# Covering the real line with translates of a zero-dimensional compact set

by

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**Abstract.** We construct a compact set $C$ of Hausdorff dimension zero such that $\mathrm{cof}(\mathcal{N})$ many translates of $C$ cover the real line. Hence it is consistent with ZFC that less than continuum many translates of a zero-dimensional compact set can cover the real line. This answers a question of Dan Mauldin.

**1. Introduction.** Given a set $A \subset \mathbb{R}$, how many of its translates are needed to cover the real line? It is clear that if $A$ is of Lebesgue measure zero or if $A$ is of the first Baire category, then countable many translates are not enough. Therefore if we assume the continuum hypothesis, then for sets of these kinds clearly we need continuum many translates to cover the real line.

Gary Gruenhage showed that it is not possible to cover $\mathbb{R}$ with less than continuum many translates of the standard middle-third Cantor set. (The proof is in ZFC, without any extra set-theoretic assumptions.) He then raised the following question (see also [2] and [3]):

Is it consistent that there exists a compact set of Lebesgue measure zero such that less than continuum many of its translates cover $\mathbb{R}$?

Since this was unsolved for some time, Dan Mauldin asked whether at least compact sets of Hausdorff dimension less than 1 possess the property that less than continuum many of their translates are not enough to cover $\mathbb{R}$. Udayan Darji and Tamás Keleti [2] showed in a beautiful simple proof that this is indeed true for compact sets of *packing* dimension less than 1.

Márton Elekes and Juris Steprāns solved the question of Gruenhage [3]. They showed that $\mathbb{R}$ can be covered by $\mathrm{cof}(\mathcal{N})$ many translates of some compact set of measure zero of Erdős and Kakutani. Here $\mathrm{cof}(\mathcal{N})$ is the so-called cofinality invariant of Lebesgue null sets, and it is consistent that $\mathrm{cof}(\mathcal{N})$ is less than continuum (see e.g. [1]).

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In this note we answer Dan Mauldin's question in the negative. We present a compact set $C$ of Hausdorff dimension zero such that $\mathbb{R}$ can be covered by $\mathrm{cof}(\mathcal{N})$ many translates of $C$ (Theorem 2.2). Thus it is consistent with ZFC that $\mathbb{R}$ can be covered by less than continuum many translates of some compact set of Hausdorff dimension zero.

We remark here that if we do not restrict ourselves to compact sets then the previous statement is clear: If $A$ is a residual subset of $\mathbb{R}$, and $B \subset \mathbb{R}$ is a set of the second Baire category, then clearly $A + B = \mathbb{R}$. Let us choose $A$ to be a residual $G_\delta$ set of Hausdorff dimension zero (e.g. the set of Liouville numbers). It is consistent that there exists a set of the second Baire category consisting of less than continuum many points (see e.g. [1]); that set should be chosen as $B$. Therefore it is consistent that $\mathbb{R}$ can be covered by less than continuum many translates of a zero-dimensional $G_\delta$ set.

In the next section we review how it is possible to write $\mathbb{R}$ as a union of $\mathrm{cof}(\mathcal{N})$ many "small" compact sets and we state our main results. These are then proved in the third section.

*Notation and definitions.* We denote the cofinality invariant of Lebesgue null sets by $\mathrm{cof}(\mathcal{N})$; it is the smallest cardinal $\kappa$ for which there exist $\kappa$ many null sets such that every null set is contained in one of them.

We denote the $d$-dimensional Hausdorff pre-measure of a set $A \subset \mathbb{R}$ by $\mathcal{H}^d_\infty(A)$, defined as

$$\mathcal{H}^d_\infty(A) = \inf\Big\{\sum_{i=1}^{\infty}(\mathrm{diam}\ I_i)^d : A \subset \bigcup_{i=1}^{\infty} I_i\Big\}.$$

The Hausdorff dimension of $A$ is then defined as $\inf\{d > 0 : \mathcal{H}^d_\infty(A) = 0\}$.

We denote the set of real numbers by $\mathbb{R}$ and the set of integers by $\mathbb{Z}$.

**2. Covering $\mathbb{R}$ with small sets.** Let $\omega$ denote the first infinite ordinal, that is, $\omega = \{0, 1, 2, \ldots\}$. Let $f : \omega \setminus \{0\} \to \omega \setminus \{0, 1\}$. A set of the form $S = \prod_{n=1}^{\infty} A_n$, where each $A_n$ is of size at most $f(n)$, is called an $f$-*slalom* (by $\prod$ we mean the direct product of sets). It is known that if $\lim f(n) = \infty$, then $\omega^\omega$ can be covered by $\mathrm{cof}(\mathcal{N})$ many $f$-slaloms (Gruenhage and Levy [4, Theorem 2.12] and [5]; see also Bartoszyński and Judah [1]).

Hence it is clear that if we fix two functions $f, g : \omega \setminus \{0\} \to \omega \setminus \{0, 1\}$ where $\lim g(n) = \infty$, then every $f$-slalom can be covered by $\mathrm{cof}(\mathcal{N})$ many $g$-slaloms. We describe below how to use this fact to cover the real line with $\mathrm{cof}(\mathcal{N})$ many "small" compact sets (here we follow Elekes and Steprāns [3]). So take the $f$-slalom $T = \prod_{n=1}^{\infty}\{0, 1, \ldots, f(n) - 1\}$, and cover it by $\mathrm{cof}(\mathcal{N})$ many $g$-slaloms $S_\alpha$ ($\alpha < \mathrm{cof}(\mathcal{N})$). We may suppose without loss of generality that each $S_\alpha$ is contained in $T$; that is, they are of the form $\prod_{n=1}^{\infty} A_n$ where

$A_n \subset \{0, 1, \ldots, f(n) - 1\}$ and $|A_n| \leq g(n)$ for each $n$. Then

$$[0,1] = \Big\{ \sum_{n=1}^{\infty} \frac{s_n}{f(1) \ldots f(n)} : (s_n)_{n=1}^{\infty} \in T \Big\}.$$

(It is worth thinking of $s_n$ as digits of a numeral system with "increasing base" corresponding to the series $f(n)$.) For a slalom $S \subset T$ define

$$(1) \qquad S^* = \Big\{ \sum_{n=1}^{\infty} \frac{s_n}{f(1) \ldots f(n)} : (s_n)_{n=1}^{\infty} \in S \Big\},$$

or equivalently, when $S = \prod_{n=1}^{\infty} A_n$,

$$(2) \quad S^* = \Big( \prod_{n=1}^{\infty} A_n \Big)^* = \frac{1}{f(1)} A_1 + \frac{1}{f(1)f(2)} A_2 + \frac{1}{f(1)f(2)f(3)} A_3 + \cdots .$$

(This infinite Minkowski sum is well-defined, and as $S^*$ is a continuous image of the compact space $\prod A_n$, it is compact. For later reference note that $(\prod_{n=1}^{\infty} A_n)^*$ makes sense even if, say, $A_n \subset \{0, 1, \ldots, 2f(n) - 1\}$.)

Since the slaloms $S_\alpha$ cover $T$, the compact sets $S_\alpha^*$ ($\alpha < \mathrm{cof}(\mathcal{N})$) cover the unit interval. The sets $S_\alpha^*$ are small if $g$ is much smaller than $f$. That is, as is easy to see, the Hausdorff dimension of $S_\alpha^*$ is at most

$$\liminf_{n \to \infty} \frac{\log(g(1) \ldots g(n))}{\log(f(1) \ldots f(n))},$$

which can be zero if we choose $g$ and $f$ appropriately. Thus $[0, 1]$ (and also of course $\mathbb{R}$) can be covered by $\mathrm{cof}(\mathcal{N})$ many compact sets of Hausdorff dimension zero. However, our point here is to find one compact set of Hausdorff dimension zero so that $\mathbb{R}$ can be covered by $\mathrm{cof}(\mathcal{N})$ many translates of this compact set. What we will do is to find a compact set $C$ of Hausdorff dimension zero such that for every $g$-slalom $S \subset T$, a translate of $C$ will cover $S^*$. Then clearly $\mathbb{R}$ can be covered by $\mathrm{cof}(\mathcal{N})$ many translates of $C$.

We will choose $g$ to be (say) $g(n) = n + 1$, and the function $f$ will be determined implicitly by the construction.

THEOREM 2.1. *Let $g(n) = n + 1$. There exists a compact set $C$ of Hausdorff dimension zero and a function $f : \omega \setminus \{0\} \to \omega \setminus \{0, 1\}$ such that for every $g$-slalom $S \subset \prod_{n=1}^{\infty} \{0, 1, \ldots, f(n) - 1\}$, $S^*$ can be covered by a translate of $C$.*

We prove this theorem in the next section. As shown above, this immediately yields our main result.

THEOREM 2.2. *There exists a compact set $C$ of Hausdorff dimension zero such that $\mathbb{R}$ can be covered by $\mathrm{cof}(\mathcal{N})$ many translates of $C$.*

**3. Construction.** Before we prove Theorem 2.1, we outline the construction of the set $C$ in an informal way. We need to construct a (small) set $C$ such that every (very small) set $S^* \subset [0,1]$ can be covered by a translate of $C$. First we will find compact sets $K_n \subset [0,2]$ (each a union of finitely many intervals) with arbitrarily small Hausdorff pre-measure (of arbitrarily small dimension) such that every $(n+1)$-point set $B \subset [0,1]$ can be covered by a translate of $K_n$. Then we associate to $K_n$ a large positive integer $f(n)$ and a set $A_n \subset \{0,1,\ldots,2f(n)-1\}$ with the property that every set $B_n \subset \{0,1,\ldots,f(n)-1\}$ with $|B_n| \leq n+1$ can be covered by a translate of $A_n$. Setting $g(n) = n+1$, we find that for every $g$-slalom $\prod B_n$ the set $(\prod B_n)^*$ (as defined by (2)) can be covered by a translate of $(\prod A_n)^*$. Finally, we show that $(\prod A_n)^*$ is of dimension zero using the fact that the sets $K_n$ can be chosen to be "arbitrarily small". Therefore we can let $C$ be $(\prod A_n)^*$.

LEMMA 3.1. *For every $d, \delta > 0$ and every positive integer $n$ there exist a compact set $K_n \subset [0,2]$ and $\varepsilon_n > 0$ such that $\mathcal{H}^d_\infty(K_n) < \delta$ and for every $B \subset [0,1]$ with $|B| = n+1$ there exists $t \in \mathbb{R}$ for which $B + [t, t+\varepsilon_n] \subset K_n$.*

*Proof.* We argue by induction on $n$. We start with the case $n = 1$.
Let $a, b > 0$ and
$$K_1 = [0,a] \cup \Big( [0,2] \cap \bigcup_{i \in \mathbb{Z}} [ia, ia+b] \Big).$$

It is easy to check that if $B \subset [0,1]$ with $|B| = 2$, then there exists $t \in \mathbb{R}$ such that $B + t \subset K_1$. Moreover, we have some freedom in choosing $t$: there exists $\varepsilon_1 > 0$ independent of $B$ such that $B + [t, t+\varepsilon_1] \subset K_1$ for some $t$ (in fact, $\varepsilon_1$ can be chosen to be $b$). The set $K_1$ can be covered by an interval of length $a$ and at most $2/a + 1$ intervals of length $b$, thus $\mathcal{H}^d_\infty(K_1) \leq a^d + (2/a+1)b^d$. Therefore if we choose $a$ small enough, and then $b$ small enough, $\mathcal{H}^d_\infty(K_1)$ can be arbitrarily small.

Now suppose that the conclusion holds for some $n$. As we want a set $K_{n+1}$ with $\mathcal{H}^d_\infty(K_n) < \delta$ for some $d$ and $\delta$, let us start by choosing $K_n$ with $\mathcal{H}^d_\infty(K_n) < \delta/2$ and an $\varepsilon_n > 0$ so that for every $B \subset [0,1]$ with $|B| = n+1$ we have $B + [t, t+\varepsilon_n] \subset K_n$ for some $t$.

Now let $0 < b < \varepsilon_n/2$ be so small that $(2/\varepsilon_n + 1)(2b)^d < \delta/2$. Let
$$K_{n+1} = K_n \cup \Big( [0,2] \cap \bigcup_{i \in \mathbb{Z}} [i\varepsilon_n - b, i\varepsilon_n + b] \Big).$$

Then we have $\mathcal{H}^d_\infty(K_{n+1}) < \delta$.

Let $B \subset [0,1]$ with $|B| = n+2$. We show that $B$ can be translated into the set $K_{n+1}$. First let $x$ be some element of $B$ which is not the smallest or the largest one. Let $B' = B \setminus \{x\}$. Then there is $t'$ such that $B' + [t', t'+\varepsilon_n] \subset K_n$.

It is easy to check that there exists $t$ such that $[t, t+b] \subset [t', t'+\varepsilon_n]$ and $x + [t, t+b] \subset K_{n+1}$. Then clearly $B + [t, t+b] \subset K_{n+1}$. Thus $\varepsilon_{n+1}$ can be chosen to be $b$. ∎

Now, fix some $K_1$ and $\varepsilon_1$ (so that the statement of Lemma 3.1 holds) with $\mathcal{H}^1_\infty(K_1) < 1$. Then fix $K_2$ and $\varepsilon_2$ so that $\mathcal{H}^{1/2}_\infty(K_2) < 100^{-1}\varepsilon_1$. If $K_1, \ldots, K_n$ and $\varepsilon_1, \ldots, \varepsilon_n$ are already fixed, then fix $K_{n+1}$ and $\varepsilon_{n+1}$ so that the statement of Lemma 3.1 holds with

(3) $$\mathcal{H}^{1/(n+1)}_\infty(K_{n+1}) < 100^{-n}\varepsilon_1 \ldots \varepsilon_n.$$

The choice of these bounds will become clear later.

To each $K_n$ and $\varepsilon_n$ we associate $A_n \subset \mathbb{Z}$ and $f(n) \in \omega$ in the following way. Let $f(n)$ be a positive integer with

(4) $$5/\varepsilon_n < f(n) < 10/\varepsilon_n.$$

Let

(5) $$A_n = \left\{ i \in \mathbb{Z} : \left[ \frac{i}{f(n)}, \frac{i+4}{f(n)} \right] \subset K_n \right\} \subset \{0, 1, \ldots, 2f(n) - 1\}.$$

(Note that this definition implies $(1/f(n))A_n + [0, 4/f(n)] \subset K_n$, which we will use (only) in Claim 3.4.)

CLAIM 3.2. *If $B_n \subset \{0, 1, \ldots, f(n) - 1\}$ and $|B_n| = n+1$, then there exists $u \in \mathbb{Z}$ such that $B_n + u \subset A_n$.*

*Proof.* By Lemma 3.1 there exists $t$ such that $B_n/f(n) + [t, t+\varepsilon_n] \subset K_n$, thus $B_n + [tf(n), tf(n) + \varepsilon_n f(n)] \subset f(n)K_n$. By (4), the length of the interval $[tf(n), tf(n) + \varepsilon_n f(n)]$ is at least 5, so it has a subinterval of the form $[u, u+4]$ with integer $u$. Thus $B_n + [u, u+4] \subset f(n)K_n$. Then $B_n + u \subset A_n$. ∎

Let

(6) $$C = \left( \prod_{n=1}^\infty A_n \right)^* = \frac{1}{f(1)}A_1 + \frac{1}{f(1)f(2)}A_2 + \frac{1}{f(1)f(2)f(3)}A_3 + \cdots;$$

then $C$ is compact, by the remark after definition (2).

CLAIM 3.3. *Let $g(n) = n+1$ ($n = 1, 2, \ldots$). Then for every $g$-slalom $S \subset \prod_{n=1}^\infty \{0, 1, \ldots, f(n) - 1\}$, $S^*$ can be covered by a translate of $C$.*

*Proof.* Let $B_n \subset \{0, 1, \ldots, f(n) - 1\}$ with $|B_n| \leq n+1$ be arbitrary ($n = 1, 2, \ldots$). Let $S$ be the $g$-slalom $\prod_{n \geq 1} B_n$. Then by definition (2), we have

$$S^* = \frac{1}{f(1)}B_1 + \frac{1}{f(1)f(2)}B_2 + \frac{1}{f(1)f(2)f(3)}B_3 + \cdots.$$

We can choose an integer $u_n$ such that $B_n + u_n \subset A_n$ by Claim 3.2. Then from (6),

$$S^* + \frac{u_1}{f(1)} + \frac{u_2}{f(1)f(2)} + \cdots \subset C. \quad ∎$$

In order to finish the proof of Theorem 2.1, what is left is to show the following.

CLAIM 3.4. *The Hausdorff dimension of $C$ is zero.*

*Proof.* For $n \geq 1$ let

(7) $$C_n = \frac{1}{f(1)\dots f(n)}A_n + \frac{1}{f(1)\dots f(n+1)}A_{n+1} + \cdots.$$

Thus $C = C_1$ and $C$ is the union of $|A_1|\dots|A_{n-1}|$ many translated copies of $C_n$. First we show that $C_n$ lies in a small neighbourhood of the first term on the right hand side of (7). For $k \geq 1$ we have

$$\sup \frac{1}{f(n)\dots f(n+k)}A_{n+k} \leq \frac{2}{f(n)\dots f(n+k-1)} \leq \frac{2}{f(n)2^{k-1}},$$

thus

$$\sup f(1)\dots f(n-1)C_{n+1} \leq \sum_{k=1}^{\infty} \frac{2}{f(n)2^{k-1}} = \frac{4}{f(n)},$$

and

(8) $$f(1)\dots f(n-1)C_{n+1} \subset \left[0, \frac{4}{f(n)}\right].$$

Since (7) implies $C_n = \frac{1}{f(1)\dots f(n)}A_n + C_{n+1}$, using (8) we obtain

$$f(1)\dots f(n-1)C_n \subset \frac{1}{f(n)}A_n + \left[0, \frac{4}{f(n)}\right].$$

From the definition (5) of $A_n$ we see that the right hand side is contained in $K_n$, thus

$$f(1)\dots f(n-1)C_n \subset K_n.$$

Hence

$$\mathcal{H}_{\infty}^d(C_n) \leq (f(1)\dots f(n-1))^{-d}\mathcal{H}_{\infty}^d(K_n) \leq \mathcal{H}_{\infty}^d(K_n).$$

As already mentioned, $C$ is the union of $|A_1|\dots|A_{n-1}|$ many translated copies of $C_n$, therefore

$$\mathcal{H}_{\infty}^{1/n}(C) \leq |A_1|\dots|A_{n-1}|\mathcal{H}_{\infty}^{1/n}(C_n) \leq |A_1|\dots|A_{n-1}|\mathcal{H}_{\infty}^{1/n}(K_n)$$
$$\leq (2f(1))\dots(2f(n-1))\mathcal{H}_{\infty}^{1/n}(K_n).$$

Using (3) and the upper bound for $f(n)$ in (4) we obtain

$$\mathcal{H}_{\infty}^{1/n}(C) \leq 20^{n-1}\frac{1}{\varepsilon_1\dots\varepsilon_{n-1}}100^{-(n-1)}\varepsilon_1\dots\varepsilon_{n-1} = 5^{-(n-1)}.$$

Thus for all $d > 0$ and all $n > 1/d$ we have $\mathcal{H}_{\infty}^d(C) \leq \mathcal{H}_{\infty}^{1/n}(C) \leq 5^{-(n-1)}$, therefore $\mathcal{H}_{\infty}^d(C) = 0$. So the Hausdorff dimension of the set $C$ is zero. ∎

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