FUNDAMENTA
MATHEMATICAE
215 (2011)

Condensation and large cardinals

by

Sy-David Friedman and **Peter Holy** (Wien)

Abstract. We introduce two generalized condensation principles: Local Club Condensation and Stationary Condensation. We show that while Strong Condensation (a generalized condensation principle introduced by Hugh Woodin) is inconsistent with an ω_1-Erdős cardinal, Stationary Condensation and Local Club Condensation (which should be thought of as weakenings of Strong Condensation) are both consistent with ω-superstrong cardinals.

This article is a contribution to the *outer model programme* (see [11]), whose aim is to show that large cardinal properties can be preserved when forcing desirable features of Gödel's constructible universe. The properties GCH, $\diamondsuit$, $\square$, definable wellordering and gap-1 morass were discussed in [8, 11, 7, 4, 5, 1]. In this article we consider Condensation. The central result of this paper is Theorem 25, which shows that Local Club Condensation is consistent with the existence of ω-superstrong cardinals, the "strongest" of large cardinals (for the definition of ω-superstrong cardinals see Definition 5 below); its main auxiliary theorem is Theorem 22, a quite different proof of which can be found in the second author's doctoral dissertation [13]. This work is also relevant to a result of Itay Neeman [17] regarding the large cardinals required to force PFA over L-like models (see the final section of the present paper).

Condensation principles. Gödel's universe **L** of constructible sets satisfies Condensation in a very strong form. There exists a sequence $\langle L_\alpha \colon \alpha \in \mathrm{Ord}\rangle$ such that:

(a) $L = \bigcup_\alpha L_\alpha$, L_α is transitive, $\mathrm{Ord}(L_\alpha) = \alpha$, $\alpha < \beta \rightarrow L_\alpha \in L_\beta$, and $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for limit λ.

2010 *Mathematics Subject Classification*: 03E35, 03E55.
Key words and phrases: condensation, large cardinals, outer model programme, generalized condensation principles, local club condensation, ω-superstrong cardinal.

DOI: 10.4064/fm215-2-3

(b) For each α: If $(M, \in)$ is elementary in $(L_\alpha, \in)$ then $(M, \in)$ is isomorphic to some $(L_{\bar{\alpha}}, \in)$.

We will give definitions of various generalized forms of Condensation; those definitions apply to models $\mathbf{M}$ of set theory with a hierarchy of levels of the form $\langle M_\alpha \colon \alpha \in \mathrm{Ord}\rangle$ with the properties that $\mathbf{M} = \bigcup_{\alpha \in \mathrm{Ord}} M_\alpha$, each M_α is transitive, $\mathrm{Ord}(M_\alpha) = \alpha$, if $\alpha < \beta$ then $M_\alpha \in M_\beta$, and if γ is a limit ordinal then $M_\gamma = \bigcup_{\alpha<\gamma} M_\alpha$. We will often use M_α to also denote the structure $(M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle)$, where context will clarify the intended meaning. If $\mathcal{B}$ has domain B and is elementary in some M_α, we say that $\mathcal{B}$ *condenses* or that $\mathcal{B}$ *has Condensation* iff $(B, \in, \langle M_\beta \colon \beta \in B\rangle)$ is isomorphic to some $(M_{\bar{\alpha}}, \in, \langle M_\beta \colon \beta < \bar{\alpha}\rangle)$. We also say that B condenses or that B has Condensation in this case.

In [19], Hugh Woodin defines the principle of Strong Condensation, which may be reformulated in the context of models with a hierarchy of levels as follows:

Total Strong Condensation is the statement that for every α, there is a structure $\mathcal{A} = (\mathcal{M}_\alpha, \in, \langle \mathcal{M}_\beta \colon \beta < \alpha\rangle, \ldots)$ for a countable language such that each of its substructures condenses [1].

Strong Condensation is the same statement with α ranging only over cardinals and with the additional assumption that for every cardinal α, $M_\alpha = H_\alpha$, the collection of sets whose transitive closure has cardinality less than α.

Strong Condensation for α is the statement of Strong Condensation for a single (fixed) cardinal α together with the assumption that $M_\kappa = H_\kappa$ for all cardinals $\kappa \le \alpha$.

Total Strong Condensation is the strongest fragment of Condensation which we will consider in this paper. Strong Condensation follows from Total Strong Condensation by Lemma 1 below. A natural weakening of Total Strong Condensation is given by the following:

Stationary Condensation is the principle that for each α and infinite cardinal $\kappa \le \alpha$, any structure $\mathcal{A} = (M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle, \ldots)$ for a countable language has a condensing substructure $\mathcal{B}$ with domain of size κ, containing κ as a subset.

As we will show later, Strong Condensation is inconsistent with the existence of an ω_1-Erdős cardinal. Since our main focus lies on condensation principles in the presence of very large cardinals, this notion is thus too

[1] As we may assume that $\mathcal{A}$ is skolemized, we could replace "substructure" by "elementary substructure" in the above. Similar remarks will apply to the definitions of all further generalized condensation principles below.

strong for our purposes. We will show that Stationary Condensation is consistent with the existence of an ω-superstrong cardinal. But there is a much stronger generalized condensation principle which we will show to be consistent with the existence of an ω-superstrong cardinal as well:

Local Club Condensation is the statement that if α has uncountable cardinality κ and $\mathcal{A} = (M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle, \ldots)$ is a structure for a countable language, then there exists a continuous chain $\langle \mathcal{B}_\gamma \colon \omega \le \gamma < \kappa\rangle$ of condensing substructures of $\mathcal{A}$ whose domains B_γ have union M_α, each B_γ has cardinality $\operatorname{card}\gamma$ (the cardinality of γ) and contains γ as a subset.

Whenever we want to work with any of the above notions, we will be in the situation that $\mathbf{M} = (\mathbf{L}[A], A)$ for some $A \subseteq \mathrm{Ord}$ and $\langle M_\alpha \colon \alpha \in \mathrm{Ord}\rangle = \langle L_\alpha[A] \colon \alpha \in \mathrm{Ord}\rangle$. In this case, we say $\mathbf{M}$ *is of the form* $\mathbf{L}[A]$ and note that $(B, \in, A) \prec (M_\alpha, \in, A)$ implies $(B, \in, \langle M_\beta \colon \beta \in B\rangle) \prec (M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle)$ and if $(B, \in, A)$ is isomorphic to $(M_{\bar\alpha}, \in, A)$ then $(B, \in, \langle M_\beta \colon \beta \in B\rangle)$ is isomorphic to $(M_{\bar\alpha}, \in, \langle M_\beta \colon \beta < \bar\alpha\rangle)$.

Acceptability is the statement that, assuming $\mathbf{M}$ is of the form $\mathbf{L}[A]$, for any ordinals $\gamma \ge \delta$, if there is a subset of δ in $M_{\gamma+1} \setminus M_\gamma$, then $H^{M_{\gamma+1}}(\delta) = M_{\gamma+1}$, where $H^{M_{\gamma+1}}(\delta)$ denotes the Skolem hull of δ in $M_{\gamma+1} = L_{\gamma+1}[A]$ using the predicate $A \cap (\gamma+1)$.

NOTE. The above property might also be referred to as "Weak Acceptability", since in the literature, "Acceptability" is often used for the following, closely related notion: If there is a subset of δ in $M_{\gamma+1} \setminus M_\gamma$, then there is a surjection of δ onto M_γ in $M_{\gamma+1}$. We will stick to the term "Acceptability" for our above-defined notion though.

LEMMA 1. *Total Strong Condensation* $\to$ *Local Club Condensation* $\to$ *Stationary Condensation* $\to$ GCH. *In fact, if Stationary Condensation holds, then for all infinite cardinals* κ, $H_\kappa = M_\kappa$ *has cardinality* κ.

Proof. We only prove the last statement, as the other implications are immediate. Suppose $\langle M_\alpha \colon \alpha \in \mathrm{Ord}\rangle$ witnesses Stationary Condensation. Let $\mathcal{A} = (M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle, f_\alpha)$ with f_α a bijection from $\operatorname{card}\alpha$ to α. Let B be the domain of a condensing substructure of $\mathcal{A}$ of size $\operatorname{card}\alpha$, containing $\operatorname{card}\alpha$ as a subset, as provided by Stationary Condensation. As f_α is contained in the structure $\mathcal{A}$, it follows that $\alpha \subseteq B$, and hence that $B = M_\alpha$ has size $\operatorname{card}\alpha$.

Thus if $\alpha < \kappa^+$, then $M_\alpha \subseteq H_{\kappa^+}$ by transitivity of M_α, hence $M_{\kappa^+} = \bigcup_{\alpha<\kappa^+} M_\alpha \subseteq H_{\kappa^+}$ for all infinite cardinals κ. Now if $x \in H_{\kappa^+}$ choose some α such that $x \in M_\alpha$. Let $f \colon \kappa \xrightarrow{\text{onto}} \mathrm{tcl}(\{x\})$ and apply Stationary Condensation to the structure $(M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle, f)$ to obtain $\bar\alpha < \kappa^+$ such that $x \in M_{\bar\alpha} \subseteq M_{\kappa^+}$. Therefore $H_{\kappa^+} \subseteq M_{\kappa^+}$; it follows that $H_\kappa = M_\kappa$ for all infinite cardinals κ. ∎

Strong Condensation

DEFINITION 2. A cardinal κ is *α-Erdős* iff $\kappa \to (\alpha)^{<\omega}$, i.e., for any $F : [\kappa]^{<\omega} \to 2$, there is a subset H of κ of order-type α such that F is constant on $[H]^n$ for each finite n.

FACT 3 (see [14]). *Let κ be the least α-Erdős cardinal, with α a limit ordinal. Then κ is strongly inaccessible and if $C \subseteq \kappa$ is CUB and $\mathcal{A}$ is a structure for a countable language whose universe includes κ, then there exists $I \subseteq C$ of order-type α such that I is a* good set of indiscernibles *for $\mathcal{A}$, i.e., whenever a, b are finite increasing sequences from I of the same length, then a, b have the same type in $\mathcal{A}$, allowing parameters less than* $\min(a \cup b)$.

THEOREM 4. *If there is an ω_1-Erdős cardinal then Strong Condensation fails* (2).

Proof. Suppose that κ is the least ω_1-Erdős cardinal. We show that Strong Condensation for κ fails. Assume for a contradiction that Strong Condensation for κ is witnessed by $\mathcal{A}$. We may assume that $\mathcal{A}$ is skolemized. Let I be a good set of indiscernibles for $\mathcal{A}$ of order-type ω_1 with I contained in the CUB set C of $\bar\kappa < \kappa$ such that $M_{\bar\kappa}$ is $\mathcal{A}$-closed. For any limit initial segment J of I let X_J be the $\mathcal{A}$-closure of J.

CLAIM.

(a) *J is cofinal in* $X_J \cap \mathrm{Ord}$.
(b) *If $J_0 \subseteq J_1$ are limit initial segments of I then $X_{J_0} \cap \mathrm{Ord}$ is an initial segment of* $X_{J_1} \cap \mathrm{Ord}$.

Proof of Claim. (a) For any α in J, $X_{J\cap\alpha}$ is a subset of M_α as J is a subset of C. So $X_J = \bigcup\{X_{J\cap\alpha} : \alpha \in J\} \subseteq \bigcup\{M_\alpha : \alpha \in J\} = M_{\sup J}$, so $X_J \cap \mathrm{Ord} \subseteq M_{\sup J} \cap \mathrm{Ord} = \sup J$.

(b) Suppose $\alpha = t^{\mathcal{A}}(\vec{j})$ with $\vec{j}$ increasing from J_1, $\alpha < \sup(X_{J_0} \cap \mathrm{Ord}) = \sup J_0$, and t a term in the language of $\mathcal{A}$. Write $\vec{j}$ as $\vec{j}_0 \cup \vec{j}_1$ where $\vec{j}_0$ is the part of $\vec{j}$ in J_0. Choose $\vec{j}_1'$ in J_0 above α so that $\vec{j}' = \vec{j}_0 \cup \vec{j}_1'$ is increasing with the same length as $\vec{j}$. Then by goodness, $\alpha = t^{\mathcal{A}}(\vec{j}) = t^{\mathcal{A}}(\vec{j}') \in X_{J_0}$. $\blacksquare_{\mathrm{Claim}}$

It follows that $(X_I, \in)$ is isomorphic to $(M_{\omega_1}, \in) = (H_{\omega_1}, \in)$. Let π be an isomorphism from $(M_{\omega_1}, \in)$ onto $(X_I, \in)$. As M_{ω_1} is an element of X_I we can choose a in M_{ω_1} such that $\pi(a) = M_{\omega_1}$. Choose a real R not in a. Then $\pi(R)$ does not belong to $\pi(a) = M_{\omega_1}$. But as $\omega + 1$ is contained in X_I, $\pi(R) = R$ so R does not belong to M_{ω_1}, a contradiction to Lemma 1. $\blacksquare_{\text{Theorem 4}}$

---

(2) A stronger result, with ω_1-Erdős replaced by weakly ω_1-Erdős, easily follows from Proposition 9 of [18], which was proven independently.

Stationary Condensation

Definition 5. Suppose that $j\colon \mathbf{V} \to \mathbf{M}$ is an elementary embedding with critical point κ. Define $j^0(\kappa) = \kappa$, $j^{n+1}(\kappa) = j(j^n(\kappa))$, $j^\omega(\kappa) = \bigcup_{n<\omega} j^n(\kappa)$. We say that j is an *α-superstrong embedding* iff $H_{j^\alpha(\kappa)} \subseteq \mathbf{M}$, and κ is *α-superstrong* iff κ is the critical point of an α-superstrong embedding.

Fact (see [15]). *There are no elementary embeddings $j\colon \mathbf{V} \to \mathbf{M}$ with critical point κ such that $H_{(j^\omega(\kappa))^+} \subseteq \mathbf{M}$. The existence of an ω-superstrong embedding is not known to be inconsistent.*

Theorem 6. *Stationary Condensation is consistent with the existence of an ω-superstrong cardinal.*

Proof. Suppose that κ is ω-superstrong. By Theorem 2 of [11], we may first force the GCH, preserving the ω-superstrength of κ. Now for each infinite cardinal α add a Cohen subset of α^+ by a reverse-Easton iteration. Let $A_\alpha \subseteq [\alpha, \alpha^+)$ be the α^+-Cohen set added (shifted up to α) and let A be the union of the A_α's. Then $V[A]$ equals $L[A]$, as any ground model set is coded into one of the A_α's. We claim that Stationary Condensation is witnessed by the M_α's, where $M_\alpha = L_\alpha[A]$ for each $\alpha \in \mathrm{Ord}$.

The forcing is cofinality-preserving and σ-closed. Also, by the argument of the proof of Theorem 2 of [11], the ω-superstrength of κ is preserved.

Claim. *For each infinite cardinal κ, any set of ordinals in $V[A] = L[A]$ of cardinality κ is covered in V by a set of ordinals of cardinality κ.*

Proof of claim. Let $\dot{x}$ be a name for a set of ordinals of cardinality κ. First suppose that κ is regular. Then $\dot{x}$ is forced to belong to an extension of V by a forcing of size κ (the iteration below κ) and the result follows easily. If $\kappa = \bigcup\{\kappa_\alpha\colon \alpha < \operatorname{cof}\kappa\}$ is singular (with each κ_α regular and greater than $\operatorname{cof}\kappa$), then inductively extend a given condition without changing it below $\operatorname{cof}\kappa$, to obtain a ground model cover of size κ_α for the first κ_α elements of $\dot{x}$; after $\operatorname{cof}\kappa$ steps, the resulting condition covers $\dot{x}$ by a ground model set of size κ. $\blacksquare_{\text{Claim}}$

Now let κ be an infinite cardinal, α an ordinal of cardinality at least κ and $\dot{S} = (L_\alpha[A], \in, A, \ldots)$ a name for a structure in $V[A]$ for a countable language. We may assume that $\dot{S}$ is skolemized (and therefore any substructure of $\dot{S}$ is isomorphic to $(L_{\bar{\alpha}}[\bar{A}], \in, \bar{A}, \ldots)$ for some $\bar{\alpha} \le \alpha$, $\bar{A} \subseteq \bar{\alpha}$). We show that below any condition p there is a condition q^* which forces Condensation for the universe of some substructure of $\dot{S}$ of size κ which contains κ as a subset. For any condition p let $p(\kappa)$ denote the κ^+-Cohen condition specified by p, whose domain we write as $[\kappa, |p(\kappa)|)$. We construct a decreasing sequence $\langle p_i\colon i < \omega\rangle$ of conditions with $p_0 = p$ and greatest lower bound q. Let $x_0 = \kappa$. Given p_i, choose $p_{i+1} \le p_i$ forcing that some $x_{i+1} \in \mathbf{V}$ of cardinality κ contains the set of ordinals in the $\dot S$-closure of $x_i \cup |p_i(\kappa)|$ as a subset, and that $A \cap x_i$ has a P_κ-*name*, i.e. a name which depends only on the generic below κ. The latter is possible using the fact that the forcing P factors as $P_\kappa * P[\kappa, \infty)$ where $P[\kappa, \infty)$ is κ^+-closed. Let $x = \bigcup_{n<\omega} x_n$. Then q forces that x is the set of ordinals of a substructure of $\dot S$ and $A \cap x$ is forced to have a P_κ-name. Therefore we are free to extend the κ^+-Cohen condition $q(\kappa)$ to $q^*(\kappa)$ so that ($\dot S$-closure of $x, \in, A$) is forced by q^* to be isomorphic to $(L_{|q^*(\kappa)|}[A], \in, A)$ ([3]). Thus we have forced Condensation for the $\dot S$-closure of x, the universe of a substructure of $\dot S$ of size κ, containing κ as a subset, as desired. $\blacksquare_{\text{Theorem 6}}$

REMARK. Actually, more than Stationary Condensation holds in the model witnessing the previous theorem: For any uncountable $\kappa \le \alpha$, κ regular, any club subset C of $[M_\alpha]^{<\kappa}$ has a condensing element M. (Stationary Condensation implies this only for uncountable successor cardinals κ.) But instead of verifying this, we show next that the stronger principle of Local Club Condensation both holds in the known fine-structural inner models for large cardinals and can be forced consistently with an ω-superstrong cardinal.

Local Club Condensation

LEMMA 7. *Local Club Condensation is equivalent to the following, seemingly weaker statement: If α has uncountable cardinality κ, then the structure $\mathcal{A} = (M_\alpha, \in, \langle M_\beta \colon \beta < \alpha \rangle, F)$ has a continuous chain $\langle \mathcal{B}_\gamma \colon \gamma \in C \rangle$ of condensing substructures with domains B_γ, $\bigcup_{\gamma \in C} B_\gamma = M_\alpha$, $C \subseteq \kappa$ is club, C consists only of cardinals if κ is a limit cardinal, each B_γ has cardinality* $\operatorname{card} \gamma$ *and contains γ as a subset, where F denotes the function $(f, x) \mapsto f(x)$ whenever $f \in M_\alpha$ is a function and $x \in \operatorname{dom}(f) \cap M_\alpha$.*

Proof. Suppose $\langle M_\alpha \colon \alpha \in \mathrm{Ord} \rangle$ witnesses the above-described, seemingly weaker property. First note that for any infinite cardinal κ, $H_{\kappa^+} \subseteq M_{\kappa^+}$: If not, let $\lambda > \kappa$ be the least cardinality of some α such that $x \in H_{\kappa^+}$ belongs to M_α. But then x belongs to the domain of some condensing $\mathcal{B} \prec \mathcal{A}$ of cardinality $< \lambda$ which contains κ as a subset and a function from κ onto $\operatorname{tcl} x$ as an element, i.e. which contains $\operatorname{tcl} x$ as a subset, using closure under F. Thus x belongs to $M_{\bar\alpha}$ for some $\bar\alpha < \lambda$, contradicting leastness of λ.

Now we prove that Local Club Condensation holds, by induction on κ: Assume α has uncountable cardinality κ and $\mathcal{E} = (M_\alpha, \in, \langle M_\beta \colon \beta < \alpha \rangle, \ldots)$

([3]) We use here the fact that $|q(\kappa)| = x \cap \kappa^+$ and that q^* may decide the values of the generic predicate A in the interval $[x \cap \kappa^+, |q^*(\kappa)|)$ according to the values of $A \cap x$ above κ^+.

is a structure for a countable language. As $\mathcal{E} \in H_{\kappa^+}$, we may choose $\alpha' > \alpha$ of cardinality κ such that $\mathcal{E} \in M_{\alpha'}$. We obtain a continuous chain $\langle \mathcal{B}_\gamma \colon \gamma \in C\rangle$ of condensing substructures of $\mathcal{A}' = (M_{\alpha'}, \in, \langle M_\beta \colon \beta < \alpha'\rangle, F)$ with domains B_γ as described in the statement of the lemma. We may assume $\mathcal{E} \in B_{\min C}$. Then we obtain a continuous chain $\langle \mathcal{D}_\gamma \colon \gamma \in C\rangle$ of condensing substructures $\mathcal{D}_\gamma = (D_\gamma, \in, \langle M_\beta \colon \beta \in D_\gamma\rangle, \ldots)$ of $\mathcal{E}$ such that $\bigcup_{\gamma \in C} D_\gamma = M_\alpha$, each D_γ has cardinality $\operatorname{card}\gamma$ and contains γ as a subset by setting $\mathcal{D}_\gamma = \mathcal{E}{\restriction}B_\gamma$, using the fact that F is part of the structure $\mathcal{A}'$.

Now if $\kappa = \delta^+$ with δ an uncountable cardinal, then by reindexing we can assume that $C = [\delta, \kappa)$, choose $\bar\alpha$ so that $(D_\delta, \in, \langle M_\beta \colon \beta \in D_\delta\rangle)$ is isomorphic to $(M_{\bar\alpha}, \in, \langle M_\beta \colon \beta < \bar\alpha\rangle)$ and define $\mathcal{D}_\gamma$ for $\gamma < \delta$ by applying Local Club Condensation inductively to $\bar\alpha$. If κ is a limit cardinal, we let $\langle \gamma_i \colon i < \operatorname{ot} C\rangle$ be the increasing enumeration of C and fill in $\langle \mathcal{D}_\gamma \colon \gamma \in C\rangle$ to $\langle \mathcal{D}_\gamma \colon \omega \le \gamma < \kappa\rangle$ by applying Local Club Condensation inductively. ∎

Theorem 8. *In known $L[E]$ models* (*see* [20]), *Local Club Condensation holds.*

Proof sketch. We verify the form of Local Club Condensation stated in Lemma 7, taking M_α to be $L_\alpha[E]$. Suppose that α has uncountable cardinality λ and let $L_\beta[E] = J_\beta[E]$, with β at least α, Σ_1-project to λ. For CUB-many ordinals $\bar\lambda < \lambda$, the Σ_1-hull in $L_\beta[E]$ of $\bar\lambda$ with p, the first standard parameter for $L_\beta[E]$, contains the witnesses for the ordinals in p. Moreover we can guarantee that this Σ_1-hull condenses to a mouse with Σ_1-projectum $\bar\lambda$, as either λ is a limit cardinal, in which case we can choose each $\bar\lambda$ to be a cardinal, or λ is a successor cardinal, in which case we can choose each $\bar\lambda$ to be a limit point of $\{\gamma < \lambda \colon \gamma = \lambda \cap \Sigma_1\text{-hull of } \gamma \cup \{p\}$ in $L_\beta[E]\}$. It follows that the Σ_1-hull of $\bar\lambda$ with p in $L_\beta[E]$ condenses to an initial segment of $L[E]$ and therefore we may take $\mathcal{B}_{\bar\lambda}$ to be the intersection of this hull with $L_\alpha[E]$. ∎

Forcing Local Club Condensation

Lemma 9. *Assume $\langle f_\gamma \colon \gamma \in [\kappa, \kappa^+)\rangle$ is so that each f_γ is a bijection from κ, a regular uncountable cardinal, to γ and $\beta \in [\kappa, \kappa^+)$. There is a club of $\delta < \kappa$ such that $f_\alpha[\delta] = f_\beta[\delta] \cap \alpha$ for all $\alpha \in f_\beta[\delta] \setminus \kappa$.*

Proof. Note that whenever $X \ni \beta$ is transitive below κ and elementary in $(H_{\kappa^+}, \in, F)$ with $F(\alpha, \gamma) = f_\alpha(\gamma)$ for $\gamma < \kappa$ and $\kappa \le \alpha < \kappa^+$, then $X \cap \kappa$ is as desired, which is easily seen using elementarity. The claim follows as $\{X \cap \kappa \colon X \prec (H_{\kappa^+}, \in, F)\}$ contains a club in κ. ∎

Definition 10. If P is a notion of forcing and η is a cardinal, we say that P is η^+-*strategically closed* iff Player I has a winning strategy in the following two-player game with perfect information: Player I and Player II alternately make moves where in each move, each player plays a condition of P. Player I has to start and play $\mathbf{1}_P$ in the first move. Player II is allowed to play any condition stronger than the condition just played by Player I in each of his moves. Player I has to play a condition stronger than all previously played conditions in each move, and has to make a move at every limit step of the game. We say that Player I wins if he can find conditions to play in any such game of length η^+ (on arriving at η^+, the game ends, no condition has to be played at stage η^+).

Now we will show how to force Local Club Condensation while preserving ω-superstrong cardinals. We assume that the universe $\mathbf{V}$ we start with satisfies GCH, that R is a predicate well-ordering $\mathbf{V}$, and that we work in the model $(\mathbf{V}, R)$. Note that the definition of the forcing iteration given below depends on the predicate R and we will see in the proof of Theorem 25 that a careful choice of R will be important for large cardinal preservation.

Definition of basic objects. For each ordinal α, fix f_α as the R-least bijection from the cardinality of α to α. Let S denote the forcing poset consisting of the conditions $\{\mathbf{1}, 0, 1\}$ where $0 \leq_S \mathbf{1}$, $1 \leq_S \mathbf{1}$, $0 \perp_S 1$. An S-generic filter simply decides for either 0 or 1. For two compatible conditions s_0 and s_1 in S, let $s_0 \cup s_1$ denote the stronger of both. Whenever $\operatorname{card} \alpha$ is regular and $g \subseteq (\alpha+1)$ ([4]), let $C_\alpha(g)$ denote the following forcing poset ([5]):

If $\operatorname{card} \alpha$ is a successor cardinal, $\operatorname{card} \alpha = \theta^+$, then q^{**} is a condition in $C_\alpha(g)$ iff

- q^{**} is a closed, bounded subset of $[\theta, \theta^+)$, and
- $\forall \eta \in q^{**}\ \ g(\operatorname{ot} f_\alpha[\eta]) = g(\alpha)$.

If $\operatorname{card} \alpha$ is inaccessible, then q^{**} is a condition in $C_\alpha(g)$ iff

- q^{**} is a closed, bounded set of cardinals below $\operatorname{card} \alpha$, and
- $\forall \eta \in q^{**}\ \ g(\operatorname{ot} f_\alpha[\eta]) = g(\alpha)$.

Conditions in $C_\alpha(g)$ are ordered by end-extension (in both cases).

Definition of the forcing. We will force with P, a reverse Easton-like iteration of $Q(\alpha)$, $\alpha \in \mathrm{Ord}$. If $\alpha < \omega$, then $Q(\alpha)$ denotes the trivial forcing. If $\operatorname{card} \alpha = \omega$ or $\operatorname{card} \alpha$ is singular, then $Q(\alpha) = Q(\alpha)(0) = S$. If $\operatorname{card} \alpha$ is regular, then $Q(\alpha) = Q(\alpha)(0) * Q(\alpha)(1)$ with $Q(\alpha)(0) = S$ and $Q(\alpha)(1) = C_\alpha(g_{\alpha+1})$, where $g_{\alpha+1}$ denotes the generic predicate obtained from the generic for $P_\alpha^\oplus$ (where $P_\alpha^\oplus = P_\alpha * Q(\alpha)(0)$, with P_α denoting the iteration P below α) as follows: $g_{\alpha+1} \restriction \omega = 0$. For any ordinal $\beta \in [\omega, \alpha]$,

([4]) We identify sets with their characteristic functions and vice versa in the following (i.e. $g(\beta) = 1 \leftrightarrow \beta \in g$).

([5]) For suitable g, $C_\alpha(g)$ will ensure that $g(\alpha)$ is coded by $g \restriction \operatorname{card} \alpha$. This "canonical function coding" was first introduced in [2] and [3].

$g_{\alpha+1}(\beta)$ is either 0 or 1, depending on whether the $P^{\oplus}_{\alpha}$-generic $G^{\oplus}_{\alpha}$ decides for either 0 or 1 at $Q(\beta)(0)$, i.e. $g_{\alpha+1}(\beta) = 1$ iff $\exists p \in G^{\oplus}_{\alpha}\ \ p{\restriction}\beta \Vdash p(\beta)(0) = 1$. To complete the definition of P, we need to specify the supports used; before doing so, we introduce some further notation:

For any notion of forcing in some forcing extension, we let $\check{\mathbf{1}}$ denote the standard name for its weakest condition $\mathbf{1}$. Assume p is a condition in some ([6]) iteration of $\langle Q(\alpha)\colon \alpha \in \delta\rangle$ for $\delta \in \mathrm{Ord} \cup \{\mathrm{Ord}\}$. If $\operatorname{card}\alpha$ is regular, we write p_α instead of $p(\alpha)(0)$ and we write p^{**}_α instead of $p(\alpha)(1)$. If $\operatorname{card}\alpha$ is singular or $\operatorname{card}\alpha = \omega$, we write p_α instead of $p(\alpha)$ and say that $p^{**}_\alpha = \check{\mathbf{1}}$ for notational simplicity. We call $\{\gamma\colon p_\gamma \neq \check{\mathbf{1}}\}$ the *string support* of p and denote it by S-supp(p); we call $\{\gamma\colon p^{**}_\gamma \neq \check{\mathbf{1}}\}$ the *club support* of p and denote it by C-supp(p).

P is a standard iteration. For every γ, $p \in P_\gamma$ iff for all $\beta < \gamma$, $p{\restriction}\beta \in P_\beta$, if $\gamma = \beta + 1$ is a successor ordinal then $\Vdash_{P_\beta} p(\beta) \in Q(\beta)$, and:

1. if γ is regular, S-supp(p) is bounded below γ,
2. C-supp$(p) \subseteq$ S-supp(p), and
3. if $\operatorname{card}\gamma$ is regular, $\operatorname{card}(\text{C-supp}(p) \cap [\operatorname{card}\gamma, \gamma)) < \operatorname{card}\gamma$.

P is the direct limit of the P_γ, $\gamma \in \mathrm{Ord}$. Note that by 2, supp(p), the support of p, is equal to S-supp(p).

For $\alpha < \beta$, $p[\alpha, \beta)$ denotes $p{\restriction}[\alpha, \beta)$ and $P[\alpha, \beta)$ denotes the iteration P restricted to the interval $[\alpha, \beta)$. Whenever we use such notation, we will tacitly assume that α is a cardinal (which will not necessarily be the case for β), and whenever we talk about properties of $P[\alpha, \beta)$, we will tacitly assume that we are in some generic extension after forcing with P_α (with generic G_α and generic predicate g_α: $g_\alpha(\gamma) = 1$ iff $\exists p \in G_\alpha\ \ p{\restriction}\gamma \Vdash p_\gamma = 1$). We will later show that forcing with P_α preserves cardinals, cofinalities and the GCH. If $\operatorname{card}\beta$ is regular, we write $p[\alpha, \beta)^{\oplus}$ to denote $p[\alpha, \beta)^\frown p(\beta)(0)$; and $P[\alpha, \beta)^{\oplus}$ denotes $P[\alpha, \beta) * Q(\beta)(0)$. If $p \in P[\alpha, \beta)$ or $P[\alpha, \beta)^{\oplus}$, we also write $p{\restriction}\gamma$ for $p[\alpha, \gamma)$ and $p{\restriction}\gamma^{\oplus}$ for $p[\alpha, \gamma)^{\oplus}$.

We will usually assume that any condition p has the following properties (possible as a dense subclass of conditions does):

A1. $\forall\gamma\ \ \mathbf{1}_{P_\gamma} \Vdash p_\gamma \in S$.
A2. $\forall\gamma\ \ \mathbf{1}_{P_\gamma^{\oplus}} \Vdash p^{**}_\gamma \in C_\gamma(g_{\gamma+1})$.
A3. $\forall\gamma\ \ ((p_\gamma = \check{\mathbf{1}}) \vee (\mathbf{1}_{P_\gamma} \Vdash p_\gamma \neq \check{\mathbf{1}}))$.

We will at some points have to temporarily withdraw from assumption A2 above. We will explicitly mention whenever we do so.

Fact 11. *If $p \parallel q$ in P (or any of its restrictions), then they have a greatest lower bound in P.*

([6]) The full support iteration for example.

Proof. Each iterand of P has canonical greatest lower bounds for its compatible conditions (namely their union), thus the same holds for P. ∎

CLAIM 12 (String extendibility). *Work in a P_α-generic extension. Assume $\beta > \alpha$ and f is a function with domain $d \subseteq [\alpha, \beta)$ which is bounded below every regular cardinal such that for every $\gamma \in d$, $f(\gamma)$ is a $P[\alpha, \gamma)$-name and $\mathbf{1}_{P[\alpha,\gamma)} \Vdash f(\gamma) \in \{0, 1\}$. Then any given $p \in P[\alpha, \beta)$ with* S-supp$(p) \cap d = \emptyset$ *can be extended to $q \leq p$ such that $q_\gamma = f(\gamma)$ whenever $\gamma \in d$.* ∎

DEFINITION 13 (upper part of a condition). Given a cardinal $\eta \in [\alpha, \beta)$ and $p \in P[\alpha, \beta)$, we define $u_\eta(p) \in P[\alpha, \beta)$ as follows:

$$(u_\eta(p))_\gamma = \begin{cases} \check{\mathbf{1}} & \text{if } \alpha \leq \gamma < \eta, \\ p_\gamma & \text{otherwise,} \end{cases} \qquad (u_\eta(p))^{**}_\gamma = \begin{cases} \check{\mathbf{1}} & \text{if } \alpha \leq \gamma < \eta^+, \\ p^{**}_\gamma & \text{otherwise,} \end{cases}$$

and call $u_\eta(p)$ the *η^+-strategically closed part* of p. We let $u_\eta(P[\alpha, \beta)) := \{u_\eta(p) \colon p \in P[\alpha, \beta)\}$ and call it the *η^+-strategically closed part* of $P[\alpha, \beta)$.

NOTE.

- The fact that $u_\eta(p) \in P[\alpha, \beta)$ heavily uses assumptions A1 and A2.
- If $\eta = \omega$ or η is a singular cardinal, then $u_\eta(P[\eta, \beta)) = P[\eta, \beta)$.
- We may think of $u_\eta(p)$ as the condition extracting from p its string of bits in the interval $[\eta, \eta^+)$ and everything at and above η^+.

DEFINITION 14 (lower part of a condition). If $\eta \in [\alpha, \beta)$ is a cardinal and $p \in P[\alpha, \beta)$, we define $l_\eta(p)$ as follows:

$$(l_\eta(p))_\gamma = \begin{cases} \check{\mathbf{1}} & \text{if } \beta > \gamma \geq \eta, \\ p_\gamma & \text{otherwise,} \end{cases} \qquad (l_\eta(p))^{**}_\gamma = \begin{cases} \check{\mathbf{1}} & \text{if } \beta > \gamma \geq \eta^+, \\ p^{**}_\gamma & \text{otherwise,} \end{cases}$$

where γ ranges over the interval $[\alpha, \beta)$, and call $l_\eta(p)$ the *η-sized part* of p. Note that $l_\eta(p)$ is in general not a condition in $P[\alpha, \beta)$. Note also that $l_\eta(p)$ complements $u_\eta(p)$ in the sense that it carries exactly all information about p not contained in $u_\eta(p)$.

NOTATION. Assume $\langle s^i \colon i < \delta \rangle$ is a decreasing sequence of conditions in S. Then $\langle s^i \colon i < \delta \rangle$ is eventually constant and we denote its limit by $\bigcup_{i<\delta} s^i$. Given a decreasing sequence of conditions in $P[\alpha, \beta)$ of limit length δ, we say that r is the *componentwise union* of $\langle p^i \colon i < \delta \rangle$ iff for every $\gamma \in [\alpha, \beta)$,

$$r_\gamma = \bigcup_{i<\delta} p^i_\gamma \quad \text{and} \quad r^{**}_\gamma = \bigcup_{i<\delta} (p^i)^{**}_\gamma.$$

r is usually not a condition in $P[\alpha, \beta)$ ([7]), but S-supp(r) and C-supp(r) can be calculated as if r were a condition by letting S-supp$(r) := \{\gamma \colon r_\gamma \neq \check{\mathbf{1}}\} = \bigcup_{i<\delta}$ S-supp(p^i) and C-supp$(r) := \{\gamma \colon r^{**}_\gamma \neq \check{\mathbf{1}}\} = \bigcup_{i<\delta}$ C-supp(p^i).

([7]) Unless $\langle (p^i)^{**}_\gamma \colon i < \delta \rangle$ is eventually constant, r^{**}_γ will not be closed.

Definition 15 (stable below η^+). Assume $\langle p^i\colon i<\delta\rangle$ is a decreasing sequence of conditions in $P[\alpha,\beta)$ of limit length $\delta<\eta^+$, where $\eta\in[\alpha,\beta)$ is a cardinal. We say that $\langle p^i\colon i<\delta\rangle$ is *stable below* η^+ iff

- $\langle l_\eta(p^i)\colon i<\delta\rangle$ is eventually constant, or
- η is singular and for every cardinal $\mu<\eta$, $\langle l_\mu(p^i)\colon i<\delta\rangle$ is eventually constant.

Fact 16. *If $\langle p^i\colon i<\delta\rangle$ is a decreasing sequence of conditions in $P[\alpha,\beta)$ of limit length $\delta<\eta^+$ which is stable below η^+ where $\eta\in[\alpha,\beta)$ is a cardinal, then the componentwise union of $\langle p^i\restriction\eta^+\colon i<\delta\rangle$ is a greatest lower bound for $\langle p^i\restriction\eta^+\colon i<\delta\rangle$.* ∎

Definition 17 (greatest lower bound). Given a cardinal $\eta\in[\alpha,\beta)$ and a decreasing sequence $\langle p^i\colon i<\delta\rangle$ of conditions in $P[\alpha,\beta)$ of limit length $\delta<\eta^+$ which is stable below η^+, form their componentwise union r. Observe that S-supp(r) is bounded below every regular cardinal and C-supp$(r)\cap[\theta,\theta^+)$ has size less than θ for every regular θ.

We want to form $q\in P[\alpha,\beta)$ by setting, for every $\gamma\in$ C-supp(r), $\gamma\ge\eta^+$:

(1) $q_{\operatorname{ot} f_\gamma[\sup r^{**}_\gamma]}:=r_\gamma$ (8).
(2) $q^{**}_\gamma:=r^{**}_\gamma\cup\{\sup r^{**}_\gamma\}$.
(3) $q_\xi=r_\xi$ for every $\xi\in$ S-supp(r), and $q^{**}_\xi=r^{**}_\xi$ for every $\xi<\eta^+$.

All other components of q should have value $\check{\mathbf{1}}$. If such a q exists, we say that q is the *greatest lower bound* for $\langle p^i\colon i<\delta\rangle$.

Fact 18. *Given a cardinal $\eta\in[\alpha,\beta)$ and a decreasing sequence $\langle p^i\colon i<\delta\rangle$ of conditions in $P[\alpha,\beta)$ as in Definition* 17, *if we can form their greatest lower bound q as above, then q is a greatest lower bound* (*in the usual sense*) *for $\langle p^i\colon i<\delta\rangle$.* ∎

Note. Definition 17 equally makes sense and Fact 18 equally holds within $P[\alpha,\beta)^\oplus$ (instead of $P[\alpha,\beta)$). It will often be the case in the following that when we give a definition or prove a statement concerning $P[\alpha,\beta)$, an analogous definition or statement will make sense or hold for $P[\alpha,\beta)^\oplus$, which we will not mention in general.

Definition 19 (cardinal predecessor). If θ is a successor cardinal, $\theta=\lambda^+$, then $\theta^-:=\lambda$. If θ is inaccessible, θ^- may be chosen to be any cardinal less than θ. If I consists only of regular cardinals, we say that $\langle\theta^-\colon\theta\in I\rangle$ is a *predecessor sequence* iff whenever $\theta_0<\theta_1$ are both in I, ${\theta_1}^-\ge\theta_0$.

---

(8) Whenever we want to do this, we will be in a situation where each $\sup r^{**}_\gamma$ will have been decided to equal an actual ordinal value (and is not just a name for an ordinal).

Definition 20 (information at θ). If $p \in P[\alpha,\beta)$ and $\theta \in [\alpha^+,\beta)$ is regular, $\theta < \zeta$, we let $\bar\zeta := \min(\zeta,\theta^+)$, $\overline{\zeta+1} := \min(\zeta+1,\theta^+)$ and define

$$i_\theta^\zeta(p) := \{\bar\zeta, p{\restriction}\overline{\zeta+1}\} \cup (\text{C-supp}(p) \cap [\theta,\bar\zeta)), \qquad i_\theta(p) := i_\theta^\beta(p).$$

Definition 21 (suitable genericity). Let $p \in P[\alpha,\beta)$, $\zeta \in [\alpha,\beta]$, $\theta \in [\alpha^+,\zeta)$ regular and assume $\operatorname{card}(i_\theta^\zeta(p)) \le \theta^- < \theta$. Assume $\langle M^i : i \le \delta\rangle$ is an increasing sequence of length $\delta < \theta$ of domains of elementary submodels of (H_ν, R) for some large, regular ν and each M_i is of size less than θ, transitive below θ, with $i_\theta^\zeta(p) \cup \{\theta^-, \langle M_i : i < \delta\rangle\} \subseteq M_\delta$. Assume $q \le p$ and $t \in P[\alpha,\beta)$. We say $t \le q$ is *suitably generic* for $P[\alpha,\zeta)$ at θ over $\langle M^i : i \le \delta\rangle$ below p with respect to θ^- iff:

1a. If $\bar\zeta < \beta$, then t meets every dense subset of $u_{\theta^-}(P[\alpha,\bar\zeta))$ which is definable in M^δ using parameters in $i_\theta^\zeta(p) \cup \{\theta^-, \langle M^i : i < \delta\rangle\}$, in the sense that for each such dense set D, there is $s \ge t{\restriction}\bar\zeta$ such that $s \in D$.

1b. If $\zeta = \bar\zeta = \beta$, then t meets every dense subset of $u_{\theta^-}(P[\alpha,\xi)^\oplus)$ which is definable in M^δ using parameters in $i_\theta(p) \cup \{\theta^-, \langle M^i : i < \delta\rangle\}$, for every $\xi < \bar\zeta$, $\xi \in M^\delta$.

2. If $\theta = \operatorname{card}\beta$ and $\beta = \zeta = \gamma+1$ is a successor ordinal, then $u_{\theta^-}(t)$ forces that $\sup(t_\gamma^{**}) \ge \sup(\text{S-supp}(p) \cap \theta)$.

Remarks.

- β is to be read off from p in Definitions 20 and 21 above.
- Note that when we require that t is suitably generic for $P[\alpha,\zeta)$ at θ over $\langle M^i : i \le \delta\rangle$ below p with respect to θ^- in the following, we implicitly require that $i_\theta^\zeta(p) \cup \{\theta^-, \langle M^i : i < \delta\rangle\} \subseteq M^\delta$, that each M^i has size less than θ, is transitive below θ and $\operatorname{card}(i_\theta^\zeta(p)) \le \theta^- < \theta$.
- Observe that if t is suitably generic for $P[\alpha,\zeta)$ at θ over $\langle M^i : i \le \delta\rangle$ below p with respect to θ^- and $t' \le t$, then t' is suitably generic for $P[\alpha,\zeta)$ at θ over $\langle M^i : i \le \delta\rangle$ below p with respect to θ^-.

Theorem 22. *Suppose $\omega \le \bar\alpha \le \eta < \alpha$, with $\bar\alpha, \eta \in$ **Card**. Then the following hold:*

1. [Greatest lower bounds] *Assume $\langle p^i : i < \gamma\rangle$ is a decreasing sequence of conditions in $P[\bar\alpha,\alpha)$ of limit length $\gamma < \eta^+$ which is stable below η^+. Let $\langle \zeta_i : i < \gamma\rangle$ be such that for each $i < \gamma$, ζ_i is least such that $p^{i+1}[\zeta_i,\alpha) = p^i[\zeta_i,\alpha)$. Let $S_i = \langle \theta_i^- :$ C-supp$(p^i) \cap [\theta,\theta^+) \ne \emptyset$, $\theta < \zeta_i\rangle$ be a predecessor sequence for every $i < \gamma$. Assume that for every $\theta \in [\eta^+,\alpha)$: if $j < \gamma$ is least such that* C-supp$(p^j \cap [\theta,\theta^+)) \ne \emptyset$, *then $\langle M_\theta^i : j \le i < \gamma\rangle$ is an increasing sequence of domains of elementary submodels of (H_ν, R) for some large (with respect to α), regular ν with union $M_\theta = \bigcup_{j \le i < \gamma} M_\theta^i$, such that for each $i \in [j,\gamma)$, if $\theta < \zeta_i$,*

then p^{i+1} is suitably generic for $P[\bar{\alpha},\zeta_i)$ at θ over $\langle M_\theta^k\colon k\le i\rangle$ (9) *below p^i with respect to θ_i^-. Then the sequence $\langle p^i\colon i<\gamma\rangle$ has a greatest lower bound.*

2. [Smallness of the iteration] *If α is regular, then $u_\eta(P[\bar{\alpha},\alpha))$ has a dense subset of size α. Otherwise $u_\eta(P[\bar{\alpha},\alpha))$ has a dense subset of size α^+.*
3. [Genericity] *Let $p\in P[\bar{\alpha},\alpha)$, $\zeta\le\alpha$ and $I\subseteq[\bar{\alpha}^+,\zeta)$ so that I consists only of regular cardinals and is bounded below every inaccessible. Assume $S=\langle\theta^-\colon\theta\in I\rangle$ is a predecessor sequence,* $\mathrm{card}(i_\theta^\zeta(p))\le\theta^-$, *$\delta<\min I$ and $\langle M_\theta^i\colon i\le\delta\rangle$ is an increasing* (10) *sequence of domains of elementary submodels of (H_ν,R) for some large (with respect to α), regular ν* (11)*, such that each M_θ^i is of size less than θ, transitive below θ and $i_\theta^\zeta(p)\cup\{\theta^-,\langle M_\theta^i\colon i<\delta\rangle\}\subseteq M_\theta^\delta$ for all $\theta\in I$. Then for every $q\le p$, there is $t\le q$ such that t is suitably generic for $P[\bar{\alpha},\zeta)$ at θ over $\langle M_\theta^i\colon i\le\delta\rangle$ below p with respect to θ^- for every $\theta\in I$ and $l_{(\min I)^-}(t)=l_{(\min I)^-}(q)$. If $\sup I\notin I$, then $t[\sup I,\alpha)=q[\sup I,\alpha)$, otherwise $t[(\sup I)^+,\alpha)=q[(\sup I)^+,\alpha)$.*
4. [Strategic closure] *$u_\eta(P[\bar{\alpha},\alpha))$ and $u_\eta(P[\bar{\alpha},\alpha)^\oplus)$ are both η^+-strategically closed.*
5. [A stronger form of genericity] *Let $p\in P[\bar{\alpha},\alpha)$ and $I\subseteq[\bar{\alpha}^+,\alpha)$ so that I consists only of regular cardinals and is bounded below every inaccessible. Assume $S=\langle\theta^-\colon\theta\in I\rangle$ is a predecessor sequence, $\langle I_\theta\colon\theta\in I\rangle$ is such that $I_\theta\supseteq i_\theta(p)$ and* $\mathrm{card}(I_\theta)\le\theta^-$ *for all $\theta\in I$, $\delta<\min I$ and $\langle M_\theta^i\colon i\le\delta\rangle$ is an increasing* (10) *sequence of domains of elementary submodels of (H_ν,R) for some large (with respect to α), regular ν* (11)*, such that each M_θ^i is of size less than θ, transitive below θ and $I_\theta\cup\{\theta^-,\langle M_\theta^i\colon i<\delta\rangle\}\subseteq M_\theta^\delta$ for all $\theta\in I$. Then for every $q\le p$, there is $t\le q$ such that $l_{(\min I)^-}(t)=l_{(\min I)^-}(q)$, t meets every dense subset of $u_{\theta^-}(P[\bar{\alpha},\alpha))$ which is definable in M_θ^δ using parameters in $I_\theta\cup\{\theta^-,\langle M_\theta^i\colon i<\delta\rangle\}$* (12)*.*
6. [Early club information] *$P[\bar{\alpha},\alpha)$ has a dense subset of conditions p for which $p{\restriction}i^\oplus$ forces that p_i^{**} has a $P[\bar{\alpha},\mathrm{card}\,i)$-name for each $i\in\mathrm{C\text{-}supp}(p)$.*
7. [Chain condition] *Assume η is regular. If J is an antichain of $P[\bar{\alpha},\alpha)$ such that whenever p and q are in J, $u_\eta(p)\parallel u_\eta(q)$, then $|J|\le\eta$.*

---

(9) Let $M_\theta^k=\emptyset$ in case $k<j$ (and thus M_θ^k was not defined).

(10) We also allow for a proper initial segment with constant value $\emptyset$.

(11) In particular, ν should be large enough so that every $p\in D_\alpha$ (as defined in the proof of 2) can be represented in H_ν.

(12) Note that if t is as described, then t is suitably generic for $P[\bar{\alpha},\alpha)$ at θ over $\langle M_\theta^i\colon i\le\delta\rangle$ below p with respect to θ^- for every $\theta\in I$.

8. [Early names]
   - *Assume η is regular. Let $\dot f$ be a $P[\bar\alpha,\alpha)$-name for an ordinal-valued function with domain η. Then any condition in $P[\bar\alpha,\alpha)$ can be strengthened to a condition q with the same η-sized part forcing that for every $i<\eta$, there is a maximal antichain of size at most η below q deciding $\dot f(i)$, where for every element a of that antichain, $u_\eta(a)=u_\eta(q)$. In particular, q forces that $\dot f$ has a $P[\bar\alpha,\gamma)$-name for some $\gamma<\eta^+$.*
   - *Assume $\eta\in[\bar\alpha,\alpha]$ is singular. Let $\dot f$ be a $P[\bar\alpha,\alpha)$-name for an ordinal-valued function with domain η. Then for any $\zeta<\eta$, any condition in $P[\bar\alpha,\alpha)$ can be strengthened to a condition q with the same ζ-sized part, forcing that for every $i<\eta$, there is a maximal antichain of size less than η below q deciding $\dot f(i)$, where for every element a of that antichain, $u_\eta(a)=u_\eta(q)$. In particular, q forces that $\dot f$ has a P_η-name.*
9. [Distributivity] *For any θ, $P[\theta,\alpha)$ is θ-distributive.*
10. [Preservation of the GCH] *After forcing with P_α,* GCH *holds.*
11. [Covering, preservation of cofinalities] *For every cardinal θ, every $p\in P_\alpha$ and every P_α-name $\dot x$ for a set of ordinals of size θ there is a set X in $\mathbf{V}$ of size θ and an extension q of p such that $q\Vdash\dot x\subseteq X$. Therefore forcing with P_α preserves all cofinalities.*
12. [Factorization] *Whenever $\alpha^*>\alpha$, $P[\bar\alpha,\alpha^*)$ is isomorphic to a dense subset of $P[\bar\alpha,\alpha)*\dot P[\alpha,\alpha^*)$.*
13. [Club Extendibility] *If $I\subseteq[\bar\alpha,\alpha)$ is such that* $\operatorname{card}(I\cap\theta)<\theta$ *for every regular θ, $I\subseteq\bigcup_{\theta\ regular}[\theta,\theta^+)$ and $\langle\bar\delta^i\colon i\in I\rangle$ is such that $\bar\delta_i<\operatorname{card} i$ for every $i\in I$, then for every $p\in P[\bar\alpha,\alpha)$, there is $q\le p$ such that $\forall i\in I\ \ q\restriction i^\oplus\Vdash\max q_i^{**}\ge\bar\delta_i$.*

Proof. By induction on α.

Proof of 1. Assume $\langle p^i\colon i<\gamma\rangle$ is as in the statement of the theorem, using predecessor sequences S_i and models M^i_θ. We want to show that $\langle p^i\colon i<\gamma\rangle$ has a greatest lower bound. Let r be the componentwise union of the p^i. Let ζ be largest such that for each $i<\gamma$ and each $\xi<\zeta$ there exists $j>i$ such that $\zeta_j>\xi$. We may assume that $\zeta=\alpha$, as the claim follows inductively otherwise. We obtain the following, using suitable genericity, for every $\theta\ge\eta^+$ with $\text{C-supp}(r)\cap[\theta,\theta^+)\ne\emptyset$: $\sup(\text{S-supp}(r)\cap\theta)=M_\theta\cap\theta$; $\text{C-supp}(r)\cap[\theta,\theta^+)=M_\theta\cap[\theta,\min(\theta^+,\alpha))$; if θ is inaccessible, $\operatorname{card} M_\theta=\sup(\text{S-supp}(r)\cap\theta)=M_\theta\cap\theta$.

If $\beta\in\text{C-supp}(r)$ and $\beta\ge\eta^+$, choose $j<\gamma$ such that $\beta\in M^j_{\operatorname{card}\beta}$. Then $\langle p^i\restriction\beta\colon j\le i<\gamma\rangle$ satisfies the hypothesis of clause 1 (at stage β), using the predecessor sequences $\langle S_i\colon j\le i<\gamma\rangle$ and models $\langle M^i_\theta\colon j\le i<\gamma\rangle$. Let q^β

denote the inductively obtained greatest lower bound of $\langle p^i \restriction \beta \colon i < \gamma \rangle$, and $(q^\beta)^\oplus$ the inductively obtained greatest lower bound of $\langle p^i \restriction \beta^\oplus \colon i < \gamma \rangle$.

Assume $\eta^+ \le \xi \in$ C-supp(r) and $\operatorname{card} \xi = \theta$. Then $(q^\xi)^\oplus$ forces that $\sup r^{**}_\xi = \sup(\text{S-supp}(r) \cap \theta)$ ([13]). Furthermore f_ξ is a bijection between θ and ξ, by elementarity of M_θ, thus $f_\xi \restriction (M_\theta \cap \theta)$ is a bijection between $M_\theta \cap \theta$ and $M_\theta \cap \xi$. Thus if we let π_θ denote the collapsing map of M_θ, it follows that $(q^\xi)^\oplus$ forces that $\pi_\theta(\xi) = \operatorname{ot}(f_\xi[\sup r^{**}_\xi])$. If θ is inaccessible, then $\pi_\theta(\xi) \ge M_\theta \cap \theta = \sup(\text{S-supp}(r) \cap \theta) = \operatorname{card} M_\theta$, thus for any $\xi_0 \ne \xi_1$ in C-supp(r) with $\operatorname{card} \xi_0 = \theta_0$ and $\operatorname{card} \xi_1 = \theta_1$, $\pi_{\theta_0}(\xi_0) \ne \pi_{\theta_1}(\xi_1)$ and we can build q out of r by setting, for every $\xi \in$ C-supp(r) with $\xi \ge \eta^+$,

$$q^{**}_\xi = r^{**}_\xi \cup \{\sup r^{**}_\xi\}, \qquad q_{\pi_\theta(\xi)} = r_\xi,$$

letting $q_\xi = r_\xi$ for every $\xi \in$ S-supp(r) and $q^{**}_\xi = r^{**}_\xi$ for every $\xi < \eta^+$, once we know that q^ξ forces r_ξ to have a $P_{\sup(\text{S-supp}(r)\cap\theta)}$-name whenever $\eta^+ \le \theta = \operatorname{card} \xi$, $\xi \in$ C-supp(r).

To see this is the case, choose $i < \gamma$ such that $\xi \in$ C-supp(p^i) and $\zeta_i \ge \xi$. The set $D := \{p \in u_{\theta_i^-}(P[\bar\alpha, \xi]) \colon p \Vdash (p^i)_\xi$ has a $P[\bar\alpha, \sup(\text{S-supp}(p)$ $\cap\theta))$-name$\}$ is dense in $u_{\theta_i^-}(P[\bar\alpha, \xi])$ using clause 8 inductively and definable in M^i_θ from parameters in $i^{\zeta_i}_\theta(p^i) \cup \{\theta_i^-\}$. The statement follows by suitable genericity of p^{i+1}.

Note. q, as obtained above, will usually not satisfy property A2. But we may replace q by an equivalent q' satisfying A2, where we say that q and q' are *equivalent* iff $q' \le q$ and $q \le q'$.

Proof of 2. Assume for simplicity of notation that $\eta = \bar\alpha$ is singular and hence $u_\eta(P[\bar\alpha, \alpha]) = P[\bar\alpha, \alpha)$. Other cases are similar. We prove that $D_\alpha := \{p \in P[\bar\alpha, \alpha) \colon \forall\theta\ \exists\gamma\ \text{S-supp}(p) \cap [\theta, \theta^+) = [\theta, \gamma)\}$ has an equivalent dense subset E_α of size α if α is regular and of size α^+ if α is singular, in the sense that for every $p \in D_\alpha$, there is $p' \in E_\alpha$ such that $p \le p' \le p$. Note that D_α itself is dense in $P[\bar\alpha, \alpha)$.

Regular cardinals. If α is regular, conditions in $P[\bar\alpha, \alpha)$ have bounded support below α, thus the claim follows by clause 2 inductively.

Successor ordinals. Assume $p \in D_\alpha$, $\alpha = \beta + 1$ and D_β has an equivalent dense subset E_β of size α^+ inductively. The condition p_β can be identified with an antichain of E_β below $p \restriction \beta$. Since for any two elements a_0, a_1 of such an antichain, $u_{\operatorname{card}\alpha}(a_0) \parallel u_{\operatorname{card}\alpha}(a_1)$, such an antichain will have size at most $\operatorname{card}\alpha$ using clause 7 inductively, thus there are α^+-many possible choices for p_β. The condition p^{**}_β can be identified with a collection of

---

([13]) If $\xi + 1 = \alpha$, then $\sup r^{**}_\xi \ge \sup(\text{S-supp}(r) \cap \theta)$ follows using clause 2 of suitable genericity. Otherwise, $\sup r^{**}_\xi \ge \sup(\text{S-supp}(r) \cap \theta)$ follows by easy density arguments and clause 1 of suitable genericity. $\sup(\text{S-supp}(r) \cap \theta) \ge \sup r^{**}_\xi$ uses similar density arguments together with clause 6 inductively, clause 1 of suitable genericity and clause 2 inductively.

$<\operatorname{card}\alpha$-many antichains of E_β below $p\restriction\beta$, each elementwise paired with ordinals below $\operatorname{card}\alpha$; thus using similar arguments to those before, there are α^+-many possible choices for p^{**}_β. Hence D_α has an equivalent dense subset of size α^+.

Singular ordinals. If α is singular and $p\in D_\alpha$, we can modify p to an equivalent p' such that for every $\gamma<\alpha$, $p'\restriction\gamma\in E_\gamma$. Hence D_α has an equivalent dense subset of size $\prod_{\gamma<\alpha}\gamma^+\le\alpha^+$.

Proof of 3

Case 1: $\zeta<\alpha$. Then clause 3 immediately follows inductively from clause 5. We may thus assume in all subsequent cases below that $\zeta=\alpha$.

Case 2: *α is a successor ordinal, $\alpha=\beta+1$.* It follows inductively from clause 5 that, given $q\le p$ we can find $q'\le q$ which satisfies clause 1 of being suitably generic for $P[\bar\alpha,\zeta)$ at θ over $\langle M^i_\theta\colon i\le\delta\rangle$ below p for every $\theta\in I$. It is easy to strengthen q' to t which also satisfies clause 2 of suitable genericity.

Case 3: *$M^\delta_{\operatorname{card}\alpha}$ is bounded in α.* Let $\mu:=\operatorname{card}\alpha$, let $\alpha^*:=\sup(M^\delta_\mu\cap\alpha)$, let $(M^\delta_\mu)^*$ be the smallest elementary submodel of (H_ν,R) which contains $M^\delta_\mu\cup\{\alpha^*\}$ as a subset and is transitive below μ, let $(M^i_\theta)^*:=M^i_\theta$ if $\theta\ne\mu$ or $i\ne\delta$, and apply clause 3 inductively to obtain $q'\le q\restriction\alpha^*$ such that q' is suitably generic for $P[\bar\alpha,\alpha^*)$ at θ over $\langle(M^i_\theta)^*\colon i\le\delta\rangle$ below p with respect to θ^- for all $\theta\in I$. Then $t:=q'^\frown q[\alpha^*,\alpha)$ is as desired. Note that case 3 covers the cases of α regular and $\operatorname{cof}\alpha=\operatorname{card}\alpha$.

Case 4: *α is a singular cardinal.* Let $\langle\theta_i\colon i<\xi\rangle$ enumerate I in increasing order. We build a decreasing sequence $\langle p^i\colon i<\xi\rangle$ of conditions in $P[\bar\alpha,\alpha)$ with $p^0=q$ so that given p^i, p^{i+1} is suitably generic for $P[\bar\alpha,\alpha)$ at θ_i over $\langle M^j_{\theta_i}\colon j\le\delta\rangle$ below p with respect to ${\theta_i}^-$, $l_{{\theta_i}^-}(p^{i+1})=l_{{\theta_i}^-}(p^i)$ and $p^{i+1}[{\theta_i}^+,\alpha)=p^i[{\theta_i}^+,\alpha)$. If $i\le\xi$ is a limit ordinal, note that we may choose p^i to be the greatest lower bound of $\langle p^j\colon j<i\rangle$ as $\langle p^j(\zeta)\colon j<i\rangle$ is eventually constant for every $\zeta\in[\bar\alpha,\alpha)$. If $\xi=\gamma+1$ is a successor ordinal, $t:=p^\gamma$ is as desired. If ξ is a limit ordinal, $t:=p^\xi$ is as desired.

Case 5: *$\operatorname{cof}\alpha<\operatorname{card}\alpha$, $\alpha\notin\mathbf{Card}$.* Let $\mu=\operatorname{card}\alpha$. Let p^0 be suitably generic for $P[\bar\alpha,\alpha)$ at θ over $\langle M^i_\theta\colon i\le\delta\rangle$ below p with respect to θ^- for every $\theta\in I\setminus\{\mu\}$ ([14]). If $\mu\notin I$, we are done by letting $t=p^0$. Assume $\mu\in I$. We may assume that $\sup(M^\delta_\mu\cap\alpha)=\alpha$, as we may use case 3 otherwise. Let $\langle\alpha_i\colon i<\operatorname{cof}\alpha\rangle$ be a cofinal, continuous and increasing sequence with limit α and $\alpha_0>\mu$. We construct a decreasing sequence $\langle p^i\colon i<\operatorname{cof}\alpha\rangle$ of

([14]) This is possible using clause 3 inductively, as $p^0\le p$ is suitably generic for $P[\bar\alpha,\alpha)$ at θ over $\langle M^i_\theta\colon i\le\delta\rangle$ below p with respect to θ^- for every $\theta\in I\setminus\{\mu\}$ iff $p^0\restriction\mu$ is suitably generic for $P[\bar\alpha,\mu)$ at θ over $\langle M^i_\theta\colon i\le\delta\rangle$ below $p\restriction\mu$ with respect to θ^- for every $\theta\in I\setminus\{\mu\}$.

conditions in $P[\bar{\alpha},\alpha)$ with greatest lower bound $t = p^{\operatorname{cof}\alpha}$ which has the desired properties of the claim:

Choose $\mu^* \geq \operatorname{cof}\alpha$ in M^δ_μ ([15]). Given p^i, choose a predecessor sequence $\langle \theta_i^- : \theta \in (\mu^*,\mu], \text{C-supp}(p^i) \cap [\theta,\theta^+) \neq \emptyset\rangle$ so that $\theta_i^- \geq \operatorname{card}(i_\theta^{\alpha_i}(p^i))$ and $\theta_i^- \geq \mu^*$ for each i, and choose $\langle N^i_\theta \colon \theta \in (\mu^*,\mu], \text{C-supp}(p^i) \cap [\theta,\theta^+) \neq \emptyset\rangle$ so that each N^i_θ is of size less than θ, transitive below θ, contains $i_\theta^{\alpha_i}(p^i) \cup \{\theta_i^-, \langle N^j_\theta : j < i\rangle\}$ and $\bigcup_{j<i} N^j_\theta$ as subsets, and $N^i_\theta \prec (H_\nu, R)$. Apply clause 3 inductively to obtain $(p^i)' \leq p^i$ which is suitably generic for $P[\bar{\alpha},\alpha_i)$ at μ over $\langle M^i_\mu \colon i \leq \delta\rangle$ below p with respect to μ^* such that $l_{\mu^*}(p^{i+1}) = l_{\mu^*}(p^i)$. Apply clause 3 inductively once more to obtain $p^{i+1} \leq (p^i)'$ which is suitably generic for $P[\bar{\alpha},\alpha_i)$ at θ over $\langle N^j_\theta \colon j \leq i\rangle$ below p^i with respect to θ_i^- for every $\theta \in (\mu^*,\mu]$ with $\text{C-supp}(p^i) \cap [\theta,\theta^+) \neq \emptyset$ such that $l_{\mu^*}(p^{i+1}) = l_{\mu^*}((p^i)')$. Note that at any limit stage $j \leq \operatorname{cof}\alpha$, we obtain a greatest lower bound of the $\langle p^i \colon i < j\rangle$ by clause 1, using the fact that $\langle l_{\mu^*}(p^i) \colon i < \operatorname{cof}\alpha\rangle$ is constant.

Proof of 4. Choose some large (relative to α), regular ν. Let $p^0 \in u_\eta(P[\bar{\alpha},\alpha))$. Given p^i, choose a predecessor sequence $\langle \theta_i^- \colon \text{C-supp}(p^i) \cap [\theta,\theta^+) \neq \emptyset\rangle$ so that $\theta_i^- \geq \operatorname{card}(i_\theta(p^i))$ and $\theta_i^- \geq \eta$ for all i, and choose $\langle M^i_\theta \colon \text{C-supp}(p^i) \cap [\theta,\theta^+) \neq \emptyset\rangle$ such that each $M^i_\theta \prec (H_\nu, R)$ is of size less than θ, transitive below θ and contains $i_\theta(p^i) \cup \{\theta^-, \langle M^j_\theta \colon j < i\rangle\}$ and $\bigcup_{j<i} M^j_\theta$ as subsets. Assume $q^i \leq p^i$ and choose $p^{i+1} \leq q^i$ such that $p^{i+1} \in u_\eta(P[\bar{\alpha},\alpha))$ is suitably generic for $P[\bar{\alpha},\alpha)$ at θ over $\langle M^j_\theta \colon j \leq i\rangle$ below p^i with respect to θ_i^- for every θ with $\text{C-supp}(p^i) \cap [\theta,\theta^+) \neq \emptyset$, using clause 3. Note that at any limit stage $< \eta^+$, we may obtain a greatest lower bound of the p^i up to that stage using clause 1.

Proof of 5. Let $p \in P[\bar{\alpha},\alpha)$, $q \leq p$ and let S, I and $\langle I_\theta, M^i_\theta \colon \theta \in I, i \leq \delta\rangle$ be as in the statement of clause 5. Let $\langle \theta_i \colon i < \xi\rangle$ enumerate I in increasing order. We build a decreasing sequence $\langle p^i \colon i < \xi\rangle$ of conditions in $P[\bar{\alpha},\alpha)$ with $p^0 = q$: Given p^i so that $i+1 < \xi$, let $\mu := (\theta_i)^-$. Choose a predecessor sequence $\langle \theta_i^- \colon \theta \in (\mu,\alpha), \text{C-supp}(p^i) \cap [\theta,\theta^+) \neq \emptyset\rangle$ ([16]) so that $\theta_i^- \geq \operatorname{card}(i_\theta(p^i))$ and $\theta_i^- \geq \mu$ for each i, and choose $\langle N^i_\theta \colon \theta \in (\mu,\alpha), \text{C-supp}(p^i) \cap [\theta,\theta^+) \neq \emptyset\rangle$ so that each N^i_θ is of size less than θ, transitive below θ, contains $i_\theta(p^i) \cup \{\theta_i^-, \langle N^j_\theta \colon j < i\rangle\}$ and $\bigcup_{j<i} N^j_\theta$ as subsets, and $N^i_\theta \prec (H_\nu, R)$. Use μ^+-strategic closure of $u_\mu(P[\bar{\alpha},\alpha))$ to find $(p^i)' \leq p^i$ which hits every dense subset of $u_\mu(P[\bar{\alpha},\alpha))$ which is definable in $M^\delta_{\theta_i}$ from parameters in $I_{\theta_i} \cup \{(\theta_i)^-, \langle M^j_{\theta i} \colon j < \delta\rangle\}$ such that $l_\mu((p^i)') = l_\mu(p^i)$. Use clause 3 to obtain $p^{i+1} \leq (p^i)'$ which is suitably generic for $P[\bar{\alpha},\alpha)$ at θ over $\langle N^j_\theta \colon j \leq i\rangle$

([15]) This is possible as $\operatorname{cof}\alpha \in M^\delta_\mu$.

([16]) To avoid possible sources of confusion, note that $(\theta_i)^-$ and θ_i^- are distinct objects.

below p^i with respect to θ_i^- whenever C-supp$(p^i) \cap [\theta, \theta^+) \neq \emptyset$ and $\theta > \mu$. Assume $i \leq \xi$ is a limit ordinal. If $\operatorname{card} i$ is regular, then $\langle l_{\operatorname{card} i}(p^j) \colon j < i\rangle$ is eventually constant ([17]), and if $\operatorname{card} i$ is singular, then for every $\mu < \operatorname{card} i$, $\langle l_\mu(p^j) \colon j < i\rangle$ is eventually constant and thus we we may, in each case, choose p^i to be the greatest lower bound of $\langle p^j \colon j < i\rangle$ using clause 1. If $\xi = \gamma + 1$ is a successor ordinal, $t := p^\gamma$ is as desired. If ξ is a limit ordinal, $t := p^\xi$ is as desired.

Proof of 6. Let $p^0 \in P[\bar{\alpha}, \alpha)$. Choose some large (relative to α), regular ν. We construct a decreasing sequence of conditions $\langle p^i \colon i < \omega\rangle$ with greatest lower bound p, which will be as desired. Given p^i, choose a predecessor sequence $\langle \theta_i^- \colon$ C-supp$(p^i) \cap [\theta, \theta^+) \neq \emptyset\rangle$ so that $\theta_i^- \geq \operatorname{card}(i_\theta(p^i))$ for all i, and choose $\langle M_\theta^i \colon$ C-supp$(p^i) \cap [\theta, \theta^+) \neq \emptyset\rangle$ such that each $M_\theta^i \prec (H_\nu, R)$ is of size less than θ, transitive below θ and contains $i_\theta(p^i) \cup \{\theta_i^-, \langle M_\theta^j : j < i\rangle\}$ and $\bigcup_{j<i} M_\theta^j$ as subsets. Choose $p^{i+1} \leq p^i$ such that p^{i+1} meets every dense subset of $u_{\theta_i^-}(P[\bar{\alpha}, \alpha))$ which is definable in M_θ^i from parameters in $i_\theta(p) \cup \{\theta_i^-, \langle M_\theta^j \colon j < i\rangle\}$ for every θ with C-supp$(p^i) \cap [\theta, \theta^+) \neq \emptyset$, using clause 5. Since if ξ has cardinality θ and $\xi \in$ C-supp(p^i), the set $D_\xi = \{t \in P[\bar{\alpha}, \xi)^\oplus \colon t \Vdash p_\xi^{**}$ has a $P[\bar{\alpha}, \zeta)$-name for some $\zeta \leq \theta\}$ is dense in $u_{\theta_i^-}(P[\bar{\alpha}, \xi)^\oplus)$ using clause 8 inductively, and is definable in M_θ^i using parameters in $i_\theta(p^i) \cup \{\theta_i^-\}$, it follows that p is as desired.

Proof of 7. Apply clause 2 inductively to obtain a dense subset $P[\bar{\alpha}, \eta)^*$ of $P[\bar{\alpha}, \eta)$ of size η and apply 6 to obtain a dense subset $P[\eta, \alpha)^*$ of $P[\eta, \alpha)$ of conditions as described in the statement of clause 6. Assume for a contradiction that $|J| > \eta$ for some antichain J of $P[\bar{\alpha}, \eta)^* * \dot{P}[\eta, \alpha)^*$ (we use clause 12 inductively here). As $P[\bar{\alpha}, \eta)^*$ has size η, $p[\bar{\alpha}, \eta)$ is the same for η^+-many conditions $p \in J$, hence there are $\bar{p} \in P[\bar{\alpha}, \eta)$ and $J' \subseteq P[\eta, \alpha)^*$ such that $\bar{p} \Vdash J'$ is an antichain of $\dot{P}[\eta, \alpha)$ of size η^+. Work in any P_η-generic extension with $\bar{p}$ contained in the P_η-generic. As GCH holds by clause 10 inductively, by a Δ-system argument, there is $W \subseteq J'$ of size η^+ and a size $< \eta$ subset A of η^+ such that C-supp$(p) \cap$ C-supp$(q) \cap [\eta, \eta^+) = A$ whenever $p \neq q$ are both in W. Again using GCH, there are only η-many possibilities for $\langle p_i^{**} \colon i \in A\rangle$ for $p \in P[\eta, \alpha)^*$. Hence for η^+-many conditions p in W, $\langle p_i^{**} \colon i \in A\rangle$ is the same (modulo equivalence). But—using the assumption that $u_\eta(p) \parallel u_\eta(q)$—any two such conditions are compatible, thus W (and hence also J) is not an antichain.

Proof of 8. Let $p \in P[\bar{\alpha}, \alpha)$. First assume $\dot{f}$ is a $P[\bar{\alpha}, \alpha)$-name for an ordinal-valued function with domain $\eta \in [\bar{\alpha}, \alpha)$ regular. Let $I := \{\dot{f}(i) \colon i \in \eta\}$

([17]) We use here the fact that $(\theta_i)_p^- \geq \sup\{\theta_j \colon j < i\}$ and for any regular cardinal κ, $\operatorname{card}\{i \colon \theta_i < \kappa\}$ is less than κ.

$\cup\, i_{\eta^+}(p)$, with $\dot{f}(i)$ being any name for the evaluation of $\dot{f}$ at i for each i. Let $M \prec (H_\nu, R)$ of size η, transitive below η^+ for some large (relative to α), regular ν such that $I \subseteq M$. Let $q \leq p$ be such that q meets every dense subset of $u_\eta(P[\bar{\alpha},\alpha))$ which is definable in M using parameters in I, using clause 5. As $D_i = \{t \in u_\eta(P[\bar{\alpha},\alpha))\colon$ there is a mac of size at most η of conditions s in $P[\bar{\alpha},\alpha)$ with $u_\eta(s) = u_\eta(t)$ deciding $\dot{f}(i)\}$ is definable in M from parameters in I and dense in $u_\eta(P[\bar{\alpha},\alpha))$ for each $i < \eta$ using clauses 4 and 7, q is as desired. To be more precise, the fact that D_i is dense in $u_\eta(P[\bar{\alpha},\alpha))$ is seen using a construction to reduce the decision about $\dot{f}(i)$ to the η-sized part of $P[\bar{\alpha},\alpha)$ as follows:

Let $p^0 \in u_\eta(P[\bar{\alpha},\alpha))$. Choose $q^0 \leq p^0$ in $P[\bar{\alpha},\alpha)$ such that q^0 decides $\dot{f}(i)$. At stage $j+1$, let $p^{j+1} \leq p^0$ be any condition in $P[\bar{\alpha},\alpha)$ incompatible with all q^k, $k \leq j$, such that $u_\eta(p^{j+1}) = u_\eta(q^j)$ (if such exists) and choose q^{j+1} as follows:

- $q^{j+1} \leq p^{j+1}$,
- q^{j+1} decides $\dot{f}(i)$, and
- $u_\eta(q^{j+1})$ is chosen with respect to the strategy for η^+-strategic closure of $u_\eta(P[\bar{\alpha},\alpha))$ below $\langle u_\eta(q^k)\colon k \leq j\rangle$, using clause 4.

At limit stages $j < \eta^+$, let $p^j \leq p^0$ be any condition in $P[\bar{\alpha},\alpha)$ incompatible with all q^k, $k < j$, such that for all $k < j$, $u_\eta(p^j) \leq u_\eta(q^k)$ (if such exists). Note that a p^j satisfying the latter condition can always be found by the strategic choice of the $u_\eta(q^k)$. Choose $q^j \leq p^j$ deciding $\dot{f}(i)$ with $u_\eta(q^j) \leq u_\eta(p^j)$.

Proceed until at some stage j no condition p^j as above can be chosen. By clause 7, this will be the case for some j of cardinality $\leq \eta$. We can then find $t \in u_\eta(P[\bar{\alpha},\alpha))$ such that $t \leq u_\eta(q^k)$ for every $k < j$. Hence we may strengthen every q^k to $\bar{q}^k$ such that $u_\eta(\bar{q}^k) = u_\eta(t)$ and $l_\eta(\bar{q}^k) = l_\eta(q^k)$. Then $\{\bar{q}^k\colon k < j\}$ is a maximal antichain of $P[\bar{\alpha},\alpha)$ below t deciding $\dot{f}(i)$.

Now assume $\dot{f}$ is a $P[\bar{\alpha},\alpha)$-name for an ordinal-valued function with domain some singular cardinal $\eta \in [\bar{\alpha},\alpha]$, and $\zeta < \eta$. Let $\eta = \bigcup_{i<\operatorname{cof}\eta} \eta_i$ with each η_i regular and η_0 greater than both $\operatorname{cof}\eta$ and ζ. Let $I_i := \{\dot{f}(j)\colon j \in \eta_i\} \cup i_{\eta_i^+}(p)$. Let $p \in P[\bar{\alpha},\alpha)$. Let $\langle M_i\colon i < \operatorname{cof}\eta\rangle$ be a sequence of elementary submodels of (H_ν, R) for some large (relative to α), regular ν such that each M_i has size η_i, is transitive below η_i^+ and contains I_i as a subset. Let $q \leq p$ be such that q meets every dense subset of $u_{\eta_i}(P[\bar{\alpha},\alpha))$ which is definable in M_i using parameters in I_i for every $i < \operatorname{cof}\eta$, using clause 5. Similar to the regular case above, it follows that q is as desired.

Proof of 9. Assume $\dot{x}$ is a P_α-name for a sequence of ordinals of length less than θ. Then by clause 8, below any $p \in P_\alpha$ there is $q \leq p$ forcing that $\dot{x}$ has a P_θ-name.

Proof of 10. P_α has a dense subset of size at most α^+ by clause 2. Thus $\Vdash_{P_\alpha} 2^\theta = \theta^+$ for $\theta > \alpha$. If $\theta = \alpha$, the claim holds using clauses 8 and 2. For $\theta < \alpha$, note that $P_\alpha \cong P_{\theta^+} * P[\theta^+, \alpha)$, where P_{θ^+} preserves $2^\theta = \theta^+$. If $\theta^+ = \alpha$, we are done. Otherwise, the result follows by clause 9.

Proof of 11. As P_α has a dense subset of size at most α^+ by clause 2, this is immediate for $\theta \geq \alpha^+$. If α is regular, P_α has a dense subset of size α and hence this is immediate for $\theta = \alpha$ in that case. If $\theta < \alpha$ and $\theta^+ < \alpha$, this follows inductively, using the fact that $P[\theta^+, \alpha)$ does not add new sets of size θ. If $\alpha = \theta^+$, we use clause 8 to obtain $q \leq p$ forcing that for every $i < \theta$, there exists an antichain of size at most θ below q deciding the ith element of $\dot{x}$, thus q forces that we can cover $\dot{x}$ by some $X \in \mathbf{V}$ of size θ. If $\theta = \alpha$ is singular, note that the "singular case" of clause 8 also holds in the case where $\eta = \alpha$ and thus we may apply clause 8 as above to obtain $q \leq p$ forcing that for every $i < \theta$, there exists an antichain of size less than θ below q deciding the ith element of $\dot{x}$.

Proof of 12. Note that whenever $(p, \dot{\sigma}) \in P[\bar{\alpha}, \alpha) * \dot{P}[\alpha, \alpha^*)$, then p forces that $\text{C-supp}(\dot{\sigma}) \cap [\alpha, \alpha^+)$ has size less than α, $\text{C-supp}(\dot{\sigma}) \cap [\alpha^+, \alpha^{++})$ has size at most α, $\text{S-supp}(\dot{\sigma}) \cap [\alpha, \alpha^+)$ has size α and each of those supports can be covered by a set of the same size in the ground model. We may strengthen p to q such that q decides those covering sets. Let $D \subseteq P[\bar{\alpha}, \alpha) * \dot{P}[\alpha, \alpha^*)$ be the dense set of conditions $(q, \dot{\sigma})$ as above. Now it can be seen as in the proof of the Factor Lemma (see [14]) that $P[\bar{\alpha}, \alpha^*)$ is isomorphic to D, using the fact that P_α has a dense subset of size at most α^+ by clause 2.

Proof of 13. Assuming I is as in the statement for a given sequence $\langle \bar{\delta}_\gamma \colon \gamma \in I \rangle$ of ordinals, we want to find $q \leq p$ such that for every $\gamma \in I$, $q \restriction \gamma^\oplus \Vdash q^{**}_\gamma \geq \bar{\delta}_\gamma$. We may assume that $p_\gamma \neq \check{\mathbf{1}}$ for every $\gamma \in I$. Choose a predecessor sequence $\langle \theta^- : I \cap [\theta, \theta^+) \neq \emptyset \rangle$ so that each $\theta^- \geq \text{card}(i_\theta(p))$, and a sequence $\langle M_\theta \colon I \cap [\theta, \theta^+) \neq \emptyset \rangle$ of domains of elementary submodels of (H_ν, R) for some large (relative to α), regular ν, so that each M_θ has size less than θ, is transitive below θ, contains $(I \cap [\theta, \theta^+)) \cup i_\theta(p)$ as a subset and $\varDelta := \langle \bar{\delta}_\gamma \colon \gamma \in I \rangle$ and θ^- as elements. Let $q \leq p$ be such that q meets every dense subset of $u_{\theta^-}(P[\bar{\alpha}, \alpha))$ which is definable in M_θ from parameters in $(I \cap [\theta, \theta^+)) \cup \{\varDelta\} \cup i_\theta(p) \cup \{\theta^-\}$ for every θ with $I \cap [\theta, \theta^+) \neq \emptyset$. As for every $\gamma \in I$, the set $D_\gamma = \{t \in u_{\theta^-}(P[\bar{\alpha}, \alpha)) \colon t \restriction \gamma \Vdash \max t^{**}_\gamma \geq \bar{\delta}_\gamma\}$ is dense in $u_{\theta^-}(P[\bar{\alpha}, \alpha))$ and definable in M_θ from parameters in $(I \cap [\theta, \theta^+)) \cup \{\varDelta\} \cup i_\theta(p) \cup \{\theta^-\}$, q is as desired.

The fact that D_γ is dense in $u_{\theta^-}(P[\bar{\alpha}, \alpha))$ is immediate if $\gamma \geq \bar{\alpha}^+$. If $\gamma < \bar{\alpha}^+$, by an easy density argument, for ϵ either 0 or 1, the set $S_\epsilon := \{\xi < \bar{\alpha} \colon G_{\bar{\alpha}}(\xi) = \epsilon\}$ intersects every unbounded ground model subset of $\bar{\alpha}$ unboundedly often below $\bar{\alpha}$. Let $\epsilon \in \{0, 1\}$ be such that $p \restriction \gamma \Vdash p_\gamma = \epsilon$, choose $\delta \geq \bar{\delta}$ such that $\text{ot}\, f_\gamma[\delta] \in S_\epsilon$ and set $q^{**}_\gamma = p^{**}_\gamma \cup \{\delta\}$. ■Theorem 22

COROLLARY 23. *P preserves* ZFC*, cofinalities, cardinals and the* GCH*.*

Proof. Note that whenever κ is singular, $P[\kappa, \infty)$, the iteration starting from κ, is κ^+-strategically closed. To verify this, we need variants of Theorem 22, clauses 1 and 3, using, instead of a single ν which is large with respect to α, a sequence $\langle \nu_\theta \colon \theta \in \mathbf{Card}\rangle$ such that each ν_θ is large with respect to θ. Those variants are proven most similarly to clauses 1 and 3 of Theorem 22. Then we can show that $P[\kappa, \infty)$ is κ^+-strategically closed most similarly to the proof of Theorem 22, clause 4, using the sequence $\langle \nu_\theta \colon \theta \in \mathbf{Card}\rangle$ instead of a single ν.

As $P[\kappa, \infty)$ is κ^+-strategically closed, it is κ^+-distributive for definable sequences of dense classes. Now it can be seen easily from [10, Section 2.2], that this suffices to show that P is tame and thus preserves ZFC. Preservation of cofinalities, cardinals and the GCH is immediate.

NOTE. For every i of regular cardinality, $\bigcup_{p \in G} p_i^{**}$ is club in $\operatorname{card} i$ for any P-generic G. This is immediate from Theorem 22, clause 13.

CLAIM 24. *P forces Local Club Condensation.*

Proof. Let G be P-generic. Let A be the generic predicate obtained from G, i.e. $\alpha \in A \leftrightarrow \exists p \in G\ \ p{\restriction}\alpha \Vdash p_\alpha = 1$. Note that $\mathbf{V}[G] = \mathbf{L}[A]$ as any set of ordinals in $\mathbf{V}$ is coded into A. We claim that $\langle M_\alpha \colon \alpha \in \mathrm{Ord}\rangle$ witnesses Local Club Condensation in $\mathbf{V}[G]$ with $M_\alpha = L_\alpha[A]$. If α has regular uncountable cardinality κ then Local Club Condensation is guaranteed by the forcing P: Note that for each $\beta \in \alpha \setminus \kappa$ we have $A(\beta) = A(\operatorname{ot} f_\beta[\delta])$ for all δ in the club $\bigcup_{p \in G} p_\beta^{**} \subseteq \kappa$. It follows that for a club C of $\delta < \kappa$, $A(\beta) = A(\operatorname{ot} f_\beta[\delta])$ and moreover $f_\beta[\delta] = f_\alpha[\delta] \cap \beta$ for all $\beta \in f_\alpha[\delta] \setminus \kappa$; this is seen using Lemma 9. Let, as in Lemma 7, F denote the function $(f, x) \mapsto f(x)$ whenever $f \in M_\alpha$ is a function with $x \in \operatorname{dom}(f)$. Now let $M_\alpha^* = (M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle, F, \ldots)$ be a skolemized structure for a countable language, and for any $X \subseteq \alpha$ let $M_\alpha^*(X)$ be the least substructure of M_α^* containing X as a subset. Consider the continuous chain $\langle M_\alpha^*(f_\alpha[\delta]) \colon \delta \in D\rangle$, where D consists of all elements δ of C such that $\delta \subseteq f_\alpha[\delta] = M_\alpha^*(f_\alpha[\delta]) \cap \mathrm{Ord}$ and $f_\alpha[\delta] \cap \kappa \in \mathrm{Ord}$. Then $M_\alpha^*(f_\alpha[\delta])$ condenses for each $\delta \in D$.

Finally we must verify Local Club Condensation for α when α has singular cardinality κ. Suppose that $\beta \geq \alpha$ and $\dot{S} \in \mathbf{V}$ is a P_β-name for a structure $(M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle, R, F, \ldots)$ for a countable language in $\mathbf{L}[A]$ such that the $\dot{S}$-closure of κ is all of M_α, with F as above, R the given well-ordering of $\mathbf{V}$ (from which in particular the canonical functions $\langle f_i \colon i \in \mathrm{Ord}\rangle$ were chosen). We show that any condition $p \in P_\beta$ has an extension q^* which forces that there is a continuous chain $\langle Y_\gamma \colon \gamma \in C\rangle$ of condensing substructures of $\dot{S}$ whose domains $\langle y_\gamma \colon \gamma \in C\rangle$ have union M_α such that $\langle y_\gamma \cap \mathrm{Ord} \colon \gamma \in C\rangle$ belongs to the ground model, where C is a closed unbounded subset of

Card $\cap\,\kappa$, each y_γ has cardinality γ and contains a γ as subset. Choose C to be any club subset of **Card** $\cap\,\kappa$ of order-type $\operatorname{cof}\kappa$ whose minimum is either ω or a singular cardinal and is at least $\operatorname{cof}\kappa$. Write C in increasing order as $\langle\gamma_i\colon i<\operatorname{cof}\kappa\rangle$. Choose some large (with respect to β), regular ν.

Let $p^0=p$. We may assume that C-supp$(p^0)\cap[\gamma_i^+,\gamma_i^{++})\neq\emptyset$ for every $i<\operatorname{cof}\kappa$. Given p^i, let $S_i=\langle\theta_i^-\colon\theta>\min C,\ \text{C-supp}(p^i)\cap[\theta,\theta^+)\neq\emptyset\rangle$ be a predecessor sequence such that $\theta_i^-\geq\operatorname{card}(i_\theta(p^i))$ and $\theta_i^-\geq\min C$ for all i, and let $\langle M_\theta^i\colon\theta>\min C,\ \text{C-supp}(p^i)\cap[\theta,\theta^+)\neq\emptyset\rangle$ be a sequence of domains of elementary submodels of (H_ν,R) such that each M_θ^i has size less than θ, is transitive below θ and contains $i_\theta(p^i)$ as a subset and p^i, θ_i^-, $\dot S$ and $\langle M_\theta^j\colon j<i\rangle$ as elements. Moreover make sure that whenever $\theta_0<\theta_1$, $M_{\theta_0}^i\subseteq M_{\theta_1}^i$ and that whenever γ is a limit point of C, $M_{\gamma^+}^i=\bigcup_{\delta\in C\cap\gamma}M_\delta$. The latter is possible as $\min C\geq\operatorname{cof}\kappa$ and we may thus sufficiently enlarge the $M_{\delta^+}^i$, $\delta\in C\cap\gamma$, after choosing $M_{\gamma^+}^i\supseteq\bigcup_{\delta\in C\cap\gamma}M_{\delta^+}^i$ in the first place.

Choose $p^{i+1}\leq p^i$ following the strategy for ω_1-strategic closure of P_β such that p^{i+1} meets every dense subset of $u_{\theta^-}(P_\beta)$ which is definable in M_θ^i using parameters in $i_\theta(p^i)\cup\{\theta_i^-,\dot S,\langle M_\theta^j\colon j<i\rangle\}$ whenever C-supp(p^i) $\cap[\theta,\theta^+)\neq\emptyset$ and θ is inaccessible. If θ is a successor cardinal and C-supp(p^i) $\cap\,[\theta,\theta^+)\neq\emptyset$, we require that p^{i+1} meets every dense subset of $u_{\theta^-}(P_\beta)$ in M_θ^i ([18]).

Finally, let r be the componentwise union of the $\langle p^i\colon i<\omega\rangle$, and let q be their greatest lower bound. Let $y_\gamma:=\bigcup_{i<\omega}M_{\gamma^+}^i$ for every $\gamma\in C$. We have obtained the following properties for every $\gamma\in C$:

(1) y_γ is transitive below γ^+,
(2) $y_\gamma\cap[\gamma,\gamma^+)=\text{S-supp}(r)\cap[\gamma,\gamma^+)$,
(3) $y_\gamma\cap[\gamma^+,\gamma^{++})=\text{C-supp}(r)\cap[\gamma^+,\gamma^{++})$,
(4) q forces that the $\dot S$-closure of y_γ intersected with Ord equals y_γ,
(5) q forces that $A\cap y_\gamma$ has a $P_{y_\gamma\cap\gamma^+}$-name, and
(6) $\langle y_\gamma\colon\gamma\in C\rangle$ is continuous and increasing.

(1) is immediate as each of the $M_{\gamma^+}^i$ is transitive below γ^+; (2) and (3) follow by suitable genericity. For (4), it suffices to show that the $\dot S$-closure of $M_{\gamma^+}^i$ intersected with the ordinals is forced by q to be contained in $M_{\gamma^+}^{i+2}$ for every $i<\omega$: We required that $M_{\gamma^+}^i\in M_{\gamma^+}^{i+1}$. Thus $D=\{t\in u_\gamma(P_\beta)\colon t\Vdash(\dot S\text{-closure of }M_{\gamma^+}^i)\cap\text{Ord is covered by a ground model set of size }\gamma\}$ is dense in P_β using clause 11 of Theorem 22, contained (as an element) in $M_{\gamma^+}^{i+1}$ and will thus be hit by p^{i+2}; (4) now follows as $p^{i+2}\in M_{\gamma^+}^{i+2}$: using elementarity, p^{i+2} forces that we can cover the $\dot S$-closure of $M_{\gamma^+}^i$ by a set in $M_{\gamma^+}^{i+2}$ of size γ;

([18]) That is, every dense subset of $u_{\theta^-}(P_\beta)$ definable in M_θ^i using parameters in M_θ^i.

as $\gamma \subseteq M^{i+2}_{\gamma^+}$, this covering set will be contained (as a subset) in $M^{i+2}_{\gamma^+}$. (5) follows similarly to (4), using easy density arguments. (6) is immediate by our requirements on the M^i_θ.

Let π_γ be the collapsing map of y_γ. If $\xi \in y_\gamma \cap [\gamma^+, \gamma^{++})$, then f_ξ is a bijection from γ^+ to ξ, hence $f_\xi\restriction(y_\gamma \cap \gamma^+)$ is a bijection from $y_\gamma \cap \gamma^+$ to $y_\gamma \cap \xi$ by elementarity, i.e. $\pi_\gamma(\xi) = \mathrm{ot}(f_\xi[y_\gamma \cap \gamma^+])$, therefore $q(\pi_\gamma(\xi)) = r(\xi)$. Now extend q to q^* such that for every $\xi \in y_\gamma$ with $\xi \geq \gamma^{++}$, we have $q^*(\pi_\gamma(\xi)) = r(\xi)$; this is possible since if γ is inaccessible then $\sup(\text{S-supp}(r) \cap \gamma) = \operatorname{card} y_\gamma$, and whenever $\text{C-supp}(r) \cap [\theta, \theta^+) \neq \emptyset$ and θ is inaccessible, $\sup r^{**}_\zeta = \sup(\text{S-supp}(r) \cap \theta) > \sup(C \cap \theta)^+$ for every $\zeta \in \text{C-supp}(r) \cap [\theta, \theta^+)$ by easy density arguments, hence when we form q out of r and have to set $q(\mathrm{ot}\, f_\zeta[\sup r^{**}_\zeta])$ to be equal to $q(\zeta)$ for $\zeta \in \text{C-supp}(r) \cap [\theta, \theta^+)$, we do not make any new requirements in the interval $[\gamma, \gamma^+)$—note that $\mathrm{ot}\, f_\zeta[\sup r^{**}_\zeta] \geq \sup r^{**}_\zeta$. We have thus made sure q^* forces Condensation for y_γ for every $\gamma \in C$. ∎

Theorem 25. *Local Club Condensation is consistent with the existence of an ω-superstrong cardinal.*

Proof. Assume κ is ω-superstrong, witnessed by the embedding $j\colon \mathbf{V} \to \mathbf{M}$. Let A be a well-ordering of V_κ (viewed as a function $A\colon \kappa \to V_\kappa$). We use A to build a well-ordering of $V_{j^\omega(\kappa)}$ as follows: By elementarity of j, $j(A)$ is a well-ordering of $M_{j(\kappa)}$ extending A. But $V_{j(\kappa)} = M_{j(\kappa)}$, hence $j(A)$ is in fact a well-ordering of $V_{j(\kappa)}$. Similarly, $j(j(A))$ is a well-ordering of $V_{j^2(\kappa)}$ extending $j(A)$. Going on like this for ω steps, using the fact that $V_{j^\omega(\kappa)} = M_{j^\omega(\kappa)}$, we obtain a well-ordering $B := \bigcup_{n\in\omega} j^n(A)$ of $V_{j^\omega(\kappa)}$ such that $j(B) = B$. Now we perform a class forcing T to add a predicate R extending B which well-orders $\mathbf{V}$: A condition in T is a function f from an ordinal into $\mathbf{V}$ extending B; f is stronger than g in T iff f extends g. Forcing with T does not add new sets and adds a predicate R which well-orders $\mathbf{V}$ with the property that $j(R\restriction j^\omega(\kappa)) = R\restriction j^\omega(\kappa)$. Since no new sets are added, j is an elementary embedding from $(\mathbf{V}, R)$ to $(\mathbf{M}, j(R))$ with $j(R) := \bigcup_{\alpha\in\mathrm{Ord}} j(R\restriction\alpha)$.

Let P be the Local Club Condensation forcing relative to R as defined at the beginning of this section, letting, for each ordinal γ, f_γ be the R-least bijection from the cardinality of γ to γ. We want to show that forcing with P preserves the ω-superstrength of κ. Let $\langle f^*_\gamma\colon \gamma \in \mathrm{Ord}\rangle$ denote the $\mathbf{M}$-version of $\langle f_\gamma\colon \gamma \in \mathrm{Ord}\rangle$—letting each f^*_γ be the $j(R)$-least bijection from the cardinality of γ in $\mathbf{M}$ to γ. Let P^* denote the $\mathbf{M}$-version of P (using the definition of P in $\mathbf{M}$ relative to $\langle f^*_\gamma\colon \gamma \in \mathrm{Ord}\rangle$). Note that by our choice of R, $f_\gamma = f^*_\gamma$ for $\gamma < j^\omega(\kappa)$ and hence we have made sure that for every $n < \omega$, $P_{j^n(\kappa)} = P^*_{j^n(\kappa)}$. We want to find a $\mathbf{V}$-generic $G \subseteq P$

with corresponding predicate $g \subseteq \mathrm{Ord}$ and an $\mathbf{M}$-generic $G^* \subseteq P^*$ such that $j''G \subseteq G^*$ and $V[G]_{j^\omega(\kappa)} \subseteq M[G^*]$. Let $G_{j(\kappa)}$ be generic for $P_{j(\kappa)}$, and let $G^*_{j(\kappa)} = G_{j(\kappa)}$. Trivially, $j''G_\kappa = G_\kappa \subseteq G_{j(\kappa)}$ and thus we may lift j to $j^*\colon \mathbf{V}[G_\kappa] \to \mathbf{M}[G_{j(\kappa)}]$. For simplicity of notation, we will denote j^* (and any further liftings of j^*) by j again. We want to show that we can arrange that for every $n \in \omega$, $j''G[j^n(\kappa), j^{n+1}(\kappa))$ has a lower bound in $P[j^{n+1}(\kappa), j^{n+2}(\kappa))$ which is contained in $G[j^{n+1}(\kappa), j^{n+2}(\kappa))$. We will then set $G^*_{j^n(\kappa)} = G_{j^n(\kappa)}$ for every $n \in \omega$. We start with $j''G[\kappa, j(\kappa))$. Let r be such that for every $\gamma \in [j(\kappa), j^2(\kappa))$,

$$r_\gamma = \bigcup_{p \in G[\kappa, j(\kappa))} j(p)_\gamma, \qquad r^{**}_\gamma = \bigcup_{p \in G[\kappa, j(\kappa))} j(p)^{**}_\gamma.$$

To simplify notation, we will abbreviate this as

$$r = \bigcup_{p \in G[\kappa, j(\kappa))} j(p),$$

an obvious abuse of notation, thinking of $\bigcup$ as the componentwise union here. We will use similar abbreviations in similar cases. As we did earlier, we write S-supp(r) for $\{\gamma\colon r_\gamma \neq \check{\mathbf{1}}\}$ and C-supp(r) for $\{\gamma\colon r^{**}_\gamma \neq \check{\mathbf{1}}\}$. We first want to show that S-supp(r) is bounded below every regular cardinal and that $\mathrm{card}(\text{C-supp}(r) \cap [\theta, \theta^+)) < \theta$ for every regular cardinal θ.

Assume $\theta \in [j(\kappa)^+, j^2(\kappa)]$ is regular. Then

$$\text{S-supp}(r) \cap \theta = \bigcup_{p \in G[\kappa, j(\kappa))} \text{S-supp}(j(p)) \cap \theta.$$

But $j(p) \in P[j(\kappa), j^2(\kappa))$ for every $p \in G[\kappa, j(\kappa))$, so S-supp$(j(p)) \cap \theta$ is bounded below θ, hence using the fact that $P[\kappa, j(\kappa))$ has a dense subset of size $j(\kappa)$ and $\theta > j(\kappa)$ is regular, it follows that S-supp$(r) \cap \theta$ is bounded in θ.

Claim 26. C-supp$(r) \cap [j(\kappa), j(\kappa)^+) = j''[\kappa, \kappa^+)$.

Proof. Assume $\gamma \in$ C-supp$(r) \cap [j(\kappa), j(\kappa)^+)$. Then $\gamma \in$ C-supp$(j(p)) \cap [j(\kappa), j(\kappa)^+) = j(\text{C-supp}(p) \cap [\kappa, \kappa^+))$ for some $p \in G[\kappa, j(\kappa))$.

But C-supp$(p) \cap [\kappa, \kappa^+)$ has order-type less than κ, thus $j(\text{C-supp}(p) \cap [\kappa, \kappa^+)) = j''(\text{C-supp}(p) \cap [\kappa, \kappa^+))$. ∎

We have thus shown that C-supp$(r) \cap j(\kappa)^+$ has size $\kappa^+ < j(\kappa)$.

Assume now that $\theta \in [j(\kappa)^{++}, j^2(\kappa))$ is a successor of a regular cardinal:

$$\text{C-supp}(r) \cap \theta = \bigcup_{p \in G[\kappa, j(\kappa))} \text{C-supp}(j(p)) \cap \theta.$$

It follows as above that $\mathrm{card}(\text{C-supp}(r) \cap \theta) < \theta^-$.

Having shown that r has appropriate supports, we want to form q^ξ out of r for every $\xi \in [j(\kappa), j^2(\kappa)]$ by setting, for every $\gamma \in$ C-supp(r) below ξ,

$$(q^\xi)^{**}_\gamma = r^{**}_\gamma \cup \{\sup r^{**}_\gamma\}, \qquad (q^\xi)_{\mathrm{ot}\, f_\gamma[\sup r^{**}_\gamma]} = r_\gamma \quad \text{if } \operatorname{card} \gamma > j(\kappa).$$

Of course we want to set $(q^\xi)_\gamma = r_\gamma$ for $\gamma < \xi$, γ in S-supp(r), and let other components have value $\check{\mathbf{1}}$. We want to show, by induction on ξ, that q^ξ is a condition in $P[j(\kappa), \xi)$ for every $\xi \in [j(\kappa), j^2(\kappa)]$. In that case, each q^ξ is a lower bound for $\{j(p){\restriction}\xi : p \in G[\kappa, j(\kappa))\}$ and $q := q^{j^2(\kappa)}$ is the desired lower bound for $j''G[\kappa, j(\kappa))$. For each ξ as above, let $(q^\xi)^\oplus$ be such that $(q^\xi)^\oplus_\xi = r_\xi$ and $(q^\xi)^\oplus{\restriction}\xi = q^\xi$. If q^ξ is a condition in $P[j(\kappa), \xi)$, then $(q^\xi)^\oplus$ is a condition in $P[j(\kappa), \xi)^\oplus$.

Claim 27. $\forall \gamma \in [\kappa, \kappa^+)\ \ \mathrm{ot}\, j(f_\gamma)[\kappa] = \gamma$.

Proof. If $\alpha < \kappa$, then $j(f_\gamma)(\alpha) = j(f_\gamma(\alpha))$, thus $j(f_\gamma)[\kappa] = j''f_\gamma[\kappa]$, which has order-type γ as j is order-preserving. ∎

First assume $\xi < j(\kappa)^+$. Given $(q^\xi)^\oplus$, note that it forces that $\sup r^{**}_\xi = \kappa$. Let γ be such that $j(\gamma) = \xi$ and $\gamma \in [\kappa, \kappa^+)$. Then $\mathrm{ot}\, f_\xi[\kappa] = \gamma$ and $(q^\xi)^\oplus_\gamma = \check{\mathbf{1}}$. Let $p \in G[\kappa, \gamma)^\oplus$ be such that $p{\restriction}\gamma$ decides p_γ. We are free to choose $q^{\xi+1}$ as desired by letting $(q^{\xi+1})_\gamma = 1$ iff $p{\restriction}\gamma \Vdash p(\gamma) = 1$. We may thus show that $q^{j(\kappa)^+}$ is a condition in $P[j(\kappa), j(\kappa)^+)$. Now assume ξ has regular cardinality $\theta \in [j(\kappa)^+, j^2(\kappa))$, $\xi \in$ C-supp(r).

Claim 28. $(q^\xi)^\oplus \Vdash \sup r^{**}_\xi \geq \sup(\operatorname{range} j \cap \theta)$.

Proof. $\exists p \in G[\kappa, j(\kappa))\ \ \xi \in$ C-supp$(j(p))$. For every δ, the set

$$D_\delta := \{t \in P[\kappa, j(\kappa)) : \forall i \geq \delta^+\ \ i \in \text{C-supp}(t) \to t \Vdash \max t^{**}_i \geq \delta\}$$

is dense in $P[\kappa, j(\kappa))$. Assume $^*\beta < \theta$, $\beta \in \operatorname{range}(j)$, and choose $t \leq p$ in $D_{j^{-1}(\beta)} \cap G[\kappa, j(\kappa))$. Then $\forall i \geq \theta\ \ i \in$ C-supp$(j(t)) \to j(t){\restriction}i^\oplus \Vdash \max j(t)^{**}_i \geq \beta$. Thus $(q^\xi)^\oplus \Vdash \sup r^{**}_\xi \geq \sup(\operatorname{range} j \cap \theta)$. ∎

Claim 29. *If $\gamma \in$ C-supp(r) has cardinality θ, $\gamma < \xi$, then $(q^\xi)^\oplus \Vdash \sup r^{**}_\gamma = \sup r^{**}_\xi$.*

Proof. Assume $\exists u \leq (q^\xi)^\oplus\ \ u \Vdash \sup r^{**}_\gamma < \sup r^{**}_\xi$. Then there is $p \in G[\kappa, j(\kappa))$ with $u \Vdash \max j(p)^{**}_\xi > \sup r^{**}_\gamma$. We may assume $\gamma \in$ C-supp$(j(p))$. The set

$$D := \{t \leq p : \forall \eta\ \forall \delta \in \text{C-supp}(p) \cap [\eta, \eta^+) \\ t \Vdash \max t^{**}_\delta > \sup\{\max p^{**}_i : i \in \text{C-supp}(p) \cap [\eta, \eta^+)\}\}$$

is dense below p. Choose $t \in D \cap G[\kappa, j(\kappa))$. Then $(q^\xi)^\oplus \leq j(t){\restriction}\xi^\oplus \Vdash \max j(t)^{**}_\gamma > j(p)^{**}_\xi$, hence $u \Vdash \max j(t)^{**}_\gamma > \sup r^{**}_\gamma$, a contradiction. Assuming that $\exists u \leq (q^\xi)^\oplus\ \ u \Vdash \sup r^{**}_\gamma > \sup r^{**}_\xi$ analogously leads to a contradiction. ∎

Claim 30. *If $\gamma \in$ C-supp(r) has cardinality θ and $\gamma < \xi$, then*

$$(q^\xi)^\oplus \Vdash \operatorname{ot} f_\gamma[\sup r^{**}_\gamma] < \operatorname{ot} f_\xi[\sup r^{**}_\xi].$$

Proof. Choose $p \in G[\kappa, j(\kappa))$ with γ, ξ both in C-supp$(j(p))$. We already know that $(q^\xi)^\oplus \Vdash \sup r^{**}_\gamma = \sup r^{**}_\xi$. Given $u' \leq (q^\xi)^\oplus$, let $u \leq u'$ decide $\sup r^{**}_\xi$ and denote that value by s. Note that for every regular cardinal η, there exists a club $C_\eta \subseteq \eta$ of ordinals ζ such that for all $\delta_0 < \delta_1$ both in C-supp$(r) \cap [\eta, \eta^+)$, $\operatorname{ot} f_{\delta_0}[\zeta] < \operatorname{ot} f_{\delta_1}[\zeta]$. We say that C_η *separates* C-supp$(r) \cap [\eta, \eta^+)$ in this case. Let $C = \langle C_\eta \colon \text{C-supp}(p) \cap [\eta, \eta^+) \neq \emptyset \rangle$. Then $j(C) = \langle E_\eta \colon \text{C-supp}(j(p)) \cap [\eta, \eta^+) \neq \emptyset \rangle$ has the property that for each η, E_η separates C-supp$(j(p)) \cap [\eta, \eta^+)$. We want to finish the proof by showing that $s \in E_\theta$ and thus u forces the desired property of the claim:

Assume for a contradiction that $s \notin E_\theta$ and thus E_θ is bounded in s by some $\alpha < s$. Choose $t \leq p$ in $G[\kappa, j(\kappa))$ such that $t \Vdash \alpha \leq \max j(t)^{**}_\gamma = \max j(t)^{**}_\xi \in E_\theta$. This is possible since there is $p' \leq p$ in $G[\kappa, j(\kappa))$ such that $\max j(p')^{**}_\gamma \geq \alpha$ and $D := \{t \colon \forall \eta\ \forall \delta_0, \delta_1 \in \text{C-supp}(t) \cap [\eta, \eta^+)\ \ t \Vdash \max t^{**}_{\delta_0} = \max t^{**}_{\delta_1} \in C_\eta\}$ is dense in $P[\kappa, j(\kappa))$, so we may choose $t \in D \cap G[\kappa, j(\kappa))$ below p'. Then t is as desired. But $u \Vdash \max j(t)^{**}_\gamma \leq \sup r^{**}_\gamma = s$, thus u forces that E_θ is not bounded by α below s, a contradiction as desired. ∎

Claim 31. $(q^\xi)^\oplus \Vdash \operatorname{ot} f_\xi[\sup r^{**}_\xi] \geq \sup(\text{S-supp}(r) \cap \theta)$.

Proof. Note that $\sup(\text{S-supp}(r) \cap \theta)$ is a limit ordinal and assume for a contradiction that $\exists u \leq (q^\xi)^\oplus\ u \Vdash \operatorname{ot} f_\xi[\sup r^{**}_\xi] < \alpha < \sup(\text{S-supp}(r) \cap \theta)$ for some α. Choose $p \in G[\kappa, j(\kappa))$ such that $\sup(\text{S-supp}(j(p)) \cap \theta) \geq \alpha$ and $\xi \in$ C-supp$(j(p))$. Now note that $D := \{t \colon t \Vdash \forall \eta\ \forall \delta \in \text{C-supp}(p) \cap [\eta, \eta^+)\ \ \max t^{**}_\delta \geq \sup(\text{S-supp}(p) \cap \eta)$ and $f_\delta[\max t^{**}_\delta] \supseteq \max t^{**}_\delta\}$ is dense in $P[\kappa, j(\kappa))$ below p. Choose $t \in D \cap G[\kappa, j(\kappa))$. Then $j(t) \Vdash \max(j(t)^{**}_\xi) \geq \sup(\text{S-supp}(j(p)) \cap \theta) \geq \alpha$ and $f_\xi[\max j(t)^{**}_\xi] \supseteq \max j(t)^{**}_\xi$. Thus $(q^\xi)^\oplus \leq j(t) \restriction \xi^\oplus \Vdash \operatorname{ot} f_\xi[\sup r^{**}_\xi] \geq \operatorname{ot} f_\xi[\max(j(t)^{**}_\xi)] \geq \alpha$, a contradiction. ∎

Claim 32. *If θ is inaccessible, then*

$$\sup(\text{S-supp}(r) \cap \theta) \geq \operatorname{card}(\text{C-supp}(r) \cap [\theta, \theta^+)).$$

Proof. The set

$$D := \{p \colon \forall \eta \text{ inaccessible } \ \sup(\text{S-supp}(p) \cap \eta) \geq \operatorname{card}(\text{C-supp}(p) \cap [\eta, \eta^+))\}$$

is dense in $P[\kappa, j(\kappa))$. Hence

$$\sup(\text{S-supp}(r) \cap \theta) = \bigcup_{p \in G[\kappa, j(\kappa))} \sup(\text{S-supp}(j(p)) \cap \theta)$$

is greater than or equal to

$$\bigcup_{p \in G[\kappa, j(\kappa))} \operatorname{card}(\text{C-supp}(j(p)) \cap [\theta, \theta^+)).$$

So for every $p \in G[\kappa, j(\kappa))$, $\sup(\text{S-supp}(r) \cap \theta) \geq \operatorname{card}(\text{C-supp}(j(p)) \cap [\theta, \theta^+))$. As $P[\kappa, j(\kappa))$ has a dense subset of size $j(\kappa)$, it suffices to show that $\sup(\text{S-supp}(r) \cap \theta) \geq j(\kappa)$, which is true as $j(\kappa) \in \text{S-supp}(r)$. ■

CLAIM 33. *q^ξ forces that r_ξ has a $P[j(\kappa), \sup(\text{S-supp}(r) \cap \theta))$-name.*

Proof. Choose $p \in G[\kappa, j(\kappa))$ such that $\xi \in \text{C-supp}(j(p))$. Note that $D := \{t \leq p : \forall \eta\ \forall \delta \in \text{C-supp}(p) \cap [\eta, \eta^+)\ \ t{\restriction}\delta \Vdash t_\delta$ has a $P[\kappa, \sup(\text{S-supp}(t) \cap \eta))$-name$\}$ is dense in $P[\kappa, j(\kappa))$ below p. Choose $t \in D \cap G[\kappa, j(\kappa))$. Then $j(t){\restriction}\xi$ forces that $j(t)_\xi = r_\xi$ has a $P[j(\kappa), \sup(\text{S-supp}(j(t)) \cap \theta))$-name. The claim follows as $\sup(\text{S-supp}(j(t)) \cap \theta) \leq \sup(\text{S-supp}(r) \cap \theta)$. ■

Now by the above claims, we may set $q_{\operatorname{ot} f_\xi[\sup r_\xi^{**}]} = r_\xi$ and $q_\xi^{**} = r_\xi^{**} \cup \{\sup r_\xi^{**}\}$, i.e. given that $(q^\xi)^\oplus$ is a condition in $P[j(\kappa), \xi)^\oplus$, we find that $q^{\xi+1}$ is a condition in $P[j(\kappa), \xi+1)$. If ξ is a limit ordinal, then q^ξ is a condition in $P[j(\kappa), \xi)$, as for each $\zeta < \xi$, $q^\xi{\restriction}\zeta$ is a condition in $P[j(\kappa), \zeta)$ inductively and q^ξ has appropriate supports. So we finally obtain $q \in P[j(\kappa), j^2(\kappa))$ which is below $j''G[\kappa, j(\kappa))$, our desired master condition. If we choose our $P[j(\kappa), j^2(\kappa))$-generic $G[j(\kappa), j^2(\kappa))$ to contain q we have ensured that $j''G[\kappa, j(\kappa)) \subseteq G[j(\kappa), j^2(\kappa))$ and we may thus lift the embedding $j\colon \mathbf{V}[G_\kappa] \to \mathbf{M}[G_{j(\kappa)}]$ to $j\colon \mathbf{V}[G_{j(\kappa)}] \to \mathbf{M}[G_{j^2(\kappa)}]$. But in order to be able to further lift the embedding j, we have to demand a little more of $G[j(\kappa), j^2(\kappa))$: We will define a condition $t \in P[j(\kappa), j^2(\kappa))$, show that t and q are compatible, demand that $G[j(\kappa), j^2(\kappa))$ contains both t and q, and show how this helps us to deduce that $j''G[j(\kappa), j^2(\kappa))$ has a lower bound in $P[j^2(\kappa), j^3(\kappa))$. This will finally enable us to lift $j\colon \mathbf{V}[G_{j(\kappa)}] \to \mathbf{M}[G_{j^2(\kappa)}]$ to $j\colon \mathbf{V}[G_{j^2(\kappa)}] \to \mathbf{M}[G_{j^3(\kappa)}]$. The further liftings of j up to $j\colon \mathbf{V}[G_{j^\omega(\kappa)}] \to \mathbf{M}[G_{j^\omega(\kappa)}]$ then work the same way (more precisely, it will be immediate to find $q \in P_{j^\omega(\kappa)}$ such that if we demand that $q \in G_{j^\omega(\kappa)}$, then $j''G_{j^\omega(\kappa)} \subseteq G_{j^\omega(\kappa)}$).

- Let $c := \bigcup\{j(A) : A \subseteq [j(\kappa), j(\kappa)^+),\ |A| < j(\kappa)\}$.
- Let $d := \sup(\operatorname{range}(j) \cap j^2(\kappa))$.

NOTE. Whichever $G[j(\kappa), j^2(\kappa))$ we choose, if we then let

$$r := \bigcup_{p \in G[j(\kappa), j^2(\kappa))} j(p),$$

it will be the case that

$$\text{C-supp}(r) \cap [j^2(\kappa), j^2(\kappa)^+) = c$$

and $\sup r_\gamma^{**} = d$ for $\gamma \in \text{C-supp}(r) \cap [j^2(\kappa), j^2(\kappa)^+)$.

Definition of t. For every $\gamma \in c$, let A_γ be a maximal antichain in $P[j(\kappa), j(\kappa)^+)$ which j-decides the bit at γ, in the sense that for every $a \in A_\gamma$, $j(a){\restriction}\gamma$ decides $j(a)_\gamma$: this is possible as the set D of conditions

p in $P[j(\kappa), j(\kappa)^+)$ such that p decides $\{p_\delta \colon \delta \in \text{C-supp}(p)\}$ and such that $\gamma \in \text{C-supp}(j(p))$ is dense in $P[j(\kappa), j(\kappa)^+)$. But for any such p, $j(p)$ decides $j(p)_\gamma$ by elementarity. Now we let, for every $\gamma \in c$,

$$t_{\operatorname{ot} f_\gamma[d]} := \{(a, \epsilon) \colon a \in A_\gamma \wedge j(a) \Vdash j(a)_\gamma = \epsilon\}.$$

Similar to Claim 30, one may show that $\operatorname{ot} f_\gamma[d]$ is different for different $\gamma \in c$. We let $t_\delta = \mathbf{1}$ for all δ which are not as above and let $t_\delta^{**} = \emptyset$ for all δ. Note that each t_δ is a $P[j(\kappa), \delta)$-name, since $d > j(\kappa)^+$. We need to show that t has sufficiently small supports in order to be a condition in $P[j(\kappa), j^2(\kappa))$. The following is clearly sufficient:

Claim 34. *$\operatorname{card}(c) \le d$.*

Proof. For each $A \subseteq [j(\kappa), j(\kappa)^+)$ of size less than $j(\kappa)$, $\operatorname{card}(j(A)) \in \operatorname{range}(j) \cap j^2(\kappa)$. There are only $j(\kappa)^+$-many possibilities for A and thus the claim follows as $d > j(\kappa)^+$. ∎

Claim 35. *$t \parallel q$.*

Proof. For $\gamma \in c$, $\operatorname{ot} f_\gamma[d] \ge d$. It suffices to note that whenever $\delta \in \text{S-supp}(q)$, then $\delta < d$. ∎

This allows us to demand that $G[j(\kappa), j^2(\kappa))$ contains both q and t.

Lifting. We want to lift $j \colon \mathbf{V}[G_{j(\kappa)}] \to \mathbf{M}[G_{j^2(\kappa)}]$ to $j \colon \mathbf{V}[G_{j^2(\kappa)}] \to \mathbf{M}[G_{j^3(\kappa)}]$. Let $r = \bigcup_{p \in G[j(\kappa), j^2(\kappa))} j(p)$. As before, one shows that r has appropriate supports. We want to form $\tilde{q}$ out of r by setting, for every $\gamma \in \text{C-supp}(r)$,

$$\tilde{q}_\gamma^{**} = r_\gamma^{**} \cup \{\sup r_\gamma^{**}\}, \qquad \tilde{q}_{\operatorname{ot} f_\gamma[\sup r_\gamma^{**}]} = r_\gamma \quad \text{if } \operatorname{card} \gamma > j(\kappa).$$

Of course we want to set $\tilde{q}_\gamma = r_\gamma$ for γ in $\text{S-supp}(r)$ and let other components have value $\check{\mathbf{1}}$. We want to show that $\tilde{q}$ is a condition in $P[j^2(\kappa), j^3(\kappa))$. In that case, $\tilde{q}$ is obviously a lower bound for $j''G[j(\kappa), j^2(\kappa))$. Note that since $t \in G[j(\kappa), j^2(\kappa))$, it follows that $g(\operatorname{ot} f_\gamma[\sup r_\gamma^{**}]) = r_\gamma$ for every $\gamma \in [j^2(\kappa), j^2(\kappa)^+)$ (to be exact, there exists $p \in G[j(\kappa), j^2(\kappa))$ such that $j(p) \restriction \gamma$ decides r_γ and thus forces the above), which shows that $\tilde{q} \restriction j^2(\kappa)^+$ is a condition in $P[j^2(\kappa), j^2(\kappa)^+)$. The rest of the proof that $\tilde{q}$ is a condition in $P[j^2(\kappa), j^3(\kappa))$ works as in the proof for q above.

Master condition. Continue as above for ω-many steps, in this way defining a master condition $u \in P_{j^\omega(\kappa)}$ with the property that $u \le j''G_{j^\omega(\kappa)}$, and choose a $P_{j^\omega(\kappa)}$-generic $G_{j^\omega(\kappa)}$ containing u. Let $G^*_{j^\omega(\kappa)} := G_{j^\omega(\kappa)} \cap P^*_{j^\omega(\kappa)}$.

Claim 36. *$G^*_{j^\omega(\kappa)}$ is $P^*_{j^\omega(\kappa)}$-generic over $\mathbf{M}$.*

Proof. Suppose $D \in \mathbf{M}$ is open dense on $P^*_{j^\omega(\kappa)}$ and write D as $j(f)(a)$ where $\operatorname{dom}(f) = V_{j^\omega(\kappa)}$ and $a \in V_{j^{n+1}(\kappa)}$ for some $n \in \omega$. We may assume that every element of $\mathbf{M}$ is of this form. Choose $p \in G_{j^\omega(\kappa)}$ such that p

reduces $f(\bar{a})$ below $j^n(\kappa)$ whenever $\bar{a}$ belongs to $V_{j^n(\kappa)}$ and $f(\bar{a})$ is open dense on $P_{j^\omega(\kappa)}$, in the sense that if q extends p then q can be further extended into $f(\bar{a})$ without changing $u_{j^n(\kappa)}(q)$. The existence of p as above is shown similarly to the proof of Theorem 22, clause 8, using the fact that $V_{j^n(\kappa)}$ has size $j^n(\kappa)$. Then $j(p)$ belongs to $j''G_{j^\omega(\kappa)} \subseteq G^*_{j^\omega(\kappa)}$ and reduces D below $j^{n+1}(\kappa)$, i.e. if $q \leq j(p)$ then $\exists r \leq q\ \ r \in D \wedge u_{j^{n+1}(\kappa)}(r) = u_{j^{n+1}(\kappa)}(q)$.

Hence $E := \{q \in P_{j^{n+2}(\kappa)} \colon q^\frown j(p)[j^{n+2}(\kappa), j^\omega(\kappa)) \in D\}$ is dense below $j(p)\restriction j^{n+2}(\kappa)$ in $P_{j^{n+2}(\kappa)}$. Since $G_{j^{n+2}(\kappa)}$ contains $j(p)\restriction j^{n+2}(\kappa)$ and is $P_{j^{n+2}(\kappa)}$-generic over $\mathbf{M}$, $G_{j^{n+2}(\kappa)} \cap E \neq \emptyset$. Choose a condition q in that intersection. Then $q^\frown j(p)[j^{n+2}(\kappa), j^\omega(\kappa)) \in D \cap G^*_{j^\omega(\kappa)}$. ∎

By the above, we obtain a lifted embedding $j\colon \mathbf{V}[G_{j^\omega(\kappa)}] \to \mathbf{M}[G^*_{j^\omega(\kappa)}]$. As $P[j^\omega(\kappa), \infty)$ is $j^\omega(\kappa)^+$-distributive by Theorem 22, we may choose an arbitrary $P[j^\omega(\kappa), \infty)$-generic $G[j^\omega(\kappa), \infty)$, assume that j is given by an ultrapower ([19]) and apply Lemma 3 of [11] to find a P^*-generic G^* extending $G^*_{j^\omega(\kappa)}$ and an elementary embedding $j\colon \mathbf{V}[G] \to \mathbf{M}[G^*]$ extending $j\colon \mathbf{V} \to \mathbf{M}$. As $\mathbf{V}[G]_{j^\omega(\kappa)} = \mathbf{V}_{j^\omega(\kappa)}[G^*_{j^\omega(\kappa)}] \subseteq \mathbf{M}[G^*]$, j witnesses ω-superstrength of κ in $\mathbf{V}[G]$. ∎Theorem 25

Consequences of Local Club Condensation. It is easy to see that Local Club Condensation implies $\diamondsuit_\kappa(E)$ for every regular κ and every stationary $E \subseteq \kappa$. The proof is very similar to the proof that $\diamondsuit_\kappa(E)$ holds in $\mathbf{L}$ for every regular κ and every stationary $E \subseteq \kappa$: see e.g. [9] for both that proof and the definition of $\diamondsuit_\kappa(E)$. $\diamondsuit_{\omega_1}$ in fact already follows from Stationary Condensation.

Local Club Condensation has interesting consequences for the existence of locally definable well-orderings. We say that a class A of ordinals *witnesses* Local Club Condensation iff the sequence $\langle L_\alpha[A] \colon \alpha \in \mathrm{Ord}\rangle$ witnesses Local Club Condensation in the sense of its original definition. The proof of Theorem 25 shows that it is consistent to have $A \subseteq \mathrm{Ord}$ witnessing Local Club Condensation in the presence of ω-superstrong cardinals.

THEOREM 37. *Suppose that $A \subseteq \mathrm{Ord}$ witnesses Local Club Condensation. Then for each limit cardinal κ (including $\aleph_0$) and each $n \in [2, \ldots, \omega]$, there is a well-ordering of $H_{\kappa^{+n}}$ which is Δ_1-definable over $H_{\kappa^{+n}}$ with parameter $A \cap \kappa^+$. If κ is inaccessible then $A \cap \kappa^+$ can be replaced by $A \cap \kappa$ and we may also allow $n = 1$.*

Proof. Suppose that $\alpha^+ \leq \beta < \alpha^{++}$ and f_β is any bijection between α^+ and β. As A witnesses Local Club Condensation, we have: $\beta \in A$ iff

([19]) To say that $j\colon \mathbf{V} \to \mathbf{M}$ is given by an ultrapower means that every element of $\mathbf{M}$ is of the form $j(f)(a)$ where f has domain $H_{j^\omega(\kappa)}$ and a belongs to $H_{j^\omega(\kappa)}$.

$\{\gamma < \alpha^+ \colon \text{ot}\, f_\beta[\gamma] \in A\}$ contains a club, and $\beta \notin A$ iff $\{\gamma < \alpha^+ \colon \text{ot}\, f_\beta[\gamma] \notin A\}$ contains a club. This gives a well-ordering of $H_{\alpha^{++}} = L_{\alpha^{++}}[A]$ which is Δ_1-definable over $H_{\alpha^{++}}$ with parameter $A \cap \alpha^+$, namely the canonical well-ordering of $L_{\alpha^{++}}[A]$. By composing these definitions we get, for any limit cardinal κ and any $n \in [2, \ldots, \omega]$, a well-ordering of $H_{\kappa^{+n}}$ which is Δ_1-definable over $H_{\kappa^{+n}}$ with parameter $A \cap \kappa^+$. If κ is inaccessible, then we can apply the same argument to show that $A \cap \kappa^+$ is Δ_1-definable over H_{κ^+} from the parameter $A \cap \kappa$. ∎

COROLLARY 38. *It is consistent with an ω-superstrong cardinal that whenever κ is regular and uncountable, H_{κ^+} has a well-ordering which is Δ_1-definable over H_{κ^+} with parameters.* ∎

In [2] and [3] the previous corollary is improved to eliminate the parameters. It is not possible to allow κ to be ω, as large cardinals imply that H_{ω_1} has no definable well-ordering, even with parameters. The case of singular κ is open.

Variations of the Local Club Condensation forcing. We can show the following, using a (much simpler) variant of the forcing used to obtain Local Club Condensation above:

THEOREM 39. *Assume κ is regular uncountable, $2^\kappa = \kappa^+$ and $\kappa^{<\kappa} = \kappa$. Then there is a κ-strategically closed, κ^+-cc forcing which forces a Δ_1-definable (from a parameter $a \subseteq \kappa$) well-order of H_{κ^+}. Moreover, one can additionally make a given ground model subset of H_{κ^+} Δ_1-definable (from the same parameter $a \subseteq \kappa$).*

Proof sketch. The idea is to construct $A \subseteq \kappa^+$ such that $H_{\kappa^+} = L_{\kappa^+}[A]$ and such that A is in fact Δ_1-definable from $A \cap \kappa$ in H_{κ^+}. The predicate A will look very much like a predicate witnessing Local Club Condensation. Our forcing S to achieve this will be an iteration of length κ^+ with supports of size less than κ. It will be similar to P_{κ^+}, the forcing P to obtain Local Club Condensation (as defined in the section "Forcing Local Club Condensation") up to κ^+, but we replace P_κ by κ-Cohen and we construct the predicate A between κ and κ^+ ourselves, where we successively choose segments of size κ (instead of letting the generic choose segments of size 1 as we did when we forced Local Club Condensation) and use a slightly enhanced version of the forcings $C_\alpha(g)$, which is capable of ensuring appropriate condensation of those κ-sized segments of the predicate into the generic below κ ($C_\alpha(g)$ only did this for a single bit of the predicate at α). When constructing the predicate between κ and κ^+, we have to take care that in the final model, $H_{\kappa^+}[A] = L_{\kappa^+}[A]$. We will describe S in more detail in the following:

At stage 0, we force with κ-Cohen forcing and let g be the generic subset of κ, and $A \cap \kappa = g$. The iterands of S will be trivial in the interval $(0,\kappa)$. Choose, for each $\beta \in [\kappa,\kappa^+)$, some bijection $f_\beta\colon \kappa \to \beta$. If $\alpha \geq \kappa$, $\alpha = \kappa + \xi$, we choose, at stage α of the iteration, a subset s of the interval $\operatorname{dom} s = [\kappa\cdot(1+\xi), \kappa\cdot(1+\xi+1))$, let $A\restriction \operatorname{dom} s = s$ and then force with $C(s,g)$ to ensure that we will be able to read off s from $A \cap \kappa$: (p^*,p^{**}) is a condition in $C(s,g)$ iff

- p^* is a subset of $\operatorname{dom} s$ of size less than κ, and
- p^{**} is a closed, bounded subset of κ.

(q^*,q^{**}) extends (p^*,p^{**}) in $C(s,g)$ iff

- $q^* \supseteq p^*$,
- q^{**} end-extends p^{**}, and
- $\forall\gamma \in p^*\ \forall\eta \in q^{**}\setminus p^{**}\ \ g(\operatorname{ot} f_\gamma[\eta]) = s(\gamma)$.

By careful book-keeping, it is easy to ensure that all relevant subsets of κ which appear in intermediate models of the iteration S are inserted into the predicate A at some stage, so that if a is any subset of κ in some intermediate model of the iteration S, then a is an element of $L[A\cap\alpha]$ for some $\alpha < \kappa^+$. The forcing S is κ-strategically closed, which is seen similarly to the proof of Theorem 22, clause 9. It is easy to see that S is κ^+-cc, using the fact that any two conditions in S which specify the same κ-Cohen condition and have the same **-components are compatible (we can just take the union of their *-components to obtain a condition stronger than both). Now by the κ^+-cc, every subset of κ in the final model after forcing with S will appear in some intermediate model of the iteration, thus we may infer that $H_{\kappa^+} = L_{\kappa^+}[A]$. Our forcing ensured that for any ordinal $\beta \in [\kappa,\kappa^+)$ and any bijection f_β between κ and β, $\beta \in A$ iff $\{\gamma < \kappa\colon \operatorname{ot} f_\beta[\gamma] \in A\}$ contains a club, and $\beta \notin A$ iff $\{\gamma < \kappa\colon \operatorname{ot} f_\beta[\gamma] \notin A\}$ contains a club. Thus, as in the proof of Theorem 37, we now conclude that H_{κ^+} has a Δ_1-definable well-order using the parameter $A\cap\kappa$.

To additionally make a given ground model subset x of κ^+ Δ_1-definable from $A\cap\kappa$ within H_{κ^+}, we may for example choose $A\cap[\kappa,\kappa^+)$ slightly more carefully in the above so that for every $\gamma < \kappa^+$, $A(\kappa\cdot(1+\gamma)) = x(\gamma)$. ∎

Remarks. (a) With more care, "κ-strategically closed" can be improved to "κ-directed closed" in the statement of Theorem 39: It can be observed by analyzing the strategy for strategic closure of the Local Club Condensation forcing given in [13] (Definition 8.5 of "strategic belowness", the forcing is the same that we used to force Local Club Condensation in the present paper, but the strategy witnessing strategic closure is quite different) that our forcing S has a κ-closed dense subset of conditions ([20]) By the nature of the extension relation on S, it is easy to see that any κ-closed subset of S is in fact κ-directed closed.

(b) A technique similar to the above is used in [12, Chapter IV], to make the club filter restricted to any ground model costationary set S Δ_1-definable over H_{κ^+} in a parameter, preserving the stationarity of subsets of S. (This argument however needs, in addition to the hypotheses of Theorem 39, the extra hypothesis that κ is not the successor of a singular cardinal or at least that $\kappa^+ \in I[\kappa^+]$, the approachability property for κ^+.)

(c) Philipp Lücke [16] has obtained a version of Theorem 39 when 2^κ is greater than κ^+. Using "Kurepa tree coding" he shows that in this case there are κ-closed, κ^+-cc forcings which add a Δ_2-definable wellorder of H_{κ^+} and which make a given ground model subset of H_{κ^+} Δ_1-definable over H_{κ^+}.

A possible future application of Local Club Condensation. We can show the following variation of a theorem in [17], using a variation of the proof given in that paper:

THEOREM 40. *Assume* $\mathbf{M}$ *is of the form* $\mathbf{L}[A]$ *and satisfies Acceptability, Local Club Condensation and* $\square$ *at small cofinalities. If there is a proper forcing extension* $\mathbf{V}$ *of* $\mathbf{M}$ *in which* $\mathrm{PFA}(\mathbf{c}^+$*-linked*$)$ *holds and* $\tau = \omega_2^{\mathbf{V}}$, *then* $[\tau, (\tau^+)^{\mathbf{M}}]$ *is* Σ^2_1*-indescribable in* $\mathbf{M}$.

For the definition of a Σ^2_1-indescribable interval of cardinals, we refer the reader to [17], and for $\square$ at small cofinalities, we refer to [11]. It is shown in [11] how to force $\square$ at small cofinalities and preserve various large cardinals. A cofinality-preserving forcing will preserve $\square$ at small cofinalities. As a corollary, we get the following:

COROLLARY 41. *Assume* $\varphi(\kappa)$ *is a large cardinal property of* κ *consistency-wise weaker than a* Σ^2_1*-indescribable interval* $[\tau, \tau^+)$, *such that we can force Local Club Condensation and Acceptability by a cofinality-preserving forcing which preserves* $\varphi(\kappa)$. *Then it is consistent that* $\varphi(\kappa)$ *holds but no proper forcing extension satisfies* $\mathrm{PFA}(\mathbf{c}^+$*-linked*$)$.

A positive answer to the following open question would not only be of central importance for the Outer Model Programme but would also show

([20]) A main observation is that the handling of separating clubs (see [13] for a definition) is unnecessarily complicated in [13]. We can choose a separating club for every $v \subseteq [\kappa, \kappa^+)$ of size less than κ in advance, by letting, for every $\{\alpha_0, \alpha_1\} \subseteq [\kappa, \kappa^+)$, $C_{\{\alpha_0,\alpha_1\}}$ be a separating club for $\{\alpha_0, \alpha_1\}$ and then for every $v \subseteq [\kappa, \kappa^+)$ of size less than κ, let $C_v := \bigcap_{\{\alpha_0,\alpha_1\}\subseteq v} C_{\{\alpha_0,\alpha_1\}}$. This gives us the property that $C_{v_1} \subseteq C_{v_0}$ whenever $v_1 \supseteq v_0$. Using this observation, it is not hard to see that the conditions $p \in S$ such that all p_i^* and p_i^{**} are ground model objects and $C_{\bigcup_{i\in\mathrm{supp}(p)\setminus\{0\}} p_i^*} \ni |p(0)| = \sup p_i^{**}$ for every $i \in \mathrm{supp}(p) \setminus \{0\}$ form a κ-closed dense subset of S.

that the hypotheses of Theorem 40 and Corollary 41 are not vacuous in the presence of very large cardinals:

QUESTION 42. *Given a model of Set Theory which satisfies* GCH *and has (very) large cardinals, can we define a cofinality-preserving forcing to obtain a model of Local Club Condensation and Acceptability while preserving certain (very) large cardinals?*

NOTE. In [13], it is shown how to force Acceptability by cofinality-preserving forcing and preserve various large cardinals. In Theorem 25 of the present paper and in [13], it is shown how to force Local Club Condensation by cofinality-preserving forcing and preserve various large cardinals. The question is whether it is possible to force both of these properties simultaneously (and witnessed by the same predicate $A \subseteq \mathrm{Ord}$) while preserving large cardinals.

QUESTION 43. *Is it possible to force a "fine structure theory", preserving ω-superstrongs? To what extent is Local Club Condensation consistent with the* failure *of the combinatorial principle* $\Box$ *and the nonexistence of morasses (at various cardinals)?*

Acknowledgements. The authors wish to thank the Austrian Research Fund (FWF) for its generous support of this research through Project P 19898-N18.

References

[1] A. Apter and J. Cummings, *An L-like model containing very large cardinals*, Arch. Math. Logic 47 (2008), 65–78.

[2] D. Asperó and S. Friedman, *Large cardinals and locally-defined well-orders of the universe*, Ann. Pure Appl. Logic 157 (2009), 1–15.

[3] —, —, *Definable wellorderings of $H(\omega_2)$ and CH*, submitted.

[4] A. Brooke-Taylor, *Large cardinals and definable well-orders on the universe*, J. Symbolic Logic 74 (2009), 641–654.

[5] A. Brooke-Taylor and S. Friedman, *Large cardinals and gap-1 morasses*, Ann. Pure Appl. Logic 159 (2009), 71–99.

[6] D. Burke, *Generic embeddings and the failure of Box*, Proc. Amer. Math. Soc. 123 (1995), 2867–2871.

[7] J. Cummings and S. Friedman, *Square on the singular cardinals*, J. Symbolic Logic 73 (2008), 1307–1314.

[8] J. Cummings and E. Schimmerling, *Indexed squares*, Israel J. Math. 131 (2002), 61–99.

[9] K. Devlin, *Constructibility*, Springer, 1984.

[10] S. Friedman, *Fine Structure and Class Forcing*, de Gruyter Ser. Logic Appl. 3, de Gruyter, 2000.

[11] —, *Large cardinals and L-like universes*, in: Set Theory: Recent Trends and Applications, A. Andretta (ed.), Quaderni Mat. 17, 2007, 93–110.

[12] S. Friedman, T. Hyttinen and V. Kulikov, *Generalized descriptive set theory and classification theory*, submitted 2010.
[13] P. Holy, *Condensation and large cardinals*, Dissertation, Univ. of Vienna, 2010.
[14] T. Jech, *Set Theory*, Springer, 2003.
[15] K. Kunen, *Elementary embeddings and infinitary combinatorics*, J. Symbolic Logic 36 (1971), 407–413.
[16] P. Lücke, *Descriptive set theory at uncountable cardinals: $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$*, submitted.
[17] I. Neeman, *Hierarchies of forcing axioms II*, J. Symbolic Logic 73 (2008), 522–542.
[18] E. Schimmerling and B. Velickovic, *Collapsing functions*, Math. Logic Quart. 50 (2004), 3–8.
[19] H. Woodin, *The Axiom of Determinacy, Forcing Axioms and the Nonstationary Ideal*, de Gruyter Ser. Logic Appl. 1, de Gruyter, 1999.
[20] M. Zeman, *Inner Models and Large Cardinals*, de Gruyter Ser. Logic Appl. 5, 2002.

Sy-David Friedman, Peter Holy
Kurt Gödel Research Center for Mathematical Logic
Universität Wien
Währinger Strasse 25
1090 Wien, Austria
E-mail: sdf@logic.univie.ac.at
peter.holy@univie.ac.at

Received 25 December 2010;
in revised form 22 August 2011