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Relative subanalytic sheaves

by

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Abstract. Given a real analytic manifold Y, denote by Y_{sa} the associated subanalytic site. Now consider a product $Y = X \times S$. We construct the endofunctor $\mathcal{F} \mapsto \mathcal{F}^S$ on the category of sheaves on Y_{sa} and study its properties. Roughly speaking, $\mathcal{F}^S$ is a sheaf on $X_{\mathrm{sa}} \times S$. As an application, one can now define sheaves of functions on Y which are tempered or Whitney in the relative sense, that is, only with respect to X.

1. Introduction. Let Y be a real analytic manifold. The subanalytic sheaf $\mathcal{D}b_Y^t$ of tempered distributions defined by Kashiwara–Schapira [10] takes its origin in Kashiwara's functor TH (see [5]) as an essential tool to establish the Riemann–Hilbert correspondence between regular holonomic $\mathcal{D}$-modules and perverse sheaves.

Let $Y = X \times S$, for some real analytic manifolds X and S. In order to study relative perversity (see [11]), it appears that a "relative" version of $\mathcal{D}b^t_{X\times S}$ is required, i.e. a sheaf $\mathcal{D}b^{t,S}_{X\times S}$ such that

$$\varGamma(U \times V; \mathcal{D}b^{t,S}_{X\times S}) \simeq \varprojlim_{W \subset\subset V} \varGamma(U \times W; \mathcal{D}b^t_{X\times S}).$$

In other words, such a sheaf "forgets" the growth conditions on S.

Let $\mathrm{Mod}(\mathbb{C}_{(X\times S)_{\mathrm{sa}}})$ be the category of subanalytic sheaves on $X \times S$. The aim of this note is to construct a functor $(\cdot)^S : \mathrm{Mod}(\mathbb{C}_{(X\times S)_{\mathrm{sa}}}) \to \mathrm{Mod}(\mathbb{C}_{(X\times S)_{\mathrm{sa}}})$ such that, given $F \in \mathrm{Mod}(\mathbb{C}_{(X\times S)_{\mathrm{sa}}})$,

$$\varGamma(U \times V; F^S) \simeq \varprojlim_{W \subset\subset V} \varGamma(U \times W; F), \tag{1.1}$$

or, more generally, when F is a bounded complex of subanalytic sheaves and G (resp. H) is a bounded complex of $\mathbb{R}$-constructible sheaves on X (resp. S), its derived version $(\cdot)^{RS}$ satisfying

$$R\mathcal{H}om(G \boxtimes H, F^{RS}) \simeq R\mathcal{H}om(\mathbb{C}_X \boxtimes H, \rho^{-1} R\mathcal{H}om(G \boxtimes \mathbb{C}_S, F)), \tag{1.2}$$

where $\rho : X \times S \to (X \times S)_{\mathrm{sa}}$ is the natural functor of sites.

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Recall that, by the definition of $\mathcal{T}$-space introduced in [10] (cf. also [3]), the usual subanalytic site $(X\times S)_{\mathrm{sa}}$ can also be regarded as the site $(X\times S)_{\mathcal{T}}$ where $\mathcal{T}$ is the family of all relatively compact subanalytic open subsets. If we let $\mathcal{T}'$ be the family of finite unions of open relatively compact subsets of the form $U \times V$, with U subanalytic in X and V subanalytic in S, then $X\times S$ becomes a $\mathcal{T}'$-space, and the associated site is the product $X_{\mathrm{sa}}\times S_{\mathrm{sa}}$. We denote by η the morphism of sites $(X \times S)_{\mathrm{sa}} \to X_{\mathrm{sa}} \times S_{\mathrm{sa}}$, by ρ the morphism of sites $X \times S \to (X \times S)_{\mathrm{sa}}$ and by ρ' the morphism of sites $X \times S \to X_{\mathrm{sa}} \times S_{\mathrm{sa}}$.

In this note, to any $\mathcal{T}$-sheaf F (that is, a sheaf on the site associated to $\mathcal{T}$, or a subanalytic sheaf) we associate canonically a $\mathcal{T}'$-sheaf $F^{S,\sharp}$ which in some way forgets the dependence of F on the subanalytic factor S_{sa}. We then define the relative sheaf F^S as the inverse image under η of the $\mathcal{T}'$-sheaf $F^{S,\sharp}$, thus obtaining a subanalytic sheaf on $(X\times S)_{\mathrm{sa}}$. This construction leads to a left exact functor $(\cdot)^S$ from the abelian category of subanalytic sheaves on $X \times S$ into itself. Denoting by $(\cdot)^{RS}$ its right derived functor, we prove in Proposition 4.7 that $(\cdot)^{RS}$ satisfies, for $F \in D^b(\mathbb{C}_{(X\times S)_{\mathrm{sa}}})$, $G \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ and $H \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_S)$, natural isomorphisms

$$\begin{aligned}\rho^{-1}R\mathcal{H}om(G\boxtimes H, F^{RS}) &\simeq \rho^{-1}R\mathcal{H}om(G\boxtimes \rho_! H, F)\\ &\simeq R\mathcal{H}om(\mathbb{C}_X\boxtimes H, \rho^{-1}R\mathcal{H}om(G\boxtimes \mathbb{C}_S, F)),\end{aligned}$$

In particular, when $G = \mathbb{C}_X$ and $H = \mathbb{C}_S$ we have $\rho^{-1}F \simeq \rho^{-1}F^{RS} \simeq \rho'^{-1}F^{RS,\sharp}$.

We then apply our construction to $\mathcal{D}b^t_{X\times S}$ and obtain the subanalytic sheaf $\mathcal{D}b^{t,S}_{X\times S}$ of relative tempered distributions with respect to a projection $f : X\times S \to S$. As the notation suggests, it is a sheaf on the subanalytic site $(X\times S)_{\mathrm{sa}}$, whose sections on open subsets of the form $U\times V$ are distributions which extend to $X \times V$.

The same procedure leads to the subanalytic sheaves $\mathcal{C}^{\infty,t,S}_X$ of relative tempered $\mathcal{C}^\infty$-functions and $\mathcal{C}^{\infty,\mathrm{w},S}_X$ of relative Whitney $\mathcal{C}^\infty$-functions on X_{sa}. Proposition 4.7 shows that taking inverse images on $X \times S$ for the usual topology, we recover respectively the classical sheaves of distributions and $\mathcal{C}^\infty$-functions forgetting the relative growth conditions.

When X and S are complex manifolds, the classical procedure of taking the Dolbeault complex applies to our constructions, thus allowing us to define the subanalytic (complexes) $\mathcal{O}^{t,S}_{X\times S}$ of relative tempered holomorphic functions and $\mathcal{O}^{\mathrm{w},S}_{X\times S}$ of relative Whitney holomorphic functions.

As the reader can naturally ask, our method applies only for products of analytic manifolds (see Remark 5.3). We conjecture that with a weaker notion of subanalytic site as in [4], a notion of relative sheaf can be given for a general smooth function but it will not suit the applications we have in mind.

However, the tools we develop here, besides their own interest, are useful for understanding the notion of relative perversity introduced in [11].

2. Complements on subanalytic $\mathcal{T}$-sheaves. The results in this section rely on the notion of $\mathcal{T}$-topology. For details we refer to [10] and [3] from which we keep the notations.

Given a topological space X and a family $\mathcal{T}$ of open subsets of X, one says that X is a *$\mathcal{T}$-space* if $\mathcal{T}$ satisfies the following conditions:

(1) $\mathcal{T}$ is a basis of the topology of X and $\emptyset \in \mathcal{T}$,
(2) $\mathcal{T}$ is closed under finite unions and intersections,
(3) each $U \in \mathcal{T}$ has finitely many $\mathcal{T}$-connected components.

To $\mathcal{T}$ one associates a Grothendieck topology in the following way: a family $\mathcal{U} = \{U_i\}_i$ in $\mathcal{T}$ is a covering of $U \in \mathcal{T}$ if it admits a finite subcover. One denotes by $X_{\mathcal{T}}$ the associated site and by $\rho : X \to X_{\mathcal{T}}$ the natural morphism of sites. There are well defined functors

$$\mathrm{Mod}(\mathbb{C}_X) \underset{\rho^{-1}}{\overset{\rho_*}{\rightleftarrows}} \mathrm{Mod}(\mathbb{C}_{X_{\mathcal{T}}}). \tag{2.1}$$

Let us consider the category $\mathrm{Mod}(\mathbb{C}_X)$ of sheaves of $\mathbb{C}_X$-modules on X, and denote by $\mathcal{K}$ the subcategory whose objects are the sheaves $\bigoplus_{i\in I} k_{U_i}$ with I finite and $U_i \in \mathcal{T}$ for each i. Let $F \in \mathrm{Mod}(\mathbb{C}_X)$. Then:

(i) F is *$\mathcal{T}$-finite* if there exists an epimorphism $G \twoheadrightarrow F$ with $G \in \mathcal{K}$.
(ii) F is *$\mathcal{T}$-pseudo-coherent* if for any morphism $\psi : G \to F$ with $G \in \mathcal{K}$, $\ker \psi$ is $\mathcal{T}$-finite.
(iii) F is *$\mathcal{T}$-coherent* if it is both $\mathcal{T}$-finite and $\mathcal{T}$-pseudo-coherent.

Note that (ii) is equivalent to the same condition with "G is $\mathcal{T}$-finite" instead of "$G \in \mathcal{K}$". We denote by $\mathrm{Coh}(\mathcal{T})$ the full subcategory of $\mathrm{Mod}(\mathbb{C}_X)$ consisting of $\mathcal{T}$-coherent sheaves. $\mathrm{Coh}(\mathcal{T})$ is additive and stable by kernels.

Moreover:

- Let $W \in \mathcal{T}$ and let $\mathbb{C}_{W_{\mathcal{T}}} \in \mathrm{Mod}(\mathbb{C}_{X_{\mathcal{T}}})$ be the constant sheaf on W. Then $\rho_*\mathbb{C}_W \simeq \mathbb{C}_{W_{\mathcal{T}}}$.
- The functor ρ_* is fully faithful. Moreover its restriction to $\mathrm{Coh}(\mathcal{T})$ is exact.
- A sheaf $F \in \mathrm{Mod}(\mathbb{C}_{X_{\mathcal{T}}})$ can be seen as a filtrant inductive limit $\varinjlim_i \rho_* F_i$ with $F_i \in \mathrm{Coh}(\mathcal{T})$.
- The functors $\mathrm{Hom}(G, \cdot)$ and $\mathcal{H}om(G, \cdot)$, with $G \in \mathrm{Coh}(\mathcal{T})$, commute with filtrant $\varinjlim$.

Finally, recall (cf. [3]) that $F \in \mathrm{Mod}(\mathbb{C}_{X_{\mathcal{T}}})$ is *$\mathcal{T}$-flabby* if the restriction morphism $\Gamma(X; F) \to \Gamma(W; F)$ is surjective for each $W \in \mathcal{T}$. $\mathcal{T}$-flabby objects are $\mathrm{Hom}(G, \cdot)$-acyclic for each $G \in \mathrm{Coh}(\mathcal{T})$.

Given a real analytic manifold Y, let $\mathrm{Op}^c(Y_{\mathrm{sa}})$ (resp. $\mathrm{Op}(Y_{\mathrm{sa}})$) denote the family of subanalytic relatively compact open subsets in Y (resp. the family of subanalytic open subsets in Y). Let Y_{sa} denote the associated subanalytic site introduced in [10]. Then Y_{sa} is the site $Y_{\mathcal{T}}$ associated to the family $\mathcal{T} = \mathrm{Op}^c(Y_{\mathrm{sa}})$ (that is, Y is a $\mathcal{T}$-space and the associated site $Y_{\mathcal{T}}$ coincides with Y_{sa}). Accordingly we shall still denote by ρ the natural functor of sites $\rho : Y \to Y_{\mathrm{sa}}$ associated to the inclusion $\mathrm{Op}(Y_{\mathrm{sa}}) \subset \mathrm{Op}(Y)$ (without reference to Y unless otherwise specified), as well as the associated functors $\rho_*, \rho^{-1}, \rho_!$ introduced in [10] (cf. also [12]).

Let us recall the following facts:

- The functor $\rho_!$ is right adjoint to ρ^{-1}. It is fully faithful and exact. Given $F \in \mathrm{Mod}(\mathbb{C}_Y)$, $\rho_! F$ is the sheaf associated to the presheaf $\mathrm{Op}(Y_{\mathrm{sa}}) \ni U \mapsto F(\overline{U})$.
- The category $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_Y)$ of $\mathbb{R}$-constructible sheaves is ρ_*-acyclic (cf. [12]).

Suppose now a subfamily $\mathcal{T}' \subset \mathrm{Op}^c(Y_{\mathrm{sa}})$ is given such that Y is still a $\mathcal{T}'$-space.

- Denoting by $Y_{\mathcal{T}'}$ the site associated to the family $\mathcal{T}'$, we shall also denote by $\rho' : Y \to Y_{\mathcal{T}'}$ the natural functor of sites. A sheaf $F \in \mathrm{Mod}(\mathbb{C}_{Y_{\mathcal{T}'}})$ can be seen as a filtrant inductive limit $\varinjlim_i \rho'_* F_i$ with $F_i \in \mathrm{Coh}(\mathcal{T}')$.
- We shall denote by η the natural functor of sites $Y_{\mathrm{sa}} \to Y_{\mathcal{T}'}$.

One obtains a commutative diagram of sites

$$
\begin{tikzcd}
Y \arrow[r, "\rho"] \arrow[d, "\rho'"'] & Y_{\mathrm{sa}} \arrow[dl, "\eta"] \\
Y_{\mathcal{T}'} &
\end{tikzcd}
\tag{2.2}
$$

Remark 2.1. One could also consider the site defined by the family of locally finite unions of elements of $\mathcal{T}$ (in the case $\mathcal{T} = \mathrm{Op}^c(Y_{\mathrm{sa}})$ these are all subanalytic open subsets) and locally finite coverings, and make the same construction using the family $\mathcal{T}'$. Since the associated categories of sheaves are respectively isomorphic to $\mathrm{Mod}(\mathbb{C}_{Y_{\mathcal{T}}})$ and $\mathrm{Mod}(\mathbb{C}_{Y_{\mathcal{T}'}})$ (see [10, Remark 6.3.6]) we will still denote by $Y_{\mathcal{T}}$ (resp. $Y_{\mathcal{T}'}$) the associated site.

Let F be a sheaf on $Y_{\mathcal{T}'}$. We define the (separated) presheaf $\eta^\dagger F$ on Y_{sa} by setting, for $W \in \mathrm{Op}(Y_{\mathrm{sa}})$,

$$\eta^\dagger F(W) = \varinjlim_{W \subset W'} F(W')$$

with $W' \in \mathrm{Op}(Y_{\mathcal{T}'})$. Let $\eta^{-1} F$ be the associated sheaf.

LEMMA 2.1. *Let* $F \simeq \varinjlim_i \rho'_* F_i \in \mathrm{Mod}(\mathbb{C}_{Y_{\mathcal{T}'}})$ *with* $F_i \in \mathrm{Coh}(\mathcal{T}')$. *Then* $\eta^{-1}F \simeq \varinjlim_i \rho_* F_i$.

Proof. Since the inverse image functor commutes with $\varinjlim$, it is enough to check that $\eta^{-1}\rho'_* F' \simeq \rho_* F'$ with $F' \in \mathrm{Coh}(\mathcal{T}')$.

Since (cf. [10, Chapter 6]) $\mathrm{Coh}(\mathcal{T}')$ is an abelian subcategory of $\mathrm{Coh}(\mathcal{T})$ and ρ_* (resp. ρ'_*) is exact on $\mathrm{Coh}(\mathcal{T})$ (resp. $\mathrm{Coh}(\mathcal{T}')$), we may restrict ourselves to the case $F' = \mathbb{C}_W$, $W \in \mathcal{T}'$.

Let $\mathbb{C}_{W_{\mathcal{T}}}$ (resp. $\mathbb{C}_{W_{\mathcal{T}'}}$) be the constant sheaf on Y_{sa} (resp. $Y_{\mathcal{T}'}$). Then, by Proposition 6.3.1 of [10] (cf. also [12]),

$$\eta^{-1}\rho'_*\mathbb{C}_W \simeq \eta^{-1}\mathbb{C}_{W_{\mathcal{T}'}} \simeq \mathbb{C}_{W_{\mathcal{T}}} \simeq \rho_*\mathbb{C}_W. \ \blacksquare$$

LEMMA 2.2. *Let* $F \in \mathrm{Mod}(\mathbb{C}_{Y_{\mathcal{T}'}})$. *Then, for any* $W \in \mathrm{Op}(Y_{\mathcal{T}'})$,

$$\Gamma(W;\eta^{-1}F) \simeq \Gamma(W;F).$$

Proof. We may write $F \simeq \varinjlim_i \rho'_* F_i$ with $F_i \in \mathrm{Coh}(\mathcal{T}')$, and by Lemma 2.1 we have $\eta^{-1}F \simeq \varinjlim_i \rho_* F_i$.

Let us first assume that $W \in \mathrm{Op}(Y_{\mathcal{T}'})$ is relatively compact. Then

$$\begin{aligned}\Gamma(W;\varinjlim_i \rho_* F_i) &\simeq \varinjlim_i \Gamma(W;\rho_* F_i) \simeq \varinjlim_i \Gamma(W;F_i)\\ &\simeq \varinjlim_i \Gamma(W;\rho'_* F_i) \simeq \Gamma\Big(W;\varinjlim_i \rho'_* F_i\Big).\end{aligned}$$

Let us now consider an arbitrary W. Then we have $W = \bigcup_n W_n$, with $W_n = U_n \cap W$, where $\{U_n\}_{n\in\mathbb{N}}$ belongs to $\mathrm{Cov}(Y_{\mathcal{T}'})$ and satisfies $U_n \subset\subset U_{n+1}$. Therefore

$$\begin{aligned}\Gamma(W;\varinjlim_i \rho_* F_i) &\simeq \varprojlim_n \Gamma\Big(W_n;\varinjlim_i \rho_* F_i\Big) \simeq \varprojlim_n \Gamma\Big(W_n;\varinjlim_i \rho'_* F_i\Big)\\ &\simeq \Gamma\Big(W;\varinjlim_i \rho'_* F_i\Big). \ \blacksquare\end{aligned}$$

The following two results are straightforward consequences of Lemma 2.2:

COROLLARY 2.3. *The adjunction morphism* $\mathrm{id} \to \eta_*\eta^{-1}$ *is an isomorphism. In particular, the functor* $\eta^{-1} : \mathrm{Mod}(\mathbb{C}_{Y_{\mathcal{T}'}}) \to \mathrm{Mod}(\mathbb{C}_{Y_{\mathrm{sa}}})$ *is fully faithful.*

COROLLARY 2.4. *Let* $W \in \mathcal{T}'$ *and let* $\mathbb{C}_{W_{\mathcal{T}}}$ *(resp.* $\mathbb{C}_{W_{\mathcal{T}'}}$*) be the constant sheaf on* Y_{sa} *(resp.* $Y_{\mathcal{T}'}$*). Then* $\mathbb{C}_{W_{\mathcal{T}'}} \simeq \eta_*\mathbb{C}_{W_{\mathcal{T}}}$.

Let $\mathcal{I}$ be the subcategory of $\mathrm{Mod}(\mathbb{C}_Y)$ consisting of all finite sums $\bigoplus_i \mathbb{C}_{W_i}$ with $W_i \in \mathcal{T}'$ connected.

LEMMA 2.5. *Let* $F, G \in \mathcal{I}$. *Then, given* $\varphi : F \to G$, *we have* $\ker\varphi \in \mathcal{I}$.

Proof. We have $F = \bigoplus_{i=1}^l \mathbb{C}_{W_i}$, $G = \bigoplus_{j=1}^k \mathbb{C}_{W'_j}$. Let p_j, $j = 1,\dots,k$, be the projections on factors of G. Then $\ker\varphi$ is the intersection of the

$\ker(p_j \circ \varphi)$ so that, if each one has the desired form, the same will hold for their intersection. Therefore it is sufficient to assume $k = 1$, say $G = \mathbb{C}_W$. A morphism $\varphi : F \to G$ is then defined by a sequence $v = (v_1, \dots, v_l)$, where v_i is the image under φ of the section of $\mathbb{C}_{W_i}$ defined by 1 on W_i, so $v_i = 0$ if $W_i \not\subset W$. More precisely, if $s = (s_1, \dots, s_l)$ is a germ of F in y, we have $\varphi(s_1, \dots, s_l) = \sum_{i=1}^{l} v_{iy} s_i$. So, given $s = (s_1, \dots, s_l) \in \ker \varphi$, if, for a given i, we have $v_{iy} s_i \neq 0$, then s defines a germ of $H_i := \bigoplus_{i' \neq i} \mathbb{C}_{W_{i'} \cap W_i}$ in y.

Accordingly, $\ker \varphi \simeq \oplus_{i=1}^{l} H_i$. ∎

Therefore, according to the definition of $\mathrm{Coh}(\mathcal{T}')$ and to Lemma 2.5, any $F \in \mathrm{Coh}(\mathcal{T}')$ admits a finite resolution

$$I^\bullet := 0 \to I_1 \to \cdots \to I_n \to F \to 0$$

consisting of objects belonging to $\mathcal{I}$.

Lemma 2.6. *Suppose that, for any $U \in \mathcal{T}'$, $\mathbb{C}_U$ is ρ'_*-acyclic. Then:*

(1) *For any $F \in \mathrm{Coh}(\mathcal{T}')$, F is ρ'_*-acyclic or, equivalently, $\rho_* F$ is η_*-acyclic.*
(2) *Let $F \in D^b(\mathbb{C}_{Y_{\mathcal{T}'}})$. Then $R\Gamma(W; \eta^{-1} F) \simeq R\Gamma(W; F)$ for each W in $\mathrm{Op}(Y_{\mathcal{T}'})$.*

Proof. (1) The equivalence of the two assertions follows from the fact that $R\rho'_* = R\eta_* \circ R\rho_*$ and $\mathbb{R}$-constructible sheaves (and hence $\mathcal{T}'$-coherent sheaves) are ρ_*-acyclic.

Note that the assumption entails that any quotient I_1/I_2 of elements of $\mathcal{I}$ is ρ'_*-acyclic, so F is ρ'_*-acyclic.

(2) By dévissage, we may reduce to $F \in \mathrm{Mod}(\mathbb{C}_{Y_{\mathcal{T}'}})$ and we can write $\eta^{-1} F \simeq \varinjlim_i \rho_* F_i$, with $F_i \in \mathrm{Coh}(\mathcal{T}')$. There exists (see [8, Corollary 9.6.7]) an inductive system of injective resolutions $I_i^\bullet$ of F_i. By (1) (resp. [12, Lemma 2.1.1]), F_i is ρ'_*-acyclic (resp. ρ_*-acyclic), hence $\rho'_* I_i^\bullet$ (resp. $\rho_* I_i^\bullet$) is an injective resolution of $\rho'_* F_i$ (resp. $\rho_* F_i$). Then, with the notations of [3], $\varinjlim_i \rho'_* I_i^\bullet$ (resp. $\varinjlim_i \rho_* I_i^\bullet$) is a $\mathcal{T}'$-flabby (resp. $\mathcal{T}$-flabby) resolution of $\varinjlim_i \rho'_* F_i$ (resp. $\varinjlim_i \rho_* F_i$) and hence $\Gamma(W; \cdot)$-acyclic. We have

$$\begin{aligned} R\Gamma\Big(W; \varinjlim_i \rho_* F_i\Big) &\simeq \Gamma\Big(W; \varinjlim_i \rho_* I_i^\bullet\Big) \simeq \Gamma\Big(W; \varinjlim_i \rho'_* I_i^\bullet\Big) \\ &\simeq R\Gamma\Big(W; \varinjlim_i \rho'_* F_i\Big), \end{aligned}$$

where the second isomorphism follows from Lemma 2.2. ∎

The following two results are straightforward consequences of Lemma 2.6:

Corollary 2.7. *Under the assumption of Lemma 2.6, the adjunction morphism* $\mathrm{id} \to R\eta_* \eta^{-1}$ *is an isomorphism. In particular, the functor $\eta^{-1} : D^b(\mathbb{C}_{Y_{\mathcal{T}'}}) \to D^b(\mathbb{C}_{Y_{\mathrm{sa}}})$ is fully faithful.*

Note that Remark 3.1 in the next section provides an example showing that the converse $\eta^{-1}R\eta_* \to \mathrm{id}$ is not in general an isomorphism.

COROLLARY 2.8. *Assume the conditions of Lemma* 2.6 *hold. Let* $W \in \mathcal{T}'$ *and let* $\mathbb{C}_{W_{\mathcal{T}}}$ *(resp.* $\mathbb{C}_{W_{\mathcal{T}'}}$*) be the constant sheaf on* Y_{sa} *(resp.* $Y_{\mathcal{T}'}$*). Then* $\mathbb{C}_{W_{\mathcal{T}'}} \simeq R\eta_*\mathbb{C}_{W_{\mathcal{T}}}$.

As a consequence of Lemma 2.1 we obtain:

COROLLARY 2.9. *Assume the conditions of Lemma* 2.6 *hold. Let* $F \in D^b(\mathrm{Coh}(\mathcal{T}'))$. *Then* $\eta^{-1}R\rho'_*F \xrightarrow{\sim} R\rho_*F$.

Proof. We have the chain of isomorphisms

$$\eta^{-1}R\rho'_*F \simeq \eta^{-1}\rho'_*F \simeq \rho_*F \simeq R\rho_*F,$$

where the first and the last isomorphisms follow since ρ'_* and ρ_* are acyclic on $\mathrm{Coh}(\mathcal{T}')$, and the second isomorphism comes from the proof of Lemma 2.1. ∎

3. The case of a product. Hereafter we will consider the case where Y is a product $X \times S$ of real analytic manifolds. On $X \times S$ it is natural to consider the family $\mathcal{T}'$ consisting of all finite unions of open relatively compact subsets of the form $U \times V$, which makes $X \times S$ a $\mathcal{T}'$-space. The associated site $Y_{\mathcal{T}'}$ is nothing other than the product of sites $X_{\mathrm{sa}} \times S_{\mathrm{sa}}$. Let $p_1 : X \times S \to X$ and $p_2 : X \times S \to S$ be the projections.

Note that $W \in \mathrm{Op}(X_{\mathrm{sa}} \times S_{\mathrm{sa}})$ is a locally finite union of relatively compact subanalytic open subsets of the form $U \times V$ with $U \in \mathrm{Op}(X_{\mathrm{sa}})$ and $V \in \mathrm{Op}(S_{\mathrm{sa}})$. According to Section 1, we denote by $\eta : (X \times S)_{\mathrm{sa}} \to X_{\mathrm{sa}} \times S_{\mathrm{sa}}$ the natural functor of sites associated to the inclusion $\mathrm{Op}(X_{\mathrm{sa}} \times S_{\mathrm{sa}}) \hookrightarrow \mathrm{Op}((X \times S)_{\mathrm{sa}})$.

We shall need the following result:

LEMMA 3.1. *Let* F, G *be objects of* $D^b(\mathrm{Coh}(\mathcal{T}'))$. *Then* $R\mathcal{H}om(F, G)$ *is an object of* $D^b(\mathrm{Coh}(\mathcal{T}'))$.

Proof. We may assume that $F \simeq \mathbb{C}_U$ and $G \simeq \mathbb{C}_V$ for some $U, V \in \mathcal{T}'$. Moreover, it is sufficient to consider U and V of the form $U = U_1 \times W_1$ and $V = U_2 \times W_2$. Then, as a consequence of Proposition 3.4.4 of [9] we have

$$R\mathcal{H}om(\mathbb{C}_U, \mathbb{C}_V) \simeq R\mathcal{H}om(\mathbb{C}_{U_1}, \mathbb{C}_{U_2}) \boxtimes R\mathcal{H}om(\mathbb{C}_{W_1}, \mathbb{C}_{W_2}).$$

Since $R\mathcal{H}om(\mathbb{C}_{U_1}, \mathbb{C}_{U_2})$ and $R\mathcal{H}om(\mathbb{C}_{W_1}, \mathbb{C}_{W_2})$ are $\mathbb{R}$-constructible complexes respectively on X and S, replacing them by almost free resolutions in the sense of [9], we conclude that $R\mathcal{H}om(\mathbb{C}_U, \mathbb{C}_V)$ belongs to $D^b(\mathrm{Coh}(\mathcal{T}'))$, and the result follows. ∎

PROPOSITION 3.2. *For any* $U \in \mathcal{T}'$, $\mathbb{C}_U$ *is* ρ'_**-acyclic.*

Proof. It is sufficient to consider $U = U_1 \times V_1$ with $U_1 \in \mathrm{Op}(X_{\mathrm{sa}})$ and $V_1 \in \mathrm{Op}(S_{\mathrm{sa}})$. The sheaf $R^j\rho'_*\mathbb{C}_U$ is associated to the presheaf $W \mapsto$

$R^j\Gamma(W;\mathbb{C}_U)$, so it is sufficient to show that $R^j\Gamma(W;\mathbb{C}_U)=0$ for $j\neq 0$ on a family of generators W of the topology of $(X\times S)_{\mathcal{T}'}$. In particular, we may assume that $W\in\mathcal{T}'$, so $W=U'\times V'$.

We use the notations of [9]. By the triangulation theorem there exist a simplicial complex (K_X,Δ_X), a simplicial complex (K_S,Δ_S), a subanalytic homeomorphism $\psi_S:|K_S|\xrightarrow{\sim} S$ compatible with U_1, and a subanalytic homeomorphism $\psi_S:|K_X|\xrightarrow{\sim} X$ compatible with V_1 such that U' is a finite union of the images under ψ_X of open stars of $|K_X|$, and V' is a a finite union of the images under ψ_S of open stars of $|K_S|$. So we may assume that U' is the image of an open star compatible with U_1, and similarly V' is the image of an open star compatible with V_1. On the other hand, it is clear by the assumption on U_1 (resp. V_1) and by the construction of an open star with a given center that $U'\setminus U_1$ always contracts to the center of U' (resp. $V'\setminus V_1$ contracts to the center of V'). Indeed, if the center of U' belongs to U_1, then $U'\subset U_1$. Otherwise, the contraction of U' to its center restricts to a contraction of $U'\setminus U_1$. Consider the distinguished triangle

$$R\Gamma(W;\mathbb{C}_{U_1\times V_1})\to R\Gamma(W;\mathbb{C}_{X\times S})\to R\Gamma(W;\mathbb{C}_{X\times S\setminus U_1\times V_1})\xrightarrow{+}.$$

It is clear that $U'\times V'$ contracts to the product of the centers of U' and V'. On the other hand, the space $(U'\times V')\setminus(U_1\times V_1)=(U'\setminus U_1)\times V'\cup U'\times(V'\setminus V_1)$ is a union of closed contractible subspaces such that the contractions coincide on the intersection, hence the space is contractible. It follows that $R\Gamma(W;\mathbb{C}_{X\times S})\simeq R\Gamma(W;\mathbb{C}_W)$ and $R\Gamma(W;\mathbb{C}_{X\times S\setminus U_1\times V_1})\simeq R\Gamma(W\setminus U_1\times V_1;\mathbb{C}_{W\setminus U_1\times V_1})$ are concentrated in degree zero. This implies that $R\Gamma(W;\mathbb{C}_{U_1\times V_1})$ is concentrated in degree zero as well. ∎

In view of Lemma 2.6 we have

Corollary 3.3. *For any $F\in\mathrm{Coh}(\mathcal{T}')$, F is ρ'_*-acyclic.*

Notation. Since every $F\in\mathrm{Coh}(\mathcal{T}')$ is ρ'_*-acyclic and ρ'_* is fully faithful, we can identify $D^b(\mathrm{Coh}(\mathcal{T}'))$ with its image in $D^b(\mathbb{C}_{(X\times S)_{\mathcal{T}'}})$. When there is no risk of confusion we will write F instead of ρ'_*F, for $F\in D^b(\mathrm{Coh}(\mathcal{T}'))$.

From Corollary 3.3 we have:

Corollary 3.4. *Let $F\in D^b(\mathbb{C}_{(X\times S)_{\mathcal{T}'}})$. Then*

$$R\Gamma(W;\eta^{-1}F)\simeq R\Gamma(W;F)\quad \textit{for each } W\in\mathrm{Op}((X\times S)_{\mathcal{T}'}).$$

In particular $\mathrm{id}\xrightarrow{\sim} R\eta_*\eta^{-1}$.

Remark 3.1. Observe that while $\mathrm{id}\xrightarrow{\sim} R\eta_*\eta^{-1}$, $\eta^{-1}R\eta_*\xrightarrow{\sim}\mathrm{id}$ does not hold in general. This can be illustrated with the following example: Let $X=S=\mathbb{R}$ and let $\overline{B}_1$ be the closed unit ball centered at the origin. It is easy to check that

$$\eta_* \mathbb{C}_{\overline{B}_1} \simeq \varinjlim_{W \supset \overline{B}_1} \rho'_* \mathbb{C}_{\overline{W}}$$

with $W \in \mathcal{T}'$. Then

$$\eta^{-1}\eta_* \mathbb{C}_{\overline{B}_1} \simeq \eta^{-1} \varinjlim_{W \supset \overline{B}_1} \rho'_* \mathbb{C}_{\overline{W}} \simeq \varinjlim_{W \supset \overline{B}_1} \rho_* \mathbb{C}_{\overline{W}} \not\simeq \mathbb{C}_{\overline{B}_1},$$

where the second isomorphism follows from Lemma 2.1.

Lemma 3.5. *Let $F \in D^b(\mathbb{C}_{(X \times S)_{\mathcal{T}'}})$ and $G \in D^b(\mathrm{Coh}(\mathcal{T}'))$. Then*

$$\eta^{-1} R\mathcal{H}om(\rho'_* G, F) \simeq R\mathcal{H}om(\rho_* G, \eta^{-1} F).$$

Proof. Let $F \in D^b(\mathbb{C}_{(X \times S)_{\mathcal{T}'}})$. By dévissage, we may reduce to the case $F \in \mathrm{Mod}(\mathbb{C}_{(X \times S)_{\mathcal{T}'}})$. So F satisfies $F \simeq \varinjlim_i \rho'_* F_i$ with $F_i \in \mathrm{Coh}(\mathcal{T}')$ and we can write $\eta^{-1} F \simeq \varinjlim_i \rho_* F_i$.

We have

$$\begin{aligned} H^j \eta^{-1} R\mathcal{H}om(\rho'_* G, F) &\simeq H^j \eta^{-1} R\mathcal{H}om\Big(\rho'_* G, \varinjlim_i \rho'_* F_i\Big) \\ &\simeq \varinjlim_i H^j \eta^{-1} \rho'_* R\mathcal{H}om(G, F_i) \simeq \varinjlim_i H^j \rho_* R\mathcal{H}om(G, F_i) \\ &\simeq H^j R\mathcal{H}om\Big(\rho_* G, \varinjlim_i \rho_* F_i\Big) \simeq H^j R\mathcal{H}om(\rho_* G, \eta^{-1} F), \end{aligned}$$

where the third isomorphism follows by Lemma 3.1 and Corollary 2.9. ∎

We end this section with a result detailing the behavior of ρ_* and ρ'_* under tensor product:

Lemma 3.6. *Let $F \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ and $G \in D^b(\mathbb{C}_S)$. Then:*

(1) $p_1^{-1} \rho_* F \otimes p_2^{-1} R\rho_* G \simeq R\rho_*(p_1^{-1} F \otimes p_2^{-1} G)$,
(2) $\rho'_* p_1^{-1} F \otimes R\rho'_* p_2^{-1} G \simeq R\rho'_*(p_1^{-1} F \otimes p_2^{-1} G)$.

Proof. Let us recall that the restriction of ρ_* (resp. ρ'_*) to $\mathbb{R}$-constructible sheaves (resp. $\mathcal{T}'$-coherent sheaves) is fully faithful, exact and commutes with $R\mathcal{H}om$, $\otimes$ and inverse image. We will often use these facts during the proof of (1) and (2).

(1) We have the chain of isomorphisms

$$\begin{aligned} R\rho_*(p_1^{-1} F \otimes p_2^{-1} G) &\simeq R\rho_* R\mathcal{H}om(p_1^{-1} D' F, p_2^{-1} G) \\ &\simeq R\mathcal{H}om(\rho_* p_1^{-1} D' F, R\rho_* p_2^{-1} G) \\ &\simeq R\mathcal{H}om(p_1^{-1} D' \rho_* F, p_2^{-1} R\rho_* G) \\ &\simeq p_1^{-1} \rho_* F \otimes p_2^{-1} R\rho_* G. \end{aligned}$$

The first isomorphism follows from Proposition 3.4.4 of [9]; the second from Proposition 2.2.1 of [12]; the third from the fact that $p_2^{-1}(\cdot) \otimes p_2^! \mathbb{C}_{X \times S} \simeq p_2^!(\cdot)$

(Proposition 2.4.9 of [12]) and that $p_2^!$ commutes with $R\rho_*$ (Proposition 2.4.5 of [12]); and the fourth from Lemma 5.3.9 of [13].

(2) We prove the assertion in several steps. Recall that $\rho' = \eta \circ \rho$, where $\eta : (X \times S)_{\mathrm{sa}} \to (X \times S)_{\mathcal{T}'}$ is the natural functor of sites.

(2a) Let $F \in \mathrm{Coh}(\mathcal{T}')$ and $G \in \mathrm{Mod}(\mathbb{C}_{S_{\mathrm{sa}}})$. Then $G = \varinjlim_i \rho_* G_i$ with $G_i \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_S)$. We have

$$\varinjlim_i p_2^{-1} \rho_* G \simeq \varinjlim_i \rho_* p_2^{-1} G_i \simeq \varinjlim_i \eta^{-1} \rho'_* p_2^{-1} G_i \simeq \eta^{-1} \varinjlim_i \rho'_* p_2^{-1} G_i.$$

The first isomorphism follows from Proposition 1.3.3 of [12], the second from Corollary 2.9, and the third since inverse images commute with $\varinjlim$. Therefore $p_2^{-1} G \simeq \eta^{-1} G'$ with $G' \simeq \varinjlim_i \rho'_* p_2^{-1} G_i \in \mathrm{Mod}(\mathbb{C}_{(X\times S)_{\mathcal{T}'}})$. We have

$$R\eta_*(\rho_* F \otimes \eta^{-1} G') \simeq R\eta_* \eta^{-1}(\rho'_* F \otimes G') \simeq \rho'_* F \otimes G'.$$

The first isomorphism follows since inverse images commute with $\otimes$ and $\rho_* \simeq \eta^{-1} \circ \rho'_*$ on $\mathrm{Coh}(\mathcal{T}')$, and the second comes from Proposition 3.2. Hence $R\eta_*(\rho_* F \otimes p_2^{-1} G)$ is concentrated in degree 0.

(2b) Let $F \in D^b(\mathrm{Coh}(\mathcal{T}'))$ and $G \in D^b(\mathbb{C}_{S_{\mathrm{sa}}})$. We shall prove that

$$\eta_*(\rho_* F \otimes p_2^{-1} G) \simeq \eta_* \rho_* F \otimes \eta_* p_2^{-1} G$$

(here we used the last assertion in (2a) to replace $R\eta_*$ by η_*). By dévissage we may reduce to F, G concentrated in degree zero. By (2a), we have $p_2^{-1} G \simeq \eta^{-1} G' \simeq \eta^{-1} \eta_* \eta^{-1} G' \simeq \eta^{-1} \eta_* p_2^{-1} G$ with $G' \in \mathrm{Mod}(\mathbb{C}_{(X\times S)_{\mathcal{T}'}})$ (the second isomorphism follows from Corollary 3.4). In view of the preceding arguments, we have the chain of isomorphisms

$$\begin{aligned}\eta_*(\rho_* F \otimes p_2^{-1} G) &\simeq \eta_*(\eta^{-1} \rho'_* F \otimes \eta^{-1} \eta_* p_2^{-1} G) \simeq \eta_* \eta^{-1}(\rho'_* F \otimes \eta_* p_2^{-1} G) \\ &\simeq \rho'_* F \otimes \eta_* p_2^{-1} G \simeq \eta_* \rho_* F \otimes \eta_* p_2^{-1} G.\end{aligned}$$

(2c) Let $F \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ and $G \in D^b(\mathbb{C}_S)$. Then

$$\begin{aligned}R\eta_* R\rho_*(p_1^{-1} F \otimes p_2^{-1} G) &\simeq R\eta_*(p_1^{-1} \rho_* F \otimes p_2^{-1} R\rho_* G) \\ &\simeq \eta_* \rho_* p_1^{-1} F \otimes R\eta_* p_2^{-1} R\rho_* G \\ &\simeq \eta_* \rho_* p_1^{-1} F \otimes R\eta_* R\rho_* p_2^{-1} G.\end{aligned}$$

The first isomorphism follows from (1), the second from (2b), and the third from the fact that p_2^{-1} commutes with $R\rho_*$ (see the proof of (1)). ∎

4. Construction of relative subanalytic sheaves. Let X and S be two real analytic manifolds. Let F be a sheaf on $(X \times S)_{\mathrm{sa}}$. We shall denote by $F^{S,\sharp}$ the sheaf on $X_{\mathrm{sa}} \times S_{\mathrm{sa}}$ associated to the presheaf

$$\begin{aligned}\mathrm{Op}(X_{\mathrm{sa}} \times S_{\mathrm{sa}}) &\to \mathrm{Mod}(\mathbb{C}),\\ U \times V &\mapsto \Gamma(X \times V; \rho^{-1}\Gamma_{U \times S}F)\\ &\simeq \mathrm{Hom}(\mathbb{C}_U \boxtimes \rho_!\mathbb{C}_V, F)\\ &\simeq \varprojlim_{W \subset\subset V,\, W \in \mathrm{Op}^c(S_{\mathrm{sa}})} \Gamma(U \times W; F).\end{aligned}$$

We set

$$F^S := \eta^{-1} F^{S,\sharp} \tag{4.1}$$

and call it the *relative sheaf* associated to F. It is a sheaf on $(X \times S)_{\mathrm{sa}}$. It is easy to check that $(\cdot)^S$ defines a left exact functor on $\mathrm{Mod}(\mathbb{C}_{(X\times S)_{\mathrm{sa}}})$.

According to Lemma 3.5 we get:

Proposition 4.1. *For each $G \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$, $H \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_S)$ and $F \in D^b(\mathbb{C}_{(X\times S)_{\mathrm{sa}}})$, we have*

$$\eta^{-1} R\mathcal{H}om(\rho'_*(G \boxtimes H), F^{RS,\sharp}) \simeq R\mathcal{H}om(\rho_*(G \boxtimes H), F^{RS}).$$

The following lemmas are steps to prove Proposition 4.7 below:

Lemma 4.2. *Let $U \in \mathrm{Op}(X_{\mathrm{sa}})$ and $V \in \mathrm{Op}(S_{\mathrm{sa}})$. Then*

$$\Gamma(U \times V; F^S) \simeq \Gamma(X \times V; \rho^{-1}\Gamma_{U\times S}F) \simeq \mathrm{Hom}(\mathbb{C}_U \boxtimes \rho_!\mathbb{C}_V, F).$$

Proof. The second isomorphism follows by adjunction. Let us prove the first one. By (2) of Lemma 2.6 it is enough to check that $\Gamma(U \times V; F^{S,\sharp}) = \Gamma(X \times V; \rho^{-1}\Gamma_{U\times S}F)$.

1) First suppose $U \times V$ is relatively compact. Let $s \in \Gamma(U \times V; F^{S,\sharp})$. Then s is defined by a finite family $s_i \in \varprojlim_{W_i \subset\subset V_i} \Gamma(U_i \times W_i; F)$, $i \in I$, where $\{U_i\}$ (resp. $\{V_i\}$) is a covering of U (resp. V) in X_{sa} (resp. S_{sa}) such that $s_i = s_j$ on $(U_i \times V_i) \cap (U_j \times V_j)$.

By Lemma 3.6 of [2], there exists a refinement $\{V'_i\}$ of $\{V_i\}$ in S_{sa} such that $\overline{V'_i} \cap V \subset V_i$. Now we have the following obvious facts:

(i) if, for a given $W'_i \in \mathrm{Op}(S_{\mathrm{sa}})$, $W'_i \subset\subset V'_i$, then $W'_i \subset\subset V$,
(ii) if, for a given $W \in \mathrm{Op}(S_{\mathrm{sa}})$, $W \subset\subset V$, then $V'_i \cap W \subset\subset V_i$.

This implies that the restriction

$$\varprojlim_{W_i \subset\subset V_i} \Gamma(U_i \times W_i; F) \to \varprojlim_{W'_i \subset\subset V'_i} \Gamma(U_i \times W'_i; F)$$

factors through $\varprojlim_{W \subset\subset V} \Gamma(U_i \times (W \cap V'_i); F)$. Therefore $s|_{U_i \times V'_i}$ extends to a section of

$$\Gamma(X \times V; \rho^{-1}\Gamma_{U_i \times V'_i}F) \simeq \varprojlim_{W \subset\subset V} \Gamma(U_i \times (W \cap V'_i); F).$$

Set $U_{ij} = U_i \cap U_j$ and $V'_{ij} = V'_i \cap V'_j$. The exact sequence

$$\bigoplus_{i \neq j \in I} \mathbb{C}_{U_{ij} \times V'_{ij}} \to \bigoplus_{k \in I} \mathbb{C}_{U_k \times V'_k} \to \mathbb{C}_{U \times V} \to 0$$

defines an exact sequence

$$\begin{aligned} 0 \to \Gamma(X \times V; \rho^{-1}\Gamma_{U \times V} F) \to \bigoplus_k \Gamma(X \times V; \rho^{-1}\Gamma_{U_k \times V'_k} F) \\ \to \bigoplus_{i \neq j} \Gamma(X \times V; \rho^{-1}\Gamma_{U_{ij} \times V'_{ij}} F). \end{aligned}$$

Then the s_i's glue to a section of

$$\Gamma(X \times V; \rho^{-1}\Gamma_{U \times V} F) \simeq \Gamma(X \times V; \rho^{-1}\Gamma_{U \times S} F)$$

as required.

2) Suppose $U \in \mathrm{Op}(X_{\mathrm{sa}})$ and $V \in \mathrm{Op}^c(S_{\mathrm{sa}})$. Then $U = \bigcup_{n \in \mathbb{N}}(U \cap U_n)$ where $\{U_n\}_{n \in \mathbb{N}}$ belongs to $\mathrm{Cov}(X_{\mathrm{sa}})$ and satisfies $U_n \subset\subset U_{n+1}$. Then

$$\begin{aligned} \Gamma(U \times V; F^{S,\sharp}) \simeq \varprojlim_n \Gamma(U_n \times V; F^{S,\sharp}) \simeq \varprojlim_n \Gamma(X \times V; \rho^{-1}\Gamma_{U_n \times S} F) \\ \simeq \Gamma\Big(X \times V; \rho^{-1} \varprojlim_n \Gamma_{U_n \times S} F\Big) \simeq \Gamma(X \times V; \rho^{-1}\Gamma_{U \times S} F). \end{aligned}$$

3) Now consider the general case. Let $s \in \Gamma(U \times V; F^{S,\sharp})$. It is defined by a countable family $s_n \in \Gamma(U \times V_n; F^{S,\sharp}) = \Gamma(X \times V_n; \rho^{-1}\Gamma_{U \times S} F)$ where $\{V_n\}_{n \in \mathbb{N}}$ is a covering of V in S_{sa} such that $\overline{V_n} \cap V \subset V_{n+1}$. Then there exists a refinement $\{V'_n\}$ of $\{V_n\}$ in S_{sa} with $V_{n-1} \subset \overline{V'_n} \cap V \subset V_n$. Arguing as in 1), the restriction $s_n|_{U \times V'_n}$ belongs to $\Gamma(X \times V; \rho^{-1}\Gamma_{U \times V'_n} F)$ and $s_n = s_{n+1}$ on $U \times V'_n$. Hence they glue to $s \in \Gamma(X \times V; \rho^{-1}\Gamma_{U \times V} F) \simeq \Gamma(X \times V; \rho^{-1}\Gamma_{U \times S} F)$ as required. ∎

With Proposition 6.5.1 of [10] (applied on X and S separately) as a tool we now prove the following result:

LEMMA 4.3. *Let $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ and $H \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_S)$, and let $F \in \mathrm{Mod}(\mathbb{C}_{(X \times S)_{\mathrm{sa}}})$. Then*

$$\mathrm{Hom}(G \boxtimes H, F^S) \simeq \mathrm{Hom}(G \boxtimes \rho_! H, F) \simeq \mathrm{Hom}(\mathbb{C}_X \boxtimes H, \rho^{-1}\mathcal{H}om(G \boxtimes \mathbb{C}_S, F)).$$

Proof. The right hand isomorphism follows by adjunction. Let us prove the left hand isomorphism.

1) Suppose at first that G and H have compact support. By Proposition 6.5.1 of [10] the functor $U \mapsto \mathrm{Hom}(\mathbb{C}_U \boxtimes \rho_! \mathbb{C}_V, F) \simeq \mathrm{Hom}(\mathbb{C}_U \boxtimes \mathbb{C}_V, F^S)$ extends uniquely to a functor $\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(\mathbb{C}_X) \to \mathrm{Mod}(\mathbb{C})$. This implies

$$\mathrm{Hom}(G \boxtimes \rho_! \mathbb{C}_V, F) \simeq \mathrm{Hom}(G \boxtimes \mathbb{C}_V, F^S).$$

Similarly, the functor $V \mapsto \mathrm{Hom}(G \boxtimes \rho_! \mathbb{C}_V, F) \simeq \mathrm{Hom}(G \boxtimes \mathbb{C}_V, F^S)$ extends uniquely to a functor $\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(\mathbb{C}_S) \to \mathrm{Mod}(\mathbb{C})$. This implies

$$\mathrm{Hom}(G \boxtimes \rho_! H, F) \simeq \mathrm{Hom}(G \boxtimes H, F^S).$$

2) Let us consider the general case. Let $\{U_n\}_{n\in\mathbb{N}}$ (resp. $\{V_n\}_{n\in\mathbb{N}}$ be a covering of X_{sa} (resp. S_{sa}) such that $U_n \subset\subset U_{n+1}$ (resp. $V_n \subset\subset V_{n+1}$) for each n. We have

$$\begin{aligned}\mathrm{Hom}(G\boxtimes H, F^S) &\simeq \varprojlim_n \mathrm{Hom}(G_{U_n}\boxtimes H_{V_n}, F^S) \simeq \varprojlim_n \mathrm{Hom}(G_{U_n}\boxtimes \rho_!(H_{V_n}), F)\\ &\simeq \varprojlim_n \mathrm{Hom}(G_{U_n}\boxtimes (\rho_! H)_{V_n}, F)\\ &\simeq \varprojlim_n \mathit{\Gamma}(U_n\times V_n; \mathcal{H}om(G\boxtimes \rho_! H, F))\\ &\simeq \mathit{\Gamma}(X\times S; \mathcal{H}om(G\boxtimes\rho_! H, F)) \simeq \mathrm{Hom}(G\boxtimes\rho_! H, F).\end{aligned}$$

The second isomorphism follows from 1). To prove the third one we remark that the morphism $(\rho_! H)_{V_n} \to (\rho_! H)_{V_{n+1}}$ factors through $\rho_!(H_{V_{n+1}}) \simeq \varinjlim_{W\subset\subset V_{n+1}} (\rho_! H)_W$. The desired isomorphism then follows by passing to the limit over $n \in \mathbb{N}$. ∎

We shall now prepare the steps to the main result of this note, Proposition 4.7 below. Recall (cf. [3]) that $F \in \mathrm{Mod}(\mathbb{C}_{(X_{\mathrm{sa}}\times S_{\mathrm{sa}})})$ is $\mathcal{T}'$-flabby if the restriction morphism $\mathit{\Gamma}(X\times S; F) \to \mathit{\Gamma}(W; F)$ is surjective for each $W \in \mathcal{T}'$. $\mathcal{T}'$-flabby objects are $\mathrm{Hom}(G, \cdot)$-acyclic for each $G \in \mathrm{Coh}(\mathcal{T}')$.

Lemma 4.4. *Let $F \in \mathrm{Mod}(\mathbb{C}_{(X\times S)_{\mathrm{sa}}})$ be injective. Then $F^{S,\sharp}$ is $\mathcal{T}'$-flabby.*

Proof. Let $W = \bigcup_{i=1}^n (U_i \times V_i)$ with $U_i \in \mathrm{Op}^c(X_{\mathrm{sa}})$ and $V_i \in \mathrm{Op}^c(S_{\mathrm{sa}})$. For $i \in \{1, \dots, n\}$, set

$$K_i = \varinjlim_{W_1\subset\subset V_1} \cdots \varinjlim_{W_i\subset\subset V_i} \rho_* \mathbb{C}_{(U_1\times W_1)\cup\cdots\cup(U_i\times W_i)}.$$

1) We first prove that $\mathit{\Gamma}\left(W; F^S\right) \simeq \mathrm{Hom}(K_n, F)$. We argue by induction on n. For $n = 1$ the result follows from Lemma 4.2.

$n-1 \Rightarrow n$: Set $K'_{n-1} = K_{n-1} \otimes (\mathbb{C}_{U_n} \boxtimes \rho_! \mathbb{C}_{V_n})$. We have

$$\begin{aligned}&K'_{n-1}\\ &\quad\simeq \varinjlim_{\substack{W_1\subset\subset V_1\\ W_n\subset\subset V_n}} \cdots \varinjlim_{\substack{W_{n-1}\subset\subset V_{n-1}\\ W_n\subset\subset V_n}} \rho_* \mathbb{C}_{((U_1\cap U_n)\times(W_1\cap W_n))\cup\cdots\cup((U_{n-1}\cap U_n)\times(W_{n-1}\cap W_n))}\\ &\quad\simeq \varinjlim_{W'_1\subset\subset V_1\cap V_n} \cdots \varinjlim_{W'_{n-1}\subset\subset V_{n-1}\cap V_n} \rho_* \mathbb{C}_{((U_1\cap U_n)\times W'_1)\cup\cdots\cup((U_{n-1}\cap U_n)\times W'_{n-1})}.\end{aligned}$$

We have an exact sequence

$$0 \to K'_{n-1} \to K_{n-1} \oplus (\mathbb{C}_{U_n} \boxtimes \rho_! \mathbb{C}_{V_n}) \to K_n \to 0.$$

Applying the functor $\mathrm{Hom}(\cdot, F)$ and using the induction hypothesis for K'_{n-1} and K_{n-1} we obtain

$$\Gamma\Big(\bigcup_{i=1}^{n-1}((U_i \cap U_n) \times (V_i \cap V_n)); F^S\Big) \simeq \mathrm{Hom}(K'_{n-1}, F),$$

$$\Gamma\Big(\bigcup_{i=1}^{n-1}(U_i \times V_i); F^S\Big) \simeq \mathrm{Hom}(K_{n-1}, F).$$

Hence $\Gamma\left(\bigcup_{i=1}^{n}(U_i \times V_i); F^S\right) \simeq \mathrm{Hom}(K_n, F)$, as required.

2) Consider the monomorphism $0 \to K_n \to \mathbb{C}_X \boxtimes \mathbb{C}_S$. Since F is injective, we obtain a surjection

$$\mathrm{Hom}(\mathbb{C}_X \boxtimes \mathbb{C}_S, F) \to \mathrm{Hom}(K_n, F) \to 0,$$

and the result follows. ∎

Corollary 4.5. *Let $G \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ and $H \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_S)$, and suppose $F \in \mathrm{Mod}(\mathbb{C}_{(X\times S)_{\mathrm{sa}}})$ is injective. Then F^S is $\mathrm{Hom}(G \boxtimes H, \cdot)$-acyclic.*

Proof. First note that, F being injective, we have $F^{RS,\sharp} \simeq F^{S,\sharp}$ and $F^{RS} \simeq F^S$. By Propositions 4.1 and 3.2,

$$\begin{aligned}\mathrm{RHom}(G \boxtimes H, F^S) &\simeq R\Gamma(X \times S; R\mathcal{H}om(G \boxtimes H, F^S))\\ &\simeq R\Gamma(X \times S; \eta^{-1} R\mathcal{H}om(G \boxtimes H, F^{S,\sharp}))\\ &\simeq R\Gamma(X \times S; R\mathcal{H}om(G \boxtimes H, F^{S,\sharp}))\\ &\simeq \mathrm{RHom}(G \boxtimes H, F^{S,\sharp}).\end{aligned}$$

Lemma 4.4 implies that $F^{S,\sharp}$ is $\mathrm{Hom}(G \boxtimes H, \cdot)$-acyclic and the result follows. ∎

Lemma 4.6. *Let $G \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$, $H \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_S)$, and $F \in D^b(\mathbb{C}_{(X\times S)_{\mathrm{sa}}})$. Then*

$$\begin{aligned}\mathrm{RHom}(G \boxtimes H, F^{RS}) &\simeq \mathrm{RHom}(G \boxtimes \rho_! H, F)\\ &\simeq \mathrm{RHom}(\mathbb{C}_X \boxtimes H, \rho^{-1} R\mathcal{H}om(G \boxtimes \mathbb{C}_S, F)).\end{aligned}$$

Proof. The second isomorphism follows by adjunction. Let us prove the first one.

By Corollary 4.5, $(\cdot)^S$ sends injective objects of $\mathrm{Mod}(\mathbb{C}_{(X\times S)_{\mathrm{sa}}})$ to $\mathrm{Hom}(G \boxtimes H, \cdot)$-acyclic objects, for $G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ and $H \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_S)$.

Therefore, we may reduce to F injective and G, H concentrated in degree 0. Then the result follows from Lemma 4.3. ∎

Observe that if $K \in \mathrm{Mod}(\mathbb{C}_{(X\times S)_{\mathrm{sa}}})$ then $\rho^{-1}K \xleftarrow{\sim} \rho^{-1}\eta^{-1}\eta_* K$. Indeed, for each $y \in X \times S$,

$$(\rho^{-1}K)_y \simeq \varinjlim_{U\times V \ni y} K(U \times V) \simeq (\rho^{-1}\eta^{-1}\eta_* K)_y$$

with $U \in \mathrm{Op}(X_{\mathrm{sa}})$ and $V \in \mathrm{Op}(S_{\mathrm{sa}})$.

Proposition 4.7. *Let $F \in D^b(\mathbb{C}_{(X\times S)_{\mathrm{sa}}})$ and $G \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$, and let $H \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_S)$. Then*

$$\begin{aligned}\rho^{-1}R\mathcal{H}om(G\boxtimes H, F^{RS}) &\simeq \rho^{-1}R\mathcal{H}om(G\boxtimes \rho_! H, F)\\ &\simeq R\mathcal{H}om(\mathbb{C}_X\boxtimes H, \rho^{-1}R\mathcal{H}om(G\boxtimes \mathbb{C}_S, F)).\end{aligned}$$

In particular, when $G=\mathbb{C}_X$ and $H=\mathbb{C}_S$ we have $\rho^{-1}F \simeq \rho^{-1}F^{RS} \simeq \rho'^{-1}F^{RS,\sharp}$.

Proof. The second isomorphism follows by adjunction. Let us prove the first one.

1) Let us first suppose that F, G, H are concentrated in degree zero. Hence, by the remark above, to any morphism

$$\eta_*\mathcal{H}om(G\boxtimes \rho_! H, F) \to \eta_*\mathcal{H}om(G\boxtimes H, F^S)$$

one associates a morphism

$$\rho^{-1}\mathcal{H}om(G\boxtimes \rho_! H, F) \to \rho^{-1}\mathcal{H}om(G\boxtimes H, F^S).$$

Note that the natural morphism $\rho_!(H_V) \to (\rho_! H)_V$ induces a morphism $\mathrm{Hom}(G_U\boxtimes(\rho_! H)_V, F) \to \mathrm{Hom}(G_U\boxtimes \rho_!(H_V), F)$, hence a morphism $\psi : \eta_*\mathcal{H}om(G\boxtimes \rho_! H, F) \to \eta_*\mathcal{H}om(G\boxtimes H, F^S)$, which defines a morphism $\rho^{-1}\mathcal{H}om(G\boxtimes \rho_! H, F) \to \rho^{-1}\mathcal{H}om(G\boxtimes H, F^S)$.

Let us check on the fibers that the latter is an isomorphism. Let $y \in X\times S$, then

$$\begin{aligned}(\rho^{-1}\mathcal{H}om(G\boxtimes \rho_! H, F))_y &\simeq \varinjlim_{U\times V\ni y} \mathrm{Hom}(G_U\boxtimes(\rho_! H)_V, F)\\ &\simeq \varinjlim_{U\times V\ni y}\varprojlim_{W\subset\subset V} \mathrm{Hom}(G_U\boxtimes(\rho_! H)_W, F)\\ &\simeq \varinjlim_{U\times V\ni y} \mathrm{Hom}(G_U\boxtimes \rho_!(H_V), F)\\ &\simeq (\rho^{-1}\mathcal{H}om(G\boxtimes H, F^S))_y\end{aligned}$$

with $U \in \mathrm{Op}(X_{\mathrm{sa}})$ and $V, W \in \mathrm{Op}(S_{\mathrm{sa}})$.

2) Suppose now that F is injective and that G, H are concentrated in degree 0. Let $U \in \mathrm{Op}(X_{\mathrm{sa}})$ and $V \in \mathrm{Op}(S_{\mathrm{sa}})$. The complex

$$R\Gamma(U\times V; R\mathcal{H}om(G\boxtimes H, F^S)) \simeq \mathrm{RHom}(G_U\boxtimes H_V, F^S)$$

is concentrated in degree 0 by Corollary 4.5. Hence F^S is $R\mathcal{H}om(G\boxtimes H,\cdot)$-acyclic.

3) Let $G \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ and $H \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_S)$. Let $F \in D^b(\mathbb{C}_{(X\times S)_{\mathrm{sa}}})$ and let $I^\bullet$ be a complex of injective objects quasi-isomorphic to F. Then

$$\rho^{-1}R\mathcal{H}om(G\boxtimes\rho_!H,F)\simeq\rho^{-1}\mathcal{H}om(G\boxtimes\rho_!H,I^\bullet)\simeq\rho^{-1}\mathcal{H}om(G\boxtimes H,(I^\bullet)^S)$$
$$\simeq\rho^{-1}R\mathcal{H}om(G\boxtimes H,F^{RS}),$$

where the second isomorphism follows from 1) and the third one from 2). ∎

We end this section with the following result on the acyclicity for the functor $(\cdot)^S$, which will be needed later:

Proposition 4.8. *Suppose that $F\in\mathrm{Mod}(\rho_!\mathcal{C}^\infty_{X\times S})$ is $\Gamma(W;\cdot)$-acyclic for each $W\in\mathrm{Op}((X\times S)_{\mathrm{sa}})$. Then for each $U\in\mathrm{Op}(X_{\mathrm{sa}})$ and $V\in\mathrm{Op}(S_{\mathrm{sa}})$ we have $R^k\Gamma(U\times V;F^{RS,\sharp})=R^k\Gamma(U\times V;F^{RS})=0$ if $k\neq 0$.*

Proof. By Lemma 4.6 we have $R\Gamma(U\times V;F^{RS,\sharp})\simeq R\Gamma(U\times V;F^{RS})\simeq R\Gamma(X\times V;\rho^{-1}R\Gamma_{U\times S}F)$. As F is $\Gamma(W;\cdot)$-acyclic for $W\in\mathrm{Op}((X\times S)_{\mathrm{sa}})$, the complex $R\Gamma_{U\times S}F$ is concentrated in degree zero. Since F is a $\rho_!\mathcal{C}^\infty_{X\times S}$-module, $\rho^{-1}\Gamma_{U\times S}F$ is a $\mathcal{C}^\infty_{X\times S}$-module, hence is c-soft and $\Gamma(X\times V;\cdot)$-acyclic. This shows the result. ∎

Corollary 4.9. *Suppose that $F\in\mathrm{Mod}(\rho_!\mathcal{C}^\infty_{X\times S})$ is $\Gamma(W;\cdot)$-acyclic for each $W\in\mathrm{Op}((X\times S)_{\mathrm{sa}})$. Then F is $(\cdot)^{S,\sharp}$-acyclic and $(\cdot)^S$-acyclic.*

Proof. As $(\cdot)^{RS}\simeq\eta^{-1}\circ(\cdot)^{RS,\sharp}$, it is enough to show that $H^kF^{RS,\sharp}=0$ if $k\neq 0$. It is enough to prove that $R^k\Gamma(W;F^{RS,\sharp})=0$ if $k\neq 0$ on a basis for the topology of $(X\times S)_{\mathcal{T}'}$. Since the products $U\times V$ with $U\in\mathrm{Op}(X_{\mathrm{sa}})$, $V\in\mathrm{Op}(S_{\mathrm{sa}})$ form a basis, the result follows from Proposition 4.8. ∎

5. The sheaves $\mathcal{C}^{\infty,t,S}_{X\times S}$, $\mathcal{D}b^{t,S}_{X\times S}$, $\mathcal{C}^{\infty,\mathrm{w},S}_{X\times S}$, $\mathcal{O}^{t,S}_{X\times S}$ and $\mathcal{O}^{\mathrm{w},S}_{X\times S}$. Let X and S be real analytic manifolds. The construction given by (4.1) allows us to introduce the following sheaves:

$\mathcal{C}^{\infty,t,S}_{X\times S}:=(\mathcal{C}^{\infty,t}_{X\times S})^S$ as the relative sheaf associated to $\mathcal{C}^{\infty,t}_{X\times S}$,

$\mathcal{D}b^{t,S}_{X\times S}:=(\mathcal{D}b^{t}_{X\times S})^S$ as the relative sheaf associated to $\mathcal{D}b^{t}_{X\times S}$,

$\mathcal{C}^{\infty,\mathrm{w},S}_{X\times S}:=(\mathcal{C}^{\infty,\mathrm{w}}_{X\times S})^S$ as the relative sheaf associated to $\mathcal{C}^{\infty,\mathrm{w}}_{X\times S}$.

We then derive from Lemma 4.2:

Proposition 5.1. *Let $U\in\mathrm{Op}(X_{\mathrm{sa}})$ and $V\in\mathrm{Op}(S_{\mathrm{sa}})$. Then*

$$\Gamma(U\times V;\mathcal{C}^{\infty,t,S}_{X\times S})\simeq\Gamma(X\times V;\rho^{-1}\Gamma_{U\times S}\mathcal{C}^{\infty,t}_{X\times S})\simeq\Gamma(X\times V;T\mathcal{H}om(\mathbb{C}_{U\times S},\mathcal{C}^\infty_{X\times S})),\tag{1}$$

$$\Gamma(U\times V;\mathcal{D}b^{t,S}_{X\times S})\simeq\Gamma(X\times V;\rho^{-1}\Gamma_{U\times S}\mathcal{D}b^{t}_{X\times S})\simeq\Gamma(X\times V;T\mathcal{H}om(\mathbb{C}_{U\times S},\mathcal{D}b_{X\times S})),\tag{2}$$

$$\Gamma(U\times V;\mathcal{C}^{\infty,\mathrm{w},S}_{X\times S})\simeq\Gamma(X\times V;\rho^{-1}\Gamma_{U\times S}\mathcal{C}^{\infty,\mathrm{w}}_{X\times S})\simeq\Gamma(X\times V;H^0D'\mathbb{C}_U\boxtimes\mathbb{C}_S\overset{\mathrm{w}}{\otimes}\mathcal{C}^\infty_{X\times S}).\tag{3}$$

We can now state:

PROPOSITION 5.2.

(i) *Suppose that* $\mathcal{F} = \mathcal{D}b^t_{X\times S}, \mathcal{C}^{\infty,t}_{X\times S}, \mathcal{C}^{\infty,\mathrm{w}}_{X\times S}$. *Then* $\mathcal{F}$ *is* $(\cdot)^{S,\sharp}$*-acyclic and hence* $(\cdot)^S$*-acyclic. Moreover* $\mathcal{D}b^{t,S}_{X\times S}$ *and* $\mathcal{C}^{\infty,t,S}_{X\times S}$ *are* $\Gamma(U\times V;\cdot)$*-acyclic for each* $U\in \mathrm{Op}(X_{\mathrm{sa}})$ *and* $V\in\mathrm{Op}(S_{\mathrm{sa}})$.

(ii) $\mathcal{C}^{\infty,\mathrm{w},S}_{X\times S}$ *is* $\Gamma(U\times V;\cdot)$*-acyclic for each* $U\in\mathrm{Op}(X_{\mathrm{sa}})$ *locally cohomologically trivial and* $V\in\mathrm{Op}(S_{\mathrm{sa}})$.

Applying Proposition 4.7 and [10, Proposition 7.2.6] we conclude:

PROPOSITION 5.3. *Let* $G\in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ *and* $H\in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_S)$. *Then*

$$\begin{aligned}(1)\quad \rho^{-1}R\mathcal{H}om(G\boxtimes H,\mathcal{C}^{\infty,t,S}_{X\times S}) &\simeq \rho^{-1}R\mathcal{H}om(G\boxtimes\rho_! H,\mathcal{C}^{\infty,t}_{X\times S})\\ &\simeq R\mathcal{H}om(\mathbb{C}_X\boxtimes H,\, T\mathcal{H}om(G\boxtimes\mathbb{C}_S,\mathcal{C}^{\infty}_{X\times S})),\\ (2)\quad \rho^{-1}R\mathcal{H}om(G\boxtimes H,\mathcal{D}b^{t,S}_{X\times S}) &\simeq \rho^{-1}R\mathcal{H}om(G\boxtimes\rho_! H,\mathcal{D}b^{t}_{X\times S})\\ &\simeq R\mathcal{H}om(\mathbb{C}_X\boxtimes H,\, T\mathcal{H}om(G\boxtimes\mathbb{C}_S,\mathcal{D}b_{X\times S})),\\ (3)\quad \rho^{-1}R\mathcal{H}om(G\boxtimes H,\mathcal{C}^{\infty,\mathrm{w},S}_{X\times S}) &\simeq \rho^{-1}R\mathcal{H}om(G\boxtimes\rho_! H,\mathcal{C}^{\infty,\mathrm{w}}_{X\times S})\\ &\simeq R\mathcal{H}om(\mathbb{C}_X\boxtimes H,\, D'G\boxtimes\mathbb{C}_S\overset{\mathrm{w}}{\otimes}\mathcal{C}^{\infty}_{X\times S}).\end{aligned}$$

In particular, when $G=\mathbb{C}_X$ *and* $H=\mathbb{C}_S$ *we have* $\rho^{-1}\mathcal{C}^{\infty,t,S}_{X\times S}\simeq\mathcal{C}^{\infty}_{X\times S}$, $\rho^{-1}\mathcal{D}b^{t,S}_{X\times S}\simeq\mathcal{D}b_{X\times S}$ *and* $\rho^{-1}\mathcal{C}^{\infty,\mathrm{w},S}_{X\times S}\simeq\mathcal{C}^{\infty}_{X\times S}$.

LEMMA 5.4. *There is a natural action of* $\rho_!\mathcal{D}_{X\times S}$ *on* $\mathcal{D}b^{t,S}_{X\times S}$, $\mathcal{C}^{\infty,t,S}_{X\times S}$ *and* $\mathcal{C}^{\infty,\mathrm{w},S}_{X\times S}$.

Proof. The proof being similar in the three cases, we just consider the first case. By Proposition 3.2.1 of [12], it is enough to check that the presheaf

$$\eta^{\dagger}\mathcal{D}b^{t,S,\sharp}_{X\times S}(W)=\varinjlim_{W\subset W'}\mathcal{D}b^{t,S,\sharp}_{X\times S}(W')$$

with $W'\in\mathrm{Op}(X_{\mathrm{sa}}\times S_{\mathrm{sa}})$, is a presheaf over the presheaf of rings $W\mapsto\Gamma(\overline{W};\mathcal{D}_{X\times S})$. Setting $W'=U\times V$, by Lemma 2.1 we have

$$\Gamma(U\times V;\mathcal{D}b^{t,S,\sharp}_{X\times S})=\Gamma(U\times V;\mathcal{D}b^{t}_{X\times V}).$$

We may assume that $W\in\mathrm{Op}^c((X\times S)_{\mathrm{sa}})$. Thus we can cover $\overline{W}$ by finitely many open subsets $\{U_i\times V_i\}$, $\{U_i'\times V_i'\}$ with $U_i\times V_i, U_i'\times V_i'\in\mathrm{Op}^c(X_{\mathrm{sa}}\times S_{\mathrm{sa}})$ sufficiently small and such that $U_i'\times V_i'\subset\subset U_i\times V_i$. Given $P\in\Gamma(\overline{W};\mathcal{D}_{X\times S})$, for a convenient covering $\{U_i\times V_i\}$ as above, P is defined on $\bigcup_i U_i\times V_i$. We then deduce the action of P on $\varinjlim_{W\subset W'}\Gamma(W';\mathcal{D}b^{t,S,\sharp}_{X\times S})$ as the image of the gluing of the actions on each $\Gamma(U_i'\times V_i';\mathcal{D}b^{t}_{X\times V_i'})$. ∎

Now assume that X and S are complex manifolds and consider the projection $f : X \times S \to S$. Denote as usual by $\overline{X} \times \overline{S}$ the complex conjugate manifold. Identifying the underlying real analytic manifold $X_{\mathbb{R}} \times S_{\mathbb{R}}$ with the diagonal of $(X \times S) \times (\overline{X} \times \overline{S})$, we have:

LEMMA 5.5. *$\rho_* f^{-1}\mathcal{O}_S$ (resp. $\rho'_* f^{-1}\mathcal{O}_S$) acts on $\mathcal{D}b^{t,S}_{X\times S}$, $\mathcal{C}^{\infty,t}_{X\times S}$ and $\mathcal{C}^{\mathrm{w},S}_{X\times S}$ (resp. on $\mathcal{D}b^{t,S,\sharp}_{X\times S}$, $\mathcal{C}^{\infty,t,S,sharp}_{X\times S}$ and $\mathcal{C}^{\mathrm{w},S,\sharp}_{X\times S}$).*

Proof. To prove the action of $\rho_* f^{-1}\mathcal{O}_S$ it is sufficient to check that $\rho_* f^{-1}\mathcal{O}_S(W)$ acts on $\mathcal{D}b^{t,S}_{X\times S}(W)$ on a basis for the topology of $(X \times S)_{\mathrm{sa}}$. Since every relatively compact subanalytic open subset of $X \times S$ can be covered by open cells (cf. [15]), we may suppose that W is an open cell such that $f|_W : W \to f(W)$ is the restriction of a composition of projections $f_j : \mathbb{R}^j \times f(W) \to \mathbb{R}^{j-1} \times f(W)$ and the fibers of f intersected with W are contractible or empty. In this case we have $\rho_* f^{-1}\mathcal{O}_S(W) = \mathcal{O}_S(f(W))$, and $\mathcal{O}_S(f(W))$ acts on $\mathcal{D}b^{t,S}_{X\times S}(W)$, since $\mathcal{D}b^{t,S}_{X\times S}(W)$ has no growth conditions on the boundary of $f^{-1}(f(W))$. The proof for $\mathcal{C}^{\infty,t}_{X\times S}$ and $\mathcal{C}^{\mathrm{w},S}_{X\times S}$ is similar.

Similarly, to prove the action of $\rho'_* f^{-1}\mathcal{O}_S$ it is sufficient to check that $\rho'_* f^{-1}\mathcal{O}_S(U \times V) \simeq \mathcal{O}_S(V)$ acts on $\mathcal{D}b^{t,S}_{X\times S}(U \times V)$ where $U \in \mathrm{Op}(Y_{\mathrm{sa}})$ is assumed to be contractible and $V \in \mathrm{Op}(S_{\mathrm{sa}})$. Since $\mathcal{D}b^{t,S}_{X\times S}(U \times V) \simeq \varprojlim_{W\subset\subset V, W\in \mathrm{Op}^c(S_{\mathrm{sa}})} \Gamma(U \times W; \mathcal{D}b^t_{X\times S})$, the statement is clear. ∎

The construction given by (4.1) allows us to introduce the following objects of $D^b(\mathbb{C}_{(X\times S)_{\mathrm{sa}}})$:

- $\mathcal{O}^{t,S}_{X\times S} := (\mathcal{O}^t_{X\times S})^{RS}$, the relative sheaf associated to $\mathcal{O}^t_{X\times S}$, that is,

$$\begin{aligned}\mathcal{O}^{t,S}_{X\times S} &\simeq (R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}\times\overline{S}}}(\rho_!\mathcal{O}_{\overline{X}\times\overline{S}}, \mathcal{D}b^t_{X\times S}))^{RS}\\ &\simeq (R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}\times\overline{S}}}(\rho_!\mathcal{O}_{\overline{X}\times\overline{S}}, \mathcal{C}^{\infty,t}_{X\times S}))^{RS},\end{aligned}$$

- $\mathcal{O}^{\mathrm{w},S}_{X\times S} := (\mathcal{O}^{\mathrm{w}}_{X\times S})^{RS}$, the relative sheaf associated to $\mathcal{O}^{\mathrm{w}}_{X\times S}$, that is,

$$\mathcal{O}^{\mathrm{w},S}_{X\times S} \simeq (R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}\times\overline{S}}}(\rho_!\mathcal{O}_{\overline{X}\times\overline{S}}, \mathcal{C}^{\infty,\mathrm{w}}_{X\times S}))^{RS}.$$

The exactness of $\rho_!$ together with Proposition 5.2 yield:

PROPOSITION 5.6. *In $D^b(\mathbb{C}_{(X\times S)_{\mathrm{sa}}})$ we have*

$$\mathcal{O}^{t,S}_{X\times S} \simeq R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}\times\overline{S}}}(\rho_!\mathcal{O}_{\overline{X}\times\overline{S}}, \mathcal{D}b^{t,S}_{X\times S}) \simeq R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}\times\overline{S}}}(\rho_!\mathcal{O}_{\overline{X}\times\overline{S}}, \mathcal{C}^{\infty,t,S}_{X\times S}),$$
$$\mathcal{O}^{\mathrm{w},S}_{X\times S} \simeq R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}\times\overline{S}}}(\rho_!\mathcal{O}_{\overline{X}\times\overline{S}}, \mathcal{C}^{\infty,\mathrm{w},S}_{X\times S}).$$

Proposition 4.7 together with [10, Proposition 7.3.2] entail:

Proposition 5.7. *Let $G \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ and $H \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_S)$. Then*

$$(1) \quad \rho^{-1}R\mathcal{H}om(G \boxtimes H, \mathcal{O}^{t,S}_{X\times S}) \simeq \rho^{-1}R\mathcal{H}om(G \boxtimes \rho_! H, \mathcal{O}^t_{X\times S}) \simeq R\mathcal{H}om(\mathbb{C}_X \boxtimes H, T\mathcal{H}om(G \boxtimes \mathbb{C}_S, \mathcal{O}_{X\times S})),$$

$$(2) \quad \rho^{-1}R\mathcal{H}om(G \boxtimes H, \mathcal{O}^{\mathrm{w},S}_{X\times S}) \simeq \rho^{-1}R\mathcal{H}om(G \boxtimes \rho_! H, \mathcal{O}^{\mathrm{w}}_{X\times S}) \simeq R\mathcal{H}om(\mathbb{C}_X \boxtimes H, D'G \boxtimes \mathbb{C}_S \overset{\mathrm{w}}{\otimes} \mathcal{O}_{X\times S}).$$

In particular, when $G = \mathbb{C}_X$ and $H = \mathbb{C}_S$ we have $\rho^{-1}\mathcal{O}^{t,S}_{X\times S} \simeq \mathcal{O}_{X\times S}$ and $\rho^{-1}\mathcal{O}^{\mathrm{w},S}_{X\times S} \simeq \mathcal{O}_{X\times S}$.

As a consequence of Lemma 4.6 together with the results in [1] we obtain the following characterization of the sections of $\mathcal{O}^{t,S}_{X\times S}$:

Proposition 5.8. *Assume that U (resp. V) is a subanalytic Stein open subset of the Stein manifold X (resp. S). Then $R\Gamma(U\times V; \mathcal{O}^{t,S}_{X\times S})$ is concentrated in degree zero and $\Gamma(U\times V; \mathcal{O}^{t,S}_{X\times S})$ is the set of holomorphic functions on $U \times V$ which are tempered on $X \times V$.*

Example 5.1. Let $U = \{z \in \mathbb{C} : \Im z > 0\}$, let V be open subanalytic in $\mathbb{C}$ and let $g(s)$ be a holomorphic function on V. Then, after choosing a determination of $\log z$ on U, $z^{g(s)}$ defines a section of $\Gamma(U \times V; \mathcal{O}^{t,S}_{\mathbb{C}\times\mathbb{C}})$.

Recall that any distribution on $\mathbb{R}^n$ is, just as any hyperfunction, the boundary value of some holomorphic function on $\Omega \cap \{(z_1,\dots,z_n) \in \mathbb{C}^n : \Im z_i \neq 0\}$, with moderate growth with respect to $\mathbb{R}^n$, for some Stein open neighborhood Ω of $\mathbb{R}^n$ in $\mathbb{C}^n$. For a precise notion of boundary value and classical hyperfunction theory we refer to the foundational work [14]. By Proposition 5.1(2) we deduce the following example:

Example 5.2. Let $U = \mathbb{R}_{>0}$ with a coordinate x, let V be a subanalytic open set in $\mathbb{R}$ and let $a(s)$ be any continuous function on V. Let $f \in \Gamma(\Omega \setminus V; \mathcal{O}_{\mathbb{C}})$, where Ω is an open neighborhood of V in $\mathbb{C}$, be such that a is the boundary value $\mathrm{vb}(f)$ of f as a hyperfunction. Then $x^a_+ := \mathrm{vb}(z^f)$, with $\arg z \in]0, 2\pi[$, is a section of $\Gamma(U \times V; \mathcal{D}b^{t,S}_{\mathbb{R}\times\mathbb{R}})$.

Remark 5.3. As the reader can naturally ask, our method applies only for products of analytic manifolds, i.e, for a projection, since the crucial trick we used here is that the allowed coverings are formed by products of open subanalytic sets and products are not kept by change of coordinates. So, if we want to treat the case of a general smooth $f : X \to S$, we have to consider on X a topology with adapted coverings which are fewer than those of the subanalytic topology. This can be illustrated with $X = \mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$ with coordinates (x, y), $S = \mathbb{R}$ and $f : X \to S$ the second projection. Consider

$U =]0,1[\times]{-1},1[$ and the open covering

$$U_1 = U \cap \{y < x\}, \qquad U_2 =]0,1[\times]0,1[$$

(so U_1 is not a product of intervals). Consider the relative tempered distributions $s_1 = 0$ on U_1 and $s_2 = \chi_{\{y=2x\}} \exp(1/y)$ on U_2 (χ_A denotes the characteristic function of A). Then $s_1 = s_2 = 0$ on $U_1 \cap U_2$, hence they glue to a distribution on U which is not relative tempered.

Hence, if we want to realize relative tempered distributions with respect to a smooth function as a sheaf on a site, we must avoid such kind of coverings. We conjecture, however, that with a weaker notion of subanalytic site as in [4] a notion of relative sheaf can be given for a general smooth function.

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