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Transfinite inductions producing coanalytic sets

by

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Abstract. A. Miller proved the consistent existence of a coanalytic two-point set, Hamel basis and MAD family. In these cases the classical transfinite induction can be modified to produce a coanalytic set. We generalize his result formulating a condition which can be easily applied in such situations. We reprove the classical results and as a new application we show that consistently there exists an uncountable coanalytic subset of the plane that intersects every C^1 curve in a countable set.

1. Introduction. A *two-point set* is a subset of the plane that intersects every line in exactly two points. Mazurkiewicz showed the existence of a two-point set using transfinite induction. Erdős asked whether a two-point set can be a Borel set. This question is still open.

A. Miller [13] proved that under certain set-theoretic assumptions (namely $V = L$, where L denotes Gödel's constructible universe) one can construct a coanalytic two-point set. Miller also proved the consistent existence of a coanalytic MAD family and a coanalytic Hamel basis. The author proves the statement solely for two-point sets and the proof uses deep set-theoretical tools. References to Miller's method appear in several papers ([4], [5], [8], etc.), sometimes omitting the proof. However, the first version of the method was published by Erdős, Kunen and Mauldin [3].

Our aim here is to make precise and prove a "black box" condition which could easily be applied without the set-theoretical machinery.

Let us remark here that in all of the above mentioned cases, except of course the two-point set, the class of coanalytic sets is best possible, since it is known that there is no analytic

(i) MAD family,
(ii) Hamel basis,

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(iii) C^1-small set (that is, an uncountable subset of the plane that intersects every C^1 curve in countably many points).

(i) is a classical result of Mathias [11] and for the proof of (iii) see [6]. (ii) can be shown with an easy computation. Moreover, assuming projective determinacy one can show that there is no projective Hamel basis or C^1-small set. It is also an interesting fact that an analytic two-point set is automatically Borel.

Now to formulate our results we first define Turing reducibility. Throughout the paper M will stand for $\mathbb{R}^n$, 2^ω, $\mathcal{P}(\omega)$ or ω^ω.

Definition 1.1. Suppose that $x, y \in M$. We say that x is *Turing reducible* to y if there exists a Turing machine that computes x with oracle y, written $x \leq_T y$. Let us say that $A \subset M$ is *cofinal in the Turing degrees* if for every $x \in M$ there exists a $y \in A$ such that $x \leq_T y$.

Roughly speaking, our theorem will state that if given a transfinite induction that picks a real x_α at each step α, the set of possible choices (described by the set F below) is nice enough and cofinal in the Turing degrees, then the induction can be realized so that it produces a coanalytic set. In most cases there will be an extra requirement that x_α has to be picked from a given set H_α. For example, in the construction of the two-point set, H_α is the αth line. Instead of the sets H_α we will use a parametrization where H_α will be coded by p_α and typically the codes will range over $\mathbb{R}$. The set of codes will be denoted by B.

Notation. If $S \subset X \times Y$ and $x \in X$, we denote by S_x the x-section of S (i.e. $\{y \in Y : (x, y) \in S\}$). Let ω denote the first infinite ordinal, and ω_1 the first uncountable ordinal. For a set H the set of countable sequences of elements of H is denoted by $H^{\leq\omega}$. Note that if M is a Polish space then there is a natural Polish structure on $M^{\leq\omega}$.

Definition 1.2. Let $F \subset M^{\leq\omega} \times B \times M$ and $X \subset M$. We say that X is *compatible* with F if there exist enumerations $B = \{p_\alpha : \alpha < \omega_1\}$ and $X = \{x_\alpha : \alpha < \omega_1\}$ and, for every $\alpha < \omega_1$, a sequence $A_\alpha \in M^{\leq\omega}$ that is an enumeration of $\{x_\beta : \beta < \alpha\}$ in type $\leq \omega$ such that $(\forall \alpha < \omega_1)(x_\alpha \in F_{(A_\alpha, p_\alpha)})$.

This definition basically says that in each step of the transfinite induction we pick an element from a set $F_{(A_\alpha, p_\alpha)}$ which depends on the set of previous choices A_α and the αth parameter p_α.

Theorem 1.3. ($V = L$) *Let B be an uncountable Borel subset of an arbitrary Polish space. Suppose that $F \subset M^{\leq\omega} \times B \times M$ is a coanalytic set, and for all $p \in B$ and $A \in M^{\leq\omega}$ the section $F_{(A,p)}$ is cofinal in the Turing degrees. Then there exists a coanalytic set X that is compatible with F.*

In fact we will prove a much stronger Theorem 3.4, which we call the Main Theorem. However, all the classical applications use Theorem 1.3, which will be an easy consequence of the Main Theorem (see Section 4). We wish to emphasize one of our further results from Section 4.

THEOREM 1.4. ($V = L$) *Suppose that $G \subset \mathbb{R} \times \mathbb{R}^n$ is a Borel set and for every countable $A \subset \mathbb{R}$ the complement of the set $\bigcup_{p \in A} G_p$ is cofinal in the Turing degrees. Then there exists an uncountable coanalytic set $X \subset \mathbb{R}^n$ that intersects the section G_p in a countable set for every $p \in \mathbb{R}$.*

Our paper is organized as follows. In Section 2 we summarize the most important facts used for the proof, and Section 3 contains the proof of the Main Theorem. In Section 4 we prove several generalizations, a partial converse, and we obtain the existence of a coanalytic Hamel basis (which slightly differs from the other applications). Finally in Section 5 we present applications of our theorem and mention some open problems. The reader only interested in how to apply the method developed in this paper may now proceed to Section 5 which does not build on Sections 2, 3 and 4.

2. Preliminaries. We will use standard notation as in [14]. If A is a set, $\mathcal{P}(A)$ denotes the power set of A. We identify ω^ω, $(\omega^\omega)^{\leq\omega}$, 2^ω, ω^ω, $\mathbb{R}^{\leq\omega}$, $\mathcal{P}(\omega)$ and their finite products, since there are recursive Borel isomorphisms between them [14, 3I.4. Theorem]. A "real" is an element of one of these spaces. For convenience we will use ω^ω in most cases. If $A \in (\omega^\omega)^{\leq\omega}$ and $n \in \omega$, let us denote by $A(n)$ the nth element of A (as a sequence).

As usual, continuous images of Borel sets are called *analytic sets* and their complements are called *coanalytic sets*. If t is a real, let us denote by $\Sigma^i_j(t)$, $\Delta^i_j(t)$ and $\Pi^i_j(t)$ ($i = 0, 1$, $j \in \omega$) the classes of the arithmetic and projective hierarchy recursive in t. Thus for example the set of coanalytic subsets of ω^ω equals $\bigcup_{t \in \omega^\omega} \Pi^1_1(t)$. For $t = \emptyset$ we will write Σ^i_j instead of $\Sigma^i_j(t)$, etc.

The theorems we will use can be found in [15] and [2], but we recall the most important facts. Let us denote by $\mathcal{S}$ the set of self-constructible reals, i.e. $\{x \in \omega^\omega : x \in L_{\omega_1^x}\}$, where ω_1^x is the first ordinal not recursive in x and L_α is the αth level of Gödel's constructible universe, L. Let $<_L$ be the standard well-ordering of L.

THEOREM 2.1 ([10, Theorem (2A-1)]). *$\mathcal{S}$ is a Π^1_1 set.*

For reals x, y let us write $x \leq_h y$ when x is hyperarithmetic in y or equivalently $x \in \Delta^1_1(y)$ (see [15] or [12, Corollary 27.4]). If A is a set, $L_\alpha[A]$ denotes the αth level of the universe constructed from A, that is, in the initial step we start from $\emptyset$ and A.

Theorem 2.2 ([15, A.II.7.3, A.III.9.11]). *$x \le_h y$ is a Π^1_1 relation and for arbitrary reals it is equivalent to $x \in L_{\omega_1^y}[y]$. Moreover, $x \le_h y$ implies $\omega_1^x \le \omega_1^y$.*

We will use the following form of the Spector–Gandy theorem:

Theorem 2.3 ([12, Corollary 29.3]). *Let $A \subset (\omega^\omega)^2$ be a $\Pi^1_1(t)$ subset of $(\omega^\omega)^2$. Then the set*

$$(\exists y \le_h x)((x, y) \in A)$$

is also $\Pi^1_1(t)$.

In [1] the authors work with a very useful alternative form. We say that a formula in the language of set theory is Σ_1 if it has just one unbounded quantifier and it is existential. In case all the quantifiers are bounded, we call it Δ_0.

Theorem 2.4. *A set A is $\Pi^1_1(t)$ if and only if there exists a Σ_1 formula θ such that*

$$x \in A \;\Leftrightarrow\; L_{\omega_1^{(x,t)}}[x, t] \models \theta(x, t).$$

Definition 2.5. We call a set $X \subset \omega^\omega$ *cofinal in the hyperdegrees* if for every $y \in \omega^\omega$ there exists an $x \in X$ such that $y \le_h x$.

Furthermore, in [1] one can find the following lemma.

Lemma 2.6. *($V = L$) Let $t \in \omega^\omega$ be arbitrary. A $\Pi^1_1(t)$ set X is cofinal in the hyperdegrees if and only if $X \cap \mathcal{S}$ is cofinal in the hyperdegrees.*

3. The main theorem. First we will prove a rather technical lemma.

Lemma 3.1. *Suppose that $\theta(s, p, q)$ is a Σ_1 formula of set theory. Then there exists a Σ_1 formula $\theta'(s, p)$ such that for every limit ordinal $\alpha > \omega$,*

$$L_\alpha \models ((\forall q <_L p)(\theta(s, p, q)) \Leftrightarrow \theta'(s, p)).$$

Proof. By [2, 3.5 Lemma, p. 75] there exists a Σ_1 formula $\zeta(x, y)$ such that for every limit ordinal $\alpha > \omega$ and $x, y \in L_\alpha$,

$$L_\alpha \models (\zeta(x, y) \Leftrightarrow y = \{t : t <_L x\}).$$

Notice that if $\alpha > \omega$ is a limit ordinal and $x \in L_\alpha$ then $\{t : t <_L x\} \in L_\alpha$. Let

$$\theta''(s, p) = (\exists y)(\zeta(p, y) \wedge (\forall q \in y)(\theta(s, p, q))).$$

Now, since θ'' contains solely existential and bounded quantifiers, by a well-known trick there exists a Σ_1 formula $\theta'(s, p)$ such that for every limit ordinal $\alpha > \omega$,

$$L_\alpha \models (\theta''(s, p) \Leftrightarrow \theta'(s, p)). \;\blacksquare$$

In the following lemma we will select a single well-ordering of ω of type α for every countable ordinal α in a "nice" way. The selection will be done by a formula $\phi(z,x)$ that intuitively means that x "knows" that z is a canonical well-ordering. Let $z \subset \omega^2$ and define $<_z$ by $m <_z n \Leftrightarrow (m,n) \in z$. Let us write $\mathrm{dom}(<_z)$ for the set $\{n \in \omega : (\exists m \in \omega)((m,n) \in z)\}$. For $z, z' \in \mathcal{P}(\omega^2)$ we say that $<_z \cong <_{z'}$ if there exists a bijection $f : \mathrm{dom}(<_z) \to \mathrm{dom}(<_{z'})$ such that

$$(\forall m, n \in \mathrm{dom}(<_z))(m <_z n \Leftrightarrow f(m) <_{z'} f(n)).$$

Now if $<_z$ is an ordering and $n \in \omega$, let us denote by $<_z|_{<_z n}$ the ordering obtained by restricting $<_z$ to the set $\{m \in \omega : m <_z n\}$.

LEMMA 3.2. ($V = L$) *There exists a formula $\phi(z,x)$ defining a Π^1_1 subset of $\mathcal{P}(\omega^2) \times \omega^\omega$ with the following properties:*

1. *if $s \subset \omega^2$ and $<_s$ is a well-ordering then there exists a unique z such that $<_z \cong <_s$, $(\exists x \in \omega^\omega)\phi(z,x)$ and $\mathrm{dom}(<_z)$ is a natural number or ω,*
2. *if $y \in \mathcal{S}$, $x \leq_h y$ and $\phi(z,x)$ then $\phi(z,y)$,*
3. *if $\phi(z,x)$ then $z \leq_h x$ and $x \in \mathcal{S}$,*
4. *if $\phi(z,x)$ and $n \in \omega$ is arbitrary then there exists a unique pair $g_n, y_n \in L_{\omega_1^x}$ such that $\phi(y_n, x)$ and $g_n \subset \omega^2$ is an isomorphism between $<_z|_{<_z n}$ and $<_{y_n}$.*

Proof. First let us denote by $\psi(z,h,\alpha)$ the conjunction of the following formulas:

- h is a function, $\mathrm{dom}(h) = \alpha$ is an ordinal, $\mathrm{ran}(h) = \mathrm{dom}(<_z)$,
- $(\forall \beta, \beta' \in \alpha)(\beta \in \beta' \Leftrightarrow h(\beta) <_z h(\beta'))$,
- $\mathrm{dom}(<_z)$ is a natural number or ω.

So $\psi(z,h,\alpha)$ says that h is an isomorphism between α and $<_z$. Notice that ψ is a Δ_0 formula (see [2, Section I]). Hence for limit ordinals $\beta > \omega$ if $z, h, \alpha \in L_\beta$ then $L \models \psi(z,h,\alpha) \Leftrightarrow L_\beta \models \psi(z,h,\alpha)$.

Let us define $\phi(z,x)$ as follows:

$$\phi(z,x) \Leftrightarrow x \in \mathcal{S} \wedge z \leq_h x \wedge$$
$$L_{\omega_1^x} \models (\exists h \exists \alpha)\big(\psi(z,h,\alpha) \wedge (\forall (z',h') <_L (z,h))(\neg\psi(z',h',\alpha))\big).$$

First, we will prove that $\phi(z,x)$ defines a Π^1_1 set. The formula

$$(\exists h \exists \alpha)\big(\psi(z,h,\alpha) \wedge (\forall (z',h') <_L (z,h))(\neg\psi(z',h',\alpha))\big)$$

by Lemma 3.1 is equivalent to a Σ_1 formula, say $\zeta(z)$, in L_β if β is a limit ordinal and $\beta > \omega$. Notice that $z \leq_h x$ implies $(x,z) \leq_h x$ so $\omega_1^{(x,z)} \leq \omega_1^x$ by Theorem 2.2. Moreover, from $(x,z) \leq_h x$ and by Theorem 2.2 we have $(x,z) \in L_{\omega_1^x}[x]$. Additionally, $x \in \mathcal{S}$ so $L_{\omega_1^x} = L_{\omega_1^x}[x]$. Thus $(x,z) \in L_{\omega_1^x}$ and

$L_{\omega_1^{(x,z)}}[x,z] = L_{\omega_1^x}$. Therefore

$$L_{\omega_1^x} \models (\exists h \exists \alpha)\big(\psi(z,h,\alpha) \wedge (\forall (z',h') <_L (z,h))(\neg\psi(z',h',\alpha))\big)$$
$$\Leftrightarrow\ L_{\omega_1^x} \models \zeta(z) \ \Leftrightarrow\ L_{\omega_1^{(x,z)}}[x,z] \models \zeta(z).$$

By Theorems 2.1 and 2.2 it is clear that $(x \in \mathcal{S}) \wedge (z \leq_h x)$ defines a Π^1_1 set. Now we can prove that the set $\{(x,z) : L_{\omega_1^{(x,z)}}[x,z] \models \zeta(z)\}$ is also Π^1_1 using Theorem 2.4 with $t = 0$ and replacing x by (x,z). Thus ϕ defines a Π^1_1 set.

Now we will prove that $\phi(z,x)$ has the required properties.

1. Let $s \subset \omega^2$ be an arbitrary well-ordering. Then $<_s$ is isomorphic to some ordinal α. There exists a $<_L$-minimal pair (z,h) such that h is an isomorphism between $<_z$ and α, and $\mathrm{dom}(<_z)$ is a natural number or ω. Therefore

$$L \models (\exists h \exists \alpha)\big(\psi(z,h,\alpha) \wedge (\forall (z',h') <_L (z,h))(\neg\psi(z',h',\alpha))\big).$$

Notice that if $\xi(s)$ is a Δ_0 formula, β is a limit ordinal such that $s \in L_\beta$ and $L \models \xi(s)$ then $L_\beta \models \xi(s)$. Therefore automatically $L_\beta \models (\exists r)(\xi(r))$. Considering this, one can conclude that

$$L_{\omega_1^x} \models (\exists h \exists \alpha)\big(\psi(z,h,\alpha) \wedge (\forall (z',h') <_L (z,h))(\neg\psi(z',h',\alpha))\big)$$

if $(z,h) \in L_{\omega_1^x}$. Since $\mathcal{S}$ is cofinal in the hyperdegrees (Lemma 2.6), there exists an $x \in \mathcal{S}$ such that $(z,h) \in L_{\omega_1^x}$. So for such an x we have $\phi(z,x)$.

2. To prove the second claim just observe that Σ_1 formulas are upward absolute for transitive sets and notice that $x \leq_h y$ implies that $L_{\omega_1^x} \subset L_{\omega_1^y}$.

3. Obvious from the definition of ϕ.

4. Let $x \in \omega^\omega$, $z \subset \omega^2$, $n \in \omega$ and assume that $\phi(z,x)$ holds. Clearly there exists a unique ordinal $\beta < \alpha$ such that $\beta \cong <_z|_{<_z n}$.

First we will prove that there exists a pair $(y'_n, h'_n) \in \mathcal{P}(\omega^2) \times \beta^{\mathrm{dom}(<_{y'_n})}$ so that $L_{\omega_1^x} \models \psi(y'_n, h'_n, \beta)$. We know that $L_{\omega_1^x} \models \psi(z,h,\alpha)$ for some $h, \alpha \in L_{\omega_1^x}$, so the same holds in L. The fact that $\psi(z,h,\alpha)$ holds implies that h is an isomorphism between $<_z$ and α, so $h' = h|_\beta$ is an isomorphism between β and $<_z|_{<_z n}$. Obviously, $h' \in L_{\omega_1^x}$, so there exists an ordinal $\gamma < \omega_1^x$ such that $h' \in L_\gamma$.

Let $e : \omega \to \mathrm{ran}(h')$ be defined as follows:

$$\langle m,k\rangle \in e \ \Leftrightarrow\ \big(k \in \mathrm{ran}(h') \wedge (\exists e')(e' : m \leftrightarrow \mathrm{ran}(h') \cap k+1)\big),$$

in other words, there exists a bijection between m and the initial segment of $\mathrm{ran}(h')$, or equivalently, $|\{l \in \mathrm{ran}(h') : l \leq k\}| = m$. Since the bijections between the finite subsets of ω are already in L_ω, we have $e \in L_{\gamma+2} \subset L_{\omega_1^x}$. The map e is clearly a one-to-one function from a finite number or ω onto $\mathrm{ran}(h')$.

Now take $\langle k,l\rangle \in y'_n \Leftrightarrow \langle e(k),e(l)\rangle \in z$ and $h'_n = e^{-1}\circ h'$. Then $L \models \psi(y'_n,h'_n,\beta)$ and of course $y'_n,h'_n,\beta \in L_{\omega_1^x}$, hence $L_{\omega_1^x} \models \psi(y'_n,h'_n,\beta)$.

Thus there exists a $<_L$-minimal pair $(y_n,h_n) \in L_{\omega_1^x}$ such that $L_{\omega_1^x} \models \psi(y_n,h_n,\beta)$. Note that the $<_L$ ordering is absolute for L_α and L if $\alpha > \omega$ is a limit ordinal, so $L_{\omega_1^x} \models$ "(y_n,h_n) is the $<_L$-minimal pair such that $\psi(y_n,h_n,\beta)$". By Theorem 2.2, if $y_n \in L_{\omega_1^x}$ then $y_n \leq_h x$. Thus $\phi(y_n,x)$ holds.

Finally recall that $h_n : \beta \to \mathrm{dom}(<_{y_n})$ and $h' : \beta \to \mathrm{dom}(<_z|_{<_z n})$ are isomorphisms in $L_{\omega_1^x}$. So the function $g_n = h_n \circ (h')^{-1}$ is in $L_{\omega_1^x}$. This is an isomorphism between two well-orderings, so it is unique. ■

Let us recall the definition of compatibility.

Definition 3.3. Let $F \subset M^{\leq\omega} \times B \times M$ and $X \subset M$. We say that X is *compatible* with F if there exist enumerations $B = \{p_\alpha : \alpha < \omega_1\}$, $X = \{x_\alpha : \alpha < \omega_1\}$ and, for every $\alpha < \omega_1$, a sequence $A_\alpha \in M^{\leq\omega}$ that is an enumeration of $\{x_\beta : \beta < \alpha\}$ in type $\leq\omega$ such that $(\forall\alpha < \omega_1)(x_\alpha \in F_{(A_\alpha,p_\alpha)})$.

Theorem 3.4 (Main Theorem). ($V = L$) *Let $t \in \omega^\omega$. Suppose that $F \subset (\omega^\omega)^{\leq\omega} \times \omega^\omega \times \omega^\omega$ is a $\Pi^1_1(t)$ set, and for all $p \in \omega^\omega$ and $A \in (\omega^\omega)^{\leq\omega}$ the section $F_{(A,p)}$ is cofinal in the hyperdegrees. Then there exists a $\Pi^1_1(t)$ set $X \subset \omega^\omega$ that is compatible with F.*

Proof of the Main Theorem. In the first step we will modify the set F. Let us define

$$F' \subset \mathcal{P}(\omega^2) \times (\omega^\omega)^{\leq\omega} \times (\omega^\omega)^{\leq\omega} \times \omega^\omega \times \omega^\omega$$

by $(z,A,P,p,x) \in F' \Leftrightarrow$

1. $\phi(z,x)$ (in particular $x \in \mathcal{S}$),
2. $A,P,p,t \leq_h x$ and $(A,p,x) \in F$,
3. $L_{\omega_1^x} \models \exists g$
   (a) g is a function, $\mathrm{dom}(g) \in \omega \cup \{\omega\}$ and $\mathrm{ran}(g) = P$,
   (b) $(\forall n,m \in \mathrm{dom}(g))(n <_z m \Leftrightarrow g(n) <_L g(m))$,
   (c) $(\forall p' <_L p)(p' \in \omega^\omega \Rightarrow (\exists n \in \omega)(g(n) = p'))$.

The role of z is that it will encode the history of the previous choices. $1 \wedge 2$ basically ensures that x is complicated enough. The clauses (a) and (b) describe that P is an enumeration in type $\leq\omega$ of the first α reals with respect to $<_L$ where $\alpha = \mathrm{tp}(<_z)$; and (c) is the formalization of $L_{\omega_1^x} \models$ "p is the αth real with respect to $<_L$".

Lemma 3.2 and Theorems 2.1 and 2.2 guarantee that items 1 and 2 define a $\Pi^1_1(t)$ set.

We can prove that item 3 defines a Π^1_1 set similarly as we did in Lemma 3.2: (a) and (b) are Δ_0 formulas, (c) is Σ_1 by Lemma 3.1. So by the well-known technical trick the conjunction is equivalent to a Σ_1 formula.

Moreover we know that for arbitrary reals, $a \leq_h b \Leftrightarrow a \in L_{\omega_1^b}[b]$ and $a \leq_h b$ implies $\omega_1^a \leq \omega_1^b$. Therefore by 1 and 2,

$$L_{\omega_1^{(z,A,P,p,t,x)}}[z, A, P, p, t, x] = L_{\omega_1^x},$$

and using the Spector–Gandy Theorem (Theorem 2.4) we can conclude that F' is a $\Pi^1_1(t)$ set.

REMARK 3.5. By absoluteness, if $(z, A, P, p, x) \in F'$ then P must be the enumeration of the first α reals given by $<_z$ in L as well. Similarly p must be the αth real with respect to $<_L$ (where $\alpha = \mathrm{tp}(<_z)$).

LEMMA 3.6. *Suppose that* $x \in F'_{(z,A,P,p)}$, $x \leq_h y$ *and* $y \in \mathcal{S} \cap F_{(A,p)}$. *Then* $y \in F'_{(z,A,P,p)}$.

Proof. Let x, y be reals satisfying the conditions above. Now considering the definition of F', the formula $\phi(z, y)$ holds by the second claim of Lemma 3.2. Of course, $A, P, p, t \leq_h x$ implies $A, P, p, t \leq_h y$. Finally, $L_{\omega_1^x} \subset L_{\omega_1^y}$, by Theorem 2.2, and the formula in 3 that must hold in $L_{\omega_1^y}$ does not depend on x, hence it is also true in $L_{\omega_1^y}$. ∎

LEMMA 3.7. *If the section* $F'_{(z,A,P,p)}$ *is non-empty then it is cofinal in the hyperdegrees.*

Proof. Fix an arbitrary $s \in \omega^\omega$ and let $x \in F'_{(z,A,P,p)}$. By the assumptions of the Main Theorem each section $F_{(A,p)}$ is cofinal in the hyperdegrees. Using Lemma 2.6 we see that there exists a $y \in F_{(A,p)} \cap \mathcal{S}$ such that $s, x \leq_h y$. Thus by the previous lemma $y \in F'_{(z,A,P,p)}$, and this proves the statement. ∎

Now we select a real from each non-empty section of F'. Let $F'' \subset F'$ be a $\Pi^1_1(t)$ uniformization of F', that is, for all $(z, A, P, p) \in \mathrm{proj}(F')$ we have $|F''_{(z,A,P,p)}| = 1$ (see [12] or [15] for the relative version of the uniformization theorem).

There may be elements $(z, A, P, p, x) \in F''$ with "wrong" history, namely $A(n)$ may not be a selected real for some $n \in \omega$. So we have to sort out the appropriate ones.

Let $F''' \subset F''$ be defined as follows: $(z, A, P, p, x) \in F''' \Leftrightarrow$

1. $(z, A, P, p, x) \in F''$,
2. $(\forall n \in \omega)(\exists g_n, y_n \leq_h x)$
   (a) $\phi(y_n, x)$,
   (b) g_n is an isomorphism between $<_z|_{<_z n}$ and y_n,
   (c) if $A_n, P_n \in (\omega^\omega)^{\leq\omega}$ is defined by $A_n(i) = A(g_n(i))$ and similarly $P_n(i) = P(g_n(i))$ then $(y_n, A_n, P_n, P(n), A(n)) \in F''$.

By the properties of ϕ, for every countable ordinal α we have a canonical enumeration of α. In the definition above, (c) ensures that for every

$(z, x, A, P, p) \in F'''$ the set A is the canonical enumeration of the previous choices given by the uniformization of F'.

The clauses (a) and (b) define a $\Pi^1_1(t)$ set. Now take the map

$$\Psi : (A, P, y_n, g_n, n) \mapsto (y_n, A \circ g_n, P \circ g_n, P(n), A(n)).$$

Observe that $\langle (A, P, y_n, g_n, n), (w_1, w_2, w_3, w_4, w_5)\rangle \in \Psi \Leftrightarrow y_n = w_1,\ w_4 = P(n),\ w_5 = A(n)$ and $(\forall m \in \omega)(w_2(m) = A(g_n(m)) \wedge w_3(m) = P(g_n(m)))$. So Ψ is a Δ^1_1 map and condition (c) describes that $(A, P, y_n, g_n, n) \in \Psi^{-1}(F'')$ thus defines a $\Pi^1_1(t)$ set. Therefore, using Theorem 2.3 we can conclude that F''' is also a $\Pi^1_1(t)$ set.

Now we will prove that F''' contains a "good selection"; then X will be the projection of F''' on the last coordinate.

More precisely, let

$$x \in X \ \Leftrightarrow\ (\exists (z, A, P, p) \leq_h x)((z, A, P, p, x) \in F''').$$

Notice that X is indeed the projection of F''' on the last coordinate: if $(z, A, P, p, x) \in F''' \subset F'$ then $(A, P, p) \leq_h x$ by the definition of F' and from the 3rd point of Lemma 3.2 we infer that $z \leq_h x$, so obviously $(z, A, P, p) \leq_h x$.

Observe that by Theorem 2.3 the set X is also $\Pi^1_1(t)$.

Proposition 3.8. *For every $\alpha \in \omega_1$ there exists a unique $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F'''$ such that $<_{z_\alpha} \cong \alpha$. Moreover, $\{A_\alpha(n) : n \in \omega\} = \{x_\beta : \beta < \alpha\}$ for every $\alpha < \omega_1$.*

Proof. Uniqueness. Let $(z, A, P, p, x), (z', A', P', p', x') \in F'''$ be such that $<_z \cong <_{z'} \cong \alpha$.

$z = z'$: follows from the 1st point of Lemma 3.2 since both $\phi(z, x)$ and $\phi(z', x')$ must hold.

$p = p'$: clear by Remark 3.5.

$P = P'$: also from Remark 3.5 we see that P and P' are enumerations of the first α reals given by $<_z = <_{z'}$.

$A = A'$: Suppose not. Then take the $<_z$-minimal $n \in \omega$ such that $A(n) \neq A'(n)$. By the definition of F''' there exist y_n, g_n and y'_n, g'_n such that $(y_n, A_n, P_n, P(n), A(n)) \in F''$ and $(y'_n, A'_n, P'_n, P'(n), A'(n)) \in F''$, g_n and g'_n are isomorphisms between $<_z|_{<_z n}$ and y_n, y'_n, and $\phi(y_n, x)$ and $\phi(y'_n, x)$ hold. Then again by Lemma 3.2, $y_n = y'_n$ and g_n is unique, so it must be equal to g'_n. We conclude that $(y_n, A_n, P_n, P(n)) = (y'_n, A'_n, P'_n, P'(n))$; but then $A(n) = A'(n)$ since F' was uniformized.

$x = x'$: also follows from the fact that F' was uniformized.

Existence. Now by transfinite induction we construct for each $\alpha \in \omega_1$ a $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F'''$ with the required properties.

Let us formulate the inductive hypothesis: let $\alpha < \omega_1$ be an ordinal and suppose that for every $\beta < \alpha$ we have $(z_\beta, A_\beta, P_\beta, p_\beta, x_\beta) \in F'''$ such that for every $\beta < \alpha$ we have $\{A_\beta(n) : n \in \omega\} = \{x_\gamma : \gamma < \beta\}$.

We will construct $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F'''$ satisfying the previous hypothesis.

z_α: using the 1st point of Lemma 3.2 there exists a unique z_α such that $<_{z_\alpha} \cong \alpha$ and $(\exists x \in \omega^\omega)\phi(z_\alpha, x)$.

p_α: let p_α be the αth real with respect to $<_L$.

A_α, P_α: The order-preserving bijection between $<_{z_\alpha}$ and α yields enumerations $\{x_\beta : \beta < \alpha\}$ and $\{p_\beta : \beta < \alpha\}$; let $A_\alpha(n)$ be the nth element of the first set's enumeration and define $P_\alpha(n)$ similarly.

By the definition of A_α we have $\{A_\alpha(n) : n \in \omega\} = \{x_\beta : \beta < \alpha\}$.

We will prove that there exists an $x_\alpha \in \omega^\omega$ such that $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F'''$. By the properties of F for every (A, p) there exist cofinally many (in the hyperdegrees) x such that $(A, p, x) \in F$, so this also holds for (A_α, p_α). From Lemma 3.7 we deduce that if the section $F'_{(z_\alpha, A_\alpha, P_\alpha, p_\alpha)}$ is non-empty then it is cofinal in the hyperdegrees.

Now we show that it is non-empty. Since $L \models$"P_α is an enumeration of the first α reals given by $<_{z_\alpha}$ and p_α is the αth real", by absoluteness arguments it holds in $L_{\omega_1^x}$ if ω_1^x is high enough. Let us choose a real x such that $x \in F_{A_\alpha, p_\alpha} \cap \mathcal{S}$, $L_{\omega_1^x} \models$"P_α is an enumeration of the first α reals given by $<_{z_\alpha}$ and p_α is the αth real" and $\phi(z_\alpha, x)$. Such an x exists by the 2nd point of Lemma 3.2 and by the fact that $F_{(A,p)} \cap \mathcal{S}$ is cofinal in the hyperdegrees. Clearly $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x) \in F'$.

Thus there exists an x_α such that $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F''$.

It remains to show that $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F'''$:

From $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F'$ it follows that $\phi(z_\alpha, x_\alpha)$. First notice that by the 4th point of Lemma 3.2, $\phi(z_\alpha, x_\alpha)$ implies the existence of y_n's and g_n's with properties 2(a) and 2(b) from the definition of F'''.

To see that 2(c) also holds for $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha)$, fix a natural number n. We know that $\phi(y_n, x_\alpha)$ holds, thus there exists a $\beta < \alpha$ such that $<_{y_n} \cong \beta$. For all $\beta < \alpha$ the formula $\phi(z_\beta, x_\beta)$ holds (by inductive hypothesis $(z_\beta, A_\beta, P_\beta, p_\beta, x_\beta) \in F''' \subset F'$ and use the 1st point of the definition of F'). Let us set $A_n = A_\alpha \circ g_n$ and $P_n = P_\alpha \circ g_n$.

We will prove that

$$(y_n, A_n, P_n, P_\alpha(n), A_\alpha(n)) = (z_\beta, A_\beta, P_\beta, p_\beta, x_\beta) \in F''.$$

By the 1st property of ϕ the equality $y_n = z_\beta$ holds.

Now using the inductive hypothesis we find that $\{A_\beta(m) : m \in \omega\} = \{x_\gamma : \gamma < \beta\}$. The latter set clearly equals $\{A_n(m) : m \in \omega\}$. Since A_β and A_n are enumerations of the same set of reals given by $<_{z_\beta} = <_{y_n}$, we have $A_n = A_\beta$.

Similarly, P_β and P_n are enumerations of the same set (namely the β-long initial segment of the reals with respect to $<_L$, see the existence part of the proof and Remark 3.5). Finally, $A_\alpha(n)$ and $P_\alpha(n)$ are defined as x_β and the βth real, respectively.

This finishes the proof of the statement that 2(c) also holds for $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha)$ and hence the proof of the existence. ∎

We have already seen that X is a $\Pi^1_1(t)$ set. Now we check that it is compatible with F. By the previous proposition, for every $\alpha < \omega_1$ there exists a unique element $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F'''$ such that $<_{z_\alpha} \cong \alpha$. This gives us the enumerations $X = \{x_\alpha : \alpha < \omega_1\}$ and $\{p_\alpha : \alpha < \omega_1\}$. Now by the 3rd point of the definition of F', if $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F''' \subset F'$ then $L_{\omega_1^{x_\alpha}} \models$"$p_\alpha$ is the αth real with respect to $<_L$" and by absoluteness the same holds in L. Thus $\omega^\omega = \{p_\alpha : \alpha < \omega_1\}$. Fix an $\alpha < \omega_1$. By the second claim of Proposition 3.8 it is clear that A_α is an enumeration of $\{x_\beta : \beta < \alpha\}$. Furthermore, $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F''' \subset F'$, so by the 2nd point of the definition of F' we have $x_\alpha \in F_{(A_\alpha, p_\alpha)}$, and we conclude that X is compatible with F. ∎

4. Generalizations and remarks. Now we will prove the following theorem.

THEOREM 4.1. $(V = L)$ *Let B be a Borel subset of an arbitrary Polish space with $|B| > \aleph_0$. Suppose that $F \subset (\omega^\omega)^{\le\omega} \times B \times \omega^\omega$ is a coanalytic set, and for all $p \in B$ and $A \in (\omega^\omega)^{\le\omega}$ the section $F_{(A,p)}$ is cofinal in the hyperdegrees. Then there exists a coanalytic set $X \subset \omega^\omega$ that is compatible with F.*

Proof. A classical result states that for every uncountable Borel subset B of a Polish space there exists a map $\Psi : \omega^\omega \to B$ that is a Borel isomorphism.

Suppose that F is a set as above. Let us define $G \subset (\omega^\omega)^{\le\omega} \times \omega^\omega \times \omega^\omega$ as follows:

$$(A, q, x) \in G \Leftrightarrow (A, \Psi(q), x) \in F.$$

Clearly, G is a coanalytic set, so there exists a $t \in \omega^\omega$ such that $G \in \Pi^1_1(t)$. Of course, each section $G_{(A,q)}$ is cofinal in the hyperdegrees. A direct application of the Main Theorem yields a $\Pi^1_1(t)$ (therefore coanalytic) set $X \subset \omega^\omega$ that is compatible with G. From compatibility we obtain an enumeration $\omega^\omega = \{q_\alpha : \alpha < \omega_1\}$. But then $\{\Psi(q_\alpha) : \alpha < \omega_1\}$ is an enumeration of B, and clearly X is compatible with F using this enumeration. ∎

We can derive an obvious but useful consequence of the previous theorem using that $x \le_T y$ implies $x \le_h y$ and omitting relativization.

THEOREM 4.2. $(V = L)$ *Let P be an uncountable Borel subset of a Polish space. Suppose that $F \subset (\omega^\omega)^{\le\omega} \times P \times \omega^\omega$ is a coanalytic set, and for all*

$p \in \omega^\omega$ and $A \in (\omega^\omega)^{\leq\omega}$ the section $F_{(A,p)}$ is cofinal in the Turing degrees. Then there exists a coanalytic set X that is compatible with F.

It is also easy to see that in the previous theorem we can replace ω^ω by $\mathbb{R}^n$ or 2^ω etc., since there are recursive Borel isomorphisms between these spaces. Thus we obtain Theorem 1.3.

With the same methods one could prove the following strengthening of the Main Theorem:

THEOREM 4.3. *($V = L$) Let B be a $\Delta^1_1(t)$ subset of ω^ω with $|B| > \aleph_0$. Suppose that $F \subset (\omega^\omega)^{\leq\omega} \times B \times \omega^\omega$ is a $\Pi^1_1(t)$ set, and for all $p \in B$ and $A \in (\omega^\omega)^{\leq\omega}$ the section $F_{(A,p)}$ is cofinal in the hyperdegrees. Then there exists an $X \in \Pi^1_1(t)$ that is compatible with F.*

Now we will examine the necessity of ($V = L$).

THEOREM 4.4. *If the conclusion of the Main Theorem holds then there exists a Σ^1_2 well-ordering of the reals. In particular, every real is constructible.*

Proof. Fix recursive Δ^1_1 bijections $\Psi_1 : \omega^\omega \to (\omega^\omega)^{\leq\omega} \times \omega^\omega$ and $\Psi_2 : \omega^\omega \to \omega^\omega \times \omega^\omega$.

Let us define a set $F \subset (\omega^\omega)^{\leq\omega} \times \omega^\omega \times \omega^\omega$ as follows:

$$(A, p, x) \in F \Leftrightarrow (A, p) = \Psi_1(\pi_1(\Psi_2(x)) \wedge (\forall n)(A(n) \neq x),$$

where π_1 is the projection of $\omega^\omega \times \omega^\omega$ on the first coordinate. So basically x codes the previous choices and the parameter in the "odd coordinates".

F is clearly Δ^1_1. Now for an arbitrary pair (A, p) and $y \in \omega^\omega$ there exist cofinally many $x \in \omega^\omega$ such that $(A, p) = \Psi_1(\pi_1(\Psi_2(x))$ and $y \leq_h x$, hence every section $F_{(A,p)}$ is cofinal in the hyperdegrees. Thus by our hypothesis there exists a Π^1_1 set $X = \{x_\alpha : \alpha < \omega_1\}$ and an enumeration $\omega^\omega = \{p_\alpha : \alpha < \omega_1\}$ such that for every $\alpha < \omega_1$ we have $x_\alpha \in F_{(A_\alpha, p_\alpha)}$, where A_α is an enumeration of $\{x_\beta : \beta < \alpha\}$.

We will define the well-ordering of ω^ω with the help of the given enumeration of X. Since every x_α codes the appropriate p_α, we can order ω^ω by the first appearance of a real p.

Now for $p, q \in \omega^\omega$ let

$(p, q) \in E \Leftrightarrow \exists x, y, A, B$

1. $x, y \in X$, $x \neq y$, $(A, p, x) \in F$ and $(B, q, y) \in F$,
2. $(\forall m)(\forall C)((C, p, A(m)) \notin F \wedge (C, q, B(m)) \notin F)$,
3. $(\exists n)(x = B(n))$.

Since F is Δ^1_1, we see that E is a Σ^1_2 relation.

Fix $p, q \in \omega^\omega$. There exist minimal ordinals α, β such that $p_\alpha = p$ and $p_\beta = q$. We will prove that $(p, q) \in E \Leftrightarrow \alpha < \beta$. We find for α and β that $(A_\alpha, p_\alpha, x_\alpha) \in F$ and $(A_\beta, p_\beta, x_\beta) \in F$.

First, if $\alpha < \beta$ choose $x = x_\alpha$, $y = x_\beta$, $A = A_\alpha$ and $B = A_\beta$. Then 1 is obvious (by the definition of F we have $x_\alpha \neq x_\beta$ if $\alpha < \beta$) and A_β is an enumeration of $\{x_\gamma : \gamma < \beta\}$ so 3 also holds. Suppose that 2 fails for p: there exists a pair m, C such that $(C, p, A(m)) \in F$ (the other case is similar). Then $A(m) = x_\gamma$ for some $\gamma < \alpha$ and $(C, p) = (A_\gamma, p_\gamma)$. This would contradict the minimality of α, and similarly for β.

For the other direction suppose that $(p, q) \in E$ and take x, y, A, B witnessing this fact. Clearly, $x = x_{\alpha'}$ for some α' so $(A_{\alpha'}, p_{\alpha'}) = (A, p)$ and similarly $(A_{\beta'}, p_{\beta'}) = (B, q)$. Using 2 we get the minimality of α' and β' so they must be equal to α and β.

Suppose that $\alpha \geq \beta$; then of course $\alpha > \beta$. By 3 there exists an $n \in \omega$ such that

$$A_\beta(n) = A_{\beta'}(n) = B(n) = x = x_{\alpha'} = x_\alpha.$$

By the assumption $\{x_\gamma : \gamma < \beta\} \subsetneq \{x_\gamma : \gamma < \alpha\}$. We have

$$\{A_\beta(m) : m \in \omega\} = \{x_\gamma : \gamma < \beta\} \subset \{A_\alpha(m) : m \in \omega\}$$

so $A_\alpha(m) = x_\alpha$ for some $m \in \omega$. But this is a contradiction, since $(\forall n)(A(n) \neq x)$ for every $(A, p, x) \in F$. Thus $\alpha < \beta$.

So we conclude that E is a Σ^1_2 well-ordering. The second claim follows from Mansfield's theorem (see [7, Theorem 25.39]). ∎

Next we show that the definability assumption on our "selection algorithm" F cannot be dropped in the Main Theorem.

Example 4.5. (CH) *There exists a family $\{A_\alpha : \alpha < \omega_1\} \subset [\omega^\omega]^{\leq\aleph_0}$ such that if for a set X there exists an enumeration $X = \{x_\alpha : \alpha < \omega_1\}$ with $(\forall\alpha < \omega_1)(x_\alpha \notin A_\alpha)$ then X is not coanalytic.*

Proof. Fix an enumeration of the reals $\{y_\alpha : \alpha < \omega_1\}$. We will define A_α by recursion. Suppose that we have defined A_β for $\beta < \alpha$, and let us choose $A_\alpha \in [\omega^\omega]^{\leq\aleph_0}$ such that for every uncountable $P \in \bigcup_{\beta\leq\alpha} \Pi^1_1(y_\beta)$ we have $|P \cap (A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta)| \geq 2$ and $\bigcup_{\beta<\alpha} A_\beta \subset A_\alpha$ and $y_\alpha \in A_\alpha$. Since $|\bigcup_{\beta<\alpha} A_\beta| \leq \aleph_0$ and $\bigcup_{\beta\leq\alpha} \Pi^1_1(y_\beta)$ is countable, there exists such an A_α.

Now suppose that $X = \{x_\alpha : \alpha < \omega_1\}$ is coanalytic and for every α we have $x_\alpha \notin A_\alpha$. Clearly, $\bigcup_\alpha A_\alpha = \omega^\omega$, so X must be uncountable. Since X is coanalytic, there exists an α_0 such that $X \in \Pi^1_1(y_{\alpha_0})$. Thus for every $\alpha \geq \alpha_0$ by the construction of A_α's, $|X \cap (A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta)| \geq 2$. Now consider the map ϕ that assigns to each $\alpha \geq \alpha_0$ the minimal index $\phi(\alpha)$ such that $x_{\phi(\alpha)} \in A_{\alpha+1} \setminus A_\alpha$. There are at least two distinct elements of X in $A_{\alpha+1} \setminus A_\alpha$ and $x_\gamma \notin A_{\alpha+1}$ for $\gamma > \alpha$ (the constructed family is increasing), hence $\phi(\alpha) < \alpha$. Moreover, ϕ is clearly injective. Therefore, ϕ is a regressive function whose domain is a co-countable subset of ω_1. This contradicts Fodor's lemma. ∎

Remark 4.6. The same holds for any projective class.

Now we will prove a general technical theorem which implies the existence of Π^1_1 Hamel basis, but could be used to prove the existence of Π^1_1 n-point sets, analogous versions for circles, etc. The situation in the following definition is that we have a relation $R(x,y)$ on finite subsets of the reals that intuitively means that x is "stronger" than y in some sense (e.g. in the case of Hamel basis all elements of y are linearly generated by x, in the case of two-point sets all lines that intersect y in at least two points intersect x in at least two points, etc.). Our goal is to find an R-independent set (all the relations are trivial) that is "stronger" than all the finite subsets of the reals. H^R_B will be the set of finite sets that can be added to B preserving its independence.

Definition 4.7. Let R be a binary relation on the finite subsets of $\mathbb{R}^n$.

- We say that a set $X \subset \mathbb{R}^n$ is *R-independent* if for all $x, y \in [X]^{<\omega}$, $R(x,y) \Rightarrow y \subset x$.
- Fix a $k \in \omega$. If for every $y \in [\mathbb{R}^n]^k$ there exists an element $x \in [X]^{<\omega}$ such that $R(x,y)$ then we say that X is a *k-generator* set for R.
- If B is an R-independent set, we define $H^R_B = \{x \in [\mathbb{R}^n]^{<\omega} : x \cup B$ is R-independent$\}$.

We use parameters n and k even though they will not be needed for the proof of the Hamel basis case.

Definition 4.8. We set $x \equiv_h y \Leftrightarrow (x \leq_h y \wedge y \leq_h x)$.

The extra difficulty in the construction of a Hamel basis is that in each step we have to put more than one real into our set, so we have to deal with finite sequences. Moreover, to use our method one have to choose reals which are high enough in $\leq_h$. Thus our strategy is to select $\leq_h$-equivalent reals in every step of the procedure.

Definition 4.9. Set

$$\mathcal{E} = \{x \in [\mathbb{R}^n]^{<\omega} : (\forall x_1, x_2 \in x)(x_1 \equiv_h x_2)\}.$$

Theorem 4.10. *($V = L$) Let $t \in \mathbb{R}$ and $n, k \in \omega$ be arbitrary. Suppose that $R \subset [\mathbb{R}^n]^{<\omega} \times [\mathbb{R}^n]^{<\omega}$ is a $\Delta^1_1(t)$ relation that has the property*

(∗) *for every countable $B \subset \mathbb{R}^n$ the set $\mathcal{E} \cap H^R_B$ is cofinal in the hyperdegrees, and if for $y \in [\mathbb{R}^n]^k$ there is no $z \in [B]^{<\omega}$ such that $R(z,y)$ then $\{x : R(x,y)\} \cap \mathcal{E} \cap H^R_B$ is cofinal in the hyperdegrees.*

Then there exists an uncountable $\Pi^1_1(t)$, R-independent set that is a k-generator for R.

Proof. Let us define a set $F \subset ([\mathbb{R}^n]^{<\omega})^{\leq\omega} \times \mathbb{R} \times [\mathbb{R}^n]^{<\omega}$ and fix a recursive Borel isomorphism $\Phi : \mathbb{R} \to [\mathbb{R}^n]^k$.

By definition, $(A, p, x) \in F \Leftrightarrow$

EITHER the conjunction of the following clauses holds:
1. $\bigcup \operatorname{ran}(A)$ is R-independent,
2. $(\forall z \in \operatorname{ran}(A))(\neg R(z, \Phi(p)))$,
3. $R(x, \Phi(p))$ holds and $x \in \mathcal{E} \cap H^R_{\bigcup \operatorname{ran}(A)}$;

OR $1 \wedge \neg 2$ holds and $x \in \mathcal{E} \cap H^R_{\bigcup \operatorname{ran}(A)}$;
OR $\neg 1$.

Since A is countable and the relation $\equiv_h$ is Π^1_1, we infer that F is $\Pi^1_1(t)$. By property $(*)$ every section $F_{(A,p)}$ is cofinal in the hyperdegrees (if $\neg 1$ then this is obvious and the cases $1 \wedge \neg 2$ and $1 \wedge 2$ are exactly described by property $(*)$) so we can apply Theorem 3.4. This gives us a $\Pi^1_1(t)$ set $Y \subset [\mathbb{R}^n]^{<\omega}$ such that $\bigcup \operatorname{ran}(Y)$ is R-independent and for every $y \in [\mathbb{R}^n]^k$ there exists an $x \in Y$ such that $R(x, y)$ holds, so $\bigcup \operatorname{ran}(Y)$ is a k-generator for R. Moreover $\operatorname{ran}(Y) \subset \mathcal{E}$. Hence it suffices to prove that $X = \bigcup \operatorname{ran}(Y)$ is a $\Pi^1_1(t)$ set. But since for every $x \in Y$ the elements of x are equivalent in hyperdegrees, we get

$$a \in X \;\Leftrightarrow\; (\exists l \in \omega)(\exists a_1, \ldots a_l \leq_h a)(\{a, a_1 \ldots a_l\} \in \operatorname{ran}(Y)).$$

Applying Theorem 2.3 we can verify that $X \in \Pi^1_1(t)$. ∎

COROLLARY 4.11. $(V = L)$ *There exists a* Π^1_1 *Hamel basis.*

Proof. Let us define a relation $R \subset [\mathbb{R}]^{<\omega} \times [\mathbb{R}]^{<\omega}$ by $R(x, y) \Leftrightarrow y \subset \langle x \rangle_{\mathbb{Q}}$, i.e. every element of y is in the linear subspace generated by the elements of x over the rationals. Notice that R is Δ^1_1. In the terminology of the previous theorem, X is a Hamel basis if it is R-independent and a 1-generator for R. So we just have to check whether property $(*)$ holds.

First, if B is a countable linearly independent subset of the reals then for all but countably many finite sets $a \in [\mathbb{R}]^{<\omega}$ we have $a \in H^R_B$. Therefore obviously H^R_B is cofinal in the hyperdegrees. So the first part of $(*)$ holds.

Now fix an element $y \in \mathbb{R}$ and a countable $B \subset \mathbb{R}$ such that there is no $z \in [B]^{<\omega}$ such that $R(z, \{y\})$ holds. We will prove that for every $s \in \mathbb{R}$ there exists a pair $w_1, w_2 \in \mathbb{R}$ satisfying $y = w_1 + w_2$, $w_1 \equiv_h w_2$, $B \cup \{w_1, w_2\}$ linearly independent and $s \leq_h w_1, w_2$. This fact indeed implies that the set $\{x : x \in \mathcal{E} \wedge R(x, y)\} \cap H^R_B$ is cofinal in the hyperdegrees, so the second part of $(*)$ also holds.

Here we repeat Miller's argument. Without loss of generality we can suppose that $y \leq_h s$ and s is not hyperarithmetic in any finite subset of $B \cup \{y\}$ because we can replace s by a more complicated real. We can choose w_1 and w_2 such that s is coded in w_1's odd and w_2's even digits so that $w_1 + w_2 = y$. Then $s \leq_h w_1, w_2$, hence $y \leq_h w_1, w_2$. But then $y = w_1 + w_2$ implies $w_1 \equiv_h w_2$. If $w_1 \in \langle B, w_2 \rangle_{\mathbb{Q}}$ then $y \in \langle B, w_2 \rangle_{\mathbb{Q}} \setminus \langle B \rangle_{\mathbb{Q}}$

and $w_2 \in \langle B, y\rangle_{\mathbb{Q}}$; but this would imply that s is hyperarithmetic in a finite subset of $B \cup \{y\}$, which is a contradiction. Thus w_1 and w_2 are the appropriate reals.

Hence property $(*)$ holds indeed, and a direct application of Theorem 4.10 produces a Π^1_1 Hamel basis. ■

Finally we will prove another variant of our theorem, considering the case where the choice at step α does not depend on the previous choices.

THEOREM 4.12. $(V = L)$ *Let $t \in \mathbb{R}$, and suppose that $G \subset \mathbb{R}^n \times \mathbb{R}$ is a $\Delta^1_1(t)$ set and for every countable $A \subset \mathbb{R}$ the complement of the set $\bigcup_{p\in A} G_p$ is cofinal in the hyperdegrees. Then there exists an uncountable $\Pi^1_1(t)$ set $X \subset \mathbb{R}^n$ that intersects every G_p in a countable set.*

Proof. By Theorem 2.4 there exists a Σ_1 formula θ such that

$$a \in G^c \Leftrightarrow L_{\omega_1^{(a,t)}}[a,t] \models \theta(a,t).$$

Now let us define the set H as follows:

$$(x,p) \in H \Leftrightarrow x \in \mathcal{S} \wedge p,t \leq_h x \wedge L_{\omega_1^x} \models (\forall p' \leq_L p)(\theta((x,p'),t)).$$

Then H is a $\Pi^1_1(t)$ set: just repeat the usual argument, i.e., $x \in \mathcal{S} \wedge p,t \leq_h x$ implies that $L_{\omega_1^x} = L_{\omega_1^{((x,p),t)}}[((x,p),t)]$, and use Theorems 2.4, 2.1, 2.2 and Lemma 3.1. Observe that for a real p,

$$H_p = \Big(\bigcap_{p' \leq_L p} G^c_{p'}\Big) \cap \mathcal{S} \cap \{z : p,t \leq_h z\}.$$

Thus the theorem's conditions imply that for every real p the section H_p is cofinal in the hyperdegrees.

Define $F \subset (\mathbb{R}^n)^{\leq\omega} \times \mathbb{R} \times \mathbb{R}^n$ by $(A,p,x) \in F \Leftrightarrow (x,p) \in H \wedge x \notin A$. Obviously, for every (A,p) the section $F_{(A,p)}$ is cofinal in the hyperdegrees and F is $\Pi^1_1(t)$. Our Main Theorem provides an uncountable $\Pi^1_1(t)$ set $X \subset \mathbb{R}^n$ and enumerations $X = \{x_\alpha : \alpha < \omega_1\}$, $\mathbb{R} = \{p_\alpha : \alpha < \omega_1\}$, and an enumeration A_α (in type $\leq \omega$) of $\{x_\beta : \beta < \alpha\}$ such that $x_\alpha \in F_{(A_\alpha,p_\alpha)} = H_{p_\alpha} \setminus \{x_\beta : \beta < \alpha\}$. Suppose that there exists a $p \in \mathbb{R}$ for which $|X \cap G_p| > \aleph_0$. Then $p_\beta >_L p$ if β is high enough, since only countably many p_α's are $<_L p$. But if $p_\beta >_L p$ then $x_\beta \in G^c_p$. ■

Now Theorem 1.4 is a trivial consequence of Theorem 4.12.

5. Applications. Theorem 1.3 can be applied in various situations. Let us remark here that one can obtain Π^1_1 sets instead of coanalytic ones by just repeating the proofs and using Theorem 3.4 in all the theorems of this section. We will prove the simpler (boldface) versions for the sake of transparency.

THEOREM 5.1. ($V = L$) *There exists a coanalytic MAD family.*

Proof. First fix a recursive partition $B = \{B_i : i \in \omega\}$ of ω to infinite sets. Define $F \subset (\mathcal{P}(\omega))^{\leq\omega} \times \mathcal{P}(\omega) \times \mathcal{P}(\omega)$ as follows: $(A, p, x) \in F \Leftrightarrow$

EITHER the conjunction of the following clauses holds:
1. $\operatorname{ran}(A) \cup B$ contains pairwise almost disjoint elements,
2. p is almost disjoint from the elements of $\operatorname{ran}(A) \cup B$,
3. $p \subset x$ and x is almost disjoint from the elements of $\operatorname{ran}(A) \cup B$;

OR $1 \wedge \neg 2$ holds and x is almost disjoint from the elements of $\operatorname{ran}(A) \cup B$;
OR $\neg 1$.

Clearly, F is Borel. We have to prove that for all pairs (A, p) the section $F_{(A,p)}$ is cofinal in the Turing degrees.

Suppose that 1 and 2 hold, and let $u \in \mathcal{P}(\omega)$ be an arbitrary real. Choose $x' = p \cup \bigcup_{i\in\omega} F_i$, where $F_i \subset B_i$ are finite and if $i > j$ then $A(j) \cap F_i = \emptyset$ and

$$|(p \cup F_i) \cap B_i| \equiv 1 \bmod 2 \;\Leftrightarrow\; u(i) = 1.$$

For every i there exists such an F_i, since the B_i's are disjoint and infinite, and $\operatorname{ran}(A) \cup B$ contains pairwise almost disjoint sets. Then x' satisfies 3 and $u \leq_T x'$.

Now in the case when $1 \wedge \neg 2$ holds our job is easier: e.g. we can repeat the previous argument omitting p.

Finally, if $\neg 1$ is true then $F_{(A,p)} = \mathcal{P}(\omega)$.

Notice that Theorem 1.3 was stated in the form where the set of parameters is $\mathbb{R}$ but we can easily replace it by $\mathcal{P}(\omega)$ using a recursive Borel isomorphism.

So we can apply Theorem 1.3 and get a coanalytic set $X = \{x_\alpha : \alpha \in \omega_1\}$ such that X is compatible with F. It is obvious by transfinite induction that the elements of X are pairwise almost disjoint. It is also clear that $X \cup B$ is maximal since for every real p there exists an $\alpha < \omega_1$ such that $p_\alpha = p$. Thus there exists an element of X that is not almost disjoint from p. ∎

THEOREM 5.2. ($V = L$) *There exists a coanalytic two-point set.*

Proof. For each real $p \in \mathbb{R}$ fix a line l_p defined by the equation $((p)_1)x + ((p)_2)y = (p)_3$, where $(p)_1, (p)_2$ and $(p)_3$ are the reals made of every $3k$th, $(3k+1)$th and $(3k+2)$th digit of p. The set l_p can be empty, however every line appears at least two times. Let us define $F \subset (\mathbb{R}^2)^{\leq\omega} \times \mathbb{R} \times \mathbb{R}^2$ by $(A, p, x) \in F \Leftrightarrow$

EITHER the conjunction of the following clauses holds:

1. there are no three collinear points in $\operatorname{ran}(A)$,
2. $|\operatorname{ran}(A) \cap l_p| < 2$ and $l_p \neq \emptyset$,

3. $x \in l_p \setminus \mathrm{ran}(A)$ and x is not collinear with any two distinct points of $\mathrm{ran}(A)$;

OR $1 \wedge \neg 2$ holds and x is not collinear with two distinct points of $\mathrm{ran}(A)$;
OR $\neg 1$.

Now F is clearly Borel. We have to check that for all (A, p) the section $F_{(A,p)}$ is cofinal in the Turing degrees. Fix a pair (A, p). If $1 \wedge 2$ holds then the section is equal to l_p minus a countable set. Every line is cofinal in the Turing degrees, because we can choose one of the coordinates arbitrarily. Now notice that if H is a set which is cofinal in the Turing degrees and H' is countable, then $H \setminus H'$ is still cofinal: To see this, let u be an arbitrary real and let s be such that $(\forall s' \in H')(s' \not\geq_T s)$. Then there exists $r \in H$ such that $s, u \leq_T r$, and clearly $r \notin H'$. So if $1 \wedge 2$ holds then $F_{(A,p)}$ is cofinal in the Turing degrees.

If $1 \wedge \neg 2$ holds then we just have to choose an arbitrary point that is not collinear with any two distinct points of A. The case when 1 is false is obvious.

Thus by Theorem 1.3 we get an uncountable coanalytic set $X = \{x_\alpha : \alpha < \omega_1\} \subset \mathbb{R}^2$. One can easily verify that X cannot contain three collinear points. Moreover, since every line l_p appears at least twice, $|l_p \cap X| = 2$. ∎

Similar statements can be formulated for n-point sets, circles, appropriate algebraic curves, etc.—the above method works in these cases.

5.1. Curves in the plane. Now we will consider the following question: What can we say about a set in the plane which intersects every "nice" curve in a countable set? Let us call a continuously differentiable $\mathbb{R} \to \mathbb{R}^2$ function a C^1 *curve*.

Definition 5.3. We say that a set $H \subset \mathbb{R}^2$ is C^1-*small* if the intersection of H with the range of every C^1 curve is a countable set.

In [6] the authors proved that under Martin's axiom and the Semi-Open Coloring Axiom, if H is C^1-small then $|H| \leq \aleph_0$. Moreover, they showed in ZFC that no perfect set is C^1-small. Thus no uncountable analytic set is C^1-small. On the other hand, the following proposition holds.

Proposition 5.4. (*CH*) *There exists an uncountable C^1-small set.*

Proof. We will prove later that the union of the ranges of countably many C^1 curves cannot cover the plane. This implies the statement by an easy transfinite induction. ∎

Thus it is of interest whether an uncountable C^1-small subset can be coanalytic. We will apply Theorem 1.4.

THEOREM 5.5. ($V = L$) *There exists an uncountable C^1-small coanalytic set.*

Proof. First we have to prove that there exists a Borel set $G \subset \mathbb{R}^2 \times \mathbb{R}$ such that if γ is a C^1 curve then there exists a $p \in \mathbb{R}$ such that $G_p = \mathrm{ran}(\gamma)$.

One can easily prove that the set B of C^1 curves as a subset of $C(\mathbb{R}, \mathbb{R}^2)$ is a Borel set (see e.g. [9, 23.D]). The set $\{((x, y), \gamma) : (x, y) \in \mathrm{ran}(\gamma)\} \subset \mathbb{R}^2 \times C(\mathbb{R}, \mathbb{R}^2)$ is clearly closed. So $(\mathbb{R}^2 \times B) \cap \{((x, y), \gamma) : (x, y) \in \mathrm{ran}(\gamma)\}$ is also a Borel set. Furthermore, there exists a Borel isomorphism $\phi : \mathbb{R} \to B$ since these two are standard Borel spaces of cardinality $\mathfrak{c}$ and we can apply the isomorphism theorem. Now we can define $G \subset \mathbb{R}^2 \times \mathbb{R}$: $((x, y), p) \in G \Leftrightarrow ((x, y), \phi(p)) \in (\mathbb{R}^2 \times B) \cap \{((x, y), \gamma) : (x, y) \in \mathrm{ran}(\gamma)\}$, which is a Borel set, and for every $\gamma \in C^1$ there exists a $p \in \mathbb{R}$ such that $G_p = \mathrm{ran}(\gamma)$.

To apply Theorem 1.4 we have to check that if we have countably many C^1 curves $\{\gamma_i : i \in \omega\}$ then the complement of the union of their ranges is cofinal in the Turing degrees. For this it is enough to show that there exists a line l such that

$$\Big|l \cap \bigcup(\{\mathrm{ran}(\gamma_i) : i \in \omega\})\Big| \leq \aleph_0.$$

Let us concentrate solely on the horizontal lines. For a curve γ_i let $f_i(x) = \pi_y(\gamma_i(x))$, the composition with the projection on the vertical axis. Since f_i is a C^1 function, by Sard's lemma the set $H_i = \{y \in \mathbb{R} : (\exists x)(f_i'(x) = 0 \wedge f_i(x) = y)\}$ has Lebesgue measure zero. Let $b \in \mathbb{R} \setminus \bigcup H_i$. Then the line $\{(x, b) : x \in \mathbb{R}\}$ intersects every curve γ_i in countably many points, since otherwise it would be the image of a critical value.

Finally, an application of Theorem 1.4 produces an uncountable C^1-small coanalytic set. ∎

5.2. Problems. In Theorem 1.3 the set of parameters is a Borel set, and this was used in the proof numerous times.

PROBLEM 5.6. *Does Theorem* 1.3 *hold if we only assume that B is coanalytic?*

As a partial converse we have proved that the conclusion of the Main Theorem implies that every real is constructible. It is natural to ask whether the converse also holds.

PROBLEM 5.7. *Does the conclusion of Theorem* 1.3 *hold if every real is constructible?*

One of the weaknesses of the method is that the constructed set X is a subset of $\mathcal{S}$. It is known (see e.g. [10]) that $\mathcal{S}$ is the largest thin (not containing a perfect subset) Π^1_1 set. Thus none of the constructed sets contain a perfect subset. In the case of C^1-small sets this cannot be expected, but how about the other constructions?

PROBLEM 5.8. *Is it consistent that there exists a Π^1_1 Hamel basis (two-point set, MAD family) that contains a perfect subset?*

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