bijection.*

Proof. First, we show that h is surjective. It suffices to consider a point $y \in Y \setminus \widetilde{\mathit{\Gamma}}_0$. Note $p(y)$ is the image of a pair $(e, t) \in \mathit{\Gamma}(x_1, x_2) \times (0, 1)$ in $|\mathit{\Gamma}|$. Fix $x_0 \in \mathit{\Gamma}_0$ and $y_0 \in p^{-1}(x_0)$, and let β be a path from y_0 to y in Y. There is a point $0 \le s < 1$ such that $p \circ \beta(s) \in \{x_1, x_2\}$. Without loss of generality, we may assume $p \circ \beta(s) = x_1$. Let $\eta = (p \circ \beta)_{[0,s]}$, and define $y_1 = \widetilde{\eta}_{y_0}(1)$ and $y_2 = \widetilde{(\alpha_e)}_{y_1}(1)$. Note $p(y_i) = x_i$ for $i = 1, 2$. Now the lift of $(\alpha_e)_{[0,t]}$ starting at y_1 ends at $h(e, t) = \widetilde{(\alpha_e)}_{y_1}(t) = y$.

For injectivity, suppose $z, z' \in |\widetilde{\Gamma}|$. If one of z or z' is a vertex and $h(z) = h(z')$, then $z, z' \in \widetilde{\Gamma}_0$ and it follows that $z = z'$. Therefore it suffices to check that h is injective on $|\widetilde{\Gamma}| \setminus \widetilde{\Gamma}_0$. Suppose z is the image of $(e,t) \in \widetilde{\Gamma}(y_1, y_2) \times (0,1)$ and z' is the image of $(f,u) \in \widetilde{\Gamma}(y_3, y_4) \times (0,1)$ under h. If $h(z) = \widetilde{(\alpha_e)}_{y_1}(t) = \widetilde{(\alpha_f)}_{y_3}(u) = h(z')$, then $\alpha_e(t) = \alpha_f(u)$ in $|\Gamma|$, however, this only occurs if $e = f$ and $t = u$, and thus $z = z'$. ∎

The main difficulty in showing that h is a homeomorphism is identifying a basis of open neighborhoods in Y.

LEMMA 5.8. *Let $x_0 \in \Gamma_0$, $y_0 \in p^{-1}(x_0)$, and $r > 0$. The subset*

$$V(y_0, r) = \{\widetilde{(\alpha_e)}_{y_0}(t) \in Y \mid 0 \le t < r,\ e \in \Gamma_{x_0}\} \cup \{\widetilde{(\alpha_f)}_{y_0}(t) \in Y \mid 1 - r < t \le 1,\ f \in \Gamma^{x_0}\}$$

is open in Y. Moreover, p maps $V(y_0, r)$ homeomorphically onto the vertex neighborhood $B(x_0, r)$ of $|\Gamma|$.

Proof. By the uniqueness of path lifting, it is clear that p maps $V(y_0, r)$ bijectively onto $B(x_0, r)$. Since p is an open map, it will follow that p maps $V(y_0, r)$ homeomorphically onto $B(x_0, r)$ once we show that $V(y_0, r)$ is open.

By Lemma 5.4, $|\Gamma|$ is locally wep-connected and therefore Y is locally wep-connected [3, Corollary 6.12]. Consequently, the evaluation map $\mathrm{ev}_1 : (\mathcal{P}Y)_{y_0} \to Y$, $\mathrm{ev}_1(\beta) = \beta(1)$, is quotient [3, Proposition 6.2]. Additionally, $\mathcal{P}p : (\mathcal{P}Y)_{y_0} \to (\mathcal{P}|\Gamma|)_{x_0}$ has a continuous inverse $L : (\mathcal{P}|\Gamma|)_{x_0} \to (\mathcal{P}Y)_{y_0}$. Thus the composition $\mathrm{ev}_1 L : (\mathcal{P}|\Gamma|)_{x_0} \to Y$, $\beta \mapsto \tilde{\beta}_{y_0}(1)$, is quotient.

Since $\mathrm{ev}_1 L$ is quotient, it suffices to show that $L^{-1}(\mathrm{ev}_1^{-1}(V(y_0, r)))$ is open in $(\mathcal{P}|\Gamma|)_{x_0}$. If $\beta \in L^{-1}(\mathrm{ev}_1^{-1}(V(y_0, r)))$, then $\tilde{\beta}_{y_0}(1) \in V(y_0, r)$, and thus $\beta(1) \in B(x_0, r)$. Let γ be the canonical arc from x_0 to $\beta(1)$ in $B(x_0, r)$, and observe that $\mathrm{Im}(\tilde{\gamma}_{y_0}) \subset V(y_0, r)$. Thus $\widetilde{(\beta \cdot \overline{\gamma})}_{y_0}$ is a loop based at y_0. It follows that $[\beta \cdot \overline{\gamma}]$ lies in the open subgroup $p_*(\pi^\tau(Y, y_0))$ of $\pi^\tau(|\Gamma|, x_0)$. Since $\pi : \Omega(|\Gamma|, x_0) \to \pi^\tau(|\Gamma|, x_0)$ is continuous, there is a basic open neighborhood $\mathcal{U} = \bigcap_{j=1}^{n} \langle [(j-1)/n, j/n], U_j \rangle$ of $\beta \cdot \overline{\gamma}$ in $\mathcal{P}|\Gamma|$ such that $\mathcal{U} \cap \Omega(|\Gamma|, x_0) \subseteq \pi^{-1}(p_*(\pi^\tau(Y, y_0)))$. Since $B(x_0, r)$ is contractible, we may assume that

(1) n is even,
(2) $U_1 = B(x_0, r)$, and
(3) $U_k = B(x_0, r)$ for $k \ge n/2$.

Now $\mathcal{V} = \mathcal{U}_{[0,1/2]} \cap (\mathcal{P}|\Gamma|)_{x_0}$ is an open neighborhood of β in $(\mathcal{P}|\Gamma|)_{x_0}$ which we claim is a subset of $L^{-1}(\mathrm{ev}_1^{-1}(V(y_0, r)))$. Suppose that $\beta' \in \mathcal{V}$. Then $\beta'(1) \in U_{n/2} = B(x_0, r)$ and, if γ' is the canonical arc from x_0 to $\beta'(1)$, then

$\beta \cdot \overline{\gamma'} \in \mathcal{U} \cap \Omega(|\Gamma|, x_0) \subseteq \pi^{-1}(p_*(\pi^\tau(Y, y_0)))$. Thus $\widetilde{(\beta \cdot \overline{\gamma'})}_{y_0}(1) = y_0$. Since $\widetilde{\gamma'}_{y_0}$ has image in $V(y_0, r)$, we have

$$\mathrm{ev}_1 L(\beta') = \widetilde{(\beta')}_{y_0}(1) = \widetilde{(\gamma')}_{y_0}(1) \in V(y_0, r).$$

It follows that $L^{-1}(\mathrm{ev}_1^{-1}(V(y_0, r)))$ is open in $(\mathcal{P}|\Gamma|)_{x_0}$. ∎

Since p is a local homeomorphism onto $|\Gamma|$, it follows that the sets $V(y_0, r)$, $r > 0$, form a neighborhood base at the point y_0 in Y.

For the following lemma, recall that if $p(y_i) = x_i$, then $\widetilde{\Gamma}(y_1, y_2)$ is an open subset of $\Gamma(x_1, x_2)$. Thus if U is an open subset of $\widetilde{\Gamma}(y_1, y_2)$ and $0 < a < b < 1$, then $U \times (a, b)$ is an edge neighborhood in $|\Gamma|$.

Lemma 5.9. *Let $y_1, y_2 \in \widetilde{\Gamma}_0$, $p(y_i) = x_i$, $U \subset \widetilde{\Gamma}(y_1, y_2)$ be an open set, and $0 < a < b < 1$. Then the subset*

$$E = \{\widetilde{(\alpha_e)}_{y_1}(t) \in Y \mid (e, t) \in U \times (a, b)\}$$

is open in Y. Moreover, p maps E homeomorphically onto the edge neighborhood $U \times (a, b)$ of $|\Gamma|$.

Proof. It is clear that p maps E bijectively onto $U \times (a, b) \subset |\Gamma|$. Since p is an open map, it suffices to show that E is open. Similarly to the previous lemma, we use the fact that $\mathrm{ev}_1 L : (\mathcal{P}|\Gamma|)_{x_1} \to Y$, $\beta \mapsto \widetilde{\beta}_{y_1}(1)$, is quotient by showing that $L^{-1}(\mathrm{ev}_1^{-1}(E))$ is open in $(\mathcal{P}X)_{x_1}$.

If $\beta \in L^{-1}(\mathrm{ev}_1^{-1}(E))$, then $\widetilde{\beta}_{y_1}(1) = \widetilde{(\alpha_e)}_{y_1}(t) \in E$ for some (e, t) in $U \times (a, b)$. Let $\gamma = (\alpha_e)_{[0,t]}$. Since $\widetilde{(\beta \cdot \overline{\gamma})}_{y_1}$ is a loop based at y_1, we have

$$\beta \cdot \overline{\gamma} \in \pi^{-1}(p_*(\pi_1(|\Gamma|, x_1))) \subset \Omega(|\Gamma|, x_1).$$

Take a basic open neighborhood $\mathcal{U} = \bigcap_{j=1}^n \langle [(j-1)/n, j/n], W_j \rangle$ of $\beta \cdot \overline{\gamma}$ in $\mathcal{P}|\Gamma|$ such that $\mathcal{U} \cap \Omega(|\Gamma|, x_1) \subseteq \pi^{-1}(p_*(\pi^\tau(|\Gamma|, x_1)))$. In particular, we may assume that

(1) n is even,
(2) $W_1 = W_n = B(x_1, r)$ is a vertex neighborhood,
(3) when $n/2 \leq k < n$, then W_k is an edge neighborhood of the form $A \times (r_k, s_k)$ for a fixed open set $A \subseteq U$, and
(4) $(r_{n/2}, s_{n/2}) = (r_{n/2+1}, s_{n/2+1}) \subseteq (a, b)$.

Now $\mathcal{V} = \mathcal{U}_{[0,1/2]} \cap (\mathcal{P}|\Gamma|)_{x_1}$ is an open neighborhood of β in $(\mathcal{P}|\Gamma|)_{x_1}$. Note that $\beta(1) \in A \times (r_{n/2}, s_{n/2})$.

We check that $\mathcal{V} \subseteq L^{-1}(\mathrm{ev}_1^{-1}(E))$. If $\beta' \in \mathcal{V}$, then $\beta'(1) \in A \times (r_{n/2}, s_{n/2}) \subseteq U \times (a, b) \subset |\Gamma|$. If $\beta'(1)$ is the image of $(f, u) \in A \times (r_{n/2}, s_{n/2}) \subseteq U \times (a, b)$ in $|\Gamma|$, then $\alpha_f(u) = \beta'(1)$. Let γ' be any reparameterization of $(\alpha_f)_{[0,u]}$ so that $\beta' \cdot \overline{\gamma'} \in \mathcal{U} \subseteq \pi^{-1}(p_*(\pi_1(|\Gamma|, x_1)))$. Since $[\beta' \cdot \overline{\gamma'}] \in p_*(\pi_1(|\Gamma|, x_1))$, we have

$$\mathrm{ev}_1 L(\beta) = \widetilde{(\beta')}_{y_1}(1) = \widetilde{(\gamma')}_{y_1}(1) = \widetilde{(\alpha_f)}_{y_1}(u) \in E.$$

Thus $\mathcal{V} \subseteq L^{-1}(\mathrm{ev}_1^{-1}(E))$. ∎

LEMMA 5.10. *$h : |\widetilde{\Gamma}| \to Y$ is continuous and open.*

Proof. Since h is induced by continuous maps $h_{y_1,y_2} : \widetilde{\Gamma}(y_1, y_2) \times [0, 1] \to Y$, h is clearly continuous when $|\widetilde{\Gamma}|$ has the quotient topology. Since we are using the coarser topology of vertex and edge neighborhoods, it suffices to show that h is continuous at each vertex of $|\widetilde{\Gamma}|$.

Fix a vertex $y_0 \in \widetilde{\Gamma}_0$, let $p(y_0) = x_0$, and let $\widetilde{B}(y_0, r)$ denote the vertex neighborhood at y_0 in $|\widetilde{\Gamma}|$. Recall that $\widetilde{B}(y_0, r)$ is the image of $(\widetilde{\Gamma}_{y_0} \times [0, r)) \sqcup (\widetilde{\Gamma}^{y_0} \times (1 - r, 1])$ in $|\widetilde{\Gamma}|$. Since $\widetilde{\Gamma}_{y_0} = \Gamma_{x_0}$ and $\widetilde{\Gamma}^{y_0} = \Gamma^{x_0}$, h maps $\widetilde{B}(y_0, r)$ bijectively onto the basic open set $V(y_0, r)$ defined in Lemma 5.8. Since the sets $V(y_0, r)$ form a neighborhood base at y_0 in Y and h is bijective, h is continuous at $y_0 \in \widetilde{\Gamma}_0$.

To see that h is an open map, first recall that we have already shown that the image $h(\widetilde{B}(y_0, r)) = V(y_0, r)$ of a vertex neighborhood $\widetilde{B}(y_0, r)$ is open in Y. Fix an open set $U \subset \widetilde{\Gamma}(y_1, y_2)$ and $0 < a < b < 1$ so that $U \times (a, b)$ is an edge neighborhood in $|\widetilde{\Gamma}|$. By definition, h maps this edge neighborhood bijectively onto the open set E defined in Lemma 5.9. Thus h is an open map. ∎

5.2. A proof of Theorem 1.1. We conclude with a proof of Theorem 1.1.

Suppose X is a space with basepoint $* \in X$, and H is an open subgroup of the free topological group $F_G(X, *)$. Let $h(X)$ be a space such that $\pi_0(h(X)) = X$ (see Remark 3.6) and Γ be the **Top**-graph with $\Gamma_0 = \{a, b\}$ (i.e. two vertices), $\Gamma(a, b) = h(X)$, and $\Gamma(b, a) = \emptyset$. Note that the edge space of $\pi_0(\Gamma)$ is precisely X. By Theorem 4.8, $\pi^\tau(\Gamma, \Gamma_0)$ is isomorphic to the free **Top**-groupoid $\mathscr{F}(\pi_0(\Gamma))$. A tree $T \subseteq \pi_0(\Gamma)$ is given by taking $T_0 = \{a, b\}$ with the edge space $T = \{*\}$. Note $\pi_0(\Gamma)/T \cong X$ as based spaces. Theorem 3.15 gives the middle isomorphism in

$$\pi_1^\tau(|\Gamma|, a) = \pi^\tau(\Gamma, \Gamma_0)(a) \cong F_G(\pi_0(\Gamma)/T, *) \cong F_G(X, *).$$

By Lemma 5.5, there is a semicovering $p : Y \to |\Gamma|$, $p(y) = a$, such that the induced homomorphism $\pi_1^\tau(Y, y) \to \pi_1^\tau(|\Gamma|, a) \cong F_G(X, *)$ is a topological embedding onto H. According to Theorem 5.6, the semicovering space Y is a **Top**-graph, i.e. $Y \cong |\widetilde{\Gamma}|$ for some **Top**-graph $|\widetilde{\Gamma}|$. Now Corollary 4.9 applies to Y to show that $\pi_1^\tau(Y, y) \cong H$ is a free Graev topological group.

Acknowledgements. The author thanks the referee for many helpful corrections and suggestions which helped to improve the exposition of this paper.

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Received 2 October 2012;
in revised form 7 February 2014