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On connectivity of Julia sets of transcendental meromorphic maps and weakly repelling fixed points II

by

Núria Fagella, Xavier Jarque and **Jordi Taixés** (Barcelona)

Abstract. Following the attracting and preperiodic cases ([FJT]), in this paper we prove the existence of weakly repelling fixed points for transcendental meromorphic maps, provided that their Fatou set contains a multiply connected parabolic basin. We use quasi-conformal surgery and virtually repelling fixed point techniques.

1. Introduction. Let f be a rational, transcendental entire or transcendental meromorphic function. We say that a p-periodic point z_0 of f is *attracting* (resp. *repelling*) if the modulus of its *multiplier* $\rho(z_0) := (f^p)'(z_0) \in \mathbb{C}$ is smaller (resp. greater) than 1, and that it is *parabolic* if $\rho(z_0) = e^{2\pi i\theta}$ with $\theta \in \mathbb{Q}$. Furthermore, z_0 is said to be *weakly repelling* if it is repelling or parabolic of multiplier 1. For any rational map of degree greater than one, the existence of at least one such fixed point is guaranteed by a theorem of Fatou [F].

As for global dynamics, points can be classified according to their long-term behaviour under iteration of the function, thus one defines the *Fatou set* $\mathcal{F}(f)$ (or simply $\mathcal{F}$ when possible) as the set of points $z_0 \in \widehat{\mathbb{C}}$ for which the family $\{f^k\}_{k\in\mathbb{N}}$ is defined and normal in a neighbourhood of z_0, and the *Julia set* as its complement, $\mathcal{J} = \mathcal{J}(f) := \widehat{\mathbb{C}} \setminus \mathcal{F}(f)$. Then a connected component of the Fatou set (Fatou component) U is called *preperiodic* if there are integers $p > q \geq 0$ such that $f^p(U) = f^q(U)$, and, more precisely, *p-periodic* when $q = 0$ and *fixed* when moreover $p = 1$. On the contrary, a Fatou component is called a *wandering domain* if it fails to be preperiodic.

According to the work of Cremer and Fatou, a p-periodic Fatou component U is necessarily one of the following: an *immediate attractive basin* if U contains an attracting p-periodic point z_0 such that $\lim_{n\to\infty} f^{np}(z) = z_0$ for

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all $z \in U$; a *parabolic basin* or *Leau domain* if ∂U contains a q-periodic point z_0, with $q \,|\, p$, such that $\lim_{n\to\infty} f^{nq}(z) = z_0$ for all $z \in U$ (and $(f^p)'(z_0) = 1$); a *Siegel disc* if there exists a holomorphic homeomorphism $\phi\colon U \to \mathbb{D}$ such that $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i\theta} z$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$; a *Herman ring* if there exist an $r > 1$ and a holomorphic homeomorphism $\phi\colon U \to \{1 < |z| < r\}$ such that $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi i\theta} z$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$; a *Baker domain* if ∂U contains a point z_0, with $f^p(z_0)$ not defined, such that $\lim_{n\to\infty} f^{np}(z) = z_0$ for all $z \in U$. Nevertheless, rational maps happen to have neither wandering domains (Sullivan [Su]) nor Baker domains (for infinity is just a regular point there).

With this setting, we are now able to describe the purpose and results of this paper. In 1990, Shishikura [Sh] proved that if the Julia set of a rational map f is disconnected, then f has at least two weakly repelling fixed points. As a consequence, the connectedness of the Julia set of the Newton's method $N_f := \mathrm{id} - f/f'$ for a non-constant polynomial f is obtained immediately (since all its fixed points but infinity are attracting), hence closing a problem for which Przytycki [P], Meier [Me] and Tan Lei [TL] among others had given partial results before.

As pointed out in [FJT], our purpose is to give the natural transcendental versions of Shishikura's results, namely: If the Julia set of a transcendental meromorphic function f is disconnected, then f has at least one weakly repelling fixed point; the connectedness of the Julia set of the Newton's method for transcendental entire functions would follow as a corollary. Notice the equivalence between such hypotheses and at least one of the Fatou components of f being *multiply connected* (as opposed to simply connected), which allows each different type of Fatou component to be treated separately.

In [FJT] we proved the following result, which covers the immediate attractive basin and preperiodic Fatou component cases.

Main Theorem 1.1. *Let f be a transcendental meromorphic function with either a multiply connected immediate attractive basin or a multiply connected Fatou component U such that $f(U)$ is simply connected. Then there exists at least one weakly repelling fixed point of f.*

On the other hand, the case of the multiply connected wandering domain was proved by Bergweiler and Terglane in the search of solutions of certain differential equations with no wandering domains (see [BT]). The present work is devoted to parabolic basins. More precisely, we prove the following.

Main Theorem 1.2. *Let f be a transcendental meromorphic function with a multiply connected parabolic basin. Then f has at least one weakly repelling fixed point.*

The main tools involved in its proof are the method of *quasi-conformal surgery* and a theorem of Buff on *virtually repelling fixed points*; we shall briefly introduce the two concepts in Section 2. Then Section 3 contains the actual proof of Main Theorem 1.2.

2. The tools

2.1. Quasi-conformal surgery. *Quasi-conformal surgery* is a powerful technique that derives from an analytical study of quasi-conformal maps and has many applications in complex dynamics, since it can produce holomorphic maps with some prescribed dynamics. In a first step, known as *topological surgery*, one takes a number of spaces and functions having a certain behaviour locally or in a suitable subspace. Roughly speaking, these can be cut and assembled together so as to construct a map with the dynamics chosen a priori. Now the measurable Riemann mapping theorem (Theorem 2.5) can be applied to that map in order to make it holomorphic via a quasi-conformal conjugation. This second step of the process is usually called *holomorphic smoothing*.

The following are just a few necessary definitions and results pertaining to the surgery process. See for example [D] for a more comprehensive text on quasi-conformal surgery.

Definition 2.1. Let $U \subset \mathbb{C}$ be an open set. A measurable function $\mu\colon U \to \mathbb{C}$ is called a *k-Beltrami coefficient* of U if $\|\mu\|_\infty = k < 1$.

Equivalently, one can associate to every k-Beltrami coefficient μ of U an *almost complex structure* σ, that is, a measurable field of (infinitesimal) ellipses in TU, defined up to multiplication by a positive real constant. More precisely, the argument of the minor axis of such an ellipse at a point $z \in U$ is $\arg(\mu(z))/2$, and its eccentricity—i.e. the ratio between its axes—equals $(1 - |\mu(z)|)/(1 + |\mu(z)|)$. Notice that this value is bounded between $(1 - \|\mu\|_\infty)/(1 + \|\mu\|_\infty) > 0$ and 1 almost everywhere.

Definition 2.2. Let U and V be open sets in $\mathbb{C}$; a homeomorphism $\phi\colon U \to V$ is said to be *k-quasi-conformal* if it has locally square integrable weak or distributional derivatives (and therefore it is differentiable almost everywhere) and the function

$$\mu_\phi(z) := \frac{\partial\phi/\partial\bar z}{\partial\phi/\partial z}(z),$$

defined almost everywhere in U, is a k-Beltrami coefficient. A *k-quasi-regular map* is the composition of a holomorphic function and a quasi-conformal map, or, equivalently, a map which is locally quasi-conformal around every point except for a discrete set of points—the critical points.

Definition 2.3. Let U and V be open sets in $\mathbb{C}$; a quasi-regular map $\phi\colon U \to V$ induces a contravariant functor $\phi^*\colon L^\infty(V) \to L^\infty(U)$ defined by

$$\phi^*\mu := \frac{\partial\phi/\partial\bar{z} + (\mu\circ\phi)\overline{(\partial\phi/\partial z)}}{\partial\phi/\partial z + (\mu\circ\phi)\overline{(\partial\phi/\partial\bar{z})}}.$$

Notice that if $\mu\colon V \to \mathbb{C}$ is a Beltrami coefficient, then so is its pull-back $\phi^*\mu\colon U \to \mathbb{C}$. Moreover, if ϕ is a holomorphic map, then $\|\phi^*\mu\|_\infty = \|\mu\|_\infty$.

When the Beltrami coefficient μ is defined in terms of a quasi-regular map ψ as above ($\mu \equiv \mu_\psi$), one can check that $\phi^*\mu_\psi = \mu_{\psi\circ\phi}$.

Definition 2.4. We call the constant Beltrami coefficient $\mu_0 := 0$, or equivalently, the associated field of circles σ_0, the *standard complex structure.*

By Weyl's Lemma, a quasi-regular map ϕ is holomorphic if, and only if, $\phi^*\mu_0 = \mu_0$.

Now, it is clear that a quasi-conformal map ϕ defines a Beltrami coefficient μ_ϕ. Conversely, given a Beltrami coefficient μ and the *Beltrami equation*

$$\frac{\partial\phi}{\partial\bar{z}} = \mu\cdot\frac{\partial\phi}{\partial z},$$

can we find an actual quasi-conformal map ϕ such that $\mu_\phi \equiv \mu$ (equivalently, $\phi^*\mu_0 = \mu$)? The celebrated measurable Riemann mapping theorem answers this question positively; the following is a version of the statement with $U = V = \mathbb{C}$ (see also [AB] or [D]).

Theorem 2.5 (Morrey, Bojarski, Ahlfors, Bers). *Let μ be a Beltrami coefficient of $\mathbb{C}$. Then there exists a unique (up to postcomposition with conformal maps of $\mathbb{C}$) quasi-conformal map $\phi\colon \mathbb{C} \to \mathbb{C}$ such that $\partial\phi/\partial\bar{z} = \mu\cdot\partial\phi/\partial z$ (or $\mu_\phi = \mu$).*

This result can be applied to complex dynamics as follows. Suppose that $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasi-regular map whose dynamics we would like to see realised by a holomorphic map of $\widehat{\mathbb{C}}$. Then Theorem 2.5 guarantees the existence of such a map as long as we can construct an appropriate f-invariant almost complex structure. The precise statement reads as follows.

Corollary 2.6. *Let μ be a Beltrami coefficient of $\mathbb{C}$ and $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ a quasi-regular map such that $f^*\mu = \mu$. Then f is quasi-conformally conjugate to a holomorphic map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.*

Proof. Applying the measurable Riemann mapping theorem to μ, there exists a quasi-conformal map ϕ with $\mu = \phi^*\mu_0$. Now, let us define $g := \phi\circ f\circ\phi^{-1}$; we just need to see that g is indeed holomorphic:

$$g^*\mu_0 = (\phi f \phi^{-1})^*\mu_0 = (\phi^{-1})^* f^* \phi^*\mu_0 = (\phi^{-1})^* f^*\mu = (\phi^{-1})^*\mu = \mu_0. \ \blacksquare$$

2.2. Rational-like maps and weakly repelling fixed points. As mentioned in the introduction, a theorem of Fatou provides the existence of a weakly repelling fixed point for every rational map of degree $d \geq 2$. This fact turns out to be true also for rational-like maps, as defined by Buff [B].

DEFINITION 2.7. A *rational-like map* is a proper holomorphic map $f\colon U \to V$ of degree $d \geq 2$, where U and V are connected open subsets of $\widehat{\mathbb{C}}$ with finite Euler characteristic and $\overline{U} \subset V$.

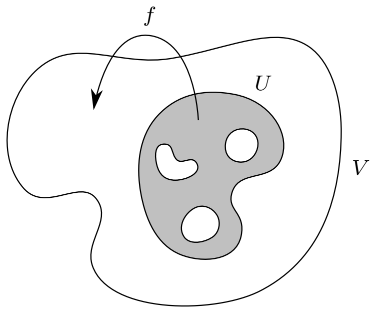

Fig. 1. A rational-like map in the special case of V simply connected

THEOREM 2.8 (Buff). *If $f\colon U \to V$ is a rational-like map, then it has at least one weakly repelling fixed point.*

REMARK 2.9. In fact, Buff's theorem is stronger, since it provides the existence of a *virtually repelling fixed point*, which is a fixed point such that $\mathrm{Re}(\iota(f,z)) < m/2$, where $m \geq 1$ is the multiplicity of the fixed point and $\iota(f,z)$ denotes the holomorphic index of f at z. Virtually repelling fixed points are in particular weakly repelling.

In this paper, we deal with transcendental meromorphic functions. Recall that a map $f\colon X \to Y$ is *proper* if the preimage set $f^{-1}(K) \subset X$ is compact for any compact set $K \subset Y$. Although our transcendental meromorphic maps are of infinite degree and therefore they fail to be proper globally, they still may be proper—and even rational-like—when restricted to appropriately chosen domains.

LEMMA 2.10. *Let f be a transcendental meromorphic function, $Y \subset \mathbb{C}$ a connected open set and X a bounded connected component of $f^{-1}(Y)$. Then the restriction $f|_X\colon X \to Y$ is a proper map. If moreover Y is simply connected and $\overline{X} \subset Y$, then $f|_X\colon X \to Y$ is rational-like.*

Proof. Let K be a compact set of Y; it follows that $\infty \notin K$. Also, $f^{-1}(K) \subset X$ is bounded, so f is locally $(f|_{f^{-1}(K)}\colon f^{-1}(K) \to K)$ holomorphic, and therefore $f^{-1}(K)$ is also compact. Therefore, $f|_X$ is proper and hence of finite degree.

If moreover Y is simply connected, then the Euler characteristic of X must be finite, since the boundary of Y has a finite number of preimages. As X is relatively compact in Y, $f|_X : X \to Y$ is rational-like, as claimed. ■

3. Multiply connected parabolic basin. In this section we prove Main Theorem 1.2 (see Section 1). Its proof involves two different techniques. The first one is based upon Shishikura's proof and applies when preimages of certain sets do not behave too wildly in the presence of an essential singularity. For the second one, the assumption that f has a pole allows us to construct some sets where the conditions of Theorem 2.8 are met.

Recall that by *p-periodic parabolic basin* B we understand a connected component of the Fatou set such that there exists a q-periodic point $\alpha \in \partial B$, $q \mid p$, with $\lim_{n\to\infty} f^{nq}(z) = \alpha$ for all $z \in B$ and, in particular, $(f^p)'(\alpha) = 1$ (i.e., the immediate basin associated to a single *petal* attached to a q-periodic parabolic point). Notice that p is the period of B, not of α, so $B, f(B), \ldots, f^{p-1}(B)$ are pairwise disjoint. Also, p/q gives the number of petals sharing α as a boundary point.

The rest of this section is the proof of Main Theorem 1.2.

First notice that if $p = 1$ (and so $q = 1$) then there exists a fixed point $\alpha \in \partial B$ such that $f'(\alpha) = 1$, i.e., there exists a weakly repelling fixed point of f and we are done. So assume from now on that $p > 1$.

Let $\langle\alpha\rangle$ be the cycle of points generated by the iteration of the q-periodic parabolic point α. We want to construct a sequence $\{U_k\}$ of open sets, starting with a simply connected one, such that $\langle\alpha\rangle \cap \partial U_k \neq \emptyset$ and $f(U_{k+1}) = U_k$ for all $k \geq 0$.

In the following we use the so-called *Fatou coordinates* (see e.g. [Mi]). Without loss of generality we can assume that $\alpha = 0$ by a coordinate change, and f^p to be in normal form $f^p(z) = z(1 + az^\nu + O(z^{\nu+1}))$ for some $a \in \mathbb{C}$ and $\nu = p/q$. Let $U_0 \subset B$ be the pull-back $U_0 := H^{-1}(\{w : \operatorname{Re} w > L\})$, where $H(z) := -1/\nu a z^\nu$ and $L > 0$ is large and to be specified later. It is easy to check that H is an actual conjugacy between f^p and

$$T(w) := (H \circ f^p \circ H^{-1})(w) = w + 1 + O(w^{-1/\nu}),$$

hence we can choose L large enough so that f^p is injective on U_0 (see Figure 2). Also, notice that $f^p(\overline{U_0}) \subset U_0 \cup \{\alpha\}$ because of the action of T.

Now define $\{U_k\}_{k\in\mathbb{N}}$ by pulling back U_0 under f, namely, U_k is the connected component of $f^{-k}(U_0)$ such that $\partial U_k \cap \langle\alpha\rangle \neq \emptyset$. Notice that $U_j \subset U_{p+j} \subset U_{2p+j} \subset \cdots$ and $f^j(B) = \bigcup_{k\geq 1} U_{kp-j}$ for all $0 \leq j < p$. Because B is multiply connected, there exists a (minimal) $n_0 \in \mathbb{N}$ such that $U := U_{n_0}$ is also multiply connected. Denote by E one of the bounded connected components of $\widehat{\mathbb{C}} \setminus U$. Notice that E is compact and full, and $\mathring{E}$ need not be connected.

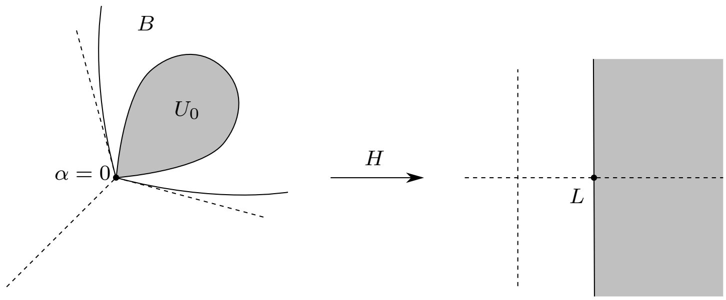

Fig. 2. Construction of U_0 as an H-pull-back of the half-plane $\{w : \mathrm{Re}\, w > L\}$ (example with $\nu = 3$, so $p = 3q$). Notice that $\overline{U_0}$ contains no critical points, since $f^p|_{U_0}$ is injective. Furthermore, we can choose L in such a way that $\partial U_0 \setminus \{\alpha\}$ does not meet the *postcritical set* (forward orbits of the critical points).

Now preimages of compact sets under transcendental meromorphic maps might become unbounded and possibly contain poles and prepoles. This fact will be an obstacle to follow Shishikura's proof of the rational case, as we will show later; so, at this point, we split the proof according to the nature of ∂E.

Case 1: *∂E contains at least one pole.* If ∂E contains a pole P, then, since $\partial E \subset \partial U$, $f(U)$ is unbounded. Because $p > 1$, $U \cap f(U) = \emptyset$ and so $f(U)$ is contained in some unbounded connected component of $\widehat{\mathbb{C}} \setminus U$. Let $V \subset \widehat{\mathbb{C}}$ be a connected, simply connected unbounded open set such that $U \subset V$ but $\underline{f(U)} \subset \widehat{\mathbb{C}} \setminus \overline{V}$. For example, V could be the connected component of $\widehat{\mathbb{C}} \setminus \overline{f(U)}$ containing E. In this case we have $E \subset V$ because V is simply connected and E is bounded. Now since E is unbounded, there exists a connected component $\widetilde{U}$ of $f^{-1}(V)$ such that $P \in \partial\widetilde{U}$. Moreover, by definition we must have $\widetilde{U} \subset E$ because points immediately outside E are in U, and $f(U) \subset \widehat{\mathbb{C}} \setminus V$ (see Figure 3). Now, by construction of the two sets, V is connected and simply connected, and $\widetilde{U}$ is bounded and relatively compact in V, since $\widetilde{U} \subset \overline{E} \subset V$. Using Lemma 2.10 and Theorem 2.8, it follows that $f|_{\widetilde{U}} : \widetilde{U} \to V$ is indeed a rational-like map and so f has a weakly repelling fixed point.

Case 2: *∂E contains no poles.* Now $f(U)$ is bounded (and simply connected by construction), therefore no other component of $\widehat{\mathbb{C}} \setminus U$ can have poles on its boundary. (Still, further images of U might be unbounded, for example, if ∂E contains prepoles.) Let us assume, without loss of generality, that $f(U), \ldots, f^{k-1}(U) \subset E$ and $f^k(U) \subset \widehat{\mathbb{C}} \setminus E$ for some $1 \le k \le p$. In that case we will use the quasi-conformal surgery technique, but must be careful with the set of preimages of α, which might intersect ∂U and make the whole process somewhat laborious.

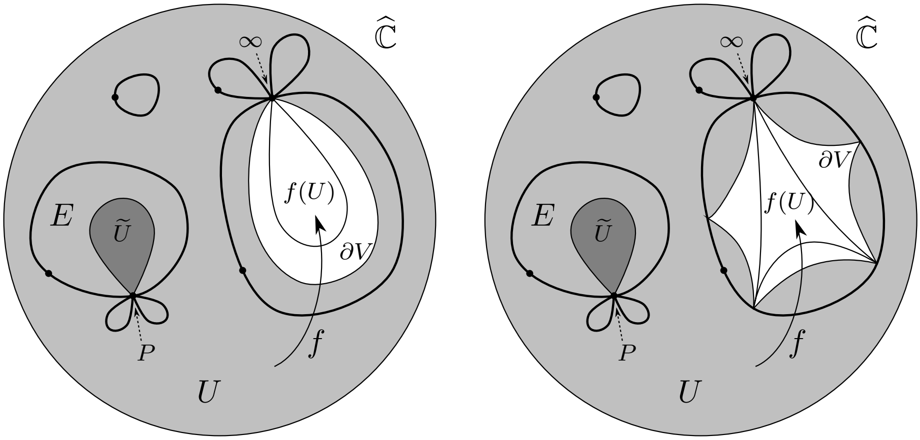

Fig. 3. If there exists a pole P on ∂E, then there exists a set $\widetilde{U} \subset E$ such that $f(\widetilde{U}) = V$, where V is an unbounded simply connected set that contains U but not $f(U)$. The thick lines correspond to ∂U, while the sets $\widetilde{U}$ and V appear dark- and light-shaded, respectively. The non-labelled points represent the different places where α can lie. On the right, a case where $\partial U \cap \partial f(U) \neq \emptyset$.

In fact, a key point during the surgery process is the construction of an interpolating map between two different functions on two disjoint closed curves. If such curves are to touch at preimages of α or at α itself, this interpolation cannot be performed and an extra step previous to surgery will be done. Since we are focusing on boundary intersections, we subdivide this case into two finer subcases as follows.

Case 2.1: *$k < p$, or $k = p$ but $\partial f^p(U) \cap \partial E = \emptyset$.* First notice that if $k < p$ then $\partial f^k(U) \cap \partial E = \emptyset$: By construction, $f^p(U)$ is the first iterate to come back inside U, so $f^k(U)$ is in some complementary component of U. The iterates $f^j(U)$, $j < k$, all lie inside E, but $f^k(U)$ is not in E so it is in a different component of $\widehat{\mathbb{C}} \setminus U$. Since two different components of $\widehat{\mathbb{C}} \setminus U$ cannot form a connected set we conclude that $\partial f^k(U) \cap \partial E = \emptyset$.

Now we apply quasi-conformal surgery as follows: Define a quasi-regular map $f_2\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that, after k iterations, maps $\widehat{\mathbb{C}} \setminus E$ strictly inside itself. More precisely, set $V_0 := \widehat{\mathbb{C}} \setminus E$ and $V_1 := f(U)$. Then, when $k > 1$, V_1 lies in E, and when $k = 1$, it lies in $\widehat{\mathbb{C}} \setminus E$. Set also $K := \overline{f^k(U)}$ and choose $b \in f(U)$ and $a = f^{k-1}(b) \in K$ (see Figure 4).

Lemma 3.1 (Interpolation Lemma). *Let V_0 and V_1 be simply connected open sets in $\widehat{\mathbb{C}}$ with $V_0 \neq \widehat{\mathbb{C}}$, and f a holomorphic map from a neighbourhood N of ∂V_0 to $\widehat{\mathbb{C}}$ such that $f(\partial V_0) = \partial V_1$ and $f(V_0 \cap N) \subset V_1$; choose a compact set K in V_0 and two points $a \in V_0$ and $b \in V_1$. Then there exists a quasi-regular mapping $f_1\colon V_0 \to V_1$ such that*

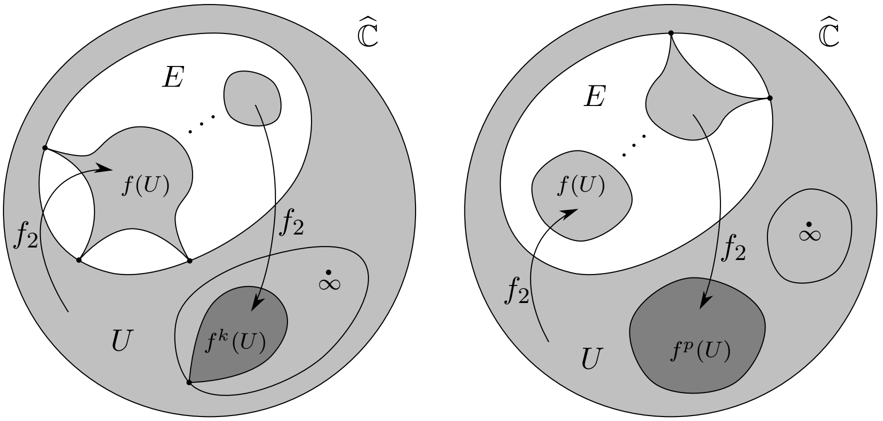

Fig. 4. In this case, $\partial f^k(U)$ and ∂E never intersect, which is crucial for Lemma 3.1 to be applied in our case. We have drawn the cases $1 < k < p$ (left) and $k = p$ (right). In both, the cycle $\widehat{\mathbb{C}} \setminus E, f(U), \ldots, f^k(U) \subset \widehat{\mathbb{C}} \setminus E$ appears in grey.

- $f_1 = f$ *in* $V_0 \cap N_1$, *where* N_1 *is a neighbourhood of* ∂V_0 *with* $N_1 \subset N$*;*
- f_1 *is holomorphic in a simply connected neighbourhood of* K*;*
- $f_1(a) = b$.

This is a standard result and the details of its proof can be found in [FJT] or [Sh]. Applied to our case, it provides us with a quasi-regular map $f_1\colon \widehat{\mathbb{C}} \setminus E \to f(U)$ which agrees with f on ∂E, is holomorphic in a neighbourhood of $f^k(U)$ and satisfies $f_1(a) = b$.

Now we construct a map f_2 by setting $f_2 = f$ on E and $f_2 = f_1$ on $\widehat{\mathbb{C}} \setminus E$, which makes it a quasi-regular map on $\widehat{\mathbb{C}}$, holomorphic in both a neighbourhood of E and a neighbourhood of $\overline{f^k(U)}$, with a k-periodic point, given that

$$f_2^k(a) = f^{k-1}(f_1(a)) = f^{k-1}(b) = a.$$

Observe also that $f_2^k(\widehat{\mathbb{C}} \setminus E) = f^k(U)$ and $\overline{f^k(U)} \subset \widehat{\mathbb{C}} \setminus E$; it follows that f_2^k is a contraction and a a global attractor for f_2^k in $\widehat{\mathbb{C}} \setminus E$.

We may define an almost complex structure σ by

$$\sigma := \begin{cases} \sigma_0 & \text{on } f(U), \\ (f_2^n)^*\sigma_0 & \text{on } f_2^{-n}(f(U)), \text{ for } n \in \mathbb{N}, \\ \sigma_0 & \text{elsewhere.} \end{cases}$$

Observe that $\sigma = \sigma_0$ on $\bigcup_{i=1}^k f^i(U)$ (see Figure 5).

Furthermore, σ is f_2-invariant by construction and has bounded distortion, since orbits pass through $\widehat{\mathbb{C}} \setminus (E \cup \overline{f^k(U)})$ (the set where f_2 is not holomorphic) at most once.

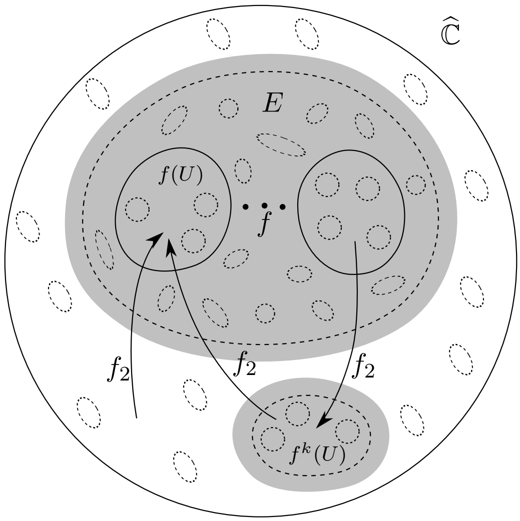

Fig. 5. Construction of the almost complex structure σ. In the grey region, f_2 is holomorphic. Orbits pass through $\widehat{\mathbb{C}} \setminus (E \cup \overline{f^k(U)})$ at most once.

Remark 3.2. At this point, notice the importance of the fact that f_2 be holomorphic on a neighbourhood of $\overline{f^k(U)}$, which was only possible because $f^k(U)$ is a relatively compact subset of $\widehat{\mathbb{C}} \setminus E$.

These are precisely the hypotheses of Corollary 2.6, so there exists a map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, holomorphic on the whole sphere—and hence rational—which is conjugate to f_2 by some quasi-conformal homeomorphism ϕ.

Now a theorem of Fatou ensures the existence of a weakly repelling fixed point z_0 of g, except when $\deg g = 1$ and g is an elliptic transformation. But $\phi(a)$ is an attracting k-periodic point of g, so this can never be the case.

Moreover, the family $\mathcal{G} = \{g^n|_{\phi(\widehat{\mathbb{C}} \setminus E)}\}_{n \geq 1}$ omits the open set $\phi(\widehat{\mathbb{C}} \setminus (E \cup \overline{f^k(U)}))$, therefore $\mathcal{G}$ is normal in $\phi(\widehat{\mathbb{C}} \setminus E)$ by Montel's Theorem, that is, $\phi(\widehat{\mathbb{C}} \setminus E) \subset \mathcal{F}(g)$. But weakly repelling fixed points belong to the Julia set, so $z_0 \in \phi(E)$. Because such points are preserved under conjugacy, also f_2 has a weakly repelling fixed point $\phi^{-1}(z_0)$ in E; and so does f, since both functions coincide precisely on this set.

Case 2.2: *$k = p$ and $\partial f^p(U) \cap \partial E \neq \emptyset$.* First rename the elements of the periodic orbit and shift the sequence $\{U_k\}$ so that $\alpha \in \partial U \equiv \partial U_{n_0}$, i.e., so that $p \mid n_0$. More precisely, it is clear that there exists $0 \leq l < p$ such that $U \subset f^l(B)$; then rename $B \equiv f^l(B)$, $\alpha \equiv f^l(\alpha)$, $U_0 \equiv f^l(U_0)$ and define $U_1, \ldots, U_{l-1}$ accordingly. Notice that $U_0, \ldots, U_l$ are all simply connected by construction, but $U_{n_0} \equiv U$ is multiply connected. Since p divides n_0, we can define $c := n_0/p \in \mathbb{N}$, which is the number of f^p-cycles from U_0 to U_{n_0} (see Figure 6).

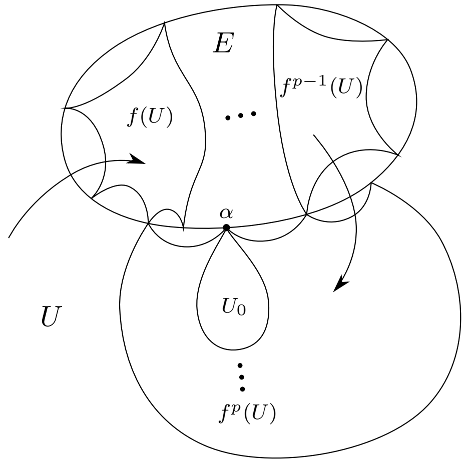

Fig. 6. The shifted sequence $\{U_k\}$. From now on, this is the primary situation we should always bear in mind. The sets $U, f(U), \ldots, f^p(U)$ are the only ones in $\{U_k\}$ that will later play a role during the quasi-conformal surgery process. Their cyclic dynamics under the action of f is also shown here.

Also, the sets $U_{kp+1}, \ldots, U_{(k+1)p-1} \subset E$ are necessarily bounded, so only those in the subsequence $U_0, U_p, U_{2p}, \ldots$ might become unbounded from a certain one on. In particular, only the sets of the form U_{kp+1} can have poles on their boundaries, and only the maps of the form $f|_{U_{kp}}: U_{kp} \to U_{kp-1}$ can be of infinite degree.

Furthermore, notice that if some intersection $\partial U_{k_1} \cap \partial U_{k_2}$ contains a preimage of some pole, then the sets U_{k_1} and U_{k_2} necessarily belong to the same subsequence $U_j \subset U_{p+j} \subset U_{2p+j} \subset \cdots$, that is, $k_1 \equiv k_2 \pmod p$ and either $U_{k_1} \subset U_{k_2}$ or $U_{k_2} \subset U_{k_1}$. In particular, only if this is the case can ∂U_{k_1} and ∂U_{k_2} share infinitely many preimages of α. This will be a key point in later arguments.

We have seen that the fact that $\partial f^p(U)$ and ∂E did not share any contact point was crucial for the quasi-conformal surgery construction of Case 2.1 (see Remark 3.2). Now, the condition $\partial f^p(U) \cap \partial E \neq \emptyset$ is exactly given by hypotheses, so some extra work must be done, in the sense of modifying slightly some sets, in order to start the surgical process proper.

On the other hand, notice that the sets $\{U_k\}$ are in some sense arbitrary, since they were constructed by repeatedly pulling back U_0, chosen arbitrarily. Also, notice that once these sets (and E) have been defined and during the process of quasi-conformal surgery (that is, from the construction of the auxiliary quasi-regular maps on), the only sets in this sequence with a role to play are $U, f(U), \ldots, f^k(U)$ (or rather $U, f(U), \ldots, f^p(U)$ for the current case).

Thus, it seems that we can modify these sets $U, f(U), \ldots, f^p(U)$ slightly and only close to the odd contact points, so that their boundaries share as few points as possible—the following result provides us with such a modification. Its proof is rather technical and will be given separately, at the end of this section.

PROPOSITION 3.3. *In the situation described hitherto, there exists a connected, multiply connected set* $\mathcal{U} \subset U$ *such that* $f^p(\mathcal{U})$ *is simply connected,* $\overline{f^p(\mathcal{U})} \subset \mathcal{U} \cup \{\alpha\}$ *and* $\partial f^p(\mathcal{U}) \cap \mathcal{J}(f) = \{\alpha\}$.

Now let $\mathcal{E}$ be the bounded component of $\widehat{\mathbb{C}} \setminus \mathcal{U}$ that contains E. The point α need not be on $\partial\mathcal{E}$, so it could happen that $\partial f^p(\mathcal{U}) \cap \partial\mathcal{E} = \emptyset$. Were that the case, notice that $\overline{f^p(\mathcal{U})} \subset \widehat{\mathbb{C}} \setminus \mathcal{E}$ and therefore we could just repeat the surgery process of Case 2.1—replacing U and E by their respective modifications—to find a weakly repelling fixed point of f.

Otherwise, we have $\partial f^p(\mathcal{U}) \cap \partial\mathcal{E} = \{\alpha\}$ and, as there seems to be no neat way to separate $\mathcal{E}$ from α, we will just work with a small extension of $\mathcal{E}$ whose interior contains α. More precisely, we first define $V_0 := \widehat{\mathbb{C}} \setminus \mathcal{E}$ and $V_1 := f(\mathcal{U}) \subset E \subset \mathcal{E}$, and use the Interpolation Lemma 3.1 to find a quasi-regular map $f_1 \colon \widehat{\mathbb{C}} \setminus \mathcal{E} \to f(\mathcal{U})$ as usual—however, notice that we have chosen no compact set K nor points a and b, since now f_1 need not be holomorphic in any subset of $\widehat{\mathbb{C}} \setminus \mathcal{E}$. Also, recall that f_1 actually agrees with f in a neighbourhood N_1 of $\partial\mathcal{E}$; let $\mathcal{N} := \mathcal{E} \cup N_1$, a neighbourhood of $\mathcal{E}$.

LEMMA 3.4. *There exist a sufficiently small neighbourhood* $\mathcal{W}^*$ *of* α *in* $f^p(\mathcal{U})$, *an open neighbourhood* $\mathcal{E}^*$ *of* $\mathcal{E} \cup \overline{f^p(\mathcal{W}^*)}$ *in* $\mathcal{N}$, *and a quasi-conformal map* $h \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *such that*

- $\overline{\mathcal{W}^*} \subset \mathcal{N}$;
- $\overline{f^p(\mathcal{W}^*)} \subset \mathcal{E}^*$ *and* $\overline{\mathcal{E}^* \cap \mathcal{W}^*} \setminus \partial f^p(\mathcal{U}) \subset \mathcal{W}^*$;
- $h = \mathrm{id}$ *in* $\mathcal{E}^*$ *and* $h(f^p(\mathcal{U})) \subset \mathcal{W}^*$.

Roughly speaking, the map h pushes the points in $f^p(\mathcal{U})$ towards $\mathcal{E}$, but will leave points in $\mathcal{E}$ untouched so that the action of any postcomposed map be preserved entirely (see Figure 7).

Proof. We define the set $\mathcal{W}^*$ as the connected component of $f^{-(c-1)p}(W_R)$ in $f^p(\mathcal{U})$ that has α on the boundary, with R so large that $\overline{\mathcal{W}^*} \subset \mathcal{N}$ (see the construction of $\mathcal{W}$ in Subsection 3.1 below). By construction, it is a neighbourhood of α in $f^p(\mathcal{U})$, i.e., $\alpha \notin \overline{f^p(\mathcal{U}) \setminus \mathcal{W}^*}$, and $f^p(\overline{\mathcal{W}^*}) \subset \mathcal{W}^* \cup \{\alpha\}$. In particular, the existence of one such $\mathcal{E}^*$ follows from the last inclusion.

Now let S be the simply connected open set $f^p(\mathcal{U}) \setminus \overline{\mathcal{E}^*}$ with a marked boundary segment at $l := \partial S \cap \partial\mathcal{E}^*$. There exists a (conformal) Riemann map $\varphi \colon S \to Q$ that sends l to one of the sides of the open unit square Q. Consider a (quasi-conformal) homothetic transformation $\tilde{h}_0 \colon \overline{Q} \to \tilde{h}_0(\overline{Q})$ such that $\tilde{h}_0|_{\varphi(l)} = \mathrm{id}$ and $\tilde{h}_0(Q) \cap \varphi(S \cap \partial\mathcal{W}^*) = \emptyset$.

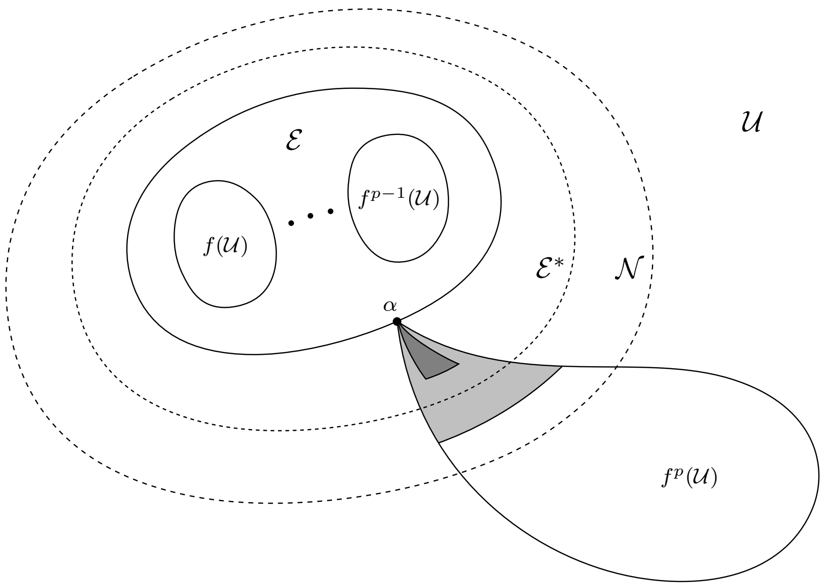

Fig. 7. The case where $\partial f^p(\mathcal{U}) \cap \partial\mathcal{E} = \{\alpha\}$, with the sets $\mathcal{N}$, $\mathcal{W}^*$ (light-shaded), $f^p(\mathcal{W}^*)$ (dark-shaded) and $\mathcal{E}^*$. Notice that points in $f^p(\mathcal{W}^*)$ will never leave $\mathcal{E}^*$ under the action of f.

Finally, define the conjugate map

$$h_0 := \varphi^{-1} \circ \tilde{h}_0 \circ \varphi\colon S \to h_0(S),$$

which is quasi-conformal (see Figure 8). Notice that $h_0|_l = \mathrm{id}$, so we can define

$$h := \begin{cases} h_0 & \text{on } S = f^p(\mathcal{U}) \setminus \overline{\mathcal{E}^*}, \\ \mathrm{id} & \text{on } \overline{\mathcal{E}^*}, \end{cases}$$

and extend it quasi-conformally to a map $h\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. ■

Now consider the quasi-regular map $f_2\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined as

$$f_2 := \begin{cases} f \circ h & \text{on } \mathcal{E}^* \\ f_1 \circ h & \text{on } \widehat{\mathbb{C}} \setminus \mathcal{E}^* \end{cases} = \begin{cases} f & \text{on } \mathcal{E}^*, \\ f_1 \circ h & \text{on } \widehat{\mathbb{C}} \setminus \mathcal{E}^*. \end{cases}$$

Also, consider the (shrinking) f_2-cycle

$$C := f(\mathcal{W}^*) \cup \cdots \cup f^p(\mathcal{W}^*) \subset \mathcal{E}^*.$$

Indeed, it is cyclic because $f_2(C) = f(C) \subset C$ (see Figure 9).

Set $X := \widehat{\mathbb{C}} \setminus \mathcal{E}^*$. Then orbits of f_2 pass through X at most twice, since

$$\cdots \xrightarrow{f_2} f_2^{-1}(X) \xrightarrow{f_2} X \xrightarrow{h} X \subset \widehat{\mathbb{C}} \setminus \mathcal{E} \xrightarrow{f_1} f(\mathcal{U}) \xrightarrow{f_2^{p-1}} f^p(\mathcal{U})$$
$$\xrightarrow{h} \mathcal{W}^* \xrightarrow{f} f(\mathcal{W}^*) \subset C \xrightarrow{f_2} C \xrightarrow{f_2} \cdots \subset \widehat{\mathbb{C}} \setminus X.$$

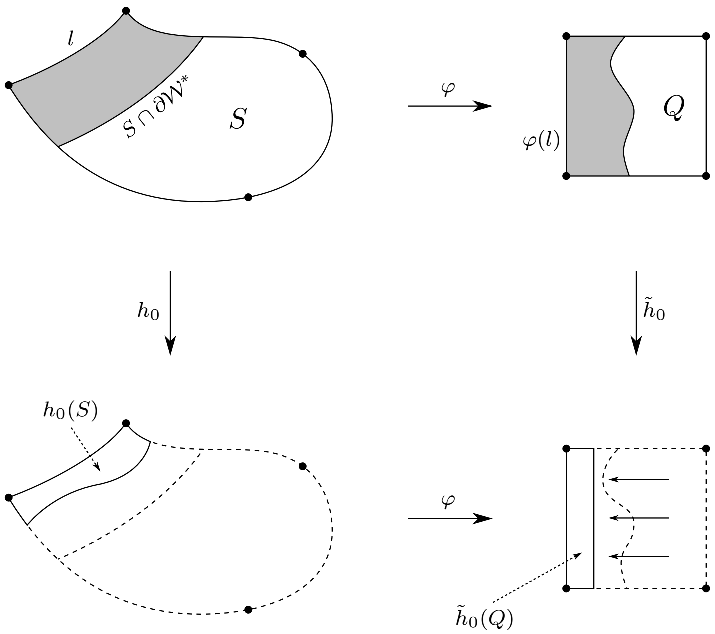

Fig. 8. For the construction of the map $h\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, we first define an auxiliary map $h_0\colon S \to h_0(S)$ as a conjugation of a quasi-conformal map on Q, where it is easy to define the desired local dynamics. In grey we find $S \cap \mathcal{W}^*$ (and its φ-image), the subset where we want $h_0(S)$ to end up.

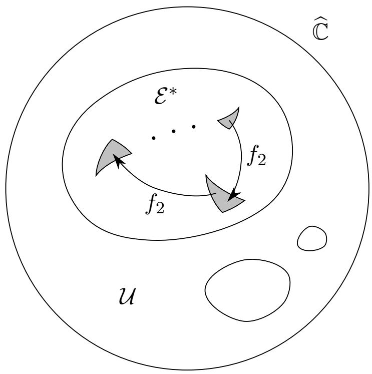

Fig. 9. The action of f_2 on the cycle C, shaded. Notice that $f^p(\mathcal{W}^*) \subset \mathcal{W}^*$, so its f_2-image falls again in $f(\mathcal{W}^*)$.

Define the almost complex structure

$$\sigma := \begin{cases} \sigma_0 & \text{on } C, \\ (f_2^n)^*\sigma & \text{on } f_2^{-n}(f(\mathcal{W}^*)),\ n \in \mathbb{N}, \\ \sigma_0 & \text{elsewhere,} \end{cases}$$

which is clearly f_2-invariant by definition, and has bounded dilatation since f_2 fails to be holomorphic only at most twice. Therefore we can use Corollary 2.6 to find a rational map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ conjugate to f_2 by a quasi-conformal homeomorphism ϕ. Thus,

$$\overline{g^{p+1}(\phi(X))} = \overline{\phi(f_2(f_2^p(X)))} \subset \overline{\phi(f_2(f^p(\mathcal{U})))} \subset \overline{\phi(C)} \subset \phi(C) \cup \langle \phi(\alpha) \rangle,$$

so $\phi(X)$ is contained in the basin of an attracting or parabolic point. By Fatou's theorem, g has a weakly repelling fixed point in $\phi(\widehat{\mathbb{C}} \setminus X) = \phi(\mathcal{E}^*)$, hence f has a weakly repelling fixed point in $\mathcal{E}^*$.

3.1. Proof of Proposition 3.3. When removing points of $\partial f^p(U) \cap \partial E$, there is a particular point we cannot ignore, namely α itself: Because its attracting dynamics in a whole petal contained in the parabolic basin (Fatou coordinates about a parabolic point), if we redefined U as some $\widetilde{U}$ in such a way that α were not on its boundary, then points close to α would become even closer under the action of f^p, and the condition $f^p(\widetilde{U}) \subset \widetilde{U}$ would be lost (see Figure 10).

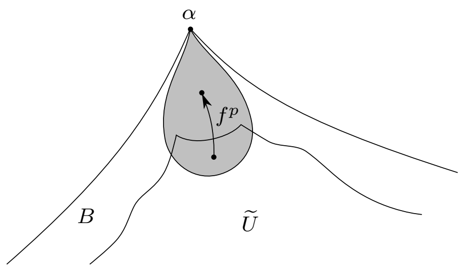

Fig. 10. If the new set $\widetilde{U}$ left out some neighbourhood of α, there would be points in it stepping outside it under f^p. The shaded set represents the attracting petal attached to α given by the Fatou coordinates.

Rather, for the construction of one such $\mathcal{U}$ we need to modify the sets $U, f(U), \ldots, f^p(U)$ close to the contact points between their boundaries except those in the cycle $\langle \alpha \rangle$ (see Figure 11).

When doing so, it is clear that if the point α does not lie in $\partial f^p(\mathcal{U}) \cap \partial E$ (Figure 11, left), the situation is identical to that of Case 2.1, and therefore we can follow an analogous surgical procedure. In case α does belong to $\partial f^p(\mathcal{U}) \cap \partial E$ (Figure 11, right), we must define another auxiliary map before we can proceed. The end of the proof then follows with a different quasi-conformal surgery argument.

Let us now construct the modification of $U, f(U), \ldots, f^p(U)$. The idea is the following: Since the ultimate aim of such a modification is to eliminate contact points between $\partial f^p(U)$ and ∂E, it suffices to modify only the set $U_{n_0-p} \equiv f^p(U)$ and redefine the sets $U_{n_0-p+1} \equiv f^{p-1}(U), \ldots, U_{n_0} \equiv U$ by

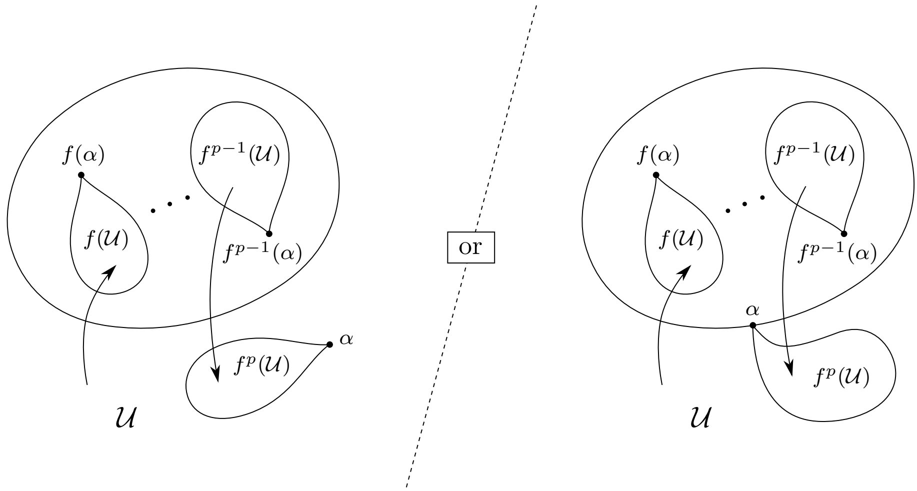

Fig. 11. The situation we want, with the points in $\langle\alpha\rangle$ marked. Notice that a priori we do not know whether α is on ∂E or not, since the set E was chosen arbitrarily as one of the bounded connected components of the complement of U; in particular, α could even happen to be on the boundary of the unbounded component of the complement of $\mathcal{U}$. It is clear that surgery cannot be used just as in Case 2.1 when $\partial f^p(\mathcal{U}) \cap \partial E$ remains nonempty (right-hand side figure).

repeatedly pulling back this first modification appropriately. Of course if the changes on these sets are arbitrarily small, and therefore the new sets are arbitrarily close to the original ones, their respective connectivities are also to be preserved (see Figure 12).

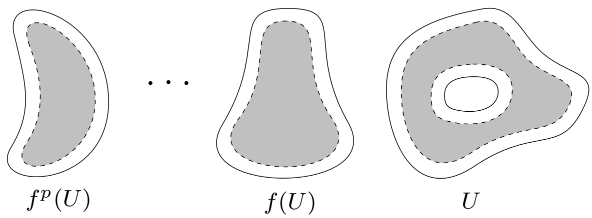

Fig. 12. The set U is multiply connected and so is its modification (shaded here) if it differs little from U. Similarly, the sets $f(U), \ldots, f^p(U)$ are simply connected and so are their modifications.

Following this reasoning, one could think that the modification of $f^p(U)$, which we can call $\mathcal{V}$, could simply be obtained by removing from $f^p(U)$ a disc of arbitrarily small radius centred at every contact point between $\partial f^p(U)$ and ∂E (see Figure 13).

But of course we want to keep the property $f^p(U) \subset U$ for the subsequent surgical work, and if we just removed those discs with no control over their

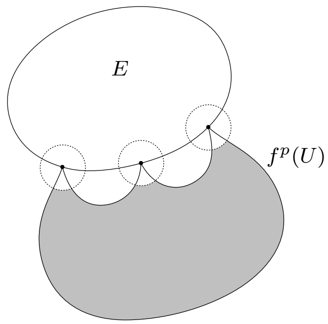

Fig. 13. The shaded set represents a first attempt towards the construction of $\mathcal{V}$.

preimages, the inclusion could be lost: Consider a point $a \in \mathcal{A} := \partial f^p(U) \cap \partial E \setminus \{\alpha\} \subset \mathcal{J}(f)$ (for instance some $a \in O^-(\alpha)$) with some preimage $b \in f^{-p}(a)$ on the same set $\mathcal{A}$. Suppose we were to remove discs $B_\varepsilon(a)$ and $B_\varepsilon(b)$ of small radius ε centred at $a, b \in \mathcal{A}$ when defining $\mathcal{V}$. If the preimage of $B_\varepsilon(a)$ under f^p were to become big enough to contain points in the complement of $B_\varepsilon(b)$, then there would be points $z_0 \in (f^{-p}(B_\varepsilon(a)) \cap f^p(U)) \setminus B_\varepsilon(b)$ such that $f^p(z_0) \in B_\varepsilon(a) \subset \widehat{\mathbb{C}} \setminus \mathcal{V}$, that is, $z_0 \notin f^{-p}(\mathcal{V})$. Then we would have $z_0 \in \mathcal{V} \setminus f^{-p}(\mathcal{V}) \neq \emptyset$, which is precisely what we want to avoid. (See Figure 14.)

This very description of the problem with the preimages of points we remove from $f^p(U)$ for the construction of $\mathcal{V}$ also provides us with a hint on how to solve it, since, in the previous example, it would have been enough to take $f^{-p}(B_\varepsilon(a)) \cap f^p(U)$ instead of $B_\varepsilon(b)$ to avoid points like z_0.

In other words, we must also exclude from $\mathcal{V}$ all points in $f^p(U)$ whose f^p-image falls on points we "already" removed from $f^p(U)$. In fact, this generates, in turn, more points whose preimage need be controlled; and so on. Regardless of what may be expected, this is not an endless recurrent process. We have $f^p(U) \equiv U_{n_0-p} \subset f^{-n_0+p}(U_0)$, and therefore after $n_0 - p$ iterations all the points in $f^p(U)$ happen to be close to α—precisely in U_0. We will see that we can make $U_0 \subset \mathcal{V}$ provided that ε is chosen small enough (see Figure 15). At the same time, we need to be careful when taking all these preimages, since they could become so big as to impede the construction of $\mathcal{V}$.

For all $\varepsilon > 0$, let

$$V_\varepsilon := f^p(U) \setminus \bigcup_{a \in \mathcal{A}} \bigcup_{k=0}^{c-2} f^{-kp}(B_{2^k\varepsilon}(a)).$$

The remaining part of the proof consists in showing that, for ε small enough, the set $\mathcal{V} \equiv V_\varepsilon$ is exactly the one we want.

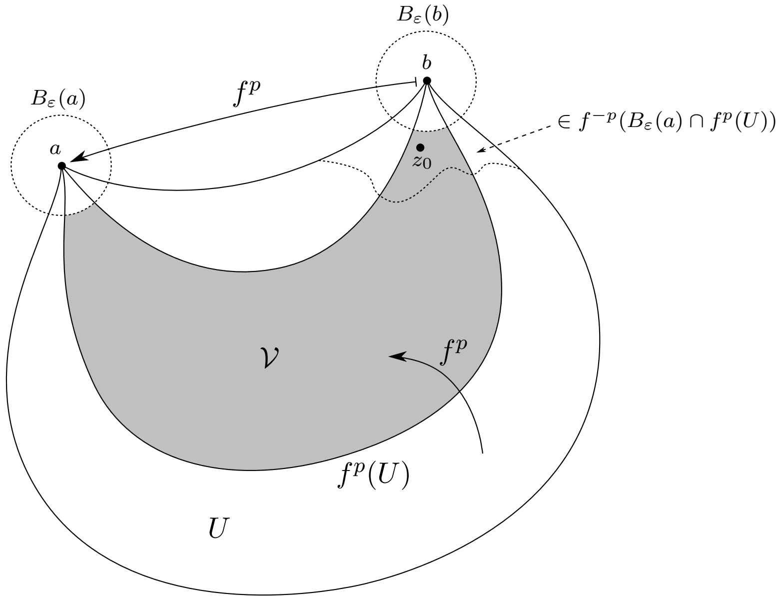

Fig. 14. We want to keep $f^p(U) \subset U$ after the modification, i.e., we want a set $\mathcal{V}$ such that $\mathcal{V} \subset f^{-p}(\mathcal{V})$. However, if we defined it as the shaded set in this figure, there would exist points $z_0 \in \mathcal{V} \setminus f^{-p}(\mathcal{V})$—so we need to have some control over the preimages of the discs we remove from $f^p(U)$.

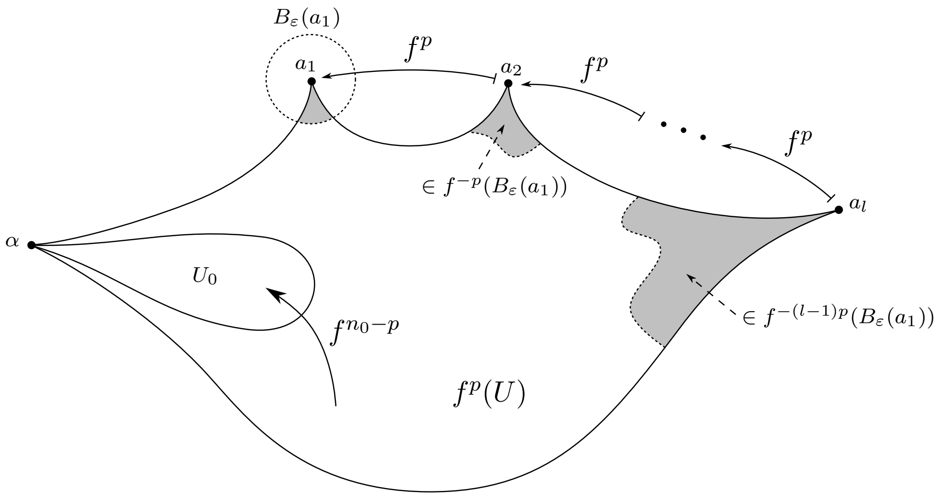

Fig. 15. Because $f^p(U) \equiv U_{n_0-p} = U_{(c-1)p}$ and $f^p(\overline{U_0}) \subset U_0 \cup \{\alpha\}$, $\partial f^p(U)$ cannot contain preimages of higher order. Thus, given a sequence of points $a_l \mapsto \cdots \mapsto a_2 \mapsto a_1$ of $\mathcal{A}$ with $1 \leq l < c$, the points $z_0 \in f^{-(l-1)p}(B_\varepsilon(a_1))$ will eventually fall inside $U_0 \subset \mathcal{V}$ and we need not worry about their preimages any more.

First of all notice that $f^p(V_\varepsilon) \subset V_\varepsilon$ by definition. Now we will show that the preimages $f^{-kp}(B_{2^k\varepsilon}(a))$ can be controlled in such a way that none of them reaches the point α, which would otherwise be excluded from V_ε. The following lemma gives sufficient conditions for this not to happen.

LEMMA 3.5. *In the situation described hitherto, there exists $\varepsilon_0 > 0$ such that $\alpha \in \partial V_\varepsilon$ for all $\varepsilon < \varepsilon_0$.*

Before its proof, we define two sets which, because of their importance, will be used also beyond this result. These sets are both neighbourhoods (in $f^p(U)$) of α and provide useful information about the dynamics of f^p close to this point.

The first set to be constructed, $\mathcal{C}$, is a neighbourhood of α whose boundary contains no points of $\mathcal{A}$. For this, notice that $\mathcal{A}$ consists only of points of $O^-(\alpha)$ and $O^-(\infty)$, since $\mathcal{A} \subset \partial f^p(U) \cap \mathcal{J}(f)$ and, by construction of $\{U_k\}$, we have $f^{(c-1)p}(\partial f^p(U)) = \partial U_0 \subset \mathcal{F}(f) \cup \{\alpha\}$. More precisely,

$$\mathcal{A} \subset \bigcup_{1 \le k < c} (f^{-kp}(\alpha) \cup f^{-(k-1)p}(\infty))$$

or, simply,

$$\mathcal{A} \subset f^{-(c-1)p}(\alpha) \cup \bigcup_{k=0}^{c-2} f^{-kp}(\infty)$$

if we take into account that α is q-periodic and so p-periodic. In particular, the set $\mathcal{A}$ has accumulation points only in $\bigcup_{k=0}^{c-2} f^{-kp}(\infty)$, and the points in $f^{-(c-1)p}(\alpha) \cap \mathcal{A}$ are all isolated in $\mathcal{A}$ (since $f^{-(c-1)p}(\alpha) \cap \bigcup_{k=0}^{c-2} f^{-kp}(\infty) = \emptyset$ because α is a periodic point). In the same way, since α is not an accumulation point of $\mathcal{A}$, there exists a simply connected open sector $\mathcal{C} \subset f^p(U)$ such that $\alpha \in \partial \mathcal{C}$, $\alpha \notin \overline{f^p(U) \setminus \mathcal{C}}$ and $\overline{\mathcal{C}} \cap \mathcal{A} = \emptyset$ (see Figure 16). Actually, we can still shrink it slightly so that $\mathcal{A}$ does not meet a whole (sufficiently small) neighbourhood of $\overline{\mathcal{C}}$—we will use this later, in order to see some technical detail.

On the other hand, we want to construct another neighbourhood of α in $f^p(U)$, to be called $\mathcal{W}$, with dynamics similar to that of U_0 in the sense that $f^p(\overline{\mathcal{W}}) \subset \mathcal{W} \cup \{\alpha\}$; in other words, the set $\mathcal{W}$ will control those points in $f^p(U)$ that happen to be already close to α. Notice that we cannot take U_0 itself as $\mathcal{W}$ because U_0 need not be a neighbourhood of α in $f^p(U)$, that is, $\alpha \in \overline{f^p(U) \setminus U_0}$ in general; but the construction of U_0 does inspire the use of Fatou coordinates in order to provide $\mathcal{W}$ with the same dynamics. More precisely, we will construct a subset of U_0 in a very similar fashion and then define $\mathcal{W}$ as an appropriate preimage of it in $f^p(U)$.

In fact, for all $R > L$, let

$$W_R := H^{-1}(\{w \in \mathbb{C} : \operatorname{Re} w > L,\ \operatorname{Re} w + |\operatorname{Im} w| > R\}) \subset U_0,$$

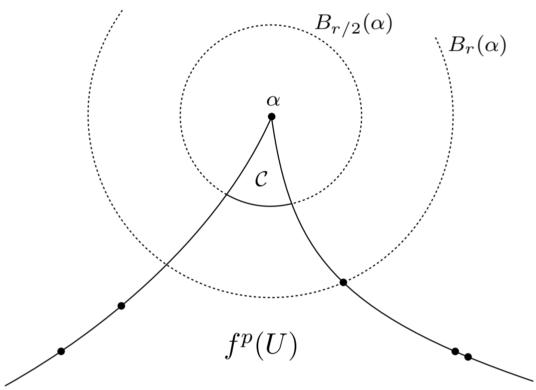

Fig. 16. The non-labelled points represent the set $\mathcal{A}$. Since they never accumulate on α, there certainly exists an open set $\mathcal{C}$ as shown. Furthermore, because α is a parabolic point, in a sufficiently small neighbourhood of it $f^p(U)$ is essentially a wedge like that of an attracting petal, so we can even take $\mathcal{C}$ as $B_r(\alpha) \cap f^p(U)$ with r so small that $\mathcal{C}$ is connected and $\overline{\mathcal{C}} \cap \mathcal{A} = \emptyset$. Even more, taking $\mathcal{C} = B_{r/2}(\alpha) \cap f^p(U)$ we ensure not only its closure but also a whole neighbourhood of $\overline{\mathcal{C}}$ free from points of $\mathcal{A}$.

where recall that $H(z) = -1/\nu a z^\nu$ conjugates the maps f^p and $T(w) = w + 1 + O(w^{-1/\nu})$, and $L > 0$ is large enough for f^p to be injective on U_0 (see Figure 17).

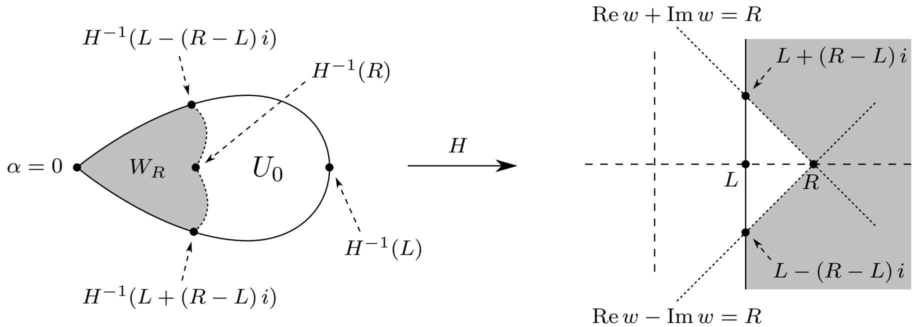

Fig. 17. Using the same Fatou coordinates setting as in the construction of U_0, we can define W_R as a subset of it in such a way that f^p keeps its injectivity also in the subset. By taking R sufficiently large, W_R can be embedded in any (arbitrarily small) neighbourhood of α.

It is clear that since we took L so large that $T(w) \approx w+1$ and $f^p(\overline{U_0}) \subset U_0 \cup \{\alpha\}$, for any $R > L$ also $f^p(\overline{W_R}) \subset W_R \cup \{\alpha\}$. Moreover, W_R is a neighbourhood of α in U_0 (i.e., $\alpha \notin \overline{U_0 \setminus W_R}$), since $H(\alpha) = \infty$ and $H(\overline{U_0 \setminus W_R}) = \{w \in \mathbb{C} : \operatorname{Re} w \geq L, \operatorname{Re} w + |\operatorname{Im} w| \leq R\}$, which is a compact set.

Consider now the connected component of $f^{-(c-1)p}(W_R)$ in $f^p(U)$ that has α on the boundary (or equivalently contains W_R). If R were close to L, then W_R would be close to U_0 and its preimage close to $f^p(U)$, so the character of neighbourhood of α would be lost. Let us show, then, that we can choose a sufficiently large R in such a way that this preimage lies even inside the neighbourhood $\mathcal{C}$ just constructed: Consider the image set $f^{(c-1)p}(\mathcal{C}) \subset U_0$; notice that $\alpha \notin \overline{U_0 \setminus f^{(c-1)p}(\mathcal{C})}$ since, by construction of $\mathcal{C}$, there are no preimages of α on $\overline{\mathcal{C}}$. Therefore, there exists $R_0 > L$ such that $W_R \subset f^{(c-1)p}(\mathcal{C}) \subset U_0$ for any $R > R_0$ (see Figure 18). Define $\mathcal{W}$ as the connected component of $f^{-(c-1)p}(W_R)$ in $f^p(U)$ that has α on the boundary, for $R > R_0$, and thus $\mathcal{W} \subset \mathcal{C}$. It follows that $f^p(\overline{\mathcal{W}}) \subset \mathcal{W} \cup \{\alpha\}$ and $\alpha \notin \overline{f^p(U) \setminus \mathcal{W}}$, since, once again, $\overline{\mathcal{C}} \cap \mathcal{A} = \emptyset$.

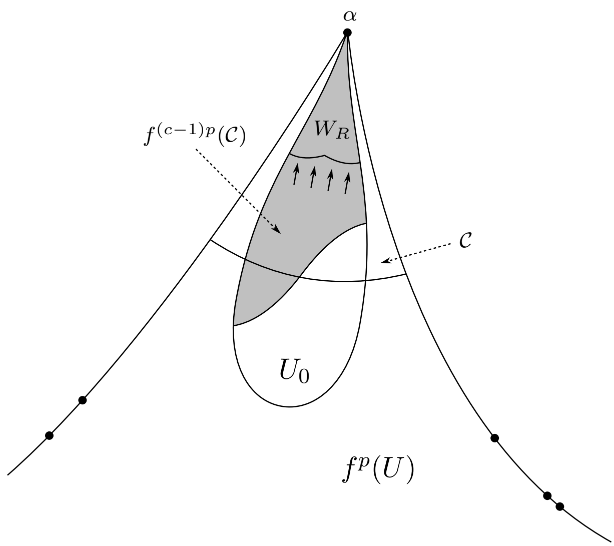

Fig. 18. Since $\alpha \notin \overline{U_0 \setminus f^{(c-1)p}(\mathcal{C})}$, the set W_R can be shrunk arbitrarily until $W_R \subset f^{(c-1)p}(\mathcal{C})$. Notice that $f^{(c-1)p}(\mathcal{C})$, shaded here, need not be contained in $\mathcal{C}$, so $\mathcal{C}$ itself or even an image of it cannot serve as W_R.

This concludes the construction of the sets $\mathcal{C}$ and $\mathcal{W}$, so we are now in a position to prove Lemma 3.5.

Proof of Lemma 3.5. Consider one of the preimages $f^{-kp}(B_{2^k\varepsilon}(a))$ and suppose that $\alpha \in f^{-kp}(B_{2^k\varepsilon}(a))$. Because $\alpha \in \partial\mathcal{W}$, we would then have $f^{-kp}(B_{2^k\varepsilon}(a)) \cap \mathcal{W} \neq \emptyset$; so let $z_0 \in f^{-kp}(B_{2^k\varepsilon}(a)) \cap \mathcal{W}$. Then

$$\begin{array}{ccccccc} \mathcal{C} \supset & \mathcal{W} & \supset & f^p(\mathcal{W}) & \supset \cdots \supset & f^{kp}(\mathcal{W}) \\ & \Cup & & \Cup & & \Cup \\ & z_0 & & f^p(z_0) & & f^{kp}(z_0) \end{array}$$

that is, $f^{kp}(z_0) \in \mathcal{C}$. On the other hand, as $z_0 \in f^{-kp}(B_{2^k\varepsilon}(a)) \cap \mathcal{W}$ it also follows that $f^{kp}(z_0) \in B_{2^k\varepsilon}(a)$; therefore, $f^{kp}(z_0) \in \mathcal{C} \cap B_{2^k\varepsilon}(a)$.

However, since $\mathcal{A}$ does not meet some neighbourhood of $\overline{\mathcal{C}}$, it is clear that there exists $\varepsilon_0 > 0$ such that $\mathcal{C} \cap B_{2^k\varepsilon}(a) = \emptyset$ for any $\varepsilon < \varepsilon_0$ and $a \in \mathcal{A}$. Therefore, it suffices to take $\varepsilon < \varepsilon_0$ to obtain $f^{kp}(z_0) \notin \mathcal{C} \cap B_{2^k\varepsilon}(a) = \emptyset$ and $\alpha \notin f^{-kp}(B_{2^k\varepsilon}(a))$. But α does belong to $\partial f^p(U)$, so, right from the definition of V_ε, we have $\alpha \in \partial V_\varepsilon$ for all $\varepsilon < \varepsilon_0$. ■

Remark 3.6. Notice that the key point of this proof is that the preimages $f^{-kp}(B_{2^k\varepsilon}(a))$ are considered only up to order $k = c - 2$. Of course, if we were to take preimages of $B_{2^k\varepsilon}(a)$ indefinitely, we would surely end up meeting $\mathcal{C}$ because $B_{2^k\varepsilon}(a)$ is a neighbourhood of $a \in \mathcal{A} \subset \mathcal{J}(f)$; but then also preimages of α would accumulate on α itself so the construction of one such $\mathcal{C}$ would never be possible.

The next step towards the construction of $\mathcal{V}$ is to ensure that $\mathcal{U}$ will keep multiple connectivity. This is precisely what the following lemma does.

Lemma 3.7. *In the situation described hitherto, there exists* $\varepsilon_1 > 0$ *such that* $f^{-p}(V_\varepsilon)$ *has a multiply connected component in* U *that separates* E *and the unbounded connected component of* $\widehat{\mathbb{C}} \setminus U$, *for all* $\varepsilon < \varepsilon_1$.

Proof. Since U is multiply connected, let $\gamma \subset U$ be a generator path of its fundamental group (as a topological space) such that E and the unbounded connected component of $\widehat{\mathbb{C}} \setminus U$ sit in different connected components of $\widehat{\mathbb{C}} \setminus \gamma$ (see Figure 19).

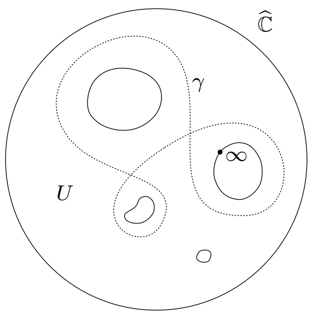

Fig. 19. One such generator path γ, as seen on the Riemann sphere. Notice that it need not separate all the connected components of $\widehat{\mathbb{C}} \setminus U$ pairwise, although it might separate components other than E and the unbounded one.

Consider now the images $\{f^{kp}(\gamma)\}_{1 \le k < c}$ in $f^p(U)$. Because γ does not accumulate on points of $\mathcal{J}(f)$, neither do the curves $f^{kp}(\gamma)$ accumulate on points of $\mathcal{A}$, and therefore there exist $\{\varepsilon_{1,k} > 0\}_{1 \le k < c}$ such that, for each $1 \le k < c$, $f^{kp}(\gamma) \cap B_{2^k\varepsilon}(a) = \emptyset$ for any $\varepsilon < \varepsilon_{1,k}$ and $a \in \mathcal{A}$ (see Figure 20).

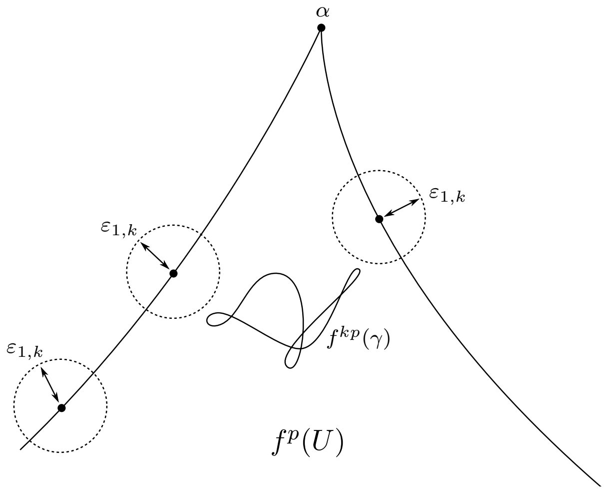

Fig. 20. For each $1 \leq k < c$, the radius $\varepsilon_{1,k}$ can be chosen in such a way that $f^{kp}(\gamma) \cap B_{2^k\varepsilon}(a) = \emptyset$ for any $\varepsilon < \varepsilon_{1,k}$ and $a \in \mathcal{A}$. Here, the set $\mathcal{A}$ is again represented by the non-labelled points, and we show just one step $1 \leq k < c$ for the sake of clarity.

In this way, if $\varepsilon < \varepsilon_1$, where

$$\varepsilon_1 := \min_{1 \leq k < c} \varepsilon_{1,k},$$

then $f^{kp}(\gamma) \cap B_{2^k\varepsilon}(a) = \emptyset$ for any $1 \leq k < c$ and $a \in \mathcal{A}$. Let us show that it follows from here that $\gamma \subset f^{-p}(V_\varepsilon)$ for all $\varepsilon < \varepsilon_1$: If $\gamma \not\subset f^{-p}(V_\varepsilon)$, then we would have $f^p(\gamma) \not\subset V_\varepsilon$, and since $f^p(\gamma) \subset f^p(U)$ and $V_\varepsilon = f^p(U) \setminus \bigcup_{a\in\mathcal{A}} \bigcup_{k=0}^{c-2} f^{-kp}(B_{2^k\varepsilon}(a))$, there would exist $0 \leq k \leq c-2$ and $a \in \mathcal{A}$ for which $f^p(\gamma) \cap f^{-kp}(B_{2^k\varepsilon}(a)) \neq \emptyset$. So let $z_0 \in f^p(\gamma) \cap f^{-kp}(B_{2^k\varepsilon}(a))$; taking f^{kp}-images we would have $f^{kp}(z_0) \in f^{(k+1)p}(\gamma) \cap B_{2^k\varepsilon}(a)$ for some $0 \leq k \leq c-2$, that is, $f^{kp}(\gamma) \cap B_{2^k\varepsilon}(a) \neq \emptyset$ for some $1 \leq k < c$, which is in contradiction with the construction of ε_1.

Finally, from the fact that $\gamma \subset f^{-p}(V_\varepsilon)$ for all $\varepsilon < \varepsilon_1$ and from the choice of $\gamma \subset U$, the lemma follows straightforwardly. ∎

Last, and in a similar spirit to that of the previous lemma, we also want to control the topology of V_ε itself, since it might happen to consist of more than one connected component due to the removal of the preimages $f^{-kp}(B_{2^k\varepsilon}(a))$ (see Figure 21).

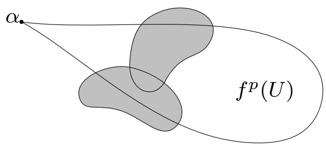

Fig. 21. When removing the preimages $f^{-kp}(B_{2^k\varepsilon}(a))$ (shaded) from $f^p(U)$, the resulting set might be disconnected.

This will pose no problem if we focus only on the connected component of V_ε that has α on its boundary, V_ε^*; but we do have to make sure that the f^p-preimage of that component will generate a multiply connected set, as expected.

Lemma 3.8. *In the situation described hitherto, there exists $\varepsilon_2 > 0$ such that $f^{-p}(V_\varepsilon^*)$ has a component like that of the previous lemma, for all $\varepsilon < \varepsilon_2$.*

Proof. The construction here is very similar to the proof of Lemma 3.7. In fact, consider $f^p(\gamma) \subset f^p(U)$, where $\gamma \subset U$ is the path which separates E and the unbounded connected component of $\widehat{\mathbb{C}} \setminus U$. Since $f^p(U)$ is simply connected and, in particular, path-connected, there exists a (continuous) path

$$\xi \colon [0,1] \to f^p(U) \cup \{\alpha\}$$

such that $\xi(0) = \alpha$ and $\xi(1) \in f^p(\gamma)$ (see Figure 22).

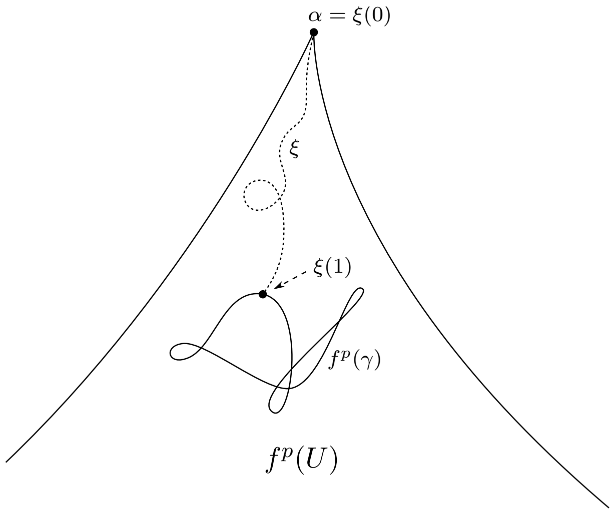

Fig. 22. We can connect α and $f^p(\gamma)$ by a path ξ in $f^p(U) \cup \{\alpha\}$.

Consider now the images $\{f^{kp}(\xi)\}_{0\le k\le c-2}$ in $f^p(U)$. Because ξ does not accumulate on points of $\mathcal{J}(f) \setminus \{\alpha\}$, neither do the curves $f^{kp}(\xi)$ accumulate on points of $\mathcal{A}$, and therefore there exist $\{\varepsilon_{2,k} > 0\}_{0\le k\le c-2}$ such that, for each $0 \le k \le c-2$, $f^{kp}(\xi) \cap B_{2^k\varepsilon}(a) = \emptyset$ for any $\varepsilon < \varepsilon_{2,k}$ and $a \in \mathcal{A}$. In this way, it is clear that if $\varepsilon < \varepsilon_2$, where

$$\varepsilon_2 := \min_{0\le k\le c-2} \varepsilon_{2,k},$$

then

$$\xi \cap f^{-kp}(B_{2^k\varepsilon}(a)) = \emptyset \quad \text{for any } 0 \leq k \leq c-2 \text{ and } a \in \mathcal{A},$$

that is, $\xi \subset V_\varepsilon$ and therefore $f^p(\gamma) \subset V_\varepsilon^*$.

Using an argument identical to that in the proof of Lemma 3.7 yields the result. ■

This completes the construction of the modification of $f^p(U)$, since now it just remains to define $\mathcal{V} \subset f^p(U)$ as V_ε^* for some $\varepsilon < \varepsilon_0, \varepsilon_1, \varepsilon_2$, and $\mathcal{U} \subset U$ as the multiply connected component of $f^{-p}(\mathcal{V})$ that separates E and the unbounded connected component of $\widehat{\mathbb{C}} \setminus U$ given by Lemma 3.8.

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Núria Fagella, Xavier Jarque, Jordi Taixés
Dept. de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona
Gran Via, 585 08007 Barcelona, Spain
E-mail: fagella@maia.ub.es
xavier.jarque@ub.edu
taixes@maia.ub.es

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