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Involutions of 3-dimensional handlebodies

by

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Abstract. We study the orientation preserving involutions of the orientable 3-dimensional handlebody H_g, for any genus g. A complete classification of such involutions is given in terms of their fixed points.

Introduction. Involutions of the 3-dimensional orientable handlebody H_g of genus g have already been classified in [6], [7], [10] and [9] for $g \leq 2$. Moreover, a classification of the orientation reversing involutions of H_g was given in [5, Theorem 3.6].

In this paper, we complete the study of the subject, by providing a classification of the orientation preserving involutions of H_g for any $g \geq 0$. Our argument is direct and elementary. The same result can also be derived from the general theory of actions on handlebodies developed in [8].

Namely, we prove the following theorem.

THEOREM. *Let $h : H_g \to H_g$ be an orientation preserving involution. If h is free, then $g = 2n+1$ for some $n \geq 0$ and h is equivalent to the involution I_g depicted in Figure 1. If h is not free, then there exist $n, m, l \geq 0$ with $1 \leq n+2m \leq n+2m+2l = g+1$ such that h is equivalent to the involution $L_g^{n,m}$ depicted in Figure 2.*

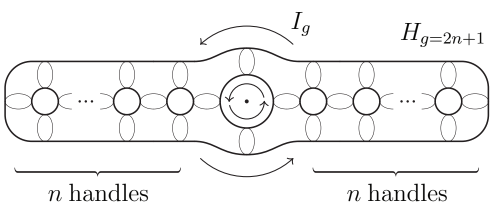

Fig. 1. The free involution I_g for $g = 2n + 1$

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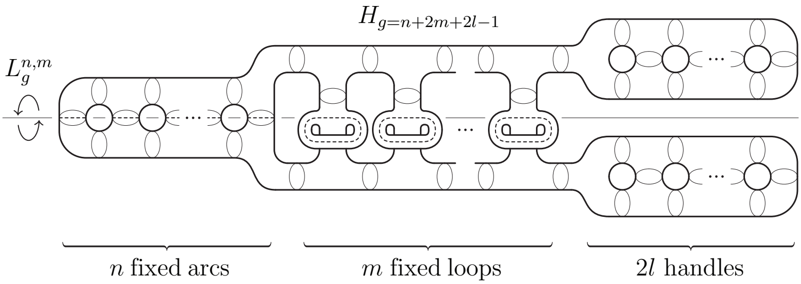

Fig. 2. The non-free involution $L_g^{n,m}$

The free involution $I_g : H_g \to H_g$ with $g = 2n + 1$ can be realized by embedding H_g in $\mathbb{R}^3$ as in Figure 1 and rotating by π radians around the axis orthogonal to the plane of the picture at the dot.

The description of the involution $L_g^{n,m} : H_g \to H_g$ with $g = n+2m+2l-1$ is a little more involved. The fixed point set $\operatorname{Fix} L_g^{n,m}$ consists of n arcs and m loops, all dashed in Figure 2. We think of H_g as $H_{n+m+2l-1}$ with m extra handles attached to it. The handlebody $H_{n+m+2l-1}$ is embedded in $\mathbb{R}^3$ in such a way that it is symmetric with respect to the median horizontal line and meets it in $n+m$ arcs, while the m extra handles are the non-symmetric ones. Then, the restriction of $L_g^{n,m}$ to $H_{n+m+2l-1}$ is given by the rotation by π radians around this axis. Of course, the fixed point set of this involution of $H_{n+m+2l-1}$ consists of $n + m$ arcs. Now, we attach each one of the m extra handles to two disks centered at the end points of a fixed arc. Finally, we extend the rotation to the extra handle as the rotation by π radians around its core. Hence, the fixed arc closes up to give a fixed loop.

We remark that $L_g^{g+1,0}$ coincides with the hyperelliptic involution of H_g.

As a consequence of our classification, we see that any orientation preserving involution of H_g is uniquely determined, up to equivalence, by its restriction to the boundary $T_g = \operatorname{Bd} H_g$. However, it is worth observing that the restrictions to T_g of non-equivalent involutions of H_g can be equivalent as involutions of T_g, by a PL homeomorphism of T_g which does not extend to H_g. Actually, two involutions of T_g are equivalent if and only if they have the same number of fixed points and they give rise to quotient surfaces of the same genus g', which follows from the Hurwitz classification of branched coverings between surfaces ([4], see also [1]).

From a different point of view, the quotient of H_g under the action of any orientation preserving involution turns out to be a handlebody $H_{g'}$. Namely, $g' = (g + 1)/2 = n + 1$ for $H_{g=2n+1}/I_g$ and $g' = (g - n + 1)/2 = m + l$ for $H_{g=n+2m+2l-1}/L_g^{n,m}$. Therefore, our result could also be reformulated in terms of double branched coverings $H_g \to H_{g'}$ between handlebodies.

1. Preliminaries. An *involution* of a PL manifold X is any PL homeomorphism $h : X \to X$ such that $h \neq \mathrm{id}_X$ and $h^2 = \mathrm{id}_X$. We denote by $\mathrm{Fix}\, h = \{x \in X \mid h(x) = x\}$ the *fixed point set* of h. The involution h is called *free* if $\mathrm{Fix}\, h = \emptyset$.

If $h' : X' \to X'$ is another involution of the PL manifold X', then we say that h and h' are *equivalent* if there exists a PL homeomorphism $\eta : X \to X'$ such that $h' = \eta \circ h \circ \eta^{-1}$.

Here, we focus on orientation preserving involutions. The 3-dimensional handlebody H_g consists of one 0-handle and g orientable 1-handles attached to it, for any $g \geq 0$. If $h : H_g \to H_g$ is such an orientation preserving involution, then $\mathrm{Fix}\, h$ is a (possibly empty) proper PL 1-submanifold of H_g. Moreover, the canonical projection $\pi : H_g \to H_g/h$ turns out to be a double branched covering.

In particular, we want to prove the theorem stated in the introduction, providing a complete classification, up to equivalence, of the orientation preserving involutions of H_g for any $g \geq 0$.

The proof proceeds by induction on the number g of 1-handles, starting from the trivial case of $g = 0$. In this case, we have $H_0 \cong B^3 \subset \mathbb{R}^3$, whose only orientation preserving involution, up to equivalence, is the symmetry $(x, y, z) \mapsto (x, -y, -z)$ with respect to the x-axis (cf. [7] and [10]), which coincides with $L_0^{1,0}$.

The following lemma, concerning involutions of 1-handles, tells us how a given orientation preserving involution of a disjoint union of orientable handlebodies can be extended to some extra 1-handles equivariantly attached to it. As an immediate consequence, such an extension is uniquely determined by the equivalence class of the involution induced on the pairs of attaching disks. This fact will be used when performing the inductive step.

Lemma 1. *The 3-dimensional 1-handle $B^1 \times B^2 \subset \mathbb{R}^3$ has only two involutions preserving the attaching disks $\{-1, 1\} \times B^2$, up to equivalence preserving such disks. Namely, they are the symmetries $(x, y, z) \mapsto (x, -y, -z)$ and $(x, y, z) \mapsto (-x, y, -z)$. The first one fixes the core $B^1 \times \{0\}$ of the handle and sends each attaching disk onto itself, while the second one fixes the diameter $\{0\} \times B^1$ of the co-core of the handle and swaps the attaching disks.*

Proof. Taking into account what we have said about involutions of B^3, the lemma can be easily derived just by considering the possible positions of the arc fixed by the involution with respect to the attaching disks. ∎

The other main tool for the inductive step is the next lemma, which allows us to split any orientation preserving involution of H_g as a boundary connected sum of involutions of simpler handlebodies.

We recall that a properly embedded PL 2-disk D in a bounded 3-manifold M is called *boundary parallel* if there exists a 2-disk $E \subset \operatorname{Bd} M$ such that $\operatorname{Bd} D = \operatorname{Bd} E$ and $D \cup E$ bounds a 3-cell in M. Moreover, if D' is another properly embedded PL 2-disk in M, then D and D' are called *parallel* if they are disjoint and there exists an annulus $A \subset \operatorname{Bd} M$ such that $\operatorname{Bd} A = \operatorname{Bd} D \cup \operatorname{Bd} D'$ and the 2-sphere $D \cup A \cup D'$ bounds a 3-cell in M.

Lemma 2. *Let $h : H_g \to H_g$ be an orientation preserving involution with $g \geq 1$. Then there exists a properly embedded PL 2-disk D in H_g which is not boundary parallel and is such that either $h(D) \cap D = \emptyset$, or $h(D) = D$ and this disk meets* $\operatorname{Fix} h$ *transversally at one point. In the first case, denoting by N a regular neighborhood of $h(D) \cup D$, we can assume that* $\operatorname{Cl}(H_g - N)$ *is PL homeomorphic to H_{g-2} or $H_{g_1} \sqcup H_{g_2}$ with $g_1 + g_2 = g - 1$. In the second case, if N denotes a regular neighborhood of $h(D) = D$, then* $\operatorname{Cl}(H_g - N)$ *is PL homeomorphic to H_{g-1} or $H_{g_1} \sqcup H_{g_2}$ with $g_1 + g_2 = g$.*

Proof. The first part of the statement follows from Theorem 3 of [2]. Concerning the second part, we first observe that $\operatorname{Cl}(H_g - N)$ is a disjoint union of handlebodies (cf. [3]) and H_g can be thought of as $\operatorname{Cl}(H_g - N)$ with one (when $h(D) = D$) or two (when $h(D) \cap D = \emptyset$) 1-handles attached to it. Hence, the only non-trivial fact to prove is that $\operatorname{Cl}(H_g - N)$ can be assumed to have at most two components. In fact, if $h(D) \cap D = \emptyset$ then $\operatorname{Cl}(H_g - N)$ could also have three components, say C_1, C_2 and C_3. It is not difficult to see that in this case h swaps two of them, say C_1 and C_2, and sends the remaining one onto itself. Since D is not boundary parallel, $C_1 \cong C_2 \cong H_{g'}$ with $g' \geq 1$. Hence, we can replace the disk D by a non-separating disk in C_1. Then we have $h(D) \cap D = \emptyset$, and $\operatorname{Cl}(H_g - N)$ turns out to be connected. ∎

By previous lemmas, one can easily determine the orientation preserving involutions of $H_1 \cong S^1 \times B^2 \subset \mathbb{C}^2$. Since these are known (cf. [9] or [6]), we limit ourselves to listing them without proof. Up to equivalence, they are $I_1 : (x, y) \mapsto (-x, y)$, $L_1^{0,1} : (x, y) \mapsto (-x, y)$ and $L_1^{2,0} : (x, y) \mapsto (\bar{x}, \bar{y})$, where the bar denotes complex conjugation, for any $(x, y) \in S^1 \times B^2$. The first involution is free, while the fixed point sets of the last two are respectively $S^1 \times \{0\}$ and $\{-1, 1\} \times [-1, 1]$.

We conclude this section by a characterization of the hyperelliptic involutions of H_g for $g \geq 2$. This will be useful in order to simplify the induction argument for the non-free case in the next section.

Lemma 3. *Let h be a non-free orientation preserving involution of H_g with $g \geq 1$. If for any 2-disk D in H_g given by Lemma 2 the union $h(D) \cup D$ (possibly coinciding with D itself) disconnects H_g, then h is equivalent to $L_g^{g+1,0}$.*

Proof. We proceed by induction on g. For $g = 0, 1$ the statement follows from the above classification of the orientation preserving involutions of H_0 and H_1.

Now, assume $g > 1$. Given a disk $D \subset H_g$ as in Lemma 2, we denote by N a regular neighborhood of $D \cup h(D)$. Then $\mathrm{Cl}(H_g - N)$ is disconnected by hypothesis, and the second part of that lemma implies that $\mathrm{Cl}(H_g - N) = C_1 \sqcup C_2$, where $C_i \cong H_{g_i}$ for $i = 1, 2$, with $g_1 + g_2 = g - 1$ if $h(D) = D$, and $g_1 + g_2 = g - 2$ if $h(D) \cap D = \emptyset$.

Since h is non-free, we deduce that each of C_1 and C_2 is sent onto itself by h. Actually, h could in principle swap C_1 and C_2 (with $g_1 = g_2$) when $h(D) \cap D = \emptyset$, but in this case it would be free. Moreover, both restrictions $h_i = h_{|C_i} : C_i \to C_i$ obviously satisfy the condition of the lemma. Therefore, by the inductive hypothesis we have $C_i \cong L_{g_i}^{g_i+1,0}$ for $i = 1, 2$.

At this point, we can easily conclude that $h \cong L_g^{g+1,0}$ by Lemma 1, after observing that N consists of one (if $h(D) = D$) or two (if $h(D) \cap D = \emptyset$) 1-handles attached to $C_1 \sqcup C_2$ to give H_g. ■

2. Proof of the theorem. Assume first that h is free. Since the Euler characteristic $\chi(H_g) = 1 - g$ is even, $g = 2n + 1$ for some $n \geq 0$. We will prove that $h \cong I_g$ by induction on n, based on the case $n = 0$, which follows from the above classification of the involutions of H_1.

So, suppose $n > 0$. Let $D \subset H_g$ be a disk as in Lemma 2. Then $h(D) \cap D = \emptyset$, since h is free. Now, denoting by N a regular neighborhood of $h(D) \cup D$ and putting $H' = \mathrm{Cl}(H_g - N)$, we have three cases.

Case 1: $H' \cong H_{g-2}$. By the inductive hypothesis, $h' = h_{|H'} \cong I_{g-2}$. Moreover, N consists of a pair of 1-handles equivariantly attached to H', which are swapped by h. Then, up to equivalence, h is the unique possible extension of h' to H_g. Since, up to equivariant PL homeomorphisms, I_g can be obtained in the same way from I_{g-2}, for example by considering as D the leftmost meridian disk in Figure 1, we have $h \cong I_g$.

Case 2: $H' = C_1 \sqcup C_2$, with $C_i \cong H_{g_i}$ and $h(C_i) = C_i$ for $i = 1, 2$. Since $g_1 < g$, by the inductive hypothesis $h_{|C_1} \cong I_{g_1}$. Now, if $g_1 > 1$ we know that there exists a disk $D' \subset C_1 \cong H_{g_1}$ such that $C_1 - (h(D') \cup D')$ is connected. Then, by replacing D with D' thought of as a disk in H_g, we are reduced to Case 1. On the other hand, if $g_1 = 1$, for any disk D' in C_1, we find that $C_1 - (h(D') \cup D')$ has two components and these are swapped by $h_{|C_1}$. Then, since also the two attaching disks of the 1-handles given by N are swapped by $h_{|C_1}$, we can easily conclude that $H_g - (h(D') \cup D')$ is connected. So, we can once again reduce ourselves to Case 1.

Case 3: $H' = C_1 \sqcup C_2$, with $C_i \cong H_{g_i}$ and $h(C_i) = C_{3-i}$ for $i = 1, 2$. In this case we have $1 \leq g_1 = g_2 < g$. Then there exists a disk $D' \subset C_1$ such that $C_1 - D'$ is connected. Since $h(D') \subset C_2$ and also $C_2 - h(D')$ is connected (being PL homeomorphic to $C_1 - D'$), we see that $H_g - (h(D') \cup D')$ is connected too. This allows the reduction to Case 1 as above.

Now, we assume that h is non-free. We will prove that $h \cong L_g^{n,m}$ by induction on g, based on the cases $g = 0, 1$, which follow from the above classification of the involutions of H_0 and H_1, and on the cases considered in Lemma 3.

So, suppose $g > 1$. Let $D \subset H_g$ be a disk as in Lemma 2. If for any such disk D the union $h(D) \cup D$ disconnects H_g, we are done by Lemma 3. Hence, we can assume that $H_g - (h(D) \cup D)$ is connected. Then, denoting by N a regular neighborhood of $h(D) \cup D$ and putting $H' = \mathrm{Cl}(H_g - N)$, we have $H' \cong H_{g-1}$ if $h(D) = D$ and $H' \cong H_{g-2}$ if $h(D) \cap D = \emptyset$. We consider these two cases separately.

Case 1: $h(D) = D$. By the inductive hypothesis, $h' = h_{|H'} \cong L_{g-1}^{n,m}$ for some n and m such that $1 \leq n + 2m \leq g$. Moreover, N consists of one 1-handle attached to H', whose attaching disks $D_1, D_2 \subset \mathrm{Bd}\, H'$ are such that $h'(D_i) = D_i$ and $D_i \cap \mathrm{Fix}\, h' = \{p_i\} \subset \mathrm{Int}\, D_i$, for $i = 1, 2$. We have the following two subcases.

Subcase 1.1: p_1 and p_2 are end points of the same arc $A \subset \mathrm{Fix}\, h'$. In this case, when attaching N to H', the arc A closes up to give a fixed loop for h. Now, if A is the rightmost fixed arc in Figure 2, then clearly $h \cong L_g^{n-1,m+1}$. On the other hand, the half-twists on the disks E and $E' = h'(E)$ on the right of Figure 3 allow us to equivariantly exchange two consecutive arcs in $\mathrm{Fix}\, h'$, hence all the arcs in $\mathrm{Fix}\, h'$ are equivalent by an equivariant PL homeomorphism. Therefore, the final result is the same for any fixed arc $A \subset \mathrm{Fix}\, h'$.

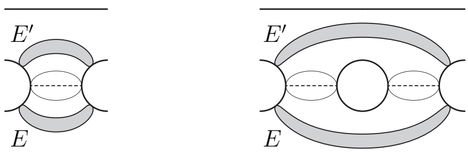

Fig. 3. Equivariantly inverting a fixed arc and exchanging two fixed arcs

Subcase 1.2: p_1 and p_2 are end points of different arcs $A_1, A_2 \subset \mathrm{Fix}\, h'$. In this case, when attaching N to H', the arcs A_1 and A_2 are joined to give one fixed arc in $\mathrm{Fix}\, h$. Now, if A_1 and A_2 are the rightmost fixed arcs in Figure 2 and the points p_1 and p_2 are the end points closest to them, then it is not difficult to see that $h \cong L_g^{n-1,m}$. On the other hand, the half-twists on the disks E and $E' = h'(E)$ on the left of Figure 3 allow us to equivariantly exchange the two end points of the same arc in $\mathrm{Fix}\, h'$. Then, using this PL homeomorphism, together with that used in the previous case to exchange two consecutive arcs in $\mathrm{Fix}\, h'$, we can always equivariantly move the points p_1 and p_2 in the preferred position described above. Hence, $h \cong L_g^{n-1,m}$ whatever p_1 and p_2 are.

CASE 2: $h(D) \cap D = \emptyset$. By the inductive hypothesis, $h' = h_{|H'} \cong L_{g-2}^{n,m}$ for some n and m such that $1 \leq n + 2m \leq g - 1$. Moreover, N consists of a pair of 1-handles equivariantly attached to H', which are swapped by h. Then, up to equivalence, h is the unique possible extension of h' to H_g. Since, up to equivariant PL homeomorphisms, $L_g^{n,m}$ can be obtained in the same way from $L_{g-2}^{n,m}$, for example by considering as D and $h(D)$ the rightmost meridian disks in Figure 2, we have $h \cong L_g^{n,m}$. ∎

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