

Non-existence of absolutely continuous invariant probabilities for exponential maps

by

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Abstract. We show that for entire maps of the form $z \mapsto \lambda \exp(z)$ such that the orbit of zero is bounded and Lebesgue almost every point is transitive, no absolutely continuous invariant probability measure can exist. This answers a long-standing open problem.

In this paper we introduce a new method to deal with the problem of existence of invariant measures for entire maps. To illustrate this method, avoiding uninteresting technical difficulties, we show the following theorem.

THEOREM 1. *Let $\lambda \in \mathbb{C} \setminus \{0\}$ be such that the Julia set of $f : z \mapsto \lambda \exp(z)$ is equal to \mathbb{C} , the forward orbit of 0 is bounded and there is a set of positive Lebesgue measure of points $z \in \mathbb{C}$ such that $\omega(z) \not\subset \mathcal{P}(f)$. Then f has a σ -finite absolutely continuous invariant measure, but it does not have an absolutely continuous invariant probability measure.*

We denote, as usual, the ω -limit set of $z \in \mathbb{C}$ by $\omega(z) \subset \overline{\mathbb{C}}$ and the post-singular set by $\mathcal{P}(f)$, here equal to the union of $\{\infty\}$ with the closure of the orbit of 0.

Theorem 1 implies, in particular, that the map $z \mapsto 2\pi i \exp(z)$ has no absolutely continuous invariant probability measure, which was a long-standing open problem (see [9]).

For a class of unimodal maps f of the interval with non-recurrent critical point, Benedicks and Misiurewicz [1] showed that there exists an absolutely continuous invariant probability measure if and only if $\int \log |f'(x)| dx > -\infty$.

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The necessity of the integrability condition was later extended by the first author (in Theorem 1 of [3]) to all $C^{1+\varepsilon}$ interval maps without any hypothesis on the critical orbits, but under the assumption that the measure has positive Lyapunov exponent. As an example, no unimodal map of the form $x \mapsto C_1 - C_2 \exp(-|x|^{-\alpha})$ with $\alpha \geq 1$ has an absolutely continuous invariant probability measure with positive Lyapunov exponent.

This paper extends the main result of [1] to the holomorphic setting. The strategy of the proof has two elements: construct a *nice set* on which the density of a hypothetical measure must be bounded away from zero; show that the return time to the nice set is not integrable. This is similar in philosophy to the proof of Benedicks and Misiurewicz, although the settings differ.

Existence of a σ -finite measure is not new; it was shown in [7] under weaker hypotheses but with a considerably more difficult proof. On the other hand, existence of absolutely continuous invariant probability measures for transcendental entire maps has been an interesting and open question for some time with a response in only one situation: J.-M. Hemke [5] proved that, for a class of entire maps for which the orbits of all asymptotic values converge to infinity sufficiently fast, $\omega(z) \subset \mathcal{P}(f)$. For these maps, $\mathcal{P}(f)$ has zero measure and it follows from the Poincaré recurrence theorem that no absolutely continuous invariant probability measure can exist. Hemke's work generalised a result proved independently by M. Rees and M. Lyubich for $z \mapsto \exp(z)$ ([11], [8]).

For non-entire maps the second author [13] has another negative result for some postcritically finite tangent maps. For a large class of transcendental non-entire maps which satisfy a Misiurewicz-type condition J. Kotus and G. Świątek [6] showed that absolutely continuous invariant probability measures can exist.

The mathematics involved in the proof has the merit of being surprisingly elementary. An important and somewhat magical technique is Juan Rivera-Letelier's construction of nice sets for rational dynamics (see [12]) which we adapt to the entire setting.

An open set U is called *nice* if $f^n(\partial U) \cap U = \emptyset$ for all $n > 0$. This implies that every pair of pullbacks (connected components of $f^{-n}(U)$, $f^{-n'}(U)$ for some $n, n' \geq 0$) is either nested or disjoint. Let us fix some $D > 0$ such that $\mathcal{P}(f) \subset B(0, D)$.

LEMMA 2. *For each sufficiently large $x \in \mathbb{R}$, there exists a connected nice set $U \subset \mathbb{C}$ satisfying $B(x, 4\pi) \subset U \subset B(x, 8\pi)$.*

Proof. There exists a $K > 1$ such that for any r and any holomorphic function g , univalent on $B(x, Kr)$, one has

$$\left| \frac{g'(z)}{g'(z')} \right| \leq 2$$

for all $z, z' \in B(x, r)$, by the Koebe distortion theorem.

Let x satisfy $x > 8K\pi + D$. Let W be a (connected) pullback of $B(x, 8\pi)$ and let $n > 0$ be such that f^n maps W univalently onto $B(x, 8\pi)$. Since $f^n|_W$ extends to map univalently onto $B(x, 8K\pi)$, it follows that the distortion of f^n restricted to W is bounded by 2. Thus there is $r > 0$ such that $B((f^n|_W)^{-1}(x), r) \subset W \subset B((f^n|_W)^{-1}(x), 2r)$. But $B((f^n|_W)^{-1}(x), r)$ must lie in a horizontal strip of height 2π , so $|W| < 4\pi$.

We shall use this to construct nice sets exactly as in [12]. We include the proof for the reader's convenience: Let $U_0 := B(x, 4\pi)$ and define U_n as the connected component of $\bigcup_{i=0}^n f^{-i}(U_0)$ containing U_0 and $U = \bigcup_{n \geq 0} U_n$. We prove by induction that $U_n \subset B(x, 8\pi)$ for all $n \geq 0$. This is clearly true for $n = 0$. So suppose it is true for all $n \leq k$. We must show it holds for $n = k + 1$.

Let X be a connected component of $U_{k+1} \setminus U_0$. Then there is a minimal $m \geq 0$ such that $f^m(z) \in U_0$ for some $z \in X$, and necessarily $m \geq 1$. Consider $f^m(X)$. This set is contained in U_{k+1-m} , and so by hypothesis is contained in $B(x, 8\pi)$. But then X , being connected, is contained in some pullback W with $|W| < 4\pi$. The result follows. ■

LEMMA 3. *There exists a $c > 0$ such that if $f^n(z) \notin B(0, 2D)$ then $n > -c \log |z|$.*

Proof. Let $M > 1$ be such that $|f'(z)| < M$ for all $z \in B(0, 2D)$. Suppose $f^n(z) \notin B(0, 2D)$. Then $|f^n(z) - f^n(0)| > D > 1$. This implies that $|z - 0| = |z| > M^{-n}$. Thus $\log |z| > -n \log M$ and $n > (-1/\log M) \log |z|$. ■

In what follows, let U be a nice set given by Lemma 2 for some $x > 8\pi + 2D$; we fix x too. In particular, $U \cap B(0, 2D) = \emptyset$. We denote by $r_U(z)$ the first return time of z to U . Also let $r, \phi \in \mathbb{R}$ be such that $\lambda = re^{i\phi}$.

LEMMA 4. *There exist $C \in \mathbb{R}$ and $c > 0$ with the following property. Suppose $z \in U$ and $\operatorname{Re}(f^k(z)) \leq -K$ for some $0 < k < r_U(z)$ and $K > 0$. Then $r_U(z) > C + cK$.*

Proof. Let c be given by Lemma 3. We have $|f^{k+1}(z)| \leq re^{-K}$. Then the time it takes for $f^{k+1}(z)$ to leave $B(0, 2D)$ is greater than $-c \log(re^{-K}) = -c(-K + \log r)$ by Lemma 3. Take $C := -c \log r$. ■

LEMMA 5. *Denote by m Lebesgue measure. Then $\int_U r_U(z) dm = \infty$.*

Proof. Define $h : \mathbb{R} \rightarrow \mathbb{R}$ by $h(y) = (r/2) \exp(y)$ and let

$$S_R := \{z : \operatorname{Re}(z) > x \text{ and } \arg f(z) \in [-\pi/4, \pi/4]\},$$

$$S_L := \{z : \operatorname{Re}(z) > x \text{ and } \arg f(z) \in [3\pi/4, 5\pi/4]\}.$$

Note that each connected component of $\{z : \arg f(z) \in [-\pi/4, \pi/4]\}$ is a horizontal strip of height $\pi/2$ and the components are periodic of period $2i\pi$. A similar statement holds for $\{z : \arg f(z) \in [3\pi/4, 5\pi/4]\}$. Let

$$P_n := \{z \in B(x, 4\pi) : f^k(z) \in S_R \text{ for all } 0 \leq k \leq n\},$$

and let $Q_n := P_{n-1} \cap f^{-n}(S_L)$. For $z \in S_R$, $\operatorname{Re}(f(z)) \geq h(\operatorname{Re}(z))$, so by induction, for all $z \in P_n$, $\operatorname{Re}(f^n(z)) \geq h^n(x)$. Then distortion arguments as in [10] show that $m(Q_n)/m(P_n)$ tends to one and

$$\lim_{n \rightarrow \infty} m(P_n)/m(P_{n+1}) = 1/4.$$

Thus there exists a $\gamma \in (0, 1/4)$ such that for all $n \geq 1$,

$$m(Q_n) \geq \gamma^n.$$

Now for $z \in Q_n$, $\operatorname{Re}(f^{n+1}(z)) < -h^{n+1}(x)$, so we have $r_U(z) > C + ch^{n+1}(x)$ where the constants c, C are given by Lemma 4. But $h^n(x)$ grows with n faster than any exponential so

$$\lim_{n \rightarrow \infty} m(Q_n) \inf\{r_U(z) : z \in Q_n\} = \infty. \blacksquare$$

Proof of Theorem 1. Let ψ denote the first return map to U . Since U is nice and disjoint from $\mathcal{P}(f)$, every connected component of the domain of ψ is mapped univalently onto U by ψ . Moreover, the branches of ψ are uniformly extendible, so the Koebe distortion theorem gives a uniform distortion bound for all branches of all iterates of ψ . Note that by Corollary 2 of [2], Lebesgue almost every point has a transitive orbit.

Denote by V_k the set of points from U whose first return time to U is k . By the Folklore Theorem (see for example [4]), there exists a unique absolutely continuous invariant probability ν for ψ and its density is bounded below by some $\varepsilon > 0$. Then the spread measure $\mu := \sum_{k=1}^{\infty} \sum_{n=0}^{k-1} f_*^n \nu|_{V_k}$ is a σ -finite absolutely continuous invariant measure for f . This gives the easy proof of its existence.

Suppose now μ is an absolutely continuous f -invariant probability measure. By transitivity of Lebesgue almost every point, $\mu(U) > 0$. Then μ is also a finite invariant measure for ψ , since ψ is a first return map. By uniqueness, the density of μ is then bounded from below on U by $\mu(U)\varepsilon > 0$. Thus

$$1 = \int_U r_U(z) d\mu \geq \mu(U)\varepsilon \int_U r_U(z) dm,$$

the first equality being Kac’s lemma. This contradicts Lemma 5, so no absolutely continuous invariant probability measure can exist. \blacksquare

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