Quotients of index two and general quotients in a space of orderings

by

Paweł Gładki (Katowice and Kraków)
and **Murray Marshall** (Saskatoon)

Abstract. We investigate quotient structures and quotient spaces of a space of orderings arising from subgroups of index two. We provide necessary and sufficient conditions for a quotient structure to be a quotient space that, among other things, depend on the stability index of the given space. The case of the space of orderings of the field $\mathbb{Q}(x)$ is particularly interesting, since then the theory developed simplifies significantly. A part of the theory firstly developed for quotients of index 2 generalizes to quotients of index 2^n for arbitrary finite n. Numerous examples are provided.

1. Introduction and notation. The theory of abstract spaces of orderings was developed in the late 1970s, and provides an abstract framework for studying orderings of fields and the reduced theory of quadratic forms in general. The monograph [16] will be of frequent use here as far as background, notation, and main results are concerned. Spaces of orderings also occur in a natural way in other more general settings: from considering maximal orderings on semi-local rings, orderings on skew fields, to orderings on ternary fields. The axioms for spaces of orderings have also been generalized in various directions—to quaternionic schemes, to spaces of signatures of higher level, or to abstract real spectra that are used to study orderings on commutative rings.

A first-order description of spaces of orderings is given in [18]. Here, we prefer to use the earlier description from [16]: a space of orderings is a pair (X, G) such that X is a non-empty set, G is a subgroup of $\{1, -1\}^X$ which contains the constant function -1, separates points of X, and satisfies two additional axioms. Considering X as a subset of the character group $\chi(G)$ (here by *characters* we mean group homomorphisms $x : G \to \{-1, 1\}$)

[255]

via the natural embedding $X \hookrightarrow \chi(G)$ obtained by identifying $x \in X$ with the character $G \ni a \mapsto a(x) \in \{-1, 1\}$, and denoting, for any pair $a, b \in G$,

$$D(a, b) = \{c \in G : \forall x \in X \ (c(x) = a(x) \vee c(x) = b(x))\},$$

the two additional axioms state

(1) if $x \in \chi(G)$ satisfies $x(-1) = -1$, and if

$$\forall a, b \in \ker(x) \ (D(a, b) \subseteq \ker(x)),$$

then x is in the image of the natural embedding $X \hookrightarrow \chi(G)$, and

(2) $\forall a_1, a_2, a_3, b, c \in G \ \exists d \in G \ [(b \in D(a_1, c) \wedge c \in D(a_2, a_3)) \Rightarrow (b \in D(d, a_3) \wedge d \in D(a_1, a_2))]$.

Spaces of orderings are easily made into a category by introducing morphisms in the following way: a morphism F from a space of orderings (X_1, G_1) to a space of orderings (X_2, G_2) is a function $F : X_1 \to X_2$ such that

$$\forall b \in G_2 \ (b \circ F \in G_1).$$

In this paper we shall investigate quotient objects in the category of spaces of orderings. They have been first studied in [11] and, up to present day, remain rather mysterious creatures: if (X, G) is a space of orderings, and G_0 is a subgroup of G containing the element -1, we denote by X_0 the set $X|_{G_0}$ of all characters from X restricted to G_0, and call the pair (X_0, G_0) a *quotient structure*. In the case when (X_0, G_0) is a space of orderings, we call it a *quotient space* of (X, G). It has been shown in [11] that quotient spaces are, indeed, quotients in the category of spaces of orderings. At the same time the problem of determining whether a given quotient structure is a quotient space proves to be surprisingly challenging. The main objective of the present paper is to address this question in some special cases. In what follows we shall introduce some more notation, and then proceed to explain our motivation for the current work as well as present our main results.

A space of orderings (X, G) has a natural topology introduced by the family of subbasic clopen Harrison sets:

$$H_X(a) = \{x \in X : a(x) = 1\}$$

for a given $a \in G$, which makes X into a Boolean space [16, Theorem 2.1.5]. Whenever it is clear from the context which space of orderings we consider, we shall simply write $H(a)$ instead of $H_X(a)$.

For any multiplicative group G of exponent 2 with distinguished element -1, we set $X = \{x \in \chi(G) : x(-1) = -1\}$ and call the pair (X, G) a *fan*. A fan is also a space of orderings [16, Theorem 3.1.1]. We can also

consider fans within a bigger space of orderings, and for this we need the notion of a subspace of a space (X, G): a subset $Y \subseteq X$ is called a *subspace* of (X, G) if Y is expressible in the form $\bigcap_{a \in S} H_X(a)$ for some subset $S \subseteq G$. For any subspace Y we will denote by $G|_Y$ the group of all restrictions $a|_Y$, $a \in G$. The pair $(Y, G|_Y)$ is a space of orderings itself ([16, Theorem 2.4.3], [10, Theorem 2.2]). Finally, if (X, G) is a space of orderings, by a fan in (X, G) we understand a subspace V such that the space $(V, G|_V)$ is a fan. One easily checks that any one- or two-element subset of a space of orderings forms a fan—thus one- or two-element fans are called *trivial fans*.

The *stability index* $\text{stab}(X, G)$ of a space of orderings (X, G) is the maximum n such that there exists a fan $V \subseteq X$ with $|V| = 2^n$ (or ∞ if there is no such n). It can be shown that the stability index of a space (X, G) is at most equal to k if every basic set in X can be expressed as an intersection of k Harrison sets ([16, Theorem 3.4.2], [12, Theorem 6.2]). Spaces of stability index 1 are also called *spaces with the strong approximation property* or *SAP spaces*.

We say that (X, G) is the *direct sum* of the spaces of orderings (X_i, G_i), $i \in \{1, \dots, n\}$, denoted $(X, G) = \coprod_{i=1}^n (X_i, G_i) = (X_1, G_1) \sqcup \cdots \sqcup (X_n, G_n)$, if X is the disjoint union of the sets $X_1, \dots, X_n$, and G consists of all functions $a : X \to \{-1, 1\}$ such that $a|_{X_i} \in G_i$, $i \in \{1, \dots, n\}$. In this case $G = G_1 \oplus \cdots \oplus G_n$, with the role of the distinguished element -1 played by $(-1, \dots, -1)$. Further, we say that (X, G) is a *group extension* of the space of orderings $(\overline{X}, \overline{G})$ if G is a group of exponent 2, $\overline{G}$ is a subgroup of G, and $X = \{x \in \chi(G) : x|_{\overline{G}} \in \overline{X}\}$. Since G decomposes as $G = \overline{G} \times H$, we shall also write $(X, G) = (\overline{X}, \overline{G}) \times H$ to denote group extensions. Both direct sums and group extensions are spaces of orderings ([16, Theorem 4.1.1], [11, Remarks 2.8, 3.7]).

For $S \subseteq G$, $\langle S \rangle$ denotes the subgroup of G generated by S. For $S \subseteq \chi(G)$, $S^\perp := \{g \in G : \sigma(g) = 1 \; \forall \sigma \in S\}$ and $\langle S \rangle := \chi(G/S^\perp)$, the closed subgroup of $\chi(G)$ generated by S.

For a space of orderings (X, G) we define the connectivity relation $\sim$ as follows: if $x_1, x_2 \in X$, then $x_1 \sim x_2$ if and only if either $x_1 = x_2$ or there exists a four-element fan V in (X, G) such that $x_1, x_2 \in V$. The equivalence classes with respect to $\sim$ are called the *connected components* of (X, G). It is known that if (X, G) is a finite space of orders and $X_1, \dots, X_n$ are its connected components, then $(X, G) = (X_1, G|_{X_1}) \sqcup \cdots \sqcup (X_n, G|_{X_n})$. Moreover, the $(X_i, G|_{X_i})$ are either one-element spaces or proper group extensions ([16, Theorem 4.2.2], [10, Theorem 4.10]).

For any space of orderings (X, G), a quadratic form with entries in G is an n-tuple $\phi = (a_1, \dots, a_n)$, $a_1, \dots, a_n \in G$; we call n the *dimension* of ϕ, and, for each $x \in X$, the *signature* of ϕ at x is $\text{sgn}_x \phi = \sum_{i=1}^n a_i(x) \in \mathbb{Z}$.

We set

$$\phi \oplus \psi = (a_1, \ldots, a_n, b_1, \ldots, b_m),$$
$$c\phi = (ca_1, \ldots, ca_n),$$
$$\phi \otimes \psi = a_1\psi \oplus \cdots \oplus a_n\psi,$$
$$((a_1, \ldots, a_n)) = (1, a_1) \otimes \cdots \otimes (1, a_n),$$
$$k \times \phi = \underbrace{\phi \oplus \cdots \oplus \phi}_{k}.$$

Two forms ϕ and ψ are *isometric*, written $\phi \cong \psi$, if they are of the same dimension and signatures for all $x \in X$. We say that ϕ and ψ are *Witt equivalent*, denoted $\phi \sim \psi$, if there exist integers $k, l \geq 0$ such that

$$\phi \oplus k \times (1, -1) \cong \psi \oplus l \times (1, -1).$$

The sum $\oplus$ and the product $\otimes$ of quadratic forms induce binary operations on the set of equivalence classes of the relation $\sim$ making it into a commutative ring with 1, which is denoted by $W(X, G)$ and called the *Witt ring* associated to the space of orderings (X, G). The ideal of the ring $W(X, G)$ additively generated by the set $\{(1, a) : a \in G\}$ is denoted by $I(X, G)$ and called the *fundamental ideal* of $W(X, G)$.

For a formally real field k denote by X_k the set of all orderings of k, and by G_k the multiplicative group $k^*/(\Sigma k^2)^*$ of all classes of sums of squares of k. Then G_k is naturally identified with a subgroup of $\{-1, 1\}^{X_k}$ via the homomorphism

$$k^* \ni a \mapsto \overline{a} \in \{-1, 1\}^{X_k}, \quad \text{where} \quad \overline{a}(\sigma) = \begin{cases} 1, & \text{if } a \in \sigma, \\ -1, & \text{if } a \notin \sigma, \end{cases} \text{ for } \sigma \in X_k,$$

whose kernel is the set $(\Sigma k^2)^*$ of all non-zero sums of squares of k, and (X_k, G_k) is a space of orderings [16, Theorem 2.1.4]. For simplicity we shall denote by the same symbol a both an element $a \in k^*$, a class of sums of squares $a \in k^*/(\Sigma k^2)^*$, and a function $a \in \{-1, 1\}^{X_k}$.

Since the invention of abstract spaces of orderings there has been a considerable interest in the question of when such a space is realized as a space of orderings of a field. It seems likely that there exist spaces of orderings that are not so realized but, so far, no such examples are known. A possible way of proving that a space of orderings is not realized in such a way is to give an example of a form $\phi \in W(X, G)$ such that

$$\forall \sigma \in X \ (\text{sgn}_\sigma \phi \equiv 0 \bmod 2^n) \quad \text{and} \quad \phi \notin I^n(X, G).$$

Here $I^n(X, G)$ denotes the nth power of the fundamental ideal of $W(X, G)$. The equivalence

$$\forall \sigma \in X \ (\text{sgn}_\sigma \phi \equiv 0 \bmod 2^n) \quad \text{if and only if} \quad \phi \in I^n(X, G)$$

is valid for any $n \in \mathbb{N}$ if (X, G) is a space of orderings of a field. This

was conjectured by Marshall and is also known as *Lam's Open Problem B* (see [9]). For $n \leq 2$ the equivalence is easy to show, but for $n \geq 3$ the proof uses a deep result from [19] and [20], and was first given by Dickmann and Miraglia [4] (see also [5] for a more general version of the result, and [17] for a short explanation of how to derive the equivalence from the results of Orlov, Vishik and Voevodsky).

The problem also has an affirmative solution for all spaces of orderings of stability index no greater than 3 [12, Theorem 6.2]. It is, therefore, desirable to seek for examples of spaces of orderings of high stability indices. A possible way of finding such spaces is to investigate quotients of already known spaces: for a given space of orderings (X, G) one can imagine some large subset of X that is itself not a fan, but, by descending to an appropriate subgroup G_0 of G, can be forced to become a fan in the quotient space $(X|_{G_0}, G_0)$, thus increasing the stability index. The main problem, of course, is that in doing so one obtains a myriad of quotient structures with no way of knowing if they are actually quotient spaces.

We begin our discussion in Section 2 with the case when G_0 is a subgroup of G of index two and provide a necessary and sufficient condition for a quotient structure of an SAP space (X, G) to be a quotient space. We then refine our condition and show that this refined condition is both necessary and sufficient for spaces of stability index two that satisfy an extra technical condition. The refinement process can be continued and provides a sequence of necessary conditions—unfortunately, other than in the two aforementioned cases and the finite case, we do not know when these conditions are sufficient.

In Section 3 we consider the special case when (X, G) is the space of orderings of the field $\mathbb{Q}(x)$. The refined condition for spaces of stability index two is here also sufficient and, moreover, is also equivalent to saying that the quotient space is profinite. This generalizes results previously obtained in [6].

Section 4 contains a handful of examples to illustrate the theory.

Finally, in Section 5, we generalize the necessary conditions to the case of subgroups of arbitrary (possibly infinite) index and discuss some of the instances when they are also sufficient. Surprisingly enough, the theory simplifies significantly in the case of the space of orderings of the field $\mathbb{Q}(x)$, and we show that a quotient structure of any finite index is a quotient space if and only if it is profinite.

2. Quotients of index two. Let (X, G) be a space of orderings, and (X_0, G_0) a quotient structure of (X, G). We search for necessary and sufficient conditions on G_0 for (X_0, G_0) to be a quotient space of (X, G).

EXAMPLE 2.1. Suppose G_0 is a subgroup of G and $-1 \in G_0$. Suppose $\sigma_0 : G_0 \to \{\pm 1\}$ is a character satisfying $\sigma_0(-1) = -1$. Let $\sigma : G \to \{\pm 1\}$

be an extension of σ_0 to a character on G. If (X, G) is a fan then $\sigma \in X$, so $\sigma_0 \in X_0$. Thus if (X, G) is a fan and $-1 \in G_0$ then (X_0, G_0) is also a fan. In particular, (X_0, G_0) is a quotient space of (X, G), and any quotient structure of a fan is a quotient space that is a fan itself.

Assume now that G_0 is a subgroup of index 2 in G, $-1 \in G_0$. Since G_0 has index 2 and $-1 \in G_0$, G_0 is determined by a character on $G/\{\pm 1\}$, i.e., there exists a unique $\gamma \in \chi(G)$, $\gamma(-1) = 1$, such that $G_0 = \ker(\gamma)$.

THEOREM 2.2.

(i) *Suppose (X_0, G_0) is a quotient space of (X, G), $(Y, G/Y^\perp)$ is a subspace of (X, G), $\gamma \in \langle Y \rangle$ and $Y_0 = Y|_{G_0}$. Then $(Y_0, G_0/Y^\perp)$ is a quotient space of $(Y, G/Y^\perp)$.*

(ii) *Suppose (X, G) is a group extension of (X', G') and $\gamma' = \gamma|_{G'}$. If $\gamma' = 1$, then (X_0, G_0) is a quotient space of (X, G). If $\gamma' \neq 1$, then (X_0, G_0) is a quotient space of (X, G) iff (X'_0, G'_0) is a quotient space of (X', G'). Here, $G'_0 := \ker(\gamma')$, $X'_0 := X'|_{G'_0}$.*

Proof. (i) $(Y_0, G_0/Y^\perp)$ is a subspace of (X_0, G_0), so the assertion is clear.

(ii) If $\gamma' = 1$, then (X_0, G_0) is a group extension of (X', G'). If $\gamma' \neq 1$, then (X_0, G_0) is a group extension of (X'_0, G'_0). In either case the result is clear, e.g., by [16, Theorems 4.11(2) and 4.1.3(2)]. ■

One needs to realize that the situation where (X_0, G_0) is a space of orderings is rather special.

EXAMPLE 2.3. Consider the space (X, G) of six orderings $\{\sigma_1, \ldots, \sigma_6\}$ and of stability index 1, so that $|G| = 2^6$. Let $\gamma = \sigma_1 \cdot \ldots \cdot \sigma_6$. The quotient structure (X_0, G_0) consists of six orderings $\overline{\sigma}_1, \ldots, \overline{\sigma}_6$ with $\overline{\sigma}_6 = \overline{\sigma}_1 \cdot \ldots \cdot \overline{\sigma}_5$, and $|G_0| = 2^5$. By the structure theorem for finite spaces of orderings ([16, Theorem 4.2.2], [10, Theorem 4.10]), (X_0, G_0) is not a space of orderings (e.g., because it is not SAP but contains no 4-element fans).

THEOREM 2.4. *A necessary condition for (X_0, G_0) to be a quotient of (X, G) is that $\gamma \in X^4$.*

Here $X^k := \{\prod_{i=1}^k \sigma_i : \sigma_i \in X, \ i = 1, \ldots, k\}$. Since the σ_i are not required to be distinct, $\{1\} \subseteq X^2 \subseteq X^4$. If (X, G) is a fan then $X^4 = X^2 = \{\gamma \in \chi(G) : \gamma(-1) = 1\}$.

Proof. Suppose first that the restriction map $r : X \to X_0$ is not injective, so there exist $\sigma, \tau \in X$, $\sigma \neq \tau$, with $r(\sigma) = r(\tau)$. In this case, $G_0 = \ker(\sigma\tau)$, so $\gamma = \sigma\tau \in X^2$. Suppose next that r is injective. Since r is continuous and injective and X is compact, r is a homeomorphism. Fix $g \in G$, $g \notin G_0$, and define $\phi : X_0 \to \{\pm 1\}$ by $\phi(r(\sigma)) = \sigma(g)$. Then ϕ is well-defined and continuous.

CLAIM. ϕ *is not in the image of* G_0 *under the natural injection* $\hat{\ }: G_0 \to$ $\mathrm{Cont}(X_0, \{\pm 1\})$.

For suppose $\phi = \hat{h}$, $h \in G_0$. Then, for any $\sigma \in X$, $\sigma(h) = r(\sigma)(h) = \hat{h}(r(\sigma)) = \phi(r(\sigma)) = \sigma(g)$, so $g = h \in G_0$, contradicting $g \notin G_0$.

Suppose now that (X_0, G_0) is a space of orderings. By the Claim and [16, Theorem 3.2.4] there exists a 4-element fan V in X_0 with $\prod_{\alpha \in V} \phi(\alpha) \neq 1$. The character $\gamma' := \prod_{\alpha \in V} r^{-1}(\alpha) \in X^4$ is $\neq 1$ (because $\gamma'(g) = \prod_{\alpha \in V} \phi(\alpha)$ $\neq 1$), but the restriction of γ' to G_0 is equal to 1 (because V is a fan). Thus $\gamma = \gamma' \in X^4$, as required. ∎

THEOREM 2.5. *If the space of orderings* (X, G) *is SAP, then the necessary condition in Theorem* 2.4 *is also sufficient.*

Proof. Suppose first that $\gamma \in X^2$, say $\gamma = \sigma_1 \sigma_2$, $\sigma_1, \sigma_2 \in X$. Let $\phi : X_0 \to \{\pm 1\}$ be continuous. Then $\phi \circ r : X \to \{\pm 1\}$ is continuous. Since (X, G) is SAP, the natural injection $\hat{\ }: G \hookrightarrow \mathrm{Cont}(X, \{\pm 1\})$ is surjective, i.e., $\phi \circ r = \hat{g}$ for some $g \in G$. Then $\sigma_1(g) = \phi(r(\sigma_1)) = \phi(r(\sigma_2)) = \sigma_2(g)$, so $g \in G_0$. This implies that the natural injection $G_0 \hookrightarrow \mathrm{Cont}(X_0, \{\pm 1\})$ is an isomorphism. Suppose next that $\gamma \in X^4$, $\gamma \notin X^2$, say $\gamma = \prod_{i=1}^4 \sigma_i$, $\sigma_i \in X$, $i = 1, \ldots, 4$. In this case one sees, by a similar argument, that the natural injection $G_0 \hookrightarrow \mathrm{Cont}(X_0, \{\pm 1\})$ identifies G_0 with

$$\Big\{ \phi \in \mathrm{Cont}(X_0, \{\pm 1\}) : \prod_{i=1}^4 \phi(r(\sigma_i)) = 1 \Big\}.$$

In either case, (X_0, G_0) can be viewed as the space of global sections of a sheaf of spaces of orderings as defined in [14, Chapter 8], so (X_0, G_0) is a space of orderings by Theorem 6.1 below, more specifically, by Corollary 6.2. Note: If $\gamma \in X^2$ then all of the stalks are singleton spaces; if $\gamma \in X^4 \setminus X^2$ then one of the stalks is a 4-element fan and the rest are singleton spaces. ∎

EXAMPLE 2.6. Theorems 2.4 and 2.5 provide a convenient description of quotients of index 2 of the space of orderings of the field $\mathbb{R}(x)$, or more generally, of the space of orderings of any formally real function field of transcendence degree 1 over a real closed field. This is because spaces of orderings of this sort are SAP.

EXAMPLE 2.7. The condition of Theorem 2.5 fails to be sufficient if the stability index of the space (X, G) is greater than 1.

(1) Consider the space (X, G), where

$$X = \{\sigma_1, \sigma_2, \sigma_3, \sigma_1 \sigma_2 \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_4 \sigma_5 \sigma_6\},$$

with $|G| = 2^6$. This is the direct sum of two 4-element fans. Let $\gamma = \sigma_1 \sigma_4$. The quotient structure (X_0, G_0) with $G_0 = \ker \gamma$ is not a quotient space.

(2) Consider the space (X, G), where

$$X = \{\sigma_1, \sigma_2, \sigma_3, \sigma_1\sigma_2\sigma_3, \sigma_4, \sigma_5, \sigma_6\},$$

with $|G| = 2^6$. This is the direct sum of a 4-element fan and three singleton spaces. Let $\gamma = \sigma_1\sigma_4\sigma_5\sigma_6$. The quotient structure (X_0, G_0) with $G_0 = \ker \gamma$ is not a quotient space.

(3) Consider the space (X, G), where

$$X = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_1\sigma_3\sigma_4, \sigma_2\sigma_3\sigma_4, \sigma_5, \sigma_6\},$$

with $|G| = 2^6$. This is the direct sum of a connected space of six elements and two singleton spaces. Let $\gamma = \sigma_1\sigma_2\sigma_5\sigma_6$. The quotient structure (X_0, G_0) with $G_0 = \ker \gamma$ is not a quotient space.

(4) Consider the space (X, G), where

$$X = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_1\sigma_3\sigma_4, \sigma_2\sigma_3\sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_5\sigma_7\sigma_8, \sigma_6\sigma_7\sigma_8\},$$

with $|G| = 2^8$. This is the direct sum of two connected spaces, each consisting of six elements. Let $\gamma = \sigma_1\sigma_2\sigma_5\sigma_6$. The quotient structure (X_0, G_0) with $G_0 = \ker \gamma$ is not a quotient space.

Details of proofs are left to the reader. In each case one uses the structure theorem ([16, Theorem 4.2.2], [10, Theorem 4.10]) for the finite spaces of orderings (X_0, G_0) and shows that the resulting quotient structure is constructed in a way contradicting the theorem.

To simplify things *we assume from now on that the space of orderings* (X, G) *contains no infinite fans.* This is the case, for example, if the stability index of (X, G) is finite. Recall that, for $\delta \in \chi(G)$, $X_\delta := \{\sigma \in X : \sigma\delta \in X\}$ $= X \cap \delta X$. Since (X, G) has no infinite fans, every connected component of (X, G) is either singleton or has the form X_δ for some $\delta \in \chi(G)$, $\delta \neq 1$, $|X_\delta| \geq 4$ ([16, Theorem 4.6.1], [13, Theorem 2.6]).

The requirement that $\gamma \in X^4$ can be substantially refined as follows:

THEOREM 2.8. *A necessary condition for* (X_0, G_0) *to be a quotient of* (X, G) *is that* $\gamma = \prod_{i=1}^{k} \sigma_i$, $\sigma_i \in X$, $k = 2$ *or* $k = 4$ *and* $\gamma \notin X^2$, *and, in the case where not all* σ_i *are in the same connected component of* (X, G) *and the connected components of the* σ_i *in* (X, G) *are not all singleton, either* $k = 2$ *and exactly one of the connected components of the* σ_i *is not singleton, or* $k = 4$, $\gamma \notin X^2$ *and, after reindexing suitably, the connected component of* σ_3 *and* σ_4 *is* $X_{\sigma_3\sigma_4}$ *and either the connected component of* σ_1 *and* σ_2 *is* $X_{\sigma_1\sigma_2}$, *or the connected component of* σ_i *is singleton for* $i = 1, 2$.

Proof. Denote by Y the union of the connected components of (X, G) which meet the set $\{\sigma_1, \ldots, \sigma_k\}$. By [15, Theorem 3.6] we see that Y, or more precisely $(Y, G/\Delta)$ where $\Delta := Y^\perp$, is a subspace of (X, G). Denote by Y_0 the set of restrictions of elements of Y to G_0. If we assume that (X_0, G_0) is

a quotient space of (X, G) then $(Y_0, G_0/\Delta)$ is a quotient space of $(Y, G/\Delta)$, by Theorem 2.2. In this way, we are reduced to the case where $X = Y$, i.e., each connected component of (X, G) meets the set $\{\sigma_1, \ldots, \sigma_k\}$.

Denote by $(Z_j, G/Z_j^\perp)$, $j \in J$, the connected components of (X, G). Each Z_j is singleton or has the form X_δ, $\delta \neq 1$, $|X_\delta| \geq 4$, and $Z_j \cap \{\sigma_1, \ldots, \sigma_k\} \neq \emptyset$ for each j, so $|J| \leq k$. By hypothesis, $2 \leq |J|$ and not all Z_j are singleton. According to [14, Corollary 7.5], (X, G) is the direct sum of the $(Z_j, G/Z_j^\perp)$, $j \in J$. In particular, $\chi(G) = \prod_{j \in J} \langle Z_j \rangle$ (direct product of groups), where $\langle Z_j \rangle$ is the closed subgroup of $\chi(G)$ generated by Z_j. Since $|J| \geq 2$ this implies in particular that $\gamma \notin \langle Z_j \rangle$ for each j.

The restriction map $r : X \to X_0$ is injective on each Z_j. This is clear if Z_j is singleton. If $Z_j = X_\delta$, $\delta \neq 1$, $|X_\delta| \geq 4$, then injectivity follows from the fact that $\gamma \notin \langle Z_j \rangle$. We also see in this latter case that $r(Z_j) \subseteq (X_0)_{\delta_0}$, where δ_0 denotes the restriction of δ to G_0, and $\delta_0 \neq 1$ (because $\gamma \notin \langle Z_j \rangle$). It follows, using [10, Lemma 4.6] repeatedly (see [13, Remark 2.1]), that the space of orderings (X_0, G_0) is connected and, moreover, that there exists $\mu_0 \in \chi(G_0)$, $\mu_0 \neq 1$ such that $X_0 = (X_0)_{\mu_0}$. This implies in turn that $X = X_\mu \cup X_{\gamma\mu}$ where μ is some fixed extension of μ_0 to a character on G. Since $|X| \geq 5$ it follows that at least one of $|X_\mu|, |X_{\gamma\mu}|$ is ≥ 4. Reindexing we can assume $|X_{\gamma\mu}| \geq 4$.

If $|X_\mu|$ is also ≥ 4 then, since X has at least two connected components, $X_\mu \cap X_{\gamma\mu} = \emptyset$ and X_μ and $X_{\gamma\mu}$ are the connected components of X, so $\chi(G) = \langle X_\mu \rangle \times \langle X_{\gamma\mu} \rangle$. If $k = 2$ then, after reindexing, $\sigma_1 \in X_\mu$ and $\sigma_2 \in X_{\gamma\mu}$ and since the two decompositions $\gamma = (\mu)(\gamma\mu)$ and $\gamma = (\sigma_1)(\sigma_2)$ must be the same (because the product is direct), $\mu = \sigma_1$ and $\gamma\mu = \sigma_2$. Since $\sigma_1(-1) = -1$ and $\mu(-1) = 1$, this is not possible. If $k = 4$ then, arguing as before with the two decompositions of γ, we see that, after reindexing suitably, $\sigma_1, \sigma_2 \in X_\mu$, $\sigma_3, \sigma_4 \in X_{\gamma\mu}$, $\mu = \sigma_1\sigma_2$ and $\gamma\mu = \sigma_3\sigma_4$.

This leaves the case $|X_\mu| = 2$, $|X_{\gamma\mu}| \geq 4$. If $k = 2$ then X has two components, one singleton and one equal to $X_{\gamma\mu}$. Suppose now that $k = 4$, $\gamma \notin X^2$. Reindexing, we can suppose $\sigma_3, \sigma_4 \in X_{\gamma\mu}$. There are two subcases: either $X_\mu \cap X_{\gamma\mu} = \emptyset$ or $X_\mu \cap X_{\gamma\mu} \neq \emptyset$. Suppose first that $X_\mu \cap X_{\gamma\mu} = \emptyset$. Then $X_\mu = \{\sigma_1, \sigma_2\}$, $\mu = \sigma_1\sigma_2$, $\gamma\mu = \sigma_3\sigma_4$. In this case the connected components are $\{\sigma_1\}$, $\{\sigma_2\}$ and $X_{\sigma_3\sigma_4}$. Suppose now that $X_\mu \cap X_{\gamma\mu} \neq \emptyset$. Reindexing we may assume $X_\mu = \{\sigma_1, \sigma_1\mu\}$, $\sigma_1\mu \in X_{\gamma\mu}$. Then $(\sigma_1\mu)(\gamma\mu) = \sigma_1\gamma = \sigma_2\sigma_3\sigma_4 \in X$, contradicting $\gamma \notin X^2$. Thus this case cannot occur. ∎

We remark that the quotient structures appearing in Example 2.7 are precisely those for which the conditions of Theorem 2.8 fail to be satisfied.

It is natural to wonder if the necessary conditions of Theorem 2.8 are sufficient when (X, G) has stability index two. We are unable to prove this in general. We are however able to prove the following:

THEOREM 2.9. *If (X, G) has stability index two and just finitely many non-singleton connected components, then the necessary conditions of Theorem 2.8 are sufficient.*

Proof. By assumption, $\gamma = \prod_{i=1}^{k} \sigma_i$, $\sigma_i \in X$, $k = 2$ or $k = 4$ and $\gamma \notin X^2$.

CLAIM. *One can reduce to the case where each connected component of (X, G) has non-empty intersection with $\{\sigma_1, \ldots, \sigma_k\}$.*

Define the subspace $(Y, G/\Delta)$ and the quotient structure $(Y_0, G_0/\Delta)$ of $(Y, G/\Delta)$ as in the proof of Theorem 2.8. Let (X_i, G_i), $i \in I_0$, be the connected components of (X, G) which do not meet $(Y, G/\Delta)$. Let $I = I_0 \cup \{i_0\}$ where i_0 is a new symbol, and let $(X_{i_0}, G_{i_0}) = (Y, G/\Delta)$. Let $\pi : X \to I$ denote the projection defined by $\pi(\sigma) = i$ if $\sigma \in X_i$, and give I the quotient topology. Since X is a Boolean space and each X_i is closed, and all but finitely many of the X_i are singleton, it follows that I is a Boolean space. From [16, Theorem 3.2.4] we deduce that (X, G) is the space of global sections of a sheaf of spaces of orderings on I with stalks (X_i, G_i), $i \in I$. Using this fact and Corollary 6.2 one checks easily that, if $(Y_0, G_0/\Delta)$ is a space of orderings, then (X_0, G_0) is the space of global sections of a sheaf of spaces of orderings on I with stalks (X_i, G_i) for $i \in I$, $i \neq i_0$, and stalk $(Y_0, G_0/\Delta)$ at $i = i_0$. This proves the Claim.

By the Claim we can assume $X = Y$. If each connected component is singleton, then either $k = 2$ and (X_0, G_0) is a singleton space, or $k = 4$, $\gamma \notin X^2$ and (X_0, G_0) is a 4-element fan. Suppose (X, G) has at least two connected components and at least one of these is not singleton. If $k = 2$ then (X, G) has exactly two connected components, one singleton and one non-singleton, and (X_0, G_0) is isomorphic to the non-singleton component of (X, G). If $k = 4$, $\gamma \notin X^2$, then either (X, G) has two connected components which, after reindexing, are $X_{\sigma_1\sigma_3}$ and $X_{\sigma_3\sigma_4}$, or three connected components which, after reindexing, are $\{\sigma_1\}$, $\{\sigma_2\}$ and $X_{\sigma_3\sigma_4}$. In either case, (X_0, G_0) is a group extension by a group of order 2 of the direct sum of the residue space of $X_{\sigma_1\sigma_2}$ associated to $\sigma_1\sigma_2$ and the residue space of $X_{\sigma_3\sigma_4}$ associated to $\sigma_3\sigma_4$. Note: In the case where $\{\sigma_1\}$ and $\{\sigma_2\}$ are connected components, we have $X_{\sigma_1\sigma_2} = \{\sigma_1, \sigma_2\}$ and the associated residue space is a singleton space. This leaves us with the case where (X, G) has just one connected component. If $\gamma X = X$ then (X_0, G_0) is the residue space of (X, G) associated to γ. Suppose $\gamma X \neq X$. Then (X, G) is a group extension of an SAP space of orderings (X', G') by a cyclic group of order two. Let γ' denote the restriction of γ to G'. The pair (X_0', G_0') associated to γ' is a quotient of (X', G'), by Theorem 2.5. We conclude that (X_0, G_0) is a group extension of (X_0', G_0') by a cyclic group of order two. ∎

REMARK 2.10. When the stability index of (X, G) is three or more, there are additional necessary conditions. Suppose $\sigma_1, \ldots, \sigma_k$ all belong to

the same connected component (Z, H) of (X, G), $\gamma Z \neq Z$, and (Z, H) is a group extension of a space of orderings (Z', H') by a cyclic group of order 2. Let $\gamma' = \prod_{i=1}^{k} \sigma'_i$, where γ', resp. σ'_i, denotes the restriction of γ, resp. σ_i, to H'. By Theorem 2.2, if (X_0, G_0) is a space of orderings then so is the associated quotient structure (Z'_0, H'_0) of (Z', H'). Replacing (X, G) and γ by (Z', H') and γ', additional necessary conditions are obtained recursively, in an obvious way. In particular, the conditions of Theorem 2.8 must hold for (Z', H') and γ'. It is not known if these recursively defined necessary conditions are sufficient. It is easy to see that they are sufficient if the space of orderings (X, G) is finite.

3. The space of orderings of $\mathbb{Q}(x)$. We consider the space of orderings of $\mathbb{Q}(x)$, the function field in a single variable x over the field $\mathbb{Q}$ of rational numbers. This space of orderings is studied in [3], [6] and [7]. We will denote this space of orderings by (X, G) for short, i.e., in this section

$$(X, G) := (X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)}).$$

For a real monic irreducible p in the polynomial ring $\mathbb{Q}[x]$, set $n_p :=$ the number of real roots of p and set $X_p :=$ the set of elements of X compatible with the discrete valuation v_p of $\mathbb{Q}(x)$ associated to p, so $|X_p| = 2n_p$. Set $X_\infty :=$ the set of orderings compatible with the discrete valuation $v_{1/x}$, so $|X_\infty| = 2$. For a transcendental real number r, set $\sigma_r :=$ the archimedean ordering of $\mathbb{Q}(x)$ corresponding to the embedding $\mathbb{Q}(x) \hookrightarrow \mathbb{R}$ given by $x \mapsto r$. Then X is the (disjoint) union of the sets X_p, p running through the real monic irreducibles in $\mathbb{Q}[x]$, X_∞, and $\{\sigma_r\}$, r running through the transcendental real numbers. The non-singleton connected components of (X, G) are the X_p, p a real monic irreducible of $\mathbb{Q}[x]$, $n_p \geq 2$.

Every monic irreducible p of $\mathbb{Q}[x]$ which is not real is positive at every element of X, i.e., it is equal to 1 in G. The set of elements

$$\{-1\} \cup \{p : p \text{ is a real monic irreducible in } \mathbb{Q}[x]\},$$

more precisely, the image of this set in G, forms a $\mathbb{Z}/2\mathbb{Z}$-basis for G, i.e., every element of G is expressible uniquely as

$$(-1)^{\delta_0} \prod_{i=1}^{k} p_i^{\delta_i},$$

$k \geq 0$, $p_1, \ldots, p_k$ distinct real monic irreducibles in $\mathbb{Q}[x]$, $\delta_0, \ldots, \delta_k \in \{0, 1\}$.

Fix A and B, where A is a finite set of real monic irreducible polynomials of $\mathbb{Q}[x]$ and B is a finite set of transcendental real numbers. Set

$$Y := \left(\bigcup_{p \in A} X_p \right) \cup X_\infty \cup \{\sigma_r : r \in B\}.$$

Consider the set $r_1 < \cdots < r_m$ of real numbers consisting of the real roots of the various polynomials $p \in A$ together with the elements of B. Clearly $m := \sum_{p \in A} n_p + |B|$. Choose rational numbers $s_1, \ldots, s_{m+1}$ such that

$$-\infty < s_1 < r_1 < s_2 < r_2 < \cdots < s_m < r_m < s_{m+1} < +\infty.$$

Set $H :=$ the subgroup of G generated by -1, the elements $p \in A$, and the elements $x - s_i$, $i = 1, \ldots, m+1$.

LEMMA 3.1.

(i) $(Y, G/Y^\perp)$ is a subspace of (X, G).
(ii) $(Y, G/Y^\perp)$ is the direct sum of the subspaces $(X_p, G/X_p^\perp)$, $p \in A$, $(X_\infty, G/X_\infty^\perp)$, and $(\{\sigma_r\}, G/\{\sigma_r\}^\perp)$, $r \in B$.
(iii) $(X|_H, H)$ is a quotient space of (X, G).
(iv) The spaces of orderings $(Y, G/Y^\perp)$ and $(X|_H, H)$ are isomorphic via the natural maps $H \hookrightarrow G \to G/Y^\perp$, $Y \hookrightarrow X \to X|_H$.

Proof. (i) and (ii) are consequences of [15, Theorem 3.6] and [14, Corollary 7.5], respectively. One can also prove (ii) using the approximation theorem for V-topologies (see e.g. [21]). (iii) is a consequence of (iv), so it suffices to prove (iv).

CLAIM 1. *The map $Y \to X|_H$ is bijective.*

Let $S_0 :=$ the set of orderings satisfying $x < s_1$, S_{m+1} be the set of orderings satisfying $x > s_{m+1}$, and S_i be the set of orderings satisfying $s_i < x < s_{i+1}$, $i = 1, \ldots, m$. Clearly $X = S_0 \cup \cdots \cup S_{m+1}$ (disjoint union) and $X|_H = S_0|_H \cup \cdots \cup S_{m+1}|_H$ (disjoint union). Each $p \in A$ has constant sign on S_0. For instance, if n_p is even (resp. odd) then p is constantly positive (resp. constantly negative) on S_0. It follows that $S_0|_H$ is a singleton set. Also, $Y \cap S_0$ is a singleton set. A similar argument shows that $S_{m+1}|_H$ and $S_{m+1} \cap Y$ are singleton sets. For $1 \le i \le m$, there are two cases. If $r_i \in B$, then each $p \in A$ has constant sign on S_i, so $S_i|_H$ is a singleton set. In this case $S_i \cap Y$ is also a singleton set. If $r_i \notin B$, then r_i is a root of some unique $p \in A$. In this case p changes sign at r_i and the other elements of A have constant sign on S_i, so $S_i|_H$ has two elements. In this case, $S_i \cap Y$ also has two elements. Moreover, different elements of $S_i \cap Y$ map to different elements of $S_i|_H$.

From the surjectivity of the map $Y \to X|_H$ it follows that the group homomorphism $H \to G/Y^\perp$ is injective. Consequently, to complete the proof it suffices to establish the following:

CLAIM 2. $|H| = |G/Y^\perp|$.

The elements of $\{-1\} \cup A \cup \{x - s_i : i = 1, \ldots, m+1\}$ form a $\mathbb{Z}/2\mathbb{Z}$-basis of H, so $|H| = 2^{|A|+m+2}$. Using (ii) we see that

$$|G/Y^\perp| = \prod_{p \in A} |G/X_p^\perp| \cdot |G/X_\infty^\perp| \cdot \prod_{r \in B} |G/\|\sigma_r\}^\perp|$$

$$= \prod_{p \in A} 2^{n_p+1} \cdot 2^2 \cdot \prod_{r \in B} 2 = 2^{\sum_{p \in A} n_p + |A| + 2 + |B|}.$$

At the same time we have $m = \sum_{p \in A} n_p + |B|$, so $|A| + m + 2 = \sum_{p \in A} n_p + |A| + 2 + |B|$. ∎

As an immediate consequence of Lemma 3.1 we obtain a result of Gładki and Jacob [6, Theorem 1].

THEOREM 3.2. *The space of orderings* (X, G) *is profinite.*

Proof. It suffices to show that for any finite subset S of G there exists a finite quotient $(X|_H, H)$ of (X, G) such that $S \subseteq H$. Define H as in the proof of Lemma 3.1, taking A to be the set of real monic irreducible polynomials appearing in the factorization of the elements of S and $B = \emptyset$. Then H contains S and $(X|_H, H)$ has the required properties. ∎

We mention another consequence of Lemma 3.1. Following the notation of Section 2, we fix a character γ of G, $\gamma \neq 1$, $\gamma(-1) = 1$, define $G_0 = \ker \gamma$, and $X_0 = X|_{G_0}$.

THEOREM 3.3. *The following are equivalent:*

(1) (X_0, G_0) *is a quotient space of* (X, G);
(2) γ *satisfies the necessary conditions of Theorem 2.8;*
(3) (X_0, G_0) *is a profinite space of orderings.*

We remark that [7, Theorem 8] already asserts (1)⇒(3).

Proof. (1)⇒(2) is a consequence of Theorem 2.8.

(3)⇒(1) is trivial (since every profinite space of orderings is, in particular, a space of orderings).

(2)⇒(3). Assume (2) holds. Let $\gamma = \prod_{i=1}^k \sigma_i$, $\sigma_i \in X$, $k = 2$ or $k = 4$ and $\gamma \notin X^4$. To prove (3) it suffices to show that for any finite subset S of G_0 there exists a finite quotient space $(X|_H, H)$ of (X, G) such that $S \subseteq H$ and $(X|_{H \cap G_0}, H \cap G_0)$ is a quotient space of $(X|_H, H)$. Define H, Y as in the preamble to Lemma 3.1, taking A to be any finite set of real monic irreducibles in $\mathbb{Q}[x]$ containing all real monic irreducible factors of elements of S together with all real monic irreducibles p such that $X_p \cap \{\sigma_1, \ldots, \sigma_k\} \neq \emptyset$, and taking B to be any finite set of transcendental real numbers such that, for each $i = 1, \ldots, k$, if σ_i is an archimedian ordering then the corresponding transcendental real number belongs to B. Obviously $S \subseteq H$. By Lemma 3.1, $(X|_H, H)$ is a quotient space of (X, G) which is naturally identified with the

subspace $(Y, G/Y^\perp)$ of (X, G). By construction, Y is a union of connected components of (X, G) and contains all components of (X, G) meeting the set $\{\sigma_1, \ldots, \sigma_k\}$. Also, $(Y, G/Y^\perp)$ is finite and has stability index 1 or 2. It follows from Theorem 2.5 if the stability index is 1, or from Theorem 2.9 if the stability index is 2, that $(Y|_{G_0}, G_0/Y^\perp)$ is a quotient space of $(Y, G/Y^\perp)$. Since $(X|_{H \cap G_0}, H \cap G_0)$ is identified with $(Y|_{G_0}, G_0/Y^\perp)$ under the isomorphism $(X|_H, H) \cong (Y, G/Y^\perp)$, this completes the proof. ∎

4. Examples of quotients of the space of orderings of $\mathbb{Q}(x)$. Theorem 3.3 provides us with an elegant criterion for checking whether a given quotient structure (X_0, G_0) of $(X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)})$ is a quotient space. In practice, however, there seems to be no good way of checking this criterion if G_0 is given in terms of generators, and we can, in fact, do this only in a few cases. We shall discuss this in some detail now.

Let (X, G) be the space of orderings $(X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)})$, and let I denote the set of all (classes of) monic irreducible polynomials in $\mathbb{Q}[x]$ with at least one real root. Let (X_0, G_0) be a fixed quotient structure of (X, G) with $(G : G_0) = 2$. Moreover, let $J \subseteq I$ be the set such that

$$G_0 = \langle \{-1\} \cup J \cup (I \setminus J)(I \setminus J) \rangle.$$

Observe also that $J = \{p \in I : p \in G_0\}$, so J determines uniquely and is uniquely determined by G_0.

EXAMPLE 4.1. If $J = I \setminus \{p\}$ for some $p \in I$, then (X_0, G_0) is a quotient space. Indeed, suppose that $r \in \mathbb{R}$ is a root of p, and that σ_r^- and σ_r^+ are the two orderings corresponding to r, one making p positive, and one making p negative. Let $\gamma = \sigma_r^- \cdot \sigma_r^+$. Then, readily, $G_0 = \ker \gamma$, and (X_0, G_0) is a space of orderings by Theorem 3.3.

EXAMPLE 4.2. If $J = I \setminus \{p_1, p_2\}$ for some $p_1, p_2 \in I$, $p_1 \neq p_2$, then (X_0, G_0) is a quotient space. As before, let $r_1, r_2 \in \mathbb{R}$ be real roots of p_1, p_2, respectively, and let $\sigma_{r_i}^-$ and $\sigma_{r_i}^+$ be the two orderings corresponding to r_i, $i \in \{1, 2\}$. Let $\gamma = \sigma_{r_1}^- \sigma_{r_1}^+ \sigma_{r_2}^- \sigma_{r_2}^+$. Then, as before, (X_0, G_0) is a quotient space by Theorem 3.3 with $G_0 = \ker \gamma$.

EXAMPLE 4.3. If $J = I \setminus \{p_1, \ldots, p_n\}$ for some $n \geq 3$, and $p_1, \ldots, p_n \in I$ pairwise distinct, then (X_0, G_0) is never a quotient space. Let $r_1, \ldots, r_n \in \mathbb{R}$ be real roots of $p_1, \ldots, p_n$, respectively, and let $\sigma_{r_i}^-$ and $\sigma_{r_i}^+$ be the two orderings corresponding to r_i, $i \in \{1, \ldots, n\}$. Then $G_0 = \ker(\sigma_{r_1}^- \sigma_{r_1}^+ \cdot \ldots \cdot \sigma_{r_n}^- \sigma_{r_n}^+)$. Suppose that (X_0, G_0) is a quotient space, and that $G_0 = \ker \gamma$ with $\gamma = \tau_1 \cdot \ldots \cdot \tau_4$ for some $\tau_1, \ldots, \tau_4 \in X$. Following an argument that will be later discussed in detail in the proof of Remark 5.3(2), we see that the presentation $\sigma_{r_1}^- \sigma_{r_1}^+ \cdot \ldots \cdot \sigma_{r_n}^- \sigma_{r_n}^+$ cannot be shortened, and thus yields a contradiction.

EXAMPLE 4.4. If J is finite, then (X_0, G_0) is never a quotient space. For suppose (X_0, G_0) is a quotient space with $G_0 = \ker \gamma$ for some $\gamma = \sigma_1 \cdot \ldots \cdot \sigma_4$, $\sigma_1, \ldots, \sigma_4 \in X$. Let S be the finite set of all points on the real line corresponding to the orderings $\sigma_1, \ldots, \sigma_4$. Take an irreducible polynomial $q \in I$, strictly positive on the set S, but not belonging to J; we note that such a q always exists, in fact, there are infinitely many such q. Then $q \in G_0$, which contradicts $q \notin J$.

The case of both J and $I \setminus J$ being infinite is widely open.

EXAMPLE 4.5. Let $r_1, \ldots, r_4$ be the complete set of distinct real roots of an irreducible polynomial $q \in \mathbb{Q}[x]$. Let

$$J = \{ p \in I : p \text{ is positive at an even number of roots } r_i, i \in \{1, \ldots, 4\} \}$$
$$\cup \{q\}.$$

Then (X_0, G_0) is a quotient space. Indeed, one checks that if $\sigma_{r_i}^-$ and $\sigma_{r_i}^+$ are the two orderings corresponding to r_i, $i \in \{1, \ldots, 4\}$, then $G_0 = \ker \gamma$ for $\gamma = \sigma_{r_1}^- \sigma_{r_2}^- \sigma_{r_3}^- \sigma_{r_4}^-$, with $\sigma_{r_1}^-, \sigma_{r_2}^-, \sigma_{r_3}^-, \sigma_{r_4}^-$ all coming from one connected component. We note that instead of $\sigma_{r_1}^- \sigma_{r_2}^- \sigma_{r_3}^- \sigma_{r_4}^-$ we can use any other combination of $\sigma_{r_i}^\epsilon$, $i \in \{1, \ldots, 4\}$, $\epsilon \in \{-, +\}$, making q positive: at the end, they all define the same γ, since $\sigma_{r_i}^- \sigma_{r_i}^+ \sigma_{r_j}^- \sigma_{r_j}^+ = 1$ for $i \neq j$, $i, j \in \{1, \ldots, 4\}$.

EXAMPLE 4.6. Let $r_1, \ldots, r_4$ be four real algebraic numbers again, but now suppose that r_1, r_2 are roots of an irreducible polynomial q_1, and r_3, r_4 are roots of an irreducible q_2. Furthermore, assume that q_1 and q_2 have no roots other than $r_1, \ldots, r_4$. Let

$$J = \{ p \in I : p \text{ is positive at an even number of roots } r_i, i \in \{1, \ldots, 4\} \}$$
$$\cup \{q_1, q_2\}.$$

Then (X_0, G_0) is a quotient space. Indeed, denote by $\sigma_{r_i}^-$ and $\sigma_{r_i}^+$ the two orderings corresponding to r_i, the first one making the minimal polynomial of r_i negative, and the second one positive, $i \in \{1, \ldots, 4\}$. Depending on how the polynomials q_1 and q_2 overlap, there are different ways of defining γ. Say, for example, that q_1 and q_2 are related as in Figure 1. Set $\gamma = \sigma_{r_1}^- \sigma_{r_3}^- \sigma_{r_2}^+ \sigma_{r_4}^+$.
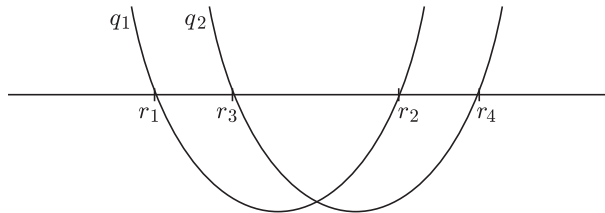

Fig. 1. Irreducible polynomials q_1 and q_2

One checks that the conditions of Theorem 3.3 are satisfied. The reader might wish to experiment with different ways of structuring roots of q_1 and q_2.

We note that if polynomials q_1 and q_2 in Example 4.6 have more than just two roots each, the quotient structure (X_0, G_0) is, in general, not a quotient space. As details at this level are becoming too technical, we are not going to discuss this any further.

5. General quotients. We continue to assume that (X, G) is a space of orderings. Fix a subgroup G_0 of G containing -1, possibly having infinite index in G, and let X_0 denote the set of all restrictions of elements of X to G_0. Denote the restriction of $\sigma \in X$ to G_0 by $\overline{\sigma}$. Let $S := X^4 \cap \chi(G/G_0)$. Theorem 2.4 generalizes as follows:

THEOREM 5.1. *A necessary condition for the quotient structure (X_0, G_0) of (X, G) to be a space of orderings is that S generates $\chi(G/G_0)$ as a topological group, i.e., $\chi(G/G_0)$ is the closure of the subgroup of $\chi(G/G_0)$ generated by S, i.e., $S^{\perp} = G_0$.*

Proof. It suffices to show that, for each $g \in G \setminus G_0$, there is a $\gamma \in S$ such that $\gamma(g) \neq 1$. Fix $g \in G \setminus G_0$. Case 1: there are $\sigma, \tau \in X$ such that $\sigma(g) \neq \tau(g)$ and $\overline{\sigma} = \overline{\tau}$. In this case we take $\gamma = \sigma\tau$. Case 2: for all $\sigma, \tau \in X$, $\overline{\sigma} = \overline{\tau} \Rightarrow \sigma(g) = \tau(g)$. In this case, the function $\phi : X_0 \to \{\pm 1\}$ given by $\phi(\overline{\sigma}) = \sigma(g)$ is well-defined and continuous, and is not in the image of the natural map $G_0 \hookrightarrow \mathrm{Cont}(X_0, \{\pm 1\})$. Thus, by [12, Theorem 7.2], there is a 4-element fan $\overline{\sigma}_1, \overline{\sigma}_2, \overline{\sigma}_3, \overline{\sigma}_4$ in X_0 such that $\prod_{i=1}^{4} \phi(\overline{\sigma}_i) \neq 1$. In this case we take $\gamma = \prod_{i=1}^{4} \sigma_i$. ∎

For each $\gamma \in S$, $\gamma \neq 1$, γ has some (not necessarily unique) minimal expression $\gamma = \prod_{i=1}^{k} \sigma_i$, $\sigma_i \in X$, $k = 2$ or 4. Denote by $(Y, G/\Delta) = (Y, G/Y^{\perp})$ the subspace of (X, G) generated by the connected components of the various σ_i, $i = 1, \ldots, k$, γ running through $S \setminus \{1\}$, and let Y_0 denote the set of restrictions of elements of Y to G_0.

THEOREM 5.2. *A necessary condition for the quotient structure (X_0, G_0) of (X, G) to be a space of orderings is that S generates $\chi(G/G_0)$ as a topological group and the quotient structure $(Y_0, G_0/\Delta)$ of $(Y, G/\Delta)$ is a space of orderings.*

Proof. If the quotient structure (X_0, G_0) of (X, G) is a space of orderings then the quotient structure $(Y_0, G_0/\Delta)$ of $(Y, G/\Delta)$ is a subspace of (X_0, G_0), so it is itself a space of orderings. ∎

The subspace $(Y, G/\Delta)$ of (X, G) defined above will be referred to as the *core of the space of orderings (X, G) with respect to the quotient structure* (X_0, G_0).

REMARK 5.3. (1) The connected components occurring in the definition of the core $(Y, G/\Delta)$ do not depend on the particular minimal presentations of the elements $\gamma \in S \setminus \{1\}$. If we have two minimal presentations

$$\gamma = \sigma_1 \ldots \sigma_k, \quad \gamma = \tau_1 \ldots \tau_\ell, \quad \sigma_i, \tau_j \in X, \; k, \ell \in \{2, 4\},$$

then, using [10, Lemma 3.2], we see that for every i there is j such that $\sigma_i \sim \tau_j$, and similarly, for every j there is i such that $\tau_j \sim \sigma_i$.

Proof. Since $\sigma_1 \ldots \sigma_k = \tau_1 \ldots \tau_\ell$ it follows that $\sigma_1 \ldots \sigma_k \tau_1 \ldots \tau_\ell = 1$. By hypothesis $\sigma_1, \ldots, \sigma_k$ are linearly independent and $k \geq 1$. After reindexing suitably, we can assume that $\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_{t-1}, \; 1 \leq t \leq \ell$, is a maximal linearly independent subset of $\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_\ell$. Then τ_t is some linear combination of $\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_{t-1}$, say $\tau_t = \prod_{i \in I} \sigma_i \prod_{j \in J} \tau_j$, $I \subseteq \{1, \ldots, k\}$, $J \subseteq \{1, \ldots, t-1\}$. Since $\tau_1, \ldots, \tau_\ell$ are linearly independent we see that $I \neq \emptyset$. According to [10, Lemma 3.2], $\sigma_i \sim \tau_t$ for each $i \in I$. If $I = \{1, \ldots, k\}$ we are done. Otherwise, after canceling, we obtain $\prod_{i \in I'} \sigma_i \prod_{j \in J'} \tau_j = 1$ where $I' = \{1, \ldots, k\} \setminus I$, $J' = \{1, \ldots, \ell\} \setminus (J \cup \{t\})$. The result follows now by induction on k. ∎

(2) If T is a maximal linearly independent subset of S then the connected components coming from the elements of $S \setminus \{1\}$ are the same as the connected components coming from the elements of T.

Proof. Suppose $\gamma \in S \setminus \{1\}$, $\gamma = \gamma_1 \ldots \gamma_m$, $\gamma_j \in T$. Choose minimal presentations $\gamma = \prod_{i=1}^k \sigma_i$, $\gamma_j = \prod_{p=1}^{k_j} \tau_{jp}$, $\sigma_i, \tau_{jp} \in X$. We want to show that for each i, $\sigma_i \sim \tau_{jp}$ for some j, p. This reduces to showing that if $\sigma_1 \ldots \sigma_k = \tau_1 \ldots \tau_\ell, \sigma_i, \tau_j \in X$, and $\sigma_1, \ldots, \sigma_k$ are linearly independent, then for each i there exists j such that $\sigma_i \sim \tau_j$. Since it is possible to reduce further to the case where $\tau_1, \ldots, \tau_\ell$ are linearly independent (by canceling whatever relations exist between the τ_j, one by one), we see that this follows by the same argument used in (1). ∎

(3) Suppose (X, G) has no infinite fans, $(G : G_0) = 2^m < \infty$, S generates $\chi(G/G_0)$ as a (topological) group, and $\gamma_1, \ldots, \gamma_m$ is some basis for $\chi(G/G_0)$ chosen so that each γ_i belongs to S, and each γ_i has a minimal presentation $\gamma_i = \prod_{j=1}^{k_i} \sigma_{ij}$, $\sigma_{ij} \in X$. Then the core of the space of orderings (X, G) with respect to the quotient structure (X_0, G_0) is the union of the connected components of the various σ_{ij}.

Proof. This follows from (2) in conjunction with the fact that any finite union of connected components is a subspace, by [15, Theorem 3.6]. ∎

Again it is natural to wonder if the necessary conditions for a quotient structure to be a quotient space given by Theorem 5.2 are sufficient. Although we are unable to prove this, we are able to show it is true in certain cases.

THEOREM 5.4. *For a space of orderings (X, G) with finitely many non-singleton connected components and no infinite fans and a quotient structure (X_0, G_0) of (X, G) of finite index, the following are equivalent:*

(1) *(X_0, G_0) is a space of orderings;*
(2) *$X^4 \cap \chi(G/G_0)$ generates $\chi(G/G_0)$ and the quotient structure $(Y_0, G_0/\Delta)$ of the core $(Y, G/\Delta)$ is a space of orderings.*

Proof. (1)⇒(2) follows from Theorem 5.2.

(2)⇒(1). Let (X_i, G_i), $i \in I_0$, be the connected components of (X, G) which do not meet $(Y, G/\Delta)$. Let $I = I_0 \cup \{i_0\}$ where i_0 is a new symbol, and let $(X_{i_0}, G_{i_0}) = (Y, G/\Delta)$. Let $\pi : X \to I$ denote the projection defined by $\pi(\sigma) = i$ if $\sigma \in X_i$, and give I the quotient topology. Arguing as in the proof of Theorem 2.9 we see that I is a Boolean space and (X_0, G_0) is the space of global sections of a sheaf of spaces of orderings on I with stalks (X_i, G_i) for $i \in I$, $i \neq i_0$, and stalk $(Y_0, G_0/\Delta)$ at $i = i_0$. ∎

Observe that Theorem 5.4 applies, in particular, to finite spaces of orderings and to SAP spaces of orderings.

THEOREM 5.5. *For the space of orderings $(X, G) = (X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)})$ and a quotient structure (X_0, G_0) of (X, G) of finite index, the following are equivalent:*

(1) *(X_0, G_0) is a space of orderings;*
(2) *$X^4 \cap \chi(G/G_0)$ generates $\chi(G/G_0)$ and the quotient structure $(Y_0, G/\Delta)$ of the core $(Y, G/\Delta)$ is a space of orderings;*
(3) *(X_0, G_0) is a profinite space of orderings.*

Proof. (1)⇒(2) is a consequence of Theorem 5.2.

(3)⇒(1) is trivial (since every profinite space of orderings is, in particular, a space of orderings).

(2)⇒(3). Assume (2) holds. Let $(G : G_0) = 2^m$, and choose $\gamma_i = \prod_{j=1}^{k_i} \sigma_{ij}$, $\sigma_{ij} \in X$, $i = 1, \ldots, m$, as in Remark 5.3(3). To prove (3) it suffices to show that for any finite subset W of G_0 there exists a finite quotient space $(X|_H, H)$ of (X, G) such that $W \subseteq H$ and $(X|_{H \cap G_0}, H \cap G_0)$ is a quotient space of $(X|_H, H)$. Define H, Y as in the preamble to Lemma 3.1, taking A to be any finite set of real monic irreducibles in $\mathbb{Q}[x]$ containing all real monic irreducible factors of elements of W together with all real monic irreducibles p such that $X_p \cap \bigcup \{\sigma_{ij} : i = 1, \ldots, m, \ j = 1, \ldots, k_i\} \neq \emptyset$, and taking B to be any finite set of transcendental real numbers such that, for each $i = 1, \ldots, m$ and each $j = 1, \ldots, k_i$, if σ_{ij} is an archimedian ordering then the corresponding transcendental real number belongs to B. Obviously $W \subseteq H$. By Lemma 3.1, $(X|_H, H)$ is a quotient space of (X, G) which is naturally identified with the quotient space $(Y, G/Y^\perp)$ of (X, G). By construction, Y is a union of connected components of (X, G) and contains all

components of (X, G) meeting the set $\{\sigma_{ij} : i = 1, \ldots, m,\ j = 1, \ldots, k_i\}$. Also, $(Y, G/Y^\perp)$ is finite and has stability index 1 or 2. It follows from Theorem 5.4 applied to the finite space of orderings $(Y, G/Y^\perp)$ and assumption (2) that $(Y|_{G_0}, G_0/Y^\perp)$ is a quotient space of $(Y, G/Y^\perp)$. Since $(X|_{H \cap G_0}, H \cap G_0)$ is identified with $(Y|_{G_0}, G_0/Y^\perp)$ under the isomorphism $(X|_H, H) \cong (Y, G/Y^\perp)$, this completes the proof. ∎

REMARK 5.6. In [2, Proposition 6] and [6, Theorem 2] it has been shown that Lam's Open Problem B holds true for any profinite spaces of orderings. Thus Theorem 5.5 shows, in particular, that a conceivable method of finding non-realizable spaces of orderings among quotients of the space of orderings of $\mathbb{Q}(x)$ of finite index will prove fruitless. At the same time, it remains an open problem whether profinite spaces of orderings are realizable. We note here that the dual question of whether direct limits of finite spaces of orderings are realizable was partially answered already in the early 1980s in [8], and recently completely resolved in [1].

6. Appendix: The sheaf construction. We recall the sheaf construction in [14, Chapter 8]. There, the results are phrased in terms of reduced Witt rings, not spaces of orderings, but the two categories are equivalent, so these results are valid for spaces of orderings.

THEOREM 6.1. *Suppose (X_i, G_i) is a space of orderings for each $i \in I$, where I is a Boolean space. Suppose $X = \dot{\bigcup}_{i \in I} X_i$ is equipped with a topology such that*

(1) *X is a Boolean space,*
(2) *the inclusion map $X_i \hookrightarrow X$ is continuous for each $i \in I$,*
(3) *the projection map $\pi : X \to I$ is continuous, and*
(4) *if $(i_\lambda)_{\lambda \in D}$ is any net in I converging to $i \in I$ and if $\sigma_1^\lambda, \sigma_2^\lambda, \sigma_3^\lambda, \sigma_4^\lambda$ is a 4-element fan in X_{i_λ} such that σ_j^λ converges to $\sigma_j \in X_i$ for each $j = 1, 2, 3, 4$, then $\sigma_1 \sigma_2 \sigma_3 \sigma_4 = 1$.*

Then (X, G) is a space of orderings, where

$$G := \{\phi \in \mathrm{Cont}(X, \{\pm 1\}) : \phi|_{X_i} \in \hat{G}_i\ \forall i \in I\}.$$

Proof. See [14, Theorem 8.5]. ∎

We need only the following special case of Theorem 6.1:

COROLLARY 6.2. *Suppose (X_i, G_i) is a space of orderings for each $i \in I$, where I is a Boolean space, and (X_i, G_i) is SAP for all but finitely many i. Suppose $X = \dot{\bigcup}_{i \in I} X_i$ is equipped with a topology such that X is a Boolean space, the inclusion map $X_i \hookrightarrow X$ is continuous for each $i \in I$, and the projection map $\pi : X \to I$ is continuous. Then (X, G) is a space of orderings,*

where

$$G := \{\phi \in \mathrm{Cont}(X, \{\pm 1\}) : \phi|_{X_i} \in \hat{G}_i \text{ for all } i \in I\}.$$

Proof. It suffices to show that condition (4) of Theorem 6.1 holds. Suppose $(i_\lambda)_{\lambda \in D}$ is a net in I satisfying the hypothesis of (4). For each $\lambda \in D$, X_{i_λ} contains a 4-element fan (so, in particular, the space of orderings $(X_{i_\lambda}, G_{i_\lambda})$ is not SAP), hence the set $\{i_\lambda : \lambda \in D\}$ is finite. Replacing the net $(i_\lambda)_{\lambda \in D}$ by a suitable subnet, we can assume $(i_\lambda)_{\lambda \in D}$ is constant. In this case, the conclusion of (iv) is obvious, using the continuity of the multiplication in the character group $\chi(G_i)$. ∎

References

[1] V. Astier and H. Mariano, *Realizing profinite reduced special groups*, Pacific J. Math. 250 (2011), 257–285.

[2] V. Astier and M. Tressl, *Axiomatization of local-global principles for pp-formulas in spaces of orderings*, Arch. Math. Logic 44 (2005), 77–95.

[3] M. Dickmann, M. Marshall and F. Miraglia. *Lattice-ordered reduced special groups*, Ann. Pure Appl. Logic 132 (2005), 27–49.

[4] M. Dickmann and F. Miraglia, *On quadratic forms whose total signature is zero mod 2^n*, Invent. Math. 133 (1998), 243–278.

[5] M. Dickmann and F. Miraglia, *Lam's conjecture*, Algebra Colloq. 10 (2003), 149–176.

[6] P. Głradki and B. Jacob, *On profinite spaces of orderings*, J. Pure Appl. Algebra 216 (2012), 2608–2613.

[7] P. Głradki and B. Jacob, *On quotients of the space of orderings of the field $\mathbb{Q}(x)$*, in: Algebra, Logic and Number Theory, Banach Center Publ., Inst. Math., Polish Acad. Sci., Warszawa, to appear.

[8] M. Kula, M. Marshall and A. Słradek, *Direct limits of finite spaces of orderings*, Pacific J. Math. 112 (1984), 391–406.

[9] T. Y. Lam, *Ten lectures on quadratic forms over fields*, in: Conference on Quadratic Forms—1976 (Kingston, 1976), G. Orzech (ed.), Queen's Papers Pure Appl. Math. 46, Queen's Univ., Kingston, ON, 1977, 1–102.

[10] M. Marshall, *Classification of finite spaces of orderings*, Canad. J. Math. 31 (1979), 320–330.

[11] M. Marshall, *Quotients and inverse limits of spaces of orderings*, Canad. J. Math. 31 (1979), 504–616.

[12] M. Marshall, *The Witt ring of a space of orderings*, Trans. Amer. Math. Soc. 258 (1980), 505–521.

[13] M. Marshall, *Spaces of orderings IV*, Canad. J. Math. 32 (1980), 603–627.

[14] M. Marshall, *Abstract Witt Rings*, Queen's Papers Pure Appl. Math. 57, Queen's Univ., Kingston, ON, 1980.

[15] M. Marshall, *Spaces of orderings: systems of quadratic forms, local structure, and saturation*, Comm. Algebra 12 (1984), 723–743.

[16] M. Marshall, *Spaces of Orderings and Abstract Real Spectra*, Lecture Notes in Math. 1636, Springer, Berlin, 1996.

[17] M. Marshall, *Review on "J.-Ph. Monnier, On Lam's conjecture concerning signatures of quadratic forms, Arch. Math. (Basel) 75 (2000), 198–206"*, Math. Rev. MR1779862 (2001g:11056).

[18] M. Marshall, *Real reduced multirings and multifields*, J. Pure Appl. Algebra 205 (2006), 452–468.

[19] D. Orlov, A. Vishik and V. Voevodsky, *An exact sequence for $K_*^M/2$ with applications to quadratic forms*, Ann. of Math. (2) 165 (2007), 1–13.

[20] V. Voevodsky, *Motivic cohomology with $\mathbf{Z}/2$-coefficients*, Publ. Math. Inst. Hautes Études Sci. 98 (2003), 59–104.

[21] H. Weber, *Zu einem Problem von H. J. Kowalsky*, Abh. Braunschweig. Wiss. Ges. 29 (1978), 127–134.

Paweł Gładki
Institute of Mathematics
University of Silesia
Bankowa 14
40-007 Katowice, Poland
and
Department of Computer Science
AGH University of Science and Technology
al. Mickiewicza 30
30-059 Kraków, Poland
E-mail: pawel.gladki@us.edu.pl

Murray Marshall
Department of Mathematics and Statistics
University of Saskatchewan
106 Wiggins Rd.
Saskatoon, SK S7N 5E6, Canada
E-mail: marshall@math.usask.ca