Rainbow Ramsey theorems for colorings establishing negative partition relations

by

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Abstract. Given a function $f$, a subset of its domain is a rainbow subset for $f$ if $f$ is one-to-one on it. We start with an old Erdős problem: Assume $f$ is a coloring of the pairs of $\omega_1$ with three colors such that every subset $A$ of $\omega_1$ of size $\omega_1$ contains a pair of each color. Does there exist a rainbow triangle? We investigate rainbow problems and results of this style for colorings of pairs establishing negative “square bracket” relations.

1. Introduction and history. Anti-Ramsey theorems appeared probably for the first time in the 1973 paper [9] of Richard Rado, claiming the existence of subsets with elements of different colors of the domain of a given coloring. Later in the game, the more expressive name of rainbow subset was coined. In this paper we will mostly consider 2-partitions, i.e. colorings $f$ of unordered pairs of a set. A subset of pairs will be called a rainbow subset (for $f$) if $f$ is one-to-one on it. Our starting point will be a problem of Paul Erdős, stated long before any of these names were coined:

Assume $f : [\omega_1]^2 \rightarrow 3$ is a 2-partition of $\omega_1$ with three colors such that each subset $A \subseteq \omega_1$ of size $\omega_1$ contains a pair of each color. Does there exist a rainbow triangle for $f$?

This is Problem 68 of [3] written in 1967. We restate it in the jargon of partition relations developed in [5]:

Problem 1.1. Assume $f : [\omega_1]^2 \rightarrow 3$ establishes $\omega_1 \not\rightarrow [\omega_1]^3_3$. Does there exist a rainbow triangle for $f$?

We knew that the answer is affirmative under some stronger conditions e.g.

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**FACT 1.2.** Assume \( f : [\omega_1]^2 \to 3 \) establishes \( \omega_1 \not\rightarrow [([\omega, \omega_1])^2_{\omega_1}]^3 \) (i.e. for \( A \in [\omega_1]^\omega \) and \( B \in [\omega_1]^\omega_1 \), \( f \) takes all three values on \([A, B]^{1,1}\)). Then there exists a rainbow triangle for \( f \).

However, in those early days, we could only construct an \( f \) satisfying the condition of 1.2 using CH.

**DEFINITION 1.3.** For a coloring \( d : [k]^2 \to \omega_1 \), \( k \leq \omega \) we write \( d \Rightarrow f \) if there is a one-to-one map \( \Phi : k \to \omega_1 \) such that
\[
d(\{n, m\}) = f(\{\Phi(n), \Phi(m)\}) \quad \text{for } n, m \in k.
\]

We could generalize 1.2 to

**FACT 1.4.** Assume \( f : [\omega_1]^2 \to \omega_1 \) establishes \( \omega_1 \not\rightarrow [([\omega, \omega_1])^2_{\omega_1}]^{\omega_1} \). Then \( d \Rightarrow f \) for an arbitrary \( d : [\omega]^2 \to \omega_1 \).

As already mentioned, we were not able to verify in ZFC that this does not hold vacuously and it bothered us that we could not lift it e.g. replacing \( \omega, \omega_1 \) by \( \omega_1, \omega_2 \) respectively. The next steps were taken in a paper of Shelah [10] written in 1975. He proved

**THEOREM 1.5 (Shelah [10]).**
1. \( CH \) implies that 1.1 fails for some \( f \) with \( \omega \) colors.
2. \( \Diamond \) implies that 1.1 fails for an \( f \) with \( \omega_1 \) colors.

Shelah also showed in [10] that a possible “lifting” of Fact 1.4 is consistently false say adding one Cohen real to a model of GCH. In more detail, he constructed a graph of size \( \omega_1 \) from the Cohen real which does not embed isomorphically into any graph of the ground model. Then any graph of the ground model establishing the partition relation \( \omega_2 \not\rightarrow [([\omega_2, \omega_1])^2_{\omega_1}]^{\omega_1} \) satisfies the same relation in the new model, and we have a graph of size \( \omega_1 \) that does not embed into it.

Knowing all this, in our 1978 paper [2] we stated implicitly a generalization of 1.4.

**THEOREM 1.6 ([2]).** Assume that \( f \) establishes \( \omega_1 \not\rightarrow [([\omega_1; \omega_1])^2_{\omega_1}]^{\omega_1} \). Then \( d \Rightarrow f \) for an arbitrary \( d : [\omega]^2 \to \omega_1 \).

The symbol with the semi-colon “;” means that all \( \omega_1 \) by \( \omega_1 \) “half-graphs” are totally multicolored, i.e. for all \( A, B \subseteq \omega_1 \) with \( |A| = |B| = \omega_1 \) and \( n < \omega \) there are \( \alpha \in A \) and \( \beta \in B \) with \( \alpha < \beta \) such that \( f(\{\alpha, \beta\}) = n \). I want to mention that [2] seems to be the first paper in print where this important concept was used. I think it was invented (discovered) by Fred Galvin. The following was proved 37 years later by Justin Moore:

**THEOREM 1.7 (Moore [7]). (ZFC) There is an \( f \) establishing**
\[
\omega_1 \not\rightarrow [([\omega_1; \omega_1])^2_{\omega_1}]^{\omega_1}.
\]
This is a byproduct of Moore’s result [7] showing the existence of $L$-spaces in ZFC. All the above justifies revisiting the old Problem 1.1.

2. $\not \supseteq$ relations. First we remark that we still do not know if the conclusions of either clauses of Theorem 1.5 can be proved under weaker conditions. Next we want to show that a Theorem 1.7 type generalization cannot hold if we only assume that each $[A]^2$ with $|A| = \omega_1$ is totally multicolored.

**Theorem 2.1.** There exist a rainbow $d : [4]^2 \rightarrow 6$ and an $f : [\omega_1]^2 \rightarrow 6$ establishing $\omega_1 \not\rightarrow [\omega_1]^2_6$ such that $d \not\supseteq f$.

*Proof.* First we define $e : [4]^2 \rightarrow W$ and $g : [\omega_1]^2 \rightarrow W$ where

$$W = \{(+,+), (+,-), (-,+), (-,-)\}.$$

Let

$$e(\{0,1\}) = (+,-), \quad e(\{1,2\}) = (-,+), \quad e(\{2,3\}) = (+,-),$$

$$e(\{0,3\}) = (-,+), \quad e(\{0,2\}) = (+,+), \quad e(\{1,3\}) = (-,-).$$

Let $<_R$ and $<_A$ be real and Aronszajn type orderings of $\omega_1$. For $\alpha < \beta < \omega_1$ let $g(\alpha,\beta) = (u,v)$ with $u,v \in \{+, -\}$, where $u = +$ iff $\alpha <_A \beta$, and $v = +$ iff $\alpha <_R \beta$.

It is a well known property of these orderings that for all $B \in [\omega_1]^{\omega_1}$ there are $C, D, E, F \in [B]^{\omega_1}$ such that $C <_A D, \ C <_R D, \ E <_A F$ and $F <_R E$. This implies that each $B \in [\omega_1]^{\omega_1}$ contains a complete $\omega_1$ by $\omega_1$ half-graph for $g$ in each of the colors in $W$.

It is an easy exercise to see that $e \not\supseteq g$. Let now $h$ be as in Moore’s Theorem 1.7. Then $k = (g,h)$ establishes $\omega_1 \not\rightarrow [\omega_1]^2_{\omega_1}$. Using $k$ and $e$ it is a matter of easy calculation to get $f$ and $d$ as required in the theorem. ■

Next we are going to investigate the cases when $f$ establishes

$$\omega_1 \not\rightarrow [([\omega_1,\omega_1])^2_{\gamma},$$

i.e. all $\omega_1$ by $\omega_1$ subgraphs are totally multicolored for some $\gamma$.

**Fact 2.2.** Assume $f$ establishes $\omega_1 \not\rightarrow [([\omega_1,\omega_1])^2_{3}]$. Let $d : [3]^2 \rightarrow 3$ be one-to-one. Then $d \supseteq f$, i.e. all possible rainbow triangles exist.

*Proof.* The assumption implies that for some $\alpha \in \omega_1$ both sets

$$\{\beta \in \omega_1 : f(\alpha,\beta) = d(0,1)\}, \quad \{\gamma \in \omega_1 : f(\alpha,\gamma) = d(0,2)\}$$

are of cardinality $\omega_1$. ■

**Fact 2.3.** There exist a rainbow $d : [5]^2 \rightarrow 10$ and an $f : [\omega_1]^2 \rightarrow 10$ establishing $\omega_1 \not\rightarrow [([\omega_1,\omega_1])^2_{10}]$ such that

$$d \not\supseteq f.$$
**Proof (outline).** Define \( e : [5]^2 \to 2 \) by the stipulation
\[
e(i, j) = 0 \quad \text{for} \quad i < 5 \quad \text{and} \quad j \equiv i + 1 \mod 5.
\]
That is, \( e \) is a “pentagon without a diagonal”. Let \( d : [5]^2 \to 10 \) be one-to-one such that \( d\{i, i + 1\} < 5 \) iff \( e\{i, i + 1\} = 0 \). Let \( <_R \) be a real type ordering of \( \omega_1 \). Let \( g(\alpha, \beta) : [\omega_1]^2 \to 2 \) be the “Sierpiński” partition, that is, \( g(\alpha, \beta) = 0 \) iff \( \alpha < R \beta \) for \( \alpha < \beta < \omega_1 \). It is well known that every complete bipartite \( \omega_1 \) by \( \omega_1 \) half-graph contains a complete bipartite \( \omega_1 \) by \( \omega_1 \) half-graph in both colors for \( g \). Again by Moore’s theorem, we can take an \( h \) establishing \( \omega_1 \not\to [\omega_1; \omega_1]_5^2 \). Set \( f = g \cdot 5 + h \). Then \( f \) establishes \( \omega_1 \not\to [\omega_1; \omega_1]_7^2 \) and \( d \Rightarrow f \) would imply \( e \Rightarrow g \), which is known to be false.

**Problem 2.4.** Can we improve 2.3 to have a \( d : [4]^2 \to 6 \) and an \( f \) establishing \( \omega_1 \not\to [\omega_1; \omega_1]_6^2 \)?

**3. Rainbow theorems**

**Theorem 3.1.** Assume \( f : [\omega_1]^2 \to \omega \) establishes \( \omega_1 \not\to [\omega_1; \omega_1]_\omega^2 \). Then there exists an infinite rainbow set.

**Proof.** We use \( A, B, C, \ldots \) to denote subsets of \( \omega_1 \) of size \( \omega_1 \), and \( N, M, \ldots \) to denote infinite subsets of \( \omega \); moreover, we set
\[
f_j(x) = \{ y \in \omega_1 : f(x, y) = f(\{x, y\}) = j \}
\]
for \( j < \omega \) and \( x \in \omega_1 \).

**3.1.1.** Assume \( B \cap C = \emptyset \) and
\[
\forall n \in M \ \forall x \in B \ (|f_n(x) \cap C| \leq \omega).
\]
Then
\[
\forall n \in M \ \forall C' \subseteq C \ \exists y \in C' \ (|f_n(y) \cap B| = \omega_1).
\]

Otherwise we could pick, by transfinite induction, a pair \((B', C'')\) omitting the color \( n \).

Let \((A, N)\) be the following property of \( A \) and \( N \): There are \( B, C \subseteq A \) and \( M \subseteq N \) such that
\[
\forall B' \subseteq B \ \forall C' \subseteq C \ \forall m \in M \ \exists x \in B' \ (|f_m(x) \cap C'| = \omega_1).
\]
When \((A, N)\) holds we denote by
\[
B(A, N), \ C(A, N), \ M(A, N)
\]
the relevant sets \( B, C, M \) respectively, with \( B \cap C = \emptyset \).

**3.1.2.** Assume that for some \( A_0, N_0 \), \((A, N)\) holds for all \( A \subseteq A_0 \) and \( N \subseteq N_0 \). Then there is an infinite rainbow subset.
Define $A_k, B_k, N_k$ by induction on $k < \omega$. Assume $A_k, N_k$ are defined. Let $B_k = B(A_k, N_k), A_{k+1} = C(A_k, N_k), N_{k+1} = N(A_k, N_k)$. Let $\{N'_k : k < \omega\}$ be a disjoint refinement of $\{N_k : k < \omega\}$ and let

$$N'_k = \{n^k_i : i < \omega\}$$

be a one-to-one enumeration of $N'_k$ for $k < \omega$. It is now easy to pick $x_i \in A_i$ for $i < \omega$ in such a way that $c(x_i, x_j) = n^i_j$ for $i < j < \omega$. This proves 3.1.2, as $\{x_i : i < \omega\}$ is an infinite rainbow set. 

Hence to finish the proof of Theorem 3.1 it is sufficient to prove

3.1.3. Assume $(*) (A, N)$ is false for some $A$ and $N$. Then $A$ has an infinite rainbow subset.

Let $N = \bigcup_{k < \omega} N_k, A = \bigcup_{k < \omega} A_k$ be disjoint partitions. To prove 3.1.3 we first prove

3.1.4. There are $x \in A_0$ and $\{n_i \in N_0 : 1 \leq i < \omega\}$ one-to-one such that

$$|f_{n_i}(x) \cap A_i| = \omega_1 \quad \text{for } 1 \leq i < \omega.$$

For an $x \in A_0$ we try to choose $n_i$, $1 \leq i < \omega$, by induction on $i$. Assume we have chosen $n_k$, $1 \leq k \leq i$, with $|f_{n_k}(x) \cap A_k| = \omega_1$. If there is always an $n$ such that

$$(+) \quad n \in N_0 \setminus \{n_k : 1 \leq k \leq i\} \quad \text{and} \quad |f_n(x) \cap A_{i+1}| = \omega_1$$

we can choose $n_{i+1}$ to be the smallest of these and 3.1.3 is true. If not, let $i(x)$ be the smallest $i$ for which $(+)$ fails. If $(+)$ fails for all $x \in A_0$ then for some $1 \leq i < \omega$ and $M = N_0 \setminus \{n_k : 1 \leq i\}$,

$$C = \{x \in A_0 : i(x) = i\}$$

has cardinality $\omega_1$. Choosing $B = A_{i+1}$ we find that

$$|f_n(x) \cap B| \leq \omega \quad \text{for } n \in M \text{ and } x \in C.$$

But then, by 3.1.2, for all $n \in M$ there is $x \in B$ with $|f_n(x) \cap C| = \omega_1$, a contradiction to the assumption that $(*) (A, N)$ is false. This shows 3.1.4. To finish the proof of 3.1.3 and Theorem 3.1, we can use 3.1.4 inductively. 

Here is a problem that has not been looked at very thoroughly:

**Problem 3.2.** Under the conditions of 3.1, is there a rainbow set containing all the colors?

**Theorem 3.3.** For every $1 < k < \omega$ there is an $n \in \omega$ with $\binom{k}{2} \leq n$ such that every $f$ satisfying $\omega_1 \not\rightarrow [\omega_1, \omega_1]^2_{\binom{k}{2}}$ has a rainbow set of size $k$.

**Proof.** We prove the following statement by induction on $2 \leq k < \omega$. There is an $n < \omega$ such that if $\text{Dom}(f) \subseteq [\omega_1]^2$ satisfies $\omega_1 \not\rightarrow [\omega_1, \omega_1]^2_{\binom{k}{2}}$ (note that this means that for all $A, B \subseteq \omega_1$ with $|A| = |B| = \omega_1$ and for all $i < n$ there are $\alpha \in A$ and $\beta \in B$ with $\{\alpha, \beta\} \in \text{Dom}(f)$ such that
of de Bruijn and Erdős, from 1951, there are \( A_i : i < n \) are pairwise disjoint subsets of \( \omega_1 \) of size \( \omega_1 \), then there is a rainbow partial transversal \( P ([P]^2 \subseteq \text{Dom}(f)) \) of size \( k \) for these sets. Just as in the proof of 3.1, put

\[
f_j(x) = \{ y \in \omega_1 : f(x, y) = f(\{x, y\}) = j \}
\]

for \( j < \omega \) and \( x \in \omega_1 \). Assume \( n \) is good for \( k \) and \( A_0, \ldots, A_{2n-1} \) are pairwise disjoint subsets of \( \omega_1 \) of size \( \omega_1 \) with union \( A \).

Let (*) denote the following statement: There are \( x, i_x, N_x, \varphi_x \) such that \( x \in A_{i_x}, N_x \subseteq 2n \setminus \{i_x\}, \varphi_x : N_x \to 2n \) is one-to-one,

\[
|f_{\varphi_x(j)}(x) \cap A_j| = \omega_1 \quad \text{for } j \in N_x,
\]

and \( |N_x| = n \). If (*) holds for an \( x \) then applying the induction hypothesis for the sets

\[
f_{\varphi_x(j)}(x) \cap A_j, \quad j \in N_x,
\]

and for the color set \( 2n \setminus \varphi[N_x] \) we get a rainbow partial transversal of size \( k \) for these sets, and adding \( x \) to it we get a rainbow transversal of size \( k+1 \) for the sets \( A_0, \ldots, A_{2n-1} \).

If (*) is false, choosing an \( N_x \) of maximal size for \( x \in A \) we will have \( |N_x| \leq n - 1 \) for \( x \in A \). By thinning out, we get sets \( B_i \subseteq A_i, i < 2n \), of size \( \omega_1 \) and \( N_i, M_i \subseteq 2n, i < 2n \), such that \( N_x = N_i \) and \( M_i = \varphi_x[N_i] \) for \( x \in B_i \) for \( i < 2n \).

Then \( i \mapsto N_i \) is a set mapping of order at most \( n - 1 \) on \( 2n \). By a theorem of de Bruijn and Erdős, from 1951, there are \( i \neq j \) such that \( i \notin N_j \) and \( j \notin N_i \). As \( |M_i \cup M_j| < 2n \) we can choose \( l \notin M_i \cup M_j \). By the maximality of \( N_i \) we know that \( |f_l(x) \cap B_j| \leq \omega \) for \( x \in B_i \) and likewise \( |f_l(x) \cap B_i| \leq \omega \) for \( x \in B_j \). We could then pick, by an easy transfinite induction, sets \( C_i \subseteq B_i \) and \( C_j \subseteq B_j \), both of size \( \omega_1 \), such that the color \( l \) is missing from the bipartite \( (\omega_1, \omega_1) \) determined by \( C_i \) and \( C_j \). This contradicts the assumption.

**Corollary 3.4.** In Theorem 3.3, \( n \) can be chosen to be \( 2^{k-2} \) for \( 2 \leq k < \omega \).

**Problem 3.5.** Can \( n \) be taken to be \( \binom{k}{2} \) in Theorem 3.3?

### 4. Resurrecting the problem for larger cardinals

We explained in Section 1 how Shelah’s example described in 1.5 forced us to consider problems only for underlying sets of size at most \( \omega_1 \). In [2] written in 1978 we tried to ask if we can get every graph of size \( \omega_1 \) as an induced subgraph provided the graph shows \( \omega_2 \not\rightarrow [((\omega_1, \omega))_\omega]^2 \), a stronger assumption that one can only make consistent. Recently Soukup showed that the simple method of adding one Cohen real gives a negative answer as well. Working through the material of this paper I realized that this trick only kills questions of \( \Rightarrow \) type. The following is probably the simplest problem I cannot solve:
Problem 4.1. Assume GCH and let $f$ establish
$$\omega_2 \not\rightarrow [\omega_1, \omega_2]_{\omega_1}^2.$$  
Does there exist a rainbow subset of size $\omega_1$ for $f$?

In fact, we do not know a single case where for some $\kappa > \lambda > \omega$ some $f : [\kappa]^2 \to \lambda$ establishes $\kappa \not\rightarrow [(\kappa, \kappa)]_{\lambda}^2$ and for all such $f$ there is an uncountable rainbow set.


Theorem 5.1 (Erdős–Hajnal [4, Theorem 1.3]). Assume $2 \leq k, s < \omega$ and $d : [k]^2 \to s$. Then there are $n_0$ and a real number $r > 0$ such that for all $f : [n]^2 \to s$ establishing
$$n \not\rightarrow \left[ e^{r \sqrt{\log n}} \right]^2_s,$$
$d \Rightarrow f$ holds.

In fact, we only wrote down the proof of this result for $s = 2$. Janos Pach kindly communicated to us that he can prove a much stronger result for a great many cases. Most relevant to this paper, he can prove:

Theorem 5.2 (Fox–Pach [6]). There are $n_0$ and $\varepsilon > 0$ such that for any $n > n_0$ and $f$ establishing
$$n \not\rightarrow \left[ n^{\varepsilon} \right]^2_3$$
there is a rainbow triangle for $f$.

References


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