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Countably convex G_δ sets

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Abstract. We investigate *countably convex G_δ* subsets of Banach spaces. A subset of a linear space is countably convex if it can be represented as a countable union of convex sets. A known sufficient condition for countable convexity of an arbitrary subset of a separable normed space is that it does not contain a *semi-clique* [9]. A semi-clique in a set S is a subset $P \subseteq S$ so that for every $x \in P$ and open neighborhood u of x there exists a finite set $X \subseteq P \cap u$ such that $\operatorname{conv}(X) \not\subseteq S$. For closed sets this condition is also necessary.

We show that for countably convex G_δ subsets of infinite-dimensional Banach spaces there are no necessary limitations on cliques and semi-cliques.

Various necessary conditions on cliques and semi-cliques are obtained for countably convex G_δ subsets of finite-dimensional spaces. The results distinguish dimension $d \le 3$ from dimension $d \ge 4$: in a countably convex G_δ subset of $\mathbb{R}^3$ all cliques are scattered, whereas in $\mathbb{R}^4$ a countably convex G_δ set may contain a dense-in-itself clique.

1. Introduction. Let S be a subset of a linear space. Let $\gamma(S)$ denote the least cardinality of a collection of convex sets whose union is equal to S. A subset of a linear space is called *countably convex* if $\gamma(S) \le \aleph_0$, that is, when $S = \bigcup_n C_n$, where each C_n is convex. We are interested in structural properties of countably convex sets, especially in characterization and classification of countably convex subsets of Banach spaces.

Historically, the characterization of unions of convex sets began with Valentine's paper [14] in which a closed planar set with no *visually independent* subset of size 3 was shown to be the union of 3 convex sets; a subset $P \subseteq S$ is visually independent in S if no two points of P "see" each other through S. Much of the work that followed Valentine dealt with connections between γ and visually independent subsets in closed planar sets ([5, 2, 1, 3, 13, 12]).

There is a natural framework for discussing convexity numbers, which we now present. With every set S in an arbitrary linear space, one associates

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a hypergraph $G(S)$, called the *convexity hypergraph*. The vertices of G are the points of S and the hyperedges of G are all finite subsets $X \subseteq S$ with $\mathrm{conv}(X) \nsubseteq S$. A hyperedge of $G(S)$ is called a *finite defected subset* of S. Much of the geometry of S is lost when passing from S to $G(S)$ (for instance, one cannot tell from $G(S)$ whether three points of S are co-linear or not), but $G(S)$ determines $\gamma(S)$ because $\gamma(S)$ is equal to the chromatic number of $G(S)$ [9]. Recall that the *chromatic number* of a hypergraph is the minimal size of a partition of the set of vertices into free sets, and a set of vertices is *free* if it does not contain hyperedges. Recall also that a set of vertices is a *k-clique* if every k-element subset of it is a hyperedge, and is a *clique* if it is a k-clique for some $k \geq 2$.

Countable convexity of closed sets in Banach spaces is characterized by *topological* properties of $G(S)$:

Definition 1 ([9]). Let S be a subset of a topological linear space. A set $P \subseteq S$ is a *semi-clique* in S if for every $p \in P$ and open neighborhood $u \ni p$, there is a finite set $X \subseteq P \cap u$ so that $\mathrm{conv}(X) \nsubseteq S$.

A clique is a semi-clique if and only if it is dense in itself, and a semi-clique does not have to be a clique or to contain one. It is readily verified that in every set S the union of all semi-cliques in S is a maximal and closed semi-clique in S, which is called the *convexity radical* of S, and is denoted by $K(S)$.

In [9] it was shown that for every subset S of a separable normed linear space, the set $S \setminus K(S)$ is countably convex. Thus, a subset of a Banach space is countably convex if and only if its radical is coverable by countably many convex subsets of the set. In a closed set S, a nonempty $K(S)$ cannot be covered by countably many convex subsets of S because of the Baire category theorem. Therefore:

Theorem 2 ([9]). *A closed subset of a Banach space is countably convex if and only if it does not contain a semi-clique.*

In the present paper, semi-cliques and cliques are examined in countably convex G_δ subsets of Banach spaces. Theorem 2 above fails badly for G_δ subsets of Banach spaces: in every infinite-dimensional Banach space there exists a G_δ set $S = \bigcup C_n$, where each C_n is convex and dense in S and so that there exists a dense-in-itself 2-clique $P \subseteq S$ which is dense in S (see Corollary 12 below). This means that although S is countably convex, all of S is contained in $K(S)$, and worse: the radical contains a dense and dense-in-itself 2-clique.

In a finite-dimensional space, however, the situation is better. The radical of a countably convex G_δ set cannot be dense in the set. In fact, it must be nowhere dense in the set (Theorem 3 below).

Can there be a dense-in-itself clique inside the radical of a countably convex G_δ subset of a finite-dimensional space? Yes, if $d \geq 4$ (Example 4 below); no, if $d \leq 3$ (Theorem 5 below).

Finally, can any limitation on the dimension make Theorem 2 hold for G_δ sets? The reader can easily check that Theorem 2 holds for G_δ subsets of $\mathbb{R}^1$, but by Example 6 below it does not hold in any higher dimension.

A word about the method. A G_δ subset of a Banach space satisfies the Baire category theorem; therefore, if it is countably convex and contains a radical [a dense-in-itself clique], one of its convex components is somewhere dense in the radical [in the dense-in-itself clique]. The interior of that convex component cannot contain any point from the radical, so the "bad" pattern is contained in the topological boundary of that convex component. This observation reduces the problem to the geometry of cliques which are contained between convex sets and their closure in some G_δ set. In a finite-dimensional space, a convex set always has a relative interior. This is the source of distinction between finite and infinite-dimensions.

The precise geometric property which differentiates the behavior of countably convex G_δ sets in $\mathbb{R}^3$ from that in $\mathbb{R}^4$ comes from the combinatorics of convex polytopes. For every simplicial convex polytope $T \subseteq \mathbb{R}^3$ (a polytope whose faces are triangles) with $\geq 4k$ vertices there exists a subset of k vertices any two of which are connected by inner diagonals. However, in $\mathbb{R}^4$ there exist simplicial polytopes with any prescribed number of vertices and with no inner diagonals at all (see [6], p. 61, or Example 4 below).

The results are presented in two sections, which can be read independently of each other. Section 2 deals with G_δ sets in finite-dimensional spaces and Section 3 is devoted to a construction of "bad" countably convex G_δ subsets in every infinite-dimensional Banach space.

Our notation is standard. In every normed linear space X, B_X and S_X denote, respectively, the unit ball and the unit sphere.

2. Finite-dimensional spaces. In this section we investigate countably convex G_δ subsets of finite-dimensional spaces. We prove several necessary conditions on the radicals of such sets. In particular, Theorem 5 and Example 4 reveal a surprising dependence of radicals on the dimension of the space.

THEOREM 3. *If a G_δ set $S \subseteq \mathbb{R}^n$ is countably convex, then the convexity radical of S is nowhere dense in S.*

Proof. Assume to the contrary that the theorem fails. Fix a countably convex $S \subseteq \mathbb{R}^n$ with radical $K = K(S)$ so that there exists an open ball u such that $K \cap u$ is dense in $S \cap u$. Since K is closed in S, $K \cap u = S \cap u$.

Since $S = \bigcup C_k$ where each C_k is convex, for some k and an open ball $v \subseteq u$, $C_k \cap v$ is dense in $S \cap v$.

Set $C = C_k \cap v$; then C is convex and dense in $S \cap v = K \cap v$.

Let H be the affine span of C. Since K has no isolated points, C contains at least two points, and therefore $\dim H > 0$. We have $S \cap v = K \cap v \subseteq \mathrm{cl}\, C \subseteq H$ (the assumption of finite-dimensionality is used here, since H is closed in $\mathbb{R}^d$). Choose an open ball w so that $\emptyset \neq w \cap C \subseteq \mathrm{int}_H C$. Since $S \cap w \subseteq H$ and $H \cap w \subseteq C$, we have $K \cap w \subseteq C$, which is a contradiction, since $C \subseteq S$ is convex and $K \cap w$ contains defected subsets. ■

The next thing to examine is whether the radical of a finite-dimensional countably convex G_δ set may contain a dense-in-itself clique.

Example 4. *There is a countably convex G_δ set in $\mathbb{R}^4$ which contains a dense-in-itself 2-clique.*

Proof. In the construction we use the moments curve in $\mathbb{R}^4$ and its supporting hyperplanes as presented in the construction of a cyclic polytope in [6], p. 61. For a real number t, let $L(t) = (t, t^2, t^3, t^4) \in \mathbb{R}^4$, $L = \{L(t) : t \in [0,1]\}$ and $S_1 = \mathrm{conv}(L)$. Clearly, L is compact, and therefore S_1 is a compact convex subset of $\mathbb{R}^4$.

For any two reals t_1, t_2, the polynomial $(t - t_1)^2 (t - t_2)^2 = \sum_{i=0}^4 a_i t^i$ takes strictly positive values at $t \neq t_1, t_2$ and 0 at $t = t_1, t = t_2$. Define an affine functional $\varphi_{t_1,t_2}(x) = a_0 + \sum_{i=1}^4 a_i x_i$, $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$.

It is clear that for $t_1, t_2 \in [0,1]$, the hyperplane $H_{t_1,t_2} = \{x \in \mathbb{R}^4 : \varphi_{t_1,t_2}(x) = 0\}$ is a supporting hyperplane for S_1 and that $H_{t_1,t_2} \cap S_1 = [L(t_1), L(t_2)]$, so

$$S_1 - [L(t_1), L(t_2)] \text{ is convex.} \tag{2.1}$$

For $t_1, t_2 \in [0,1]$ let $y_{t_1,t_2} = (L(t_1) + L(t_2))/2$. Define

$$S = S_1 - \{y_{t_1,t_2} : t_1, t_2 \in \mathbb{Q} \cap [0,1],\ t_1 \neq t_2\}. \tag{2.2}$$

Then S is a G_δ set, since it is obtained by subtracting a countable set from a closed set. The set

$$\begin{aligned} S - \bigcup \{[L(t_1), L(t_2)] &: t_1, t_2 \in \mathbb{Q} \cap [0,1]\} \\ &= \bigcap \{S_1 - [L(t_1), L(t_2)] : t_1, t_2 \in \mathbb{Q} \cap [0,1]\} \end{aligned} \tag{2.3}$$

is convex by (2.1) and therefore the set

$$\begin{aligned} S = S - \bigcup &\{[L(t_1), L(t_2)] : t_1, t_2 \in \mathbb{Q} \cap [0,1]\} \\ &\cup \bigcup_{t_1, t_2 \in \mathbb{Q} \cap [0,1]} [L(t_1), y_{t_1,t_2}) \end{aligned} \tag{2.4}$$

is countably convex.

Finally, put $P = \{L(t) : t \in \mathbb{Q} \cap [0,1]\}$. From the construction it is clear that $P \subseteq S$, that P is dense in itself and that P is a 2-clique in S. ∎

THEOREM 5. *If $d \le 3$ then a countably convex G_δ subset of $\mathbb{R}^d$ does not contain a clique which is dense in itself.*

Proof. By induction on $d \le 3$. If $d = 0$ then no subset of $\mathbb{R}^d$ is dense in itself.

Suppose now that $0 < d \le 3$ and, to the contrary, that $S \subseteq \mathbb{R}^d$ is a G_δ set, $S = \bigcup_n C_n$ where each C_n is convex and that $P \subseteq S$ has no isolated points and is a k-clique for some $k \ge 2$.

Set $\overline{P} = \mathrm{cl}_S P$, the closure of P in S. Since $\overline{P}$ is closed in S and S is G_δ, $\overline{P}$ satisfies the Baire category theorem, and therefore for some n and an open ball u, the set $u \cap C_n \cap \overline{P}$ is dense in $u \cap \overline{P}$. Define $C = C_n \cap u$. Thus C is convex and $C \cap \overline{P}$ is dense in $\overline{P} \cap u$. Therefore, $P \cap u \subseteq \mathrm{cl}_S C$.

Let H be the affine span of C, and consider $S' = S \cap H$. This is again a countably convex G_δ set. Since H is closed, $\mathrm{cl}\, C \subseteq H$, so $S \cap H$ contains a dense-in-itself clique. If $\dim H < d$, then a contradiction to the induction hypothesis is reached. Assume, then, that the affine span of C is $\mathbb{R}^d$ (so C has nonempty interior in $\mathbb{R}^d$). The interior of C contains no points from P, so P is contained in the boundary of C.

If $d = 1$ then again a contradiction is reached, since the boundary of C is finite and cannot contain a dense-in-itself set. Suppose, then, that $d > 1$.

We next argue that P is contained in a finite union of hyperplanes in $\mathbb{R}^d$. Suppose this is not so. Then we can choose a sufficiently large subset $\{x_0, \ldots, x_{l-1}\} \subseteq P$ in general position (and, since all points belong to ∂C, it will also be in convex position) and set $T = \mathrm{conv}(x_0, \ldots, x_{l-1})$. Then T is a convex polytope in $\mathbb{R}^d$ whose vertices are $\{x_0, \ldots, x_{l-1}\}$. If $d = 2$ then T is a polygon and if $d = 3$ then T is a simplicial polytope (that is, every face of T is a triangle). To contradict the assumption that P is a k-clique it is enough to find a k-element subset $\{y_0, \ldots, y_{k-1}\}$ of vertices so that $\mathrm{conv}(y_0, \ldots, y_{k-1}) - \{y_0, \ldots, y_{k-1}\} \subseteq \mathrm{int}\, T$.

If $d = 2$ then $l = 2k$ suffices, since for a polygon with $2k$ vertices, taking every second vertex in a cyclical ordering gives a polygon that, except for its vertices, is contained in the interior of the original polygon. If $d = 3$ then let $l = 4k$. Since the graph of T is planar, the vertices of T are colorable by 4 colors so that no two vertices which are joined by an edge have the same color (by the 4-colors theorem). Find k vertices with the same color, and observe that in a simplicial polytope in $\mathbb{R}^3$, two vertices which are not joined by an edge are joined by an inner diagonal.

Since P is contained in a finite union of hyperplanes, there exists a hyperplane F so that $P \cap F$ contains a dense-in-itself set. Now the induction hypothesis is violated by $S \cap F$. This completes the proof. ∎

Remarks. Our first remark is that the full power of the 4-colors theorem is not really needed. All we need is *some* fixed bound on the chromatic number of all graphs of 3-polytopes—and a bound of 5 follows easily from Euler's equation and is known since the middle of the 19th century. Our second remark is that lower and upper bounds on the number of inner diagonals in a polytope were computed in [4].

We conclude this section with an example that shows that even in $\mathbb{R}^2$ there are countably convex G_δ sets with nonempty radicals. It is easy to check that in $\mathbb{R}^1$ no such example exists.

Example 6. *There exists a G_δ set in $\mathbb{R}^2$ which is countably convex and has a nonempty radical.*

Proof. Let D be the closed unit disk in $\mathbb{R}^2$ and let C be the upper half of its boundary. Subtract from F an open half-plane that contains exactly the middle third of C. Continue inductively, where at stage n, for each of the remaining 2^n subarcs C_i^n, $i < 2^n$, of C, we subtract from D an open half-plane which contains the middle third of C_i^n. Let S_n be the set which is obtained at stage n and set $S = \bigcap_n S_n$.

Let P be the Cantor set constructed by this process inside C. Clearly, $P \subseteq S$. Put $P = P_1 \cup P_2$ where P_1 is the set of all end-points of all C_i^n, $n < \infty$, $i < 2^n$, and $P_2 = P - P_1$.

Clearly S is a G_δ set, since $S = \operatorname{int} S \cup P \cup C'$, where C' is the lower half of ∂D: a union of an open set and two closed sets. Further, P is easily verified to be a semi-clique in S. Finally, S is countably convex, since $\operatorname{int} S \cup P_2 \cup C'$ is convex, while P_1 is countable. ∎

3. Infinite-dimensional spaces. All Banach spaces in this section are infinite-dimensional. We will see that the infinite-dimensional case is completely different from the finite-dimensional one. We start with two standard auxiliary results (which we prove just for the completeness of presentation) and then prove Theorem 9 which is the main ingredient for the main Corollary 12.

Recall that for a Banach space E, a subspace $Y \subseteq E^*$ is 1-*norming* if $\|x\| = \sup\{f(x) : f \in B_Y\}$ for each $x \in E$.

Lemma 7. *A subspace $Y \subset E^*$ is* 1*-norming iff B_Y is w^*-dense in B_{E^*}.*

Proof. Clearly, if B_Y is w^*-dense in B_{E^*} then $Y \subset E^*$ is 1-norming. Conversely, assume that $Y \subset E^*$ is 1-norming and suppose to the contrary that there is an $f \in B_{E^*} \setminus w^*\text{-cl}\, B_Y$. Apply a separating theorem to find an $x \in E$ such that $f(x) > 1 > \sup\{g(x) : g \in w^*\text{-}\operatorname{cl} B_Y\}$, a contradiction. ∎

Lemma 8. *Let E be a Banach space with separable dual. Then each norm-closed subset F of B_{E^*} is a w^*-G_δ set.*

Proof. For each $f \in B_{E^*} \setminus F$ set $d_f = d(f, F)$. It is clear that

$$B_{E^*} \setminus F = \bigcup_{f \in B_{E^*} \setminus F} \big(\big(f + \tfrac{1}{2} \operatorname{int} B_{E^*}\big) \cap B_{E^*} \big).$$

By the Lindelöf theorem there is a countable subcovering, i.e.

$$B_{E^*} \setminus F = \bigcup_{i=1}^{\infty} \big(\big(f_i + \tfrac{1}{2} \operatorname{int} B_{E^*}\big) \cap B_{E^*} \big).$$

By passing to closed balls we get

$$B_{E^*} \setminus F = \bigcup_{i=1}^{\infty} \big(\big(f_i + \tfrac{1}{2} B_{E^*}\big) \cap B_{E^*} \big).$$

Since B_{E^*} is w^*-G_δ and each $\big(f_i + \frac{1}{2} B_{E^*}\big) \cap B_{E^*}$ is w^*-closed, it follows that F is a w^*-G_δ set. ∎

We denote by $\operatorname{Ker} F$, for $F \in E^*$, the null space of the functional F.

THEOREM 9. *Let E be a Banach space with separable dual E^* such that there is a 2-dimensional subspace $M \subset E^{**}$ with the property: for each $F \in M$,* $\operatorname{Ker} F$ *is* 1*-norming. Then there is a sequence $\{C_i\}_{i=1}^{\infty}$ of convex subsets of B_{E^*} such that*

(i) *The set $S = \bigcup_{i=1}^{\infty} C_i$ is w^*-G_δ.*
(ii) *Each C_i is w^*-dense in S.*
(iii) *There is a subset $P = \{f_i\} \subset S$ which is w^*-dense in S, does not contain w^*-isolated points and the interval $[f_i, f_j]$ is not contained in S for $i < j$.*

Proof. Let the functionals $F, G \in S_M$ form a basis of M. Put

$$F_1 = F, \quad F_n = F + \frac{1}{n} G, \quad n = 2, 3, \ldots, \quad C_n = \operatorname{Ker} F_n \cap B_{E^*}, \quad S = \bigcup_{i=1}^{\infty} C_i.$$

To check that S is w^*-G_δ in B_{E^*}, in view of Lemma 8 it is enough to show that S is norm-closed. Let $\lim h_k = h$, $h_k \in S$. Since $S = \bigcup_{i=1}^{\infty} C_i$ and each C_i is norm-closed, we may assume that $h_k \in C_{n_k}$, $n_k \neq n_m$, $k \neq m$. However $\| \cdot \|$-$\lim F_{n_k} = F_1$ and hence $0 = F_{n_k}(h_{n_k}) \to F_1(f)$. Thus $h \in C_1 \subset S$, which proves that S is $\| \cdot \|$-closed, and w^*-G_δ by Lemma 8.

By Lemma 7 each C_n is w^*-dense in B_{E^*}. Next put $L = \bigcap_{n=1}^{\infty} C_n$. By using the definition of C_n it is easy to see that for each pair $p \neq q$,

$$L = C_p \cap C_q. \tag{3.5}$$

For each n it is clear that $C_n \setminus L$ is norm-dense in C_n and thus $C_n \setminus L$ is w^*-dense in B_{E^*}.

We now construct the set $P = \{f_i\}$ as in (iii). Fix a sequence $\{g_i\} \subset B_{E^*}$ which is w^*-dense in B_{E^*}. Denote by d a metric which generates the w^*-topology on B_{E^*}.

Take $f_1 \in C_1 \setminus L$ with $d(g_1, f_1) < 1$. Next, using the density of $C_2 \setminus L$ in B_{E^*} find $f_2 \in C_2 \setminus L$ with $d(f_1, f_2) < 1$.

Now choose $f_3 \in C_3 \setminus L$ with $d(g_2, f_3) < 1/2$. Next we approximate f_1, f_2, f_3. Choose $f_4 \in C_4 \setminus L$, $f_5 \in C_5 \setminus L$, $f_6 \in C_6 \setminus L$ so that

$$d(f_1, f_4) < 1/2, \quad d(f_2, f_5) < 1/2, \quad d(f_3, f_6) < 1/2.$$

Now choose $f_7 \in C_7 \setminus L$ with $d(g_3, f_7) < 1/3$. The rest of the construction is clear.

In this way we construct a dense-in-itself sequence $P = \{f_i\}_{n=1}^{\infty}$, $f_i \in C_i \setminus L$, such that each point in S (even in B_{E^*}) is a w^*-cluster point of P.

We check that for $i < j$ the interval $[f_i, f_j]$ is not contained in S. Indeed, from (3.5) it follows that for each p the intersection $C_p \cap [f_i, f_j]$ is either a singleton or empty. Thus $S \cap [f_i, f_j]$ is countable, which completes the proof of the theorem. ∎

REMARK 10. We now show how to get an example of a Banach space E with the property required by Theorem 9. Let X be a Banach space with separable dual X^* and such that $\dim X^{**}/X = \infty$, i.e. X is not quasi-reflexive (e.g. $X = c_0$). Let $M \subset X^{**}$ be a 2-dimensional subspace of X^{**} with $M \cap X = \{0\}$. Define a new norm in X as follows:

$$|||x||| = \sup\{f(x) : f \in M_\perp,\ \|f\| = 1\}$$

where $M_\perp = \{f \in l_1 : F(f) = 0,\ F \in M\}$. We now check that the new norm is equivalent to the original one. First, from $M \cap X = \{0\}$ and $\dim M < \infty$ it follows that

$$a = \inf\{d(x, M) : x \in S_X\} > 0.$$

Take an $x \in S_X$ and by the Hahn–Banach theorem find an $\widetilde{f} \in S_{M^\perp}$ with $\widetilde{f}(x) = d(x, M) \geq a$. Now by using the w^*-density of $M_\perp$ in $M^\perp$ we find for each $\delta > 0$ a functional $f_\delta \in S_{M_\perp}$ so that $|(\widetilde{f} - f_\delta)(x)| < \delta$. Now it is clear that $a\|x\| \leq |||x||| \leq \|x\|$ for each $x \in X$.

It is trivial that the space $E = (X, |||\cdot|||)$ has the property required by Theorem 9.

Next we slightly change the formulation of Theorem 9. More precisely, we want to transfer the set S into some Banach space X in such a way that the w^*-properties of the sets S and P in E^* coincide with the norm-properties in X.

THEOREM 11. *There exists a separable Banach space X which contains a subset $\widetilde{S}$ with the following properties*:

(i) $K = \operatorname{cl} \widetilde{S}$ *is compact.*
(ii) $\widetilde{S}$ *is a* G_δ *set.*
(iii) $\widetilde{S} = \bigcup_{i=1}^\infty \widetilde{C}_i$ *where each* $\widetilde{C}_i$ *is convex and dense in* $\widetilde{S}$.
(iv) *There is a dense-in-itself* 2*-clique* $\widetilde{P} \subset \widetilde{S}$ *which is dense in* $\widetilde{S}$.

Proof. Let $\{x_i\}_{i=1}^\infty$ be a dense subset of S_E. Define a weaker norm on E^* as follows:

$$|\!|\!| f |\!|\!| = \sum_{k=1}^\infty 2^{-k} |f(x_k)|.$$

Then the set $K = (B_{E^*}, |\!|\!| \cdot |\!|\!|)$ is homeomorphic to (B_{E^*}, w^*) and hence is compact. In particular, $(E^*, |\!|\!| \cdot |\!|\!|) = \bigcup_{n=1}^\infty nB_{E^*}$ is separable. Denote by X the completion of $(E^*, |\!|\!| \cdot |\!|\!|)$. Thus X is a separable Banach space. Let $J : E^* \to X$ be a natural embedding. Now use the notation of Theorem 9. Put $\widetilde{C}_i = J(C_i)$, $\widetilde{S} = J(S)$ and $\widetilde{P} = J(P)$. All the properties (i)–(iv) are clear. ∎

Corollary 12. *Every infinite-dimensional Banach space* Y *contains subsets* S *and* P *which have all the properties of* $\widetilde{S}$ *and* $\widetilde{P}$ *from Theorem* 11.

Proof. Let X be the Banach space from Theorem 11 and let $T : X \to Y$ be a compact 1-1 operator. Since $K \subset X$ is compact, it follows that $K_1 = T(K)$ is compact and hence G_δ in Y. The restriction $T{\restriction}_K$ is an affine homeomorphism of K onto K_1. Clearly, the sets $S = T(\widetilde{S})$ and $P = T(\widetilde{P})$ have the same properties as the sets $\widetilde{S}$ and $\widetilde{P}$ from Theorem 11. ∎

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