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Embedding of a planar rational compactum into a planar continuum with the same rim-type

by

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Abstract. We prove that every planar rational compactum with rim-type $\le \alpha$, where α is a countable ordinal greater than 0, can be topologically embedded into a planar rational (locally connected) continuum with rim-type $\le \alpha$.

1. Introduction. All spaces under consideration are separable and metrizable. A space is said to be *planar* if it is homeomorphic to a subset of the plane. A space is said to be *rational* if it has a basis consisting of open sets with countable boundaries.

Let X be a space. For every ordinal α the *α-derivative* $X^{(\alpha)}$ of X is defined by induction as follows: $X^{(0)} = X$, $X^{(\alpha+1)}$ is the set of all limit points of $X^{(\alpha)}$ (in $X^{(\alpha)}$), and $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$ for α a limit ordinal ([5], IV, §24). If $X^{(\alpha)} = \emptyset$ for some ordinal α, then the least such ordinal is called the *type* of X and is denoted by $\mathrm{type}(X)$. Obviously, $\mathrm{type}(X) = 0$ iff $X = \emptyset$.

It is easy to show that a compactum X is countable iff it has a type. Note that the type of any countable compactum is an isolated countable ordinal and there exists a compactum of type α for any isolated countable ordinal α ([6]).

The following statements are easily proved and will be used in what follows:

(1.1) *If M and N are compacta*, $\mathrm{type}(M) \le \alpha$ *and* $\mathrm{type}(N) \le \alpha$, *then* $\mathrm{type}(M \cup N) \le \alpha$.

(1.2) *If f is a continuous map of a countable compactum M onto N*, *then* $\mathrm{type}(N) \le \mathrm{type}(M)$.

(1.3) *If f is a continuous map of a countable compactum M onto N such that $f^{-1}(x)$ is finite for every $x \in N$*, *then* $\mathrm{type}(N) = \mathrm{type}(M)$.

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By the *rim-type* of a rational compactum X is meant the least ordinal α for which there exists a basis of X consisting of open sets whose boundaries are of type $\leq \alpha$. Obviously the rim-type of any rational compactum is a countable ordinal. From now on, α denotes a countable ordinal greater than 0. Clearly rim-type$(X) = 0$ iff X is zero-dimensional.

We quote the following results concerning embeddings of a rational compactum into a rational continuum:

(a) For every countable ordinal α there exists a planar rational (locally connected) continuum with rim-type $\leq \alpha + 1$ containing topologically any planar rational compactum with rim-type $\leq \alpha$ ([7]).

(b) Every rational compactum (not necessarily planar) with rim-type $\leq \alpha$ is contained topologically in a rational continuum with rim-type $\leq \alpha$ ([2]), Theorem 3).

In this paper we prove that every planar rational compactum with rim-type $\leq \alpha$ can be topologically embedded into a planar rational (locally connected) continuum with rim-type $\leq \alpha$.

2. Definitions and notations. Let E^2 be the plane with a system Oxy of orthogonal coordinates. By a *disk* we mean a subset of E^2 homeomorphic to $\{(x,y) \mid x^2 + y^2 \leq 1\}$. An *arc* is a subset A of E^2 for which there exists a homeomorphism h of the segment $I \equiv [0,1]$ onto A. The points $h(0)$ and $h(1)$ are the *endpoints* of A.

By L_n, $n = 1, 2, \ldots$, we denote the set of all ordered n-tuples $i_1 \ldots i_n$, where $i_t = 0$ or $i_t = 1$, for every $t = 1, \ldots, n$. We set $L_0 = \{\emptyset\}$. For $\bar{i} = i_1 \ldots i_n \in L_n$, $n \geq 1$, we set $\bar{i}0 = i_1 \ldots i_n 0$ and $\bar{i}1 = i_1 \ldots i_n 1$. For $\bar{i} = \emptyset \in L_0$ we define $\bar{i}0 = 0$ and $\bar{i}1 = 1$.

By $I_{\bar{i}}$, where $\bar{i} = i_1 \ldots i_n \in L_n$, $n \geq 1$, we denote the set of all points of I for which the tth digit of the dyadic expansion, $t = 1, \ldots, n$, is i_t. For $\bar{i} = \emptyset \in L_0$ we set $I_{\bar{i}} = I_{\emptyset} = I$. We denote by $m(\bar{i})$ the midpoint of $I_{\bar{i}}$.

Let $\mathcal{W}_n = \{I_{\bar{i}} \times I_{\bar{j}} \mid \bar{i}, \bar{j} \in L_n\}$, $n = 0, 1, \ldots$ Obviously for every $n = 0, 1, \ldots$ the family $\mathcal{W}_n$ is a finite closed covering of I^2 by disks. We set $\mathrm{Bd}(\mathcal{W}_n) = \bigcup\{\mathrm{Bd}(F) \mid F \in \mathcal{W}_n\}$. Obviously, $\mathrm{Bd}(\mathcal{W}_n) \subseteq \mathrm{Bd}(\mathcal{W}_{n+1})$. Note that

$$\mathrm{Bd}(\mathcal{W}_{n+1}) \setminus \mathrm{Bd}(\mathcal{W}_n) = \bigcup_{\bar{i}, \bar{j} \in L_n} ((I_{\bar{i}} \times \{m(\bar{j})\}) \cup (\{m(\bar{j})\} \times I_{\bar{i}})).$$

By π_1 and π_2 we denote the first and the second projection, respectively, of I^2 onto I. We set

$$\mathcal{V} = \{(m/2^n, k/2^n) \in I^2 \mid n, m, k = 0, 1, \ldots\}.$$

Let C be the Cantor ternary set and $C(1)$ be the set of all points of C which are the endpoints of the components of $[0,1] \setminus C$.

For $\overline{i} = i_1 \ldots i_n \in L_n$, $n \geq 1$, we denote by $C_{\overline{i}}$ the set of all points of C for which the tth digit of their ternary expansion, $t = 1, \ldots, n$, is 0 if $i_t = 0$, and 2 if $i_t = 1$. For $\overline{i} = \emptyset \in L_0$ we set $C_{\overline{i}} = C_\emptyset = C$. Also we write $a(\overline{i}) = \max\{x \mid x \in C_{\overline{i}0}\}$ and $b(\overline{i}) = \min\{x \mid x \in C_{\overline{i}1}\}$ for every $\overline{i} \in \bigcup_{n=0}^{\infty} L_n$.

For every $c \in C$ and $n = 0, 1, \ldots$ we denote by $\overline{i}(c, n)$ the unique $\overline{i} \in L_n$ such that $c \in C_{\overline{i}}$. Obviously, $\{c\} = \bigcap_{n=0}^{\infty} C_{\overline{i}(c,n)}$.

3. Lemma. *Let D be a disk in the plane, $a, b \in \operatorname{Bd}(D)$, $a \neq b$, and $X \subseteq D \setminus \{a, b\}$ be a rational compactum with* rim-type$(X) \leq \alpha$. *Then there exists an arc $A \subseteq D$ with endpoints a, b such that* type$(A \cap X) \leq \alpha$.

Proof. Let A_1 and A_2 be the arcs of D with endpoints a, b such that $A_1 \cup A_2 = \operatorname{Bd}(D)$. It is clear that $X \cap A_1$ and $X \cap A_2$ are closed disjoint subsets of $X \cap D$. Since rim-type$(X \cap D) \leq \alpha$, there exists a closed subset F of $X \cap D$ such that type$(F) \leq \alpha$ and F separates the sets $X \cap A_1$ and $X \cap A_2$ in $X \cap D$ ([3], Theorem 6).

The rest of the proof which provides an arc $A \subseteq D$ with endpoints a, b such that $A \cap X \subseteq F$ is the same as the corresponding part of the proof of Lemma 5 in [1].

4. Lemma. *For every planar rational compactum X with* rim-type(X) $\leq \alpha$ *there exists a homeomorphism h of X onto a subset of I^2 such that*:

(1h) $h(X) \subseteq \operatorname{Int}(I^2)$,
(2h) $h(X) \cap \mathcal{V} = \emptyset$,
(3h) type$(h(X) \cap \operatorname{Bd}(\mathcal{W}_n)) \leq \alpha$ *for every* $n = 1, 2, \ldots$

Proof. Let X be a planar rational compactum with rim-type$(X) \leq \alpha$. We set

$$Q_\Delta = \{m/2^n \in I \setminus \{0,1\} \mid m, n = 0, 1, \ldots\},$$
$$Q_T = \{m/3^n \in I \mid m, n = 0, 1, \ldots\}.$$

Observe that $x \in \operatorname{Int}(I^2) \cap \bigcup_{n=0}^{\infty} \operatorname{Bd}(\mathcal{W}_n)$ iff either $\pi_1(x) \in Q_\Delta$ or $\pi_2(x) \in Q_\Delta$, and $x \in \operatorname{Int}(I^2) \cap \mathcal{V}$ iff $\pi_1(x), \pi_2(x) \in Q_\Delta$.

Let $Y = I^2 \setminus (((I \setminus Q_T) \times Q_\Delta) \cup (Q_\Delta \times (I \setminus Q_T)))$.

It is proved in [1] that Y is a rational containing space for the family of all planar rational compacta. Since X is a planar rational compactum, there exists a homeomorphism h of X onto a subspace of Y. Moreover, in [1] the above homeomorphism is constructed in such a manner that condition (1h) is satisfied. Since $Y \cap \operatorname{Int}(I^2) \cap \mathcal{V} = \emptyset$, condition (2h) holds.

Since $Y \cap \operatorname{Bd}(\mathcal{W}_n)$ is countable for every $n = 1, 2, \ldots$, it follows that $h(X) \cap \operatorname{Bd}(\mathcal{W}_n)$ is countable for every $n = 1, 2, \ldots$

It remains to prove that h satisfies condition (3h). To do that, we slightly modify the construction of h in [1] applying Lemma 3 of the present paper instead of Lemma 5 of [1].

5. Theorem. *Every planar rational compactum X with* rim-type$(X) \le \alpha$ *can be homeomorphically embedded into a planar rational locally connected continuum D with* rim-type$(D) \le \alpha$.

Proof. Let X be a planar rational compactum with rim-type$(X) \le \alpha$. By Lemma 4 we can assume that:

(i) $X \subseteq \mathrm{Int}(I^2)$,
(ii) $X \cap \mathcal{V} = \emptyset$,
(iii) $\mathrm{type}(X \cap \mathrm{Bd}(\mathcal{W}_n)) \le \alpha$ for every $n = 1, 2, \ldots$

We define a map f of C^2 onto I^2 as follows: if $c = (c_1, c_2) \in C^2$ and $\{c\} = \bigcap_{n=0}^{\infty}(C_{\bar{i}(c_1,n)} \times C_{\bar{i}(c_2,n)})$ then $f(c) = \bigcap_{n=0}^{\infty}(I_{\bar{i}(c_1,n)} \times I_{\bar{i}(c_2,n)})$. It is easily seen that f has the following properties:

(1f) $f(C_{\bar{i}} \times C_{\bar{j}}) = I_{\bar{i}} \times I_{\bar{j}}$ for all $\bar{i}, \bar{j} \in L_n$, $n = 0, 1, \ldots,$
(2f) $f^{-1}(x)$ is a singleton iff neither $\pi_1(x)$ nor $\pi_2(x)$ is in Q_Δ,
(3f) $f^{-1}(x)$ consists of two points iff exactly one of $\pi_1(x)$ and $\pi_2(x)$ is in Q_Δ,
(4f) $f^{-1}(x)$ consists of four points iff $\pi_1(x), \pi_2(x) \in Q_\Delta$,
(5f) f is a continuous map.

Let $x \in X$. Since $X \subseteq \mathrm{Int}(I^2) \setminus \mathcal{V}$, it follows that either $\pi_1(x) \notin Q_\Delta$ or $\pi_2(x) \notin Q_\Delta$. From (2f) and (3f) we conclude that $f^{-1}(x)$ consists of at most two points. In particular, $f^{-1}(x)$ is a singleton iff $x \notin \bigcup_{n=0}^{\infty} \mathrm{Bd}(\mathcal{W}_n)$, and $f^{-1}(x)$ is a two-element subset of C^2 iff $x \in \bigcup_{n=0}^{\infty} \mathrm{Bd}(\mathcal{W}_n)$.

It is easy to verify that

$$\mathrm{Int}(I^2) \cap \bigcup_{n=0}^{\infty} \mathrm{Bd}(\mathcal{W}_n) = \bigcup_{n=0}^{\infty} \bigcup_{\bar{i},\bar{j} \in L_n} ((I_{\bar{i}} \times \{m(\bar{j})\}) \cup (\{m(\bar{j})\} \times I_{\bar{i}})).$$

Thus $X \cap \bigcup_{n=0}^{\infty} \mathrm{Bd}(\mathcal{W}_n)$ is the union of the sets of the form $X \cap (I_{\bar{i}} \times \{m(\bar{j})\})$ and $X \cap (\{m(\bar{j})\} \times I_{\bar{i}})$, where $\bar{i}, \bar{j} \in L_n$, $n = 0, 1, \ldots$

Let $\bar{i}, \bar{j} \in L_n$, $n = 0, 1, \ldots$ From the definition of f it follows that if $x \in (I_{\bar{i}} \times \{m(\bar{j})\}) \setminus \mathcal{V}$, then $f^{-1}(x) = \{c_1, c_2\}$, where $\pi_1(c_1) = \pi_1(c_2) \in C_{\bar{i}} \setminus C(1)$ and $\{\pi_2(c_1), \pi_2(c_2)\} = \{a(\bar{j}), b(\bar{j})\}$.

Therefore $f^{-1}((I_{\bar{i}} \times \{m(\bar{j})\}) \setminus \mathcal{V}) = (C_{\bar{i}} \setminus C(1)) \times \{a(\bar{j}), b(\bar{j})\}$. Similarly, $f^{-1}((\{m(\bar{j})\} \times I_{\bar{i}}) \setminus \mathcal{V}) = \{a(\bar{j}), b(\bar{j})\} \times (C_{\bar{i}} \setminus C(1))$.

Since $X \cap (I_{\bar{i}} \times \{m(\bar{j})\})$ and $X \cap (\{m(\bar{j})\} \times I_{\bar{i}})$ are subsets of $\mathrm{Bd}(\mathcal{W}_{n+1})$, from the property (iii) of X it follows that $\mathrm{type}(X \cap (I_{\bar{i}} \times \{m(\bar{j})\})) \le \alpha$ and $\mathrm{type}(X \cap (\{m(\bar{j})\} \times I_{\bar{i}})) \le \alpha$.

Let $\bar{i} \in L_n$, $n = 0, 1, \ldots$ We set

$$P_{\bar{i}}^X = \bigcup_{\bar{j} \in L_n} (\pi_1(f^{-1}(X \cap (I_{\bar{i}} \times \{m(\bar{j})\}))) \cup \pi_2(f^{-1}(X \cap (\{m(\bar{j})\} \times I_{\bar{i}})))),$$

where π_1 (resp. π_2) is the projection of C^2 onto the first (resp. second) coordinate.

Since $X \cap \mathcal{V} = \emptyset$, we have $P_{\bar{i}}^X \subseteq C_{\bar{i}} \setminus C(1)$. From (1.1)–(1.3) it follows that $\operatorname{type}(P_{\bar{i}}^X) \le \alpha$. Since $\alpha > 0$, there exists a compact subset $P_{\bar{i}}$ of $C_{\bar{i}} \setminus C(1)$ such that

(i) $P_{\bar{i}}^X \subseteq P_{\bar{i}}$,
(ii) $\operatorname{type}(P_{\bar{i}}) = \operatorname{type}(P_{\bar{i}}^X)$ (therefore $\operatorname{type}(P_{\bar{i}}) \le \alpha$),
(iii) $P_{\bar{i}} \cap C_{\bar{i}0} \ne \emptyset$ and $P_{\bar{i}} \cap C_{\bar{i}1} \ne \emptyset$.

For every $\bar{i}, \bar{j} \in L_n$, $n = 0, 1, \ldots$, we define a collection $D(\bar{i}, \bar{j})$ of two-element subsets of C^2 as follows: $\{c_1, c_2\} \in D(\bar{i}, \bar{j})$ iff either $\pi_1(c_1) = \pi_1(c_2) \in P_{\bar{i}}$ and $\{\pi_2(c_1), \pi_2(c_2)\} = \{a(\bar{j}), b(\bar{j})\}$ or $\pi_2(c_1) = \pi_2(c_2) \in P_{\bar{j}}$ and $\{\pi_1(c_1), \pi_1(c_2)\} = \{a(\bar{i}), b(\bar{i})\}$.

We set $D(1) = \bigcup_{n=0}^{\infty} \{D(\bar{i}, \bar{j}) \mid \bar{i}, \bar{j} \in L_n\}$.

Let D be the partition of C^2 consisting of all elements of $D(1)$ and all singletons $\{c\}$, where $c \in C^2$ and $c \notin \bigcup \{d \mid d \in D(1)\}$. It has been proved ([4], Lemma 4) that D is an upper semicontinuous partition of C^2 and that the corresponding quotient space D is a planar locally connected continuum with rim-type$(D) \le \alpha$.

Let $p : C^2 \to D$ be the quotient mapping. Since the set $f(p^{-1}(d))$ is a singleton for every $d \in D$, we can define a mapping $g : D \to I^2$ by letting $g(d) = f(p^{-1}(d))$ for $d \in D$. The mapping g is continuous since the composition $f = g \circ p$ is continuous and p is a quotient map. It is easy to see that $g|_{g^{-1}(X)} : g^{-1}(X) \to X$ is one-to-one, hence $g^{-1}(X)$ is homeomorphic to X. It follows that X embeds in D.

The proof of the theorem is complete.

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