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Well-quasi-ordering Aronszajn lines

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Abstract. We show that, assuming PFA, the class of all Aronszajn lines is well-quasi-ordered by embeddability.

1. Introduction. A rough classification result for a given class $\mathcal{K}$ of mathematical structures usually depends on a reflexive and transitive relation $\preceq$, i.e. a quasi-ordering, where for A and B in $\mathcal{K}$ the relation $A \preceq B$ means that, in some sense, A is simpler than B. The strength of the rough classification result depends not only on how fine the corresponding equivalence relation is ($A \equiv B$ iff $A \preceq B$ and $B \preceq A$) but also on the information about the quasi-ordering $(\mathcal{K}, \preceq)$ it gives. One of the most prominent global conditions, generally considered as giving a satisfactory rough classification result, is the requirement of being well-quasi-ordered. Recall that a class $(\mathcal{K}, \preceq)$ is *well-quasi-ordered* (or *wqo*) if for every infinite sequence A_n ($n \in \omega$) of elements of $\mathcal{K}$ there exists $n < m$ such that $A_n \preceq A_m$. The sense of strength of such a rough classification result comes from the fact that whenever $(\mathcal{K}, \preceq)$ is well-quasi-ordered then the complete invariants of the equivalence relation $\equiv$ on $\mathcal{K}$ are only slightly more complicated than the ordinals.

In this note we are interested in proving such a rough classification result for a class $\mathcal{K}$ of linear orderings. Recall that in this context the quasi-ordering $\preceq$ is usually taken to be isomorphic embedding, i.e., $A \preceq B$ iff there is a strictly increasing map $f : A \to B$. The first result of this sort is a result of Laver [13] who showed, verifying an old conjecture of Fraïssé [8], that the class of countable linear orderings is well-quasi-ordered. Some restrictions on the linear orderings in Laver's result are needed in view of a classical result of Dushnik and Miller [7] who proved that the class of separable

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linear orderings of size continuum (more precisely, suborders of the real line) fails badly to be well-quasi-ordered. The idea behind Dushnik and Miller's construction combined with the more recent constructions using the weak-diamond principle of Devlin and Shelah (see [6]) shows that under CH there is basically no room for extending Laver's theorem to a larger class of linear orderings. It is still an open problem whether Laver's result is sharp (see [11] and [15]). It is for this reason that one is naturally led to examine this possibility using some alternative to CH. It turns out that the right alternative to CH is a strong form of the Baire category theorem known as the Proper Forcing Axiom, PFA.

However, even assuming PFA the situation is not so simple as there are natural restrictions that do not depend on CH or any other additional axioms. For example, in [3], Baumgartner showed that the class of non-σ-scattered linear orderings A of size $\aleph_1$ with the property that every uncountable subset of A contains an uncountable well-ordered set (an isomorphic copy of ω_1) is not well-quasi-ordered. Thus one is led to consider only restricted classes of uncountable linear orderings. One such class is $\mathcal{R}_{\aleph_1}$ which consists of the suborders of the real line of cardinality $\aleph_1$. These were completely classified by Baumgartner [4].

Theorem 1.1 ([4]). (PFA) *Any two $\aleph_1$-dense suborders of $\mathbb{R}$ are order-isomorphic. In particular, any two elements of $\mathcal{R}_{\aleph_1}$ are equivalent.*

Another class which arises naturally in this context is the class $\mathcal{A}$ of those linear orderings which do not contain uncountable separable suborders or ω_1 or ω_1^*. These linear orders are known as *Aronszajn lines* (or *A-lines*) and were first discovered by Kurepa [12] using Aronszajn trees constructed by Aronszajn in the same paper (see also [21]). They were originally motivated by Suslin's problem [20]. The following theorem is the main result of the present article.

Theorem 1.2. (PFA) *The class $\mathcal{A}$ of Aronszajn lines is well-quasi-ordered by embeddability.*

In fact, there is a very close relationship between Aronszajn trees and Aronszajn lines: any lexicographical ordering of an Aronszajn tree is an Aronszajn line, and conversely, any binary partition tree of an Aronszajn line is an Aronszajn tree (see [21]). It is therefore surprising that there is a discrepancy between Aronszajn trees and Aronszajn lines when it comes to the wqo-theory. Our Theorem 1.2 is in contrast with a result of Todorcevic from [23], where it is proved that the class of Aronszajn trees is not wqo under embeddability. The question of whether $\mathcal{A}$ is well-quasi-ordered appears in print in the survey article by Moore [16].

Besides the ideas of Nash-Williams and Laver from the wqo-theory, while proving Theorem 1.2, we shall also rely on some ideas behind the deep results obtained by Todorcevic [23] and Moore [17].

In 1970 Countryman [5] made a brief but important contribution to the subject by asking whether there is an uncountable linear order C whose square is the union of countably many chains. Such an order is called *Countryman* (or *C-line*). It can be seen that every C-line is Aronszajn and that if C is a C-line and C^* denotes its reverse, then no uncountable linear order can be embedded simultaneously into both C and C^*. Shelah [19] proved that Countryman orders exists in ZFC, and Todorcevic [22] produced a number of concrete representations of C-lines. Moore [14] proved, solving a longstanding conjecture of Shelah, the following deep result.

THEOREM 1.3 ([14]). (PFA) *The class of A-lines contains a two-element basis consisting of C and C^* where C is any Countryman line.*

Furthermore, in [17] Moore proved that, assuming PFA, there is a universal A-line η_C.

THEOREM 1.4 ([17]). (PFA) *Every A-line is isomorphic to a suborder of η_C.*

Moreover, η_C can be easily described in terms of a fixed C-line C. Let $D = C^* + 1 + C$; then η_C consists of all elements of D^ω which are eventually zero, ordered lexicographically.

It is worth mentioning that in view of the following result of Abraham and Shelah [1] some extra assumptions, such as PFA, are needed in our main result.

THEOREM 1.5 ([1]). ($2^{\aleph_0} < 2^{\aleph_1}$) *There is a collection $\mathcal{F}$ of pairwise incomparable A-lines of cardinality $2^{\aleph_1}$.*

Furthermore, this theorem can be easily modified, by using the tree $T(\rho_1)$ (see [23]), to obtain a family $\mathcal{F}$ of pairwise incomparable C-lines of cardinality $2^{\aleph_1}$.

The paper is organized as follows. Section 2 provides some background on the theory of well-quasi-orderings. Section 3 gives a review of the basic properties of Aronszajn lines as well as the deep results of Moore [14] and [16]. The main results of the article are proved in Section 4.

2. Well-quasi-orderings. It will be helpful to fix some notation and review some basic facts from the theory of well-quasi-orderings. First, recall that a *quasi-order* is a structure of the form $(Q, \preceq)$ where $\preceq$ is a transitive and reflexive binary relation.

Definition 2.1. A quasi-order $(Q, \preceq)$ is a *well-quasi-order* (or *wqo*) if it satisfies any of the following two equivalent conditions:

(1) For any function $f : \omega \to Q$ there exist $i < j$ such that $f(i) \preceq f(j)$.
(2) Any strictly decreasing sequence of members of Q is finite, and every antichain of members of Q is finite.

The equivalence between the two definitions is an immediate consequence of Ramsey's theorem.

Definition 2.2. Given a quasi-ordering $(A, \prec)$ we define its *reverse* $A^* = (A, \prec^*)$ by $a \prec^* b$ iff $b \prec a$.

Definition 2.3. Let $(A, \prec_A), (B, \prec_B)$ be quasi-orderings.

(1) Define $A \times B$ as the *lexicographical ordering* on the cartesian product, i.e., $(a_1, b_1) \prec (a_2, b_2)$ if $(a_1 \prec_A a_2)$ or $(a_1 = a_2$ and $b_1 \prec_B b_2)$.
(2) Define $A + B$ as the quasi-ordering on $(A \times \{0\}) \cup (B \times \{1\})$ given by $a \prec_{A+B} b$ if $(\pi_1(a) \prec \pi_1(b))$ or $(\pi_1(a) = \pi_1(b)$ and $\pi_0(a) \prec \pi_0(b))$.
(3) Let $(I, \prec_I)$ be a quasi-ordering and let $(A_i, \prec_{A_i})$ $(i \in I)$ be a collection of quasi-orderings. We define the *sum* $\sum_{i \in I} A_i$ to be the quasi-ordering $(C, \prec_C)$ where $C = \bigcup_{i \in I} A_i \times \{i\}$ and $x \prec_C y$ if $(\pi_1(x) \prec_I \pi_1(y))$ or $(\pi_1(x) = \pi_1(y)$ and $\pi_0(x) \prec_{A_{\pi_1(x)}} \pi_0(y))$.

In order to make the induction hypothesis in our main theorem go through we will need a generalization of a quasi-ordering called Q-type. Intuitively a Q-type is a quasi-ordered set whose points are labeled by members of Q.

Definition 2.4. Let Q be a quasi-ordering. A pair (A, f) is a *Q-type* if A is a linearly ordered set and f is a function from A into Q.

We quasi-order the class of Q-types by the following embeddability relation: $(A_1, f_1) \preceq (A_2, f_2)$ if there is a strictly increasing function $f : A_1 \to A_2$ such that

$$f_1(x) \preceq f_2(f(x)) \quad \text{for all } x \in A_1.$$

One way to motivate this definition is as follows: Given two linear orderings A, B formed by sums of linear orderings, say

$$A = \sum_{x \in X} A_x \quad \text{and} \quad B = \sum_{y \in Y} B_y,$$

associate to A, B a natural Q-type structure given by $(A, x \mapsto A_x)$ and $(B, y \mapsto B_y)$, respectively. We take $Q = \{A_x : x \in X\} \cup \{B_y : y \in Y\}$ quasi-ordered by embeddability. Observe that if $(A, x \mapsto A_x) \preceq (B, y \mapsto B_y)$ as Q-types then A embeds into B.

Definition 2.5. If $\mathcal{M}$ is a class of linear orders and Q is a quasi-order, let $Q^{\mathcal{M}} = \{(A, f) : A \in \mathcal{M},\ f : A \to Q\}$

In order to prove our result we will need a generalization of the theory of well-quasi-orderings called better-quasi-orderings (bqo), a concept introduced by Nash-Williams [18].

Remember that $[\omega]^\omega$ represents the set of infinite subsets of natural numbers. We will consider $[\omega]^\omega$ as a topological space with the *Ellentuck* topology, which has the basic open sets of the form

$$[s, A] = \{X \in [\omega]^\omega : s \sqsubset X \subset A\}$$

for $A \in [\omega]^\omega$ and $s \in [\omega]^{<\omega}$.

We are now in a position to state the concept of better-quasi-order.

DEFINITION 2.6. Let Q be a quasi-order. Q is a *better-quasi-ordering* (or *bqo*) if for every Borel map $f : [\omega]^\omega \to Q$ there exists $X \in [\omega]^\omega$ such that $f(X) \preceq f(X \setminus \{\min(X)\})$.

Even though the concept of bqo might appear unintuitive at first, it is more natural in the sense that the property of being bqo is preserved by almost all operations on quasi-orders as opposed to the case of well-quasi-orders.

It is worth mentioning that our definition of better-quasi-ordering is not the original one given by Nash-Williams [18]. Our definition comes from [2].

It is easy to see that every better-quasi-ordered set is well-quasi-ordered but the reverse implication does not necessarily hold as shown by the next example:

EXAMPLE 2.7 (Rado). Let Q denote the set of all pairs (i, j) with $i \leq j$ quasi-ordered by $(i, j) \prec (k, l)$ if either $i = k$ and $j < l$ or $j < k$. It can be easily seen that Q is an example of a well-quasi-order which is not a better-quasi-order. Moreover, the set Q^ω is not well-quasi-ordered by embeddability as a Q-type.

Using the concept of better-quasi-order we can state the main theorem of Laver.

THEOREM 2.8 ([13]). *Let $\mathcal{S}$ denote the class of σ-scattered linear orderings, and let Q be a better-quasi-order. Then $Q^{\mathcal{S}}$ is better-quasi-ordered by embeddability as a Q-type. In particular $Q^{\mathcal{C}}$ is bqo where $\mathcal{C}$ denotes the class of countable linear orders.*

Observe that in view of Example 2.7 the requirement of being better-quasi-ordered in Laver's theorem is essential.

3. A-lines. In this section we collect some standard facts about the class of A-lines.

The notion of a *tree* in this paper is to be interpreted in its order-theoretic sense, i.e., it is a partially ordered set $(T, \leq_T)$ with the property that for every *node* $t \in T$ the set $\{s \in T : s <_T t\}$ is a well-order. The *height* of a node t in T, written $\mathrm{ht}(t)$, is the order-type of $\{x \in T : x <_T t\}$. The αth *level* of T is the set $T_\alpha = \{t \in T : \mathrm{ht}(t) = \alpha\}$. We shall identify $(T, \leq_T)$ with its domain T. The *height* of the tree, $\mathrm{ht}(T)$, is the ordinal $\min\{\alpha : T_\alpha = \emptyset\}$. For any node $t \in T$ and any $\xi < \mathrm{ht}(t)$ let $t\restriction\xi$ denote the unique node $s \in T_\xi$ so that $s \leq_T t$. For any two incompatible nodes s, t in T let

$$\Delta(t, s) = \mathrm{ot}\{\xi < \omega_1 : \xi < \mathrm{ht}(t) \text{ and } \xi < \mathrm{ht}(s)\}.$$

DEFINITION 3.1. Let $\mathcal{A}$ denote the class of Aronszajn lines.

Let T be an A-tree. For every $\alpha < \omega_1$, we fix a linear ordering $\leq_\alpha$ of T_α. Then the *lexicographical ordering* $\preceq_l$ of T induced by$\{\leq_\alpha : \alpha < \omega_1\}$ is defined by $t \preceq_l s$ iff

(i) $t \leq_T s$ or
(ii) t, s are incomparable and $t_{\Delta(s,t)} \leq_{\Delta(t,s)} s_{\Delta(t,s)}$.

The following are standard facts about A-lines and A-trees (see [21]).

FACT 3.2. *Every lexicographical ordering of an Aronszajn tree is an Aronszajn line.*

Let us now define an inverse operation which connects A-lines and A-trees. This operation is called a *process of atomization* of an A-line $(A, \leq_A)$, and it is an inductive construction of families T^α, $\alpha \in \mathrm{ON}$, of non-empty convex subsets of A such that:

(i) If $\alpha = 0$, then $T^\alpha = \{A\}$.
(ii) If $\alpha = \beta + 1$, then for each non-trivial interval $I \in T^\beta$ there exist disjoint $I_0, I_1 \in T^\alpha$ such $I_0 \cup I_1 = I$, and

$$T^\alpha = \{\{I_0, I_1\} : I \in T^\beta \text{ and } |I| \geq 2\}.$$

(iii) If α is a limit ordinal, then

$$T^\alpha = \Big\{\bigcap b : b \subset \bigcup_{\beta<\alpha} T^\beta,\ b \cap T^\beta \neq \emptyset \text{ for all } \beta < \alpha, \text{ and } \bigcap b \neq \emptyset\Big\}.$$

It is clear that for some α, $T^\alpha = \emptyset$, hence we may define

$$\mathrm{ht}(T) = \min\{\alpha : T^\alpha = \emptyset\} \quad \text{and} \quad T = \bigcup_{\alpha<\mathrm{ht}(T)} T^\alpha.$$

Then $(T, \supseteq)$ is a tree and T^α is the αth level of T for all $\alpha < \mathrm{ht}(T)$. Any tree which is a result of an atomization process of A is called a *partition tree* of A.

FACT 3.3. *Every partition tree of an Aronszajn line is an Aronszajn tree.*

As we can see from Facts 3.2 and 3.3 there is a strong duality relation between the Aronszajn lines and the Aronszajn trees. At this point we recall the following definitions.

DEFINITION 3.4. A *Suslin tree* is a tree T such that $|T| = \aleph_1$ and every chain and every antichain of T has countable cardinality.

DEFINITION 3.5. A *Suslin line* is a non-separable linear ordering A with the countable chain condition (or ccc), i.e., every family of pairwise disjoint non-empty open intervals of A is countable,

FACT 3.6.

(i) *Every lexicographical ordering of a Suslin tree is a Suslin line. Moreover, every Suslin line A is isomorphic to a lexicographical ordering of a Suslin tree.*

(ii) *Every partition tree of a Suslin line is a Suslin tree.*

DEFINITION 3.7. A linear ordering A is $\aleph_1$-*dense* if it has cardinality $\aleph_1$, it has no end-points, and between any two elements of A there are exactly $\aleph_1$ elements of A.

DEFINITION 3.8. An uncountable linear ordering C is a *Countryman line* (*C-line*, for short) if its lexicographical square is a countable union of chains. C^2 is quasi-ordered by $(a_1, b_1) \preceq (a_2, b_2)$ if $a_1 \preceq a_2$ and $b_1 \preceq b_2$. Note that this order differs from the lexicographical ordering of Definition 2.4.

The C-lines play a prominent role in the structure theory of the class of A-lines, under PFA, as they constitute the building blocks of the class of A-lines. This will be explained in detail in Section 4.

FACT 3.9. *If C is Countryman, then C does not contain a Suslin suborder.*

Proof. Observe that if C is Countryman then C remains Countryman in any forcing extension which preserves $\aleph_1$. It suffices to show that any Suslin line A fails to be Countryman in some ccc forcing extension. Let A be a Suslin line. Using Fact 3.6 we can find a Suslin tree $(T, \leq_T)$ so that A is isomorphic to a lexicographical ordering of T. Forcing with the ccc poset $(T, \geq_T)$ we add a copy of ω_1 to A, which implies that A is not Countryman in $V[G]$. ■

For the rest of the paper we fix an $\aleph_1$-dense Countryman line which we denote by C. For example, to be specific, we fix an $\aleph_1$-dense subordering C of $C(\rho_0)$ (see [24, p. 25]).

Let us recall that $A \equiv B$ iff $A \preceq B$ and $B \preceq A$.

FACT 3.10 ([24]). (MA_{ω_1}) *C is equivalent to any uncountable suborder A of C.*

FACT 3.11. (MA_{ω_1}) *C is equivalent to $C \times C$.*

Proof. It should be clear that $C \preceq C \times C$. So let us show that $C \times C \preceq C$. By Fact 3.9, C is not Suslin so we can find a family I_α ($\alpha \in \omega_1$) of pairwise disjoint non-empty open intervals of C. For each $\alpha < \omega_1$, we fix an element $x_\alpha \in I_\alpha$ and set $X = \{x_\alpha : \alpha < \omega_1\}$. By Fact 3.10 and since C is $\aleph_1$-dense, we can find a strictly increasing map $f_\alpha : C \to I_\alpha$ for all $\alpha < \omega_1$ and a strictly increasing map $f : C \to X$. Define a map $F : C \times C \to C$ by

$$F(x, y) = f_\alpha(y) \quad \text{where} \quad f(x) = x_\alpha.$$

It is easy to check that F is a strictly increasing map. ∎

The following result of Moore [17] is a generalization of the existence of a two-element basis for the class of Aronszajn lines.

THEOREM 3.12 ([17]). (PFA) *If A is an A-line, then either A is equivalent to η_C or else A contains an interval equivalent to C or C^*.*

4. Fine structure theory of A-lines under PFA. Observe that if A is an Aronszajn line which contains η_C, then A is equivalent to η_C. This leads us to the following definition.

DEFINITION 4.1. An A-line A is *fragmented* if $\eta_C \not\preceq A$.

In order to show that the class $\mathcal{A}$ is wqo we need to introduce a notion of rank. Theorem 3.12 gives us a hint on how to associate a rank to each fragmented Aronszajn line: roughly speaking, the rank corresponds to how many applications of a derivative operation are necessary in order to trivialize it. This will be explained in detail below.

Given $A \in \mathcal{A}_F$ consider the following relation:

$$x \sim y \quad \text{iff} \quad [x, y] \preceq \sum_{i \in I} A_i$$

where $I \preceq \mathbb{Q}$ and the linear orderings A_i ($i \in I$) belong to $\{C, C^*\}$. It is clear that $\sim$ is an equivalence relation and that each equivalence class is convex. For each $x \in A$ let $[x]$ denote the equivalence class of x, i.e., $[x] = \{y \in A : x \sim y\}$.

There is a natural map associated to this equivalence relation:

$$c : A \to A^1$$

where $A^1 = \{[x] : x \in A\}$ ordered by $[x] < [y]$ if $x < y$. We call A^1 a *condensation* of A. Thus a condensation map is a map from A into a partition of A into convex intervals. Since every partition of a fragmented A-line into convex intervals is itself a fragmented A-line we can iterate this process as follows:

Definition 4.2. For every ordinal α and any A-line A we construct a condensation map $c^\alpha : A \to A^\alpha$ recursively as follows:

(1) For $\alpha = 0$ let $c^0 = \mathrm{Id}$ and let $A^0 = A$.
(2) For $\alpha = \beta + 1$ set $c^{\beta+1}(x) = \{y : c(c^\beta(x)) = c(c^\beta(y))\}$ and $A^{\beta+1} = (A^\beta)^1$.
(3) For a non-zero limit ordinal α, let $c^\alpha(x) = \bigcup\{c^\beta(x) : \beta < \alpha\}$ where $A^\alpha = \{c^\alpha(x) : x \in A\}$.

We have the following:

Theorem 4.3. *Let $A \in \mathcal{A}_F$. Then there is an ordinal $\alpha < \omega_2$ such that $c^\beta(x) = c^\alpha(x)$ for all $x \in A$ and $\beta \geq \alpha$. The least such α will be called the* C-rank *of A.*

Proof. Note that $c^\alpha(x) \subseteq c^\beta(x)$ for $\alpha \leq \beta$. Since A has size $\aleph_1$ this process must stop for some ordinal less than ω_2. ∎

Note that the condensation of an A-line is itself an A-line. If α is equal to the C-rank of A then A^α is either 1 or does not contain a non-trivial interval embeddable in either C or C^*. In the latter case we find, by virtue of Theorem 3.12, that $\eta_C \preceq A$.

Definition 4.4. For every $\alpha < \omega_2$, recursively define the classes $\mathcal{A}_\alpha$ as follows. Suppose $\alpha < \omega_2$ is given and $\mathcal{A}_\beta$ has been defined for all $\beta < \alpha$.

(1) For $\alpha = 0$, let $\mathcal{A}^0$ denote the class of Countryman lines.
(2) For a non-zero ordinal α, let $\mathcal{A}_\alpha$ be the class of all linear orderings which are equivalent to ones of the form
$$\sum_{i \in I} A_i,$$
where $I \preceq C$ or $I \preceq C^*$ and the linear orderings A_i $(i \in I)$ are from $\bigcup_{\xi<\alpha} \mathcal{A}^\xi$.

We obtain the following theorem which is an analogue of the well-known result of Hausdorff [10] about (countable) scattered linear orders.

Theorem 4.5. (PFA) *The class $\mathcal{A}_F$ of fragmented A-lines admits a decomposition*
$$\mathcal{A}_F = \bigcup_{\xi<\omega_2} \mathcal{A}^\xi.$$

Proof. We will prove by induction on α that any A-line A of C-rank α is in $\bigcup_{\xi<\alpha+2} \mathcal{A}^\xi$. Suppose α is given and that any A-line of C-rank $\beta < \alpha$ is in $\bigcup_{\xi<\beta+2} \mathcal{A}^\xi$. Let A be an A-line of C-rank α.

If $\alpha = 0$, then C has cardinality one, which is impossible.

If $\alpha = \beta + 1$, then $c^\alpha(x) = A$. Since $A^\beta = \{c^\beta(x) : x \in A\}$ is Aronszajn, it has both countable cofinality and countable coinitiality. So let x_n $(n \in \mathbb{Z})$

be such that $x_n < x_m$ for $n < m$ and cofinal and coinitial in A^β. We are focusing on the case where A does not have a first or last element since the argument applies with routine modifications to the degenerate cases. Thus, $[c^\beta(x_n), c^\beta(x_{n+1}))$ $(n \in \mathbb{Z})$ is a partition of A^β. It is sufficient to show that each interval $[x_n, x_{n+1})$ is in $\bigcup_{\xi<\alpha+2} \mathcal{A}^\xi$. For each $n \in \mathbb{Z}$, we fix a set $X_n \subset [x_n, x_{n+1})$ so that $|X_n \cap c^\beta(x)| = 1$ for all $x_n \le x < x_{n+1}$, i.e., X_n is a set of representatives of the interval $[c^\beta(x_n), c^\beta(x_{n+1}))$. We have

$$[x_n, x_{n+1}) = \bigcup_{x \in X_n} c^\beta(x).$$

As $c^\beta(x)$ has C-rank β for all $x \in A$ and $c^\beta(x_n) \sim c^\beta(x_{n+1})$ for all $n \in \mathbb{Z}$, we obtain $X_n \preceq \sum_{i \in I} A_i$, where $I \preceq C$ or $I \preceq C^*$ and A_i is Countryman for each $i \in I$, and $c^\beta(x)$ is in $\bigcup_{\xi<\beta+2} \mathcal{A}^\xi$ for all $x \in A$. Thus, we infer that $[x_n, x_{n+1}) \in \bigcup_{\xi<\alpha+2} \mathcal{A}^\xi$.

If α is a non-zero limit ordinal, then $A = \bigcup_{\beta<\alpha} c^\beta(x)$ for some (any) $x \in A$. First note that if $\operatorname{cof}(\alpha) = \omega_1$ and α_ξ $(\xi \in \omega_1)$ is strictly increasing and cofinal in α then picking an element $x_\xi \in (c^{\alpha_{\xi+1}}(x) \cap [x, \infty)) \setminus c^{\alpha_\xi}(x)$ or $x_\xi \in (c^{\alpha_{\xi+1}}(x) \cap (-\infty, x]) \setminus c^{\alpha_\xi}(x)$ we obtain a copy of ω_1 or ω_1^*, respectively. Therefore, $\operatorname{cof}(\alpha) = \omega$. Let α_n be strictly increasing and cofinal in α. Then

$$A = \bigcup_{n\in\omega} [(c^{\alpha_{n+1}}(x) \cap [x, \infty)) \setminus c^{\alpha_n}(x)] \cup \bigcup_{n\in\omega} [(c^{\alpha_{n+1}}(x) \cap [x, \infty)) \setminus c^{\alpha_n}(x)].$$

Since the C-rank of the intervals $(c^{\alpha_{n+1}}(x) \cap [x, \infty)) \setminus c^{\alpha_n}(x)$ and $(c^{\alpha_{n+1}}(x) \cap (-\infty, x]) \setminus c^{\alpha_n}(x)$ is α_n we infer that $A \in \bigcup_{\xi<\alpha+2} \mathcal{A}^\xi$. ∎

Note that if $\alpha < \beta$, then $\mathcal{A}^\alpha \subseteq \mathcal{A}^\beta$ and we have a natural rank:

Definition 4.6. Given $A \in \mathcal{A}_F$ let $\operatorname{rank}(A) = \min\{\alpha : A \in \mathcal{A}^\alpha\}$.

Note that $A \preceq B$ implies $\operatorname{rank}(A) \le \operatorname{rank}(B)$. We are now ready to prove an important structural result about the class $\mathcal{A}_F$ of fragmented Aronszajn lines.

Lemma 4.7. (MA_{ω_1}) *For every ordinal $\alpha < \omega_2$ there exist two incomparable A-lines D_α^+ and D_α^- of rank α such that:*

1. $C \times D_\alpha^+ \equiv D_\alpha^+$, $C^* \times D_\alpha^- \equiv D_\alpha^-$,
2. $D_\alpha^- \preceq C^* \times D_\alpha^+$, $D_\alpha^+ \preceq C \times D_\alpha^-$,
3. *for every $A \in \mathcal{A}^\alpha$ either $A \equiv D_\alpha^+$ or $A \equiv D_\alpha^-$ or else both $A \prec D_\alpha^+$ and $A \prec D_\alpha^-$.*

Proof. The proof is by induction on α. Suppose that α is given and that D_β^+ and D_β^- satisfying clauses (1)–(3) have been defined for all $\beta < \alpha$.

If $\alpha = 0$, then let $D_0^+ = C$ and $D_0^- = C^*$. Clause (1) follows from Fact 3.11, clause (2) is trivial and clause (3) follows from Fact 3.10.

If $\alpha = \beta + 1$, then let $D^+_\alpha = C \times D^-_\beta$ and $D^-_\alpha = C^* \times D^+_\beta$. Clause (1) follows from Fact 3.11. For (2) note that $D^-_\alpha \preceq C^* \times D^+_\alpha$ is equivalent to $C^* \times D^+_\beta \preceq C^* \times (C \times D^-_\beta)$ which follows from the induction hypothesis $D^+_\beta \preceq C^* \times D^-_\beta$.

In order to prove (3), let $A \in \mathcal{A}^\alpha$ be given. We may assume that $A = \sum_{x\in C} A_x$ where $\operatorname{rank}(A_x) \leq \beta$ for all $x \in C$. We will show that either $A \equiv D^+_\alpha$ or both $A \preceq D^+_\alpha$ and $A \preceq D^-_\alpha$. Let $X = \{x : A_x \equiv D^-_\beta\}$ (note $X \neq \emptyset$, otherwise $A \preceq D^+_\beta$, which has rank $< \alpha$). If X is uncountable, then by Fact 3.10 we have $X \equiv C$. Using the embedding of C into X we obtain $D^+_\alpha \preceq A$, and since $A \preceq D^+_\alpha$ it follows that they are equivalent.

So suppose $D^+_\alpha \not\preceq A$ and hence X is countable.

Consider the following relation on $C \setminus X$:

$$x \sim y \quad \text{iff} \quad [x, y] \cap X = \emptyset.$$

It is easy to see that $\sim$ is an equivalence relation with convex classes.

Since C does not contain uncountable real types we have

$$|\{[a] : a \in C \setminus X\}| = \omega.$$

By the induction hypothesis we see that for each $a \in C \setminus X$ we can write $B_a = \sum_{x\in[a]} A_x$, where $A_x \preceq D^+_\beta$ for all $x \in [a]$.

Since $[a]$ has countable cofinality and coinitiality, we have $B_a \preceq D^+_\beta$. Let I be a set such that $|I \cap [a]| = 1$ for all $a \in C$. Therefore $A = \sum_{i\in(I\cup X)} X_i$, where $X_i = B_i$ for $i \in I$ and $X_i = A_i$ for $i \in X$. Hence X is a countable sum of linear orders which are embeddable into either D^+_β or D^-_β and therefore A is a countable sum of linear orders which embed into both D^+_α and D^-_α.

If α is a non-zero limit ordinal, then let us note that properties (1)–(3) imply that every $A \in \mathcal{A}^{\beta+1} \setminus \mathcal{A}^\beta$ must contain a copy of both D^+_β and D^-_β for all $\beta < \alpha$. Thus, by clause (3) again we infer that A contains every line of smaller rank. Observe that if A has rank α then for each $\beta < \alpha$ there exists an A-line embedded into A with rank greater than β. Thus, we have the following useful property: $\operatorname{rank}(A) < \operatorname{rank}(B) \leq \alpha$ implies $A \preceq B$.

Fix a strictly increasing sequence (α_j) converging to α (we use the convention that $j \in \omega$ or $j \in \omega_1$ depending on whether $\operatorname{cof}(\alpha) = \omega$ or $\operatorname{cof}(\alpha) = \omega_1$, respectively). Let

$$D^+_\alpha = \sum_{x\in C} A_x, \qquad D^-_\alpha = \sum_{x\in C^*} A_x,$$

where $A_x = D^+_{\alpha_j}$ for some j. Moreover, for all j the set $\{x : A_x = D^+_{\alpha_j}\}$ is dense in both C and C^*. By Fact 3.9, C is not Suslin, so let $\mathcal{I} = \{I_\alpha : \alpha < \omega_1\}$ be an uncountable family of pairwise disjoint non-empty intervals of C. We order $\mathcal{I}$ by

$$I < J \quad \text{iff} \quad (\forall x \in I)(\forall y \in J)\ \ x <_C y.$$

Since $\mathcal{I}$ is isomorphic to an uncountable subset of C, we deduce by Fact 3.10 that $C \preceq \mathcal{I}$. This gives us an embedding of $C \times D^+_\alpha$ into D^+_α, i.e., (1) holds. Part (2) should be clear from the definition of D^+_α and D^-_α.

We shall prove that (3) holds. Let $A \in \mathcal{A}^\alpha$ be given; we may assume that $A = \sum_{x\in C} A_x$ where $\operatorname{rank}(A_x) \le \alpha$ for all $x \in C$. Define the following relation on C:

$$a \sim b \quad \text{iff} \quad \sup\{\operatorname{rank}(A_x) : x \in [a,b]\} < \alpha.$$

It is easy to see that $\sim$ is an equivalence relation with convex equivalence classes. Since $[a]$ has countable coinitiality and cofinality it follows that the ordering

$$A'_a = \sum_{x\in[a]} A_x$$

has rank $\le \alpha$. Let $C' \subset C$ be such that $|C' \cap [a]| = 1$ for all $a \in C$. Then we also have the equality

$$A = \sum_{a\in C'} A'_a.$$

The point of this new representation is that if $a, b \in C'$, $a < b$ and $(a, b) \ne \emptyset$ then

$$(\forall j)(\exists x \in (a,b)) \ \ \operatorname{rank}(A_x) > \alpha_j.$$

If this were not the case then we would get $a \sim b$, which is impossible.

Observe that if C' is countable then A is a countable sum of linear orders which embed into both D^+_α and D^-_α. So it suffices to show that if C' is uncountable then $A \equiv D^+_\alpha$. By going to a subset of C', we may assume that C' is $\aleph_1$-dense; using Fact 3.9 we can find an uncountable family

$$I_\xi := \{(a_\xi, b_\xi) : \xi < \omega_1\}$$

of pairwise disjoint non-empty intervals of C'. As before we order $X = \{I_\xi : \xi < \omega_1\}$ by

$$I_\xi \prec I_\eta \quad \text{iff} \quad (\forall x \in I_\xi)(\forall y \in I_\eta)\ x <_C y.$$

By Fact 3.10 there is a strictly increasing map $F : C \to X$. We will use F to construct a map $f : C \to C'$ with the property that $f(x) \in F(x)$. Note that this guarantees that f is strictly increasing. Given $x \in C$ find $a \in F(x)$ such that $\operatorname{rank}(A_x) < \operatorname{rank}(A'_a)$. Thus, f provides an embedding of D^+_α into A; by a similar argument we can obtain the reverse embedding. Therefore A is equivalent to D^+_α, which concludes the proof of the lemma. ■

THEOREM 4.8. (PFA) *The class of Aronszajn lines is bqo under embeddability.*

Proof. We will prove by induction on α that $\mathcal{A}^\alpha$ is bqo. Suppose that α is given and that $\mathcal{A}^\beta$ is bqo for all $\beta < \alpha$.

If $\alpha = 0$, then the result follows from Theorem 1.3 and Fact 3.10.

If $\alpha = \beta + 1$, then let

$$f : [\omega]^\omega \to \mathcal{A}^\alpha$$

be a given Borel map. Consider the partition

$$[\omega]^\omega = X_1 \cup X_2 \cup X_3$$

where

$$\begin{aligned}
X_1 &= \{A \in [\omega]^\omega : f(A) \equiv D^+_\alpha\},\\
X_2 &= \{A \in [\omega]^\omega : f(A) \equiv D^-_\alpha\},\\
X_3 &= \{A \in [\omega]^\omega : f(A) \prec D^+_\alpha \wedge f(A) \prec D^-_\alpha\}.
\end{aligned}$$

By the Galvin–Prikry theorem (see [9]) there is $X \in [\omega]^\omega$ such that $f"[X]^\omega \subset X_i$ for some $i \in \{1,2,3\}$. If $i = 1, 2$ then

$$f(X) \equiv f(X \setminus \{\min(X)\})$$

and the result holds. If $i = 3$ then f maps X into $\mathcal{X} = \mathcal{A}^\alpha \setminus \{B \in \mathcal{A}^\alpha : B \equiv D^+_\alpha \vee B \equiv D^-_\alpha\}$. By the previous lemma every element of $\mathcal{X}$ is a countable sum of linear orders which are embeddable into both D^+_β and D^-_β. Thus, $\mathcal{X}$ can be identified with the set $(\mathcal{A}^\beta)^{\mathcal{C}}$ of Q-types. By Theorem 2.8, $\mathcal{X}$ is bqo and therefore there exists a $Y \in [X]^\omega$ so that

$$f(Y) \preceq f(Y \setminus \{\min(Y)\}).$$

If α is a non-zero limit ordinal, then it follows from Lemma 4.7 that

$$\mathcal{A}^\alpha = \{A : A \equiv D^+_\alpha\} \cup \{A : A \equiv D^-_\alpha\} \cup \Big(\bigcup_{\xi<\alpha} \mathcal{A}^\xi\Big)^{\mathcal{C}}.$$

By Theorem 2.8 and the Galvin–Prikry theorem it is enough to show that $\bigcup_{\xi<\alpha} \mathcal{A}^\xi$ is bqo. Let $f : [\omega]^\omega \to \bigcup_{\xi<\alpha} \mathcal{A}^\xi$ be a given Borel map. Consider the following partition of $[\omega]^\omega$:

$$\begin{aligned}
X_1 &= \{A \in [\omega]^\omega : \operatorname{rank}(f(A)) > \operatorname{rank}(f(A \setminus \{\min(A)\}))\},\\
X_2 &= \{A \in [\omega]^\omega : \operatorname{rank}(f(A)) < \operatorname{rank}(f(A \setminus \{\min(A)\}))\},\\
X_3 &= \{A \in [\omega]^\omega : \operatorname{rank}(f(A)) = \operatorname{rank}(f(A \setminus \{\min(A)\}))\}.
\end{aligned}$$

Again by the Galvin–Prikry theorem there is an $X \in [\omega]^\omega$ such that $f"[X]^\omega \subset X_i$ for some $i \in \{1,2,3\}$. The case $i = 1$ is impossible as it would give us a strictly decreasing sequence of ordinals. Since the case $i = 2$ gives us the desired conclusion, we may assume $i = 3$. By going to an infinite subset X' of X we can assume that $\operatorname{rank}(f(Y))$ is constant for all $Y \in [X']^\omega$; then the conclusion follows from the induction hypothesis. ∎

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