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Some model theory of $\mathrm{SL}(2,\mathbb{R})$

by

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Abstract. We study the action of $G = \mathrm{SL}(2,\mathbb{R})$, viewed as a group definable in the structure $M = (\mathbb{R}, +, \times)$, on its type space $S_G(M)$. We identify a minimal closed G-flow I and an idempotent $r \in I$ (with respect to the Ellis semigroup structure $*$ on $S_G(M)$). We also show that the "Ellis group" $(r * I, *)$ is nontrivial, in fact it is the group with two elements, yielding a negative answer to a question of Newelski.

1. Introduction and preliminaries. Abstract topological dynamics concerns the actions of (often discrete) groups G on compact Hausdorff spaces X. Newelski has suggested in a number of papers [6], [7] that the notions of topological dynamics may be useful for "generalized stable group theory", namely the understanding of definable groups in unstable settings, but informed by methods of stable group theory.

Given a structure M and a group G definable in M, we have the (left) action of G on its type space $S_G(M)$. When $\mathrm{Th}(M)$ is stable, there is a unique minimal closed G-invariant subset I of $S_G(M)$ which is precisely the set of generic types of G. Moreover (still in the stable case) $S_G(M)$ is equipped with a semigroup structure $*$: $p * q = \mathrm{tp}(a \cdot b/M)$ where a, b are independent realizations of p, q respectively, and $(I, *)$ is a compact Hausdorff topological group which turns out to be isomorphic to $G(\overline{M})/G(\overline{M})^0$ where $\overline{M}$ is a saturated elementary extension of M. In fact this nice situation is more or less characteristic of the stable case, so will not extend as such to unstable settings (other than what has been called in [4] "generically stable groups").

However, it was shown in [10] that for the much larger class of so-called *fsg* groups definable in *NIP* theories, the situation is not so far from the stable case. In the *o*-minimal context the *fsg* groups are precisely the definably compact groups; for example working in the structure $(\mathbb{R}, +, \times)$, these will be the semialgebraic compact Lie groups. However, there is no general model-

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theoretic machinery (of a stability-theoretic nature) for understanding *simple noncompact real Lie groups* (and their interpretations in arbitrary real closed fields).

In this paper we try to initiate such a study, focusing on $G = \mathrm{SL}(2,\mathbb{R})$. The reason we work over the standard model $(\mathbb{R},+,\times)$ rather than an arbitrary or saturated model is that all types over the standard model are definable, hence externally definable sets correspond to definable sets, and the type space is equipped with an "Ellis semigroup structure" $*$. We expect that analogues of our results hold over arbitrary models, expanded by the externally definable sets, and also for arbitrary semialgebraic semisimple Lie groups in place of $\mathrm{SL}(2,\mathbb{R})$. But we leave this for others to investigate.

In any case our main objective is to identify a minimal closed G-invariant subset I of $S_G(M)$, to identify an idempotent element $r \in I$ and to describe the "Ellis group" $r * I$. Now $r * I$ as an abstract group does not depend on the choice of I or r. Newelski asked in [6] whether for groups G definable in *NIP* theories, $G(\overline{M})/G(\overline{M})^{00}$ is isomorphic to this $r * I$ as an abstract group. In [10] we gave a positive answer for so-called *fsg* groups in *NIP* theories. When $G = \mathrm{SL}(2,-)$ and K is a saturated real closed field then $G(K)$ is simple (modulo its finite centre) as an abstract group, whereby $G(K) = G(K)^{00}$. However, we will show that in the case of $\mathrm{SL}(2,\mathbb{R})$ acting on its type space, the Ellis group $r * I$ is the group with two elements, in particular nontrivial, so giving a negative answer to Newelski's question.

Our idempotent will be obtained as an "independent" (with respect to nonforking) product of realizations of a generic type of T^{00} over $\mathbb{R}$ and an $H(\mathbb{R})$-invariant type of H where T is a maximal compact and H is the connected component of the standard Borel subgroup of $\mathrm{SL}(2,\mathbb{R})$. These results have additional interest in the light of the theory of "definable" topological dynamics, discussed briefly in the next paragraph.

After a preliminary version of the current paper was written, the three authors developed a theory of definable topological dynamics [2]. That is, given a first order structure M and a group G definable in M we gave an appropriate definition of a "definable" action of G on a compact space, and developed relative analogues of the classical theory [1] of the topological dynamics of a discrete group. A definability of types assumption was needed, which explains the restriction to the field $\mathbb{R}$ in the current paper. The analogue of the Stone–Čech compactification βG of G (from the discrete case) is the type space $S_G(M)$ in the definable case. Moreover any minimal closed G-invariant subspace of $S_G(M)$ will be the universal minimal definable G-flow. So from the point of view of this definable topological dynamics, we have described in this paper, among other things, the universal minimal definable G-flow, where G is $\mathrm{SL}(2,\mathbb{R})$ considered as a group definable in the real field

$M = (\mathbb{R}, +, \times)$. Let us emphasize that the type space $S_G(M)$ is equipped with its Stone space topology, so in particular it is totally disconnected. When $\mathrm{SL}(2,\mathbb{R})$ is given its Euclidean topology, its action on $S_G(M)$ is not (jointly) continuous. Nevertheless the invariants we obtain are related to invariants emanating from the dynamics of $\mathrm{SL}(2,\mathbb{R})$ as a topological group, which we will mention at the end of the paper. In any case, to our knowledge the current paper is the first study, from the points of view of model theory, dynamics, or even semialgebraic geometry, of semisimple Lie groups acting on their type spaces.

We will assume a basic knowledge of model theory (types, saturation, definable types, heirs, coheirs, ...). References are [11] and [8]. Let us fix a complete 1-sorted theory T, a saturated model $\overline{M}$ of T, and a model M which is an elementary substructure of $\overline{M}$. In the body of the paper, T will be RCF, the theory of real closed fields, in the language of rings, and M will be the "standard model" $(\mathbb{R}, +, \times, 0, 1)$. By a definable set in M we mean a subset of M^n definable (with parameters) in M, namely by a formula $\phi(x_1, \ldots, x_n, \bar{b})$ where we exhibit the parameters $\bar{b}$ from M. $S_n(M)$ is the space of complete n-types over M, equivalently, ultrafilters on the Boolean algebra of definable subsets of M^n (which we identify with the Boolean algebra of formulas $\phi(x_1, \ldots, x_n)$ with parameters from M, up to equivalence). This is a compact Hausdorff space, under the Stone space topology. Although not strictly needed, we will now discuss externally definable sets and types, to situate our results in a broader context which allows for generalizations.

Definition 1.1.

(i) A subset $X \subseteq M^n$ is *externally definable* if there is a formula $\phi(x_1, \ldots, x_n, \bar{b})$, where the parameters $\bar{b}$ are from $\overline{M}$, such that

$$X = \{\bar{a} \in M^n : \overline{M} \models \phi(\bar{a}, \bar{b})\}.$$

(ii) By $S_{\mathrm{ext},n}(M)$ we mean the space of ultrafilters on the Boolean algebra of externally definable subsets of M^n.

Fact 1.2. *For any $p(\bar{x}) \in S_{\mathrm{ext},n}(M)$, there is a unique $p'(\bar{x}) \in S_n(\overline{M})$ which is finitely satisfiable in M and such that the "trace on M" of any formula in p' is in p. This sets up a homeomorphism ι between $S_{\mathrm{ext},n}(M)$ and the closed subspace of $S_n(\overline{M})$ consisting of all types finitely satisfiable in M.*

Note that if all types over M are definable, then externally definable subsets of M^n are definable and $S_{\mathrm{ext},n}(M)$ coincides with $S_n(M)$.

Lemma 1.3. *Suppose that all types over M are definable, and let $p(x) \in S_n(M)$.*

(i) *For any* $B \supseteq M$, *p has a unique coheir* $p'(x) \in S_n(B)$, *namely an extension of p to a complete type over B which is finitely satisfiable in M.*
(ii) *For any* $B \supseteq M$, *p has a unique heir over B, which we write as* $p|B$ *and which can also be characterized as the unique extension of p to B which is definable over M. Moreover* $p|B$ *is simply the result of applying the defining schema for p to the set of parameters B.*
(iii) *For any tuples* b, c *from* $\overline{M}$, $\mathrm{tp}(b/M, c)$ *is definable over M if and only if* $\mathrm{tp}(c/M, b)$ *is finitely satisfiable in M.*

Now suppose G is a group definable over M. We identify G with the group $G(\overline{M})$ and write $G(M)$ for the points in the model M. We have the spaces of types $S_G(M)$, $S_{\mathrm{ext},G}(M)$ and $S_G(\overline{M})$. For $g, h \in G$ we write gh for the product. $G(M)$ acts (on the left) by homeomorphisms on $S_G(M)$ and $S_{\mathrm{ext},G}(M)$.

Definition 1.4. Let $p(x), q(x) \in S_{\mathrm{ext},G}(M)$. Let b realize q in G, and let a realize the unique $p' \in S_G(\overline{M})$ given by 1.2. We define $p * q$ to be the (external) type of ab over M. So in the case when all types over M are definable, this just means: Let $b \in G$ realize q and let $a \in G$ realize the unique coheir of p over M, b; then $p * q = \mathrm{tp}(ab/M)$.

The following is contained in [6, Section 4] and [7]. Everything can be proved directly, but it is a special case of the theory of abstract topological dynamics, as treated in [1, Chapter 6] for example.

Lemma 1.5.

(i) $(S_{\mathrm{ext},G}(M), *)$ *is a semigroup, which we call the Ellis semigroup, and* $*$ *is continuous in the first coordinate, namely for any* $q \in S_{\mathrm{ext},G}(M)$ *the map taking* $p \in S_{\mathrm{ext},G}(M)$ *to* $p * q \in S_{\mathrm{ext},G}(M)$ *is continuous* [6, Section 4], [7, Lemma 1.5].
(ii) *Left ideals of* $S_{\mathrm{ext},G}(M)$ *(with respect to* $*$*) coincide with subflows, that is, closed* $G(M)$*-invariant subsets* [7, Sections 1, 4].
(iii) *If* $I \subseteq S_{\mathrm{ext},G}(M)$ *is a minimal subflow, then I contains an idempotent r such that* $r*r = r$, *and* $(r*I, *)$ *is a group, whose isomorphism type does not depend on I or r* [6, Section 4], [7, Sections 1, 4].

As mentioned earlier, in the stable case there is a unique minimal subflow, the space of generic types of G over M. We will, below, consider the case where T is the theory of real closed fields, $M = (\mathbb{R}, +, \times)$ is the standard model and $G = \mathrm{SL}(2, -)$. Sometimes we write $\mathbb{R}$ for M to be consistent with standard notation. So $G(\mathbb{R})$ is the interpretation of G in M, namely $\mathrm{SL}(2, \mathbb{R})$, and $\mathbb{R}$ as a structure is $(\mathbb{R}, +, \times)$. It is well-known that all types over $\mathbb{R}$ are definable [5, 9], hence Lemma 1.5 applies to $G(\mathbb{R})$ acting on $S_G(\mathbb{R})$.

2. SL$(2,\mathbb{R})$**.** We review some basic and well-known facts about SL$(2,\mathbb{R})$, the group of 2×2 matrices over $\mathbb{R}$ with determinant 1. All the objects, maps etc. we mention will be semialgebraic and so pass over to SL$(2,K)$ where K is a saturated real closed field. We sometimes write G for SL(2), so $G(\mathbb{R})$ for SL$(2,\mathbb{R})$. Write I for the identity matrix. The centre of SL$(2,\mathbb{R})$ is $\{I,-I\}$. The quotient of SL$(2,\mathbb{R})$ by this centre is denoted PSL$(2,\mathbb{R})$.

$H(\mathbb{R})$ will denote the connected component of the standard Borel subgroup of $G(\mathbb{R})$, namely the subgroup consisting of all matrices $\left(\begin{smallmatrix} b & c\\ 0 & b^{-1}\end{smallmatrix}\right)$ where $b\in\mathbb{R}_{>0}$ and $c\in\mathbb{R}$. The group $H(\mathbb{R})$ is precisely the semidirect product of $(\mathbb{R}_{>0},\times)$ with $(\mathbb{R},+)$. We let $T(\mathbb{R})$ denote SO$(2,\mathbb{R})$, the subgroup of $G(\mathbb{R})$ consisting of all matrices $\left(\begin{smallmatrix} x & -y\\ y & x\end{smallmatrix}\right)$ with $x,y\in\mathbb{R}$ and $x^2+y^2=1$. The symbol T here stands for torus. We have $H(\mathbb{R})\cap T(\mathbb{R})=\{I\}$ and any element of G can be uniquely written in the form ht (as well as t_1h_1) for $t,t_1\in T$ and $h,h_1\in H$. The group $T(\mathbb{R})$ is a maximal compact subgroup of $G(\mathbb{R})$. Note that $-I\in T(\mathbb{R})$.

We write $V(\mathbb{R})$ for the homogeneous space $G(\mathbb{R})/H(\mathbb{R})$ (space of left cosets $\{gH(\mathbb{R}) : g\in G(\mathbb{R})\}$), and π (or $\pi(\mathbb{R})$) for the projection $G(\mathbb{R})\to V(\mathbb{R})$. Note that $\pi_{|T(\mathbb{R})}\colon T(\mathbb{R})\to V(\mathbb{R})$ is a homeomorphism. We indicate the action of $G(\mathbb{R})$ on $V(\mathbb{R})$ by $\cdot$. Understanding this action will be quite important for us. The usual action of $G(\mathbb{R})$ on the real projective line by Möbius transformations factors through the action of $G(\mathbb{R})$ on $V(\mathbb{R})$, and we will try to describe what is going on.

Remark 2.1. The standard action of $G(\mathbb{R})$ on $\mathbb{P}^1(\mathbb{R})$ is

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\cdot\begin{bmatrix} x\\ y\end{bmatrix}=\begin{bmatrix} ax+by\\ cx+dy\end{bmatrix},$$

where $\left[\begin{smallmatrix} x\\ y\end{smallmatrix}\right]$ is a representative of an element of $\mathbb{P}^1(\mathbb{R})$. We identify $\left[\begin{smallmatrix} x\\ 1\end{smallmatrix}\right]$ with $x\in\mathbb{R}$, and treat $\left[\begin{smallmatrix} 1\\ 0\end{smallmatrix}\right]=\infty$ as the "point at infinity". If $c=0$, then

$$\begin{pmatrix} a & b\\ 0 & a^{-1}\end{pmatrix}\cdot\begin{bmatrix} x\\ 1\end{bmatrix}=\begin{bmatrix} a^2x+ab\\ 1\end{bmatrix}\quad\text{and}\quad\begin{pmatrix} a & b\\ 0 & a^{-1}\end{pmatrix}\cdot\begin{bmatrix} 1\\ 0\end{bmatrix}=\begin{bmatrix} 1\\ 0\end{bmatrix}.$$

It is easy to prove the following fact.

Remark 2.2.

(i) $\mathrm{Stab}_{G(\mathbb{R})}\left[\begin{smallmatrix} 1\\ 0\end{smallmatrix}\right]=H_1(\mathbb{R})$, where $H_1(\mathbb{R})=H(\mathbb{R})\times\{I,-I\}$.
(ii) $Z(G(\mathbb{R}))=\{I,-I\}$ acts trivially on $\mathbb{P}^1(\mathbb{R})$, and the resulting action of PSL$(2,\mathbb{R})=G(\mathbb{R})/Z(G(\mathbb{R}))$ on $\mathbb{P}^1(\mathbb{R})$ is the usual faithful action.

Let π_1 denote the map from $G(\mathbb{R})$ to $\mathbb{P}^1(\mathbb{R})$ taking g to $g\cdot\left[\begin{smallmatrix} 1\\ 0\end{smallmatrix}\right]$. So by Remark 2.2(i), π_1 induces an isomorphism of $G(\mathbb{R})$-homogeneous spaces $G(\mathbb{R})/H_1(\mathbb{R})$ and $\mathbb{P}^1(\mathbb{R})$. Moreover we have:

REMARK 2.3. The restriction of π_1 to $T(\mathbb{R})$ induces a homeomorphism between $T(\mathbb{R})/\{I,-I\}$ and $\mathbb{P}^1(\mathbb{R})$ such that the identity of $T(\mathbb{R})/\{I,-I\}$ goes to $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$.

Finally, by virtue of the homeomorphism $\pi_{|T(\mathbb{R})}$ between $T(\mathbb{R})$ and $V(\mathbb{R})$ and the action of $G(\mathbb{R})$ on $V(\mathbb{R})$, we have an action (also written $\cdot$) of $G(\mathbb{R})$ on $T(\mathbb{R})$. Note that $g\cdot t$ is the unique $t_1\in T(\mathbb{R})$ such that $gt=t_1h_1$ for some (unique) $h_1\in H(\mathbb{R})$. Likewise by virtue of Remark 2.3, and the action of $G(\mathbb{R})$ on $\mathbb{P}^1(\mathbb{R})$, we obtain an action $\cdot_1$ of $G(\mathbb{R})$ on $T(\mathbb{R})/\{I,-I\}$. We clearly have:

REMARK 2.4. The action $\cdot_1$ of $G(\mathbb{R})$ on $T(\mathbb{R})/\{I,-I\}$ is induced by the action $\cdot$ of $G(\mathbb{R})$ on $T(\mathbb{R})$. In particular, for any $g\in G(\mathbb{R})$ and $t\in T(\mathbb{R})$, we have $g\cdot t\in g\cdot_1 (t/\{I,-I\})$.

As remarked above, all this passes to a saturated model K of RCF in place $\mathbb{R}$. We write G for $G(K)=\mathrm{SL}(2,K)$, H for $H(K)$, V for $V(K)$ etc. But now our groups and homogeneous spaces contain nonstandard points, and the study of their types and interaction is what this paper is about.

3. Main results. We follow the conventions at the end of the last section. ($G=\mathrm{SL}(2,-)$, K a saturated real closed field, etc.) We say that $a\in K$ is *infinite* if $a>\mathbb{R}$, and *negative infinite* if $a<\mathbb{R}$. We denote by $\mathrm{Fin}(K)$ the elements of K which are neither infinite nor negative infinite. Any $a\in \mathrm{Fin}(K)$ has a standard part $\mathrm{st}(a)\in\mathbb{R}$. Also given $B\subset K$, a is *infinite* (resp. *negative infinite*) *over* B if $a>\mathrm{dcl}(B)$ (resp. $a<\mathrm{dcl}(B)$). Call $a\in K$ *positive infinitesimal* if $a>0$ and $a<r$ for all positive $r\in\mathbb{R}$. Likewise for *negative infinitesimal* and for *infinitesimal over* B. Note that if for example $a\in K$ is positive infinitesimal, and $p(x)=\mathrm{tp}(a/\mathbb{R})$ and $B\subset K$, then $p|B$ is the type of an element which is positive infinitesimal over B.

We sometimes write g/H for the left coset gH. The projection $\pi\colon G\to V=G/H$ induces a surjective continuous map, which we also call π, from $S_G(\mathbb{R})$ to $S_V(\mathbb{R})$. Both these type spaces are acted on (by homeomorphisms) by $G(\mathbb{R})$, and we clearly have:

LEMMA 3.1. *π is $G(\mathbb{R})$-invariant: for any $p\in S_G(\mathbb{R})$ and $g\in G(\mathbb{R})$, $\pi(gp)=g\cdot\pi(p)$.*

DEFINITION 3.2. Let $p_1\in S_G(\mathbb{R})$, and $q\in S_V(\mathbb{R})$. Define p_1*q to be $\mathrm{tp}(g\cdot b/\mathbb{R})\in S_V(\mathbb{R})$ where b realizes q, and g realizes the unique coheir of p_1 over $\mathbb{R},b$.

With the above notation, the following extends Lemma 3.1.

LEMMA 3.3. *For any $p,p_1\in S_G(\mathbb{R})$, $\pi(p_1*p)=p_1*\pi(p)$.*

Proof. Fix $p, p_1 \in S_G(\mathbb{R})$. Then $p_1 * p = \mathrm{tp}(g_1 g/\mathbb{R})$ where g_1 realizes p_1, g realizes p and $\mathrm{tp}(g_1/\mathbb{R}, g)$ is finitely satisfiable in $\mathbb{R}$. But then $\pi(p_1 * p) = \mathrm{tp}((g_1 g/H)/\mathbb{R}) = \mathrm{tp}(g_1 \cdot (g/H)/\mathbb{R})$. Now $\mathrm{tp}(g_1/\mathbb{R}, g/H)$ is finitely satisfiable in $\mathbb{R}$ and g/H realizes $\pi(p)$. Hence $\mathrm{tp}(g_1 \cdot (g/H)/\mathbb{R}) = p_1 * \pi(p)$, as required. ∎

As above, the symbol g will range over elements of G. Also h ranges over elements of H, and t over elements of T. If $h = \left(\begin{smallmatrix} b & c \\ 0 & b^{-1} \end{smallmatrix}\right)$ is in H we identify it with the pair $(b, c) \in \mathbb{R}_{>0} \times \mathbb{R}$. And if $t = \left(\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right)$ is an element of T we identify it with the pair (x, y) (so T is identified with the unit circle under complex multiplication).

We now fix some canonical types: $p_0 = \mathrm{tp}(b, c/\mathbb{R})$ where b is infinite and c is infinite over b. It is easy to check that p_0 is *left* $H(\mathbb{R})$*-invariant*: if h realizes p_0 and $h_1 \in H(\mathbb{R})$, then $h_1 h$ also realizes p_0.

Note that all nonalgebraic types (over $\mathbb{R}$) of elements of T are generic in the sense of [4]. In fact, T is the simplest possible *fsg* group in RCF. Let $q_0 = \mathrm{tp}(x, y/\mathbb{R})$ (as the type of an element of T) where y is positive infinitesimal and $x > 0$ (so x is the positive square root of $1 - y^2$). We call q_0 the type of a "positive infinitesimal" of T: it is infinitesimally close to the identity, on the "positive" side.

Likewise, for any $t \in T(\mathbb{R})$ and $t_1 \in T$, we will say that t_1 is "infinitesimally close, on the positive side" to t if $t_1 t^{-1}$ realizes q_0.

The bijection (homeomorphism) between T and V given by $\pi_{|T}$ induces a homeomorphism (still called π) between $S_T(\mathbb{R})$ and $S_V(\mathbb{R})$, so we will sometimes identify them below, although we distinguish between q and $\pi(q)$ (for $q \in S_T(\mathbb{R})$).

Definition 3.4. We define r_0 to be $\mathrm{tp}(th/\mathbb{R}) \in S_G(\mathbb{R})$ where $h \in H$ realizes p_0 and $t \in T$ realizes the unique coheir of q_0 over $\mathbb{R}, h$.

Note that $\pi(r_0) = \pi(q_0)$. Our first aim is to show that $\mathrm{cl}(G(\mathbb{R}) r_0) = I$ is a minimal $G(\mathbb{R})$-flow, and that r_0 is an idempotent. Note that $\mathrm{cl}(G(\mathbb{R}) r_0)$ is precisely the set of $p * r_0 \in S_G(\mathbb{R})$ for p ranging over $S_G(\mathbb{R})$. Likewise for $\mathrm{cl}(G(\mathbb{R}) \cdot \pi(r_0))$.

Lemma 3.5. *For any* $p \in S_G(\mathbb{R})$, $p * \pi(q_0) = \pi(q_0)$ *if and only if* p *is of the form* $\mathrm{tp}(t_1 h_1/\mathbb{R})$ *with* $h_1 \in H$ *and* $t_1 \in T$ *the identity or a realization of* q_0.

Proof. Let $\mathrm{tp}(t_1, h_1/\mathbb{R}, t)$ be finitely satisfiable in $\mathbb{R}$ with $h_1 \in H$, $t_1 \in T$ and t realizing q_0 (so t/H realizes $\pi(q_0)$). Then $\mathrm{tp}(t/\mathbb{R}, t_1, h_1)$ is the unique heir of q_0 over $(\mathbb{R}, t_1, h_1)$ (Lemma 1.3(iii)). In particular $t \in T$ is positive infinitesimal over $(\mathbb{R}, t_1, h_1)$ as is $t/H \in V$. Now $h_1 \cdot (1/H) = 1/H$, hence clearly $h_1 \cdot (t/H)$ is also infinitesimally close (over $\mathbb{R}, t_1, h_1$) to $1/H$.

Claim. $h_1 \cdot (t/H)$ *is on the "positive" side of* $1/H$ *and realizes the unique heir of* $\pi(q_0)$ *over* $\mathbb{R}, t_1, h_1$.

Proof of Claim. When we mention "positive side" we are identifying V and T. Now the map π_1 from T to $\mathbb{P}^1$ is a "local homeomorphism" taking the identity to $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = \infty$ (Remarks 2.1–2.3) and taking positive infinitesimals in T to infinite $x \in K$ (because by Remark 2.1, $\left(\begin{smallmatrix}x & -y\\ y & x\end{smallmatrix}\right) \cdot \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = \left[\begin{smallmatrix}x/y\\1\end{smallmatrix}\right]$, where x/y is infinite) and (by definition) respects the action of G. Hence it suffices to show that for $h_1 = (b,c) \in H$ and $x \in K$ infinite such that $\mathrm{tp}(h_1/\mathbb{R}, x)$ is finitely satisfiable in $\mathbb{R}$, the element $h_1 \cdot x = b^2x + bc$ is (positive) infinite over $\mathbb{R}, b, c$. This is clear: Firstly, x is infinite over $\mathbb{R}, b, c$ (as $\mathrm{tp}(x/\mathbb{R}, b, c)$ is definable over $\mathbb{R}$, 1.3(iii)). Now as $b^2 > 0$, b^2x is positive infinite over $\mathbb{R}, b, c$, as is $b^2x + bc$. ∎

By the claim, $\mathrm{tp}(h_1 \cdot (t/H)/\mathbb{R}, t_1, h_1) = \mathrm{tp}((t/H)/\mathbb{R}, t_1, h_1)$. So without loss of generality $h_1 = 1$. So we are in the situation where $t, t_1 \in T$, t realizes q_0 and $\mathrm{tp}(t_1/\mathbb{R}, t)$ is finitely satisfiable in $\mathbb{R}$. It is then clear that t_1t realizes q_0 if and only if t_1 is the identity, or itself realizes q_0. As $t_1 \cdot (t/H) = (t_1t/H)$, and by virtue of π inducing a homeomorphism between $S_T(\mathbb{R})$ and $S_V(\mathbb{R})$, we see that $t_1 \cdot (t/H)$ realizes $\pi(q_0)$ if and only if t_1 is the identity or a realization of q_0. This proves the lemma.

COROLLARY 3.6. *Let $t \in T$ realize q_0 and let $h_1 \in H$ be such that $\mathrm{tp}(h_1/\mathbb{R}, t)$ is finitely satisfiable in $\mathbb{R}$. Then $h_1t = t_1h_2$ for $t_1 \in T$ realizing q_0 and $h_2 \in H$.*

Proof. We have just seen in the first part of the proof of 3.5 that $h_1 \cdot (t/H)$ realizes $\pi(q_0)$, which suffices. ∎

LEMMA 3.7. *For any $p \in S_G(\mathbb{R})$, $p * r_0 = r_0$ if and only if $p = \mathrm{tp}(t_1h_1/\mathbb{R})$ with $h_1 \in H$ and $t_1 \in T$ the identity or a realization of q_0.*

Proof. If $p * r_0 = r_0$ then by Lemma 3.3, $p * \pi(r_0) = \pi(r_0)$. As $\pi(r_0) = \pi(q_0)$, by Lemma 3.5, p is of the required form.

Now let $p = \mathrm{tp}(t_1h_1/\mathbb{R})$ with t_1 the identity or a realization of q_0. Suppose th realizes r_0 and $\mathrm{tp}(t_1, h_1/\mathbb{R}, t, h)$ is finitely satisfiable in $\mathbb{R}$. Note that (as $\mathrm{tp}(t/\mathbb{R}, h)$ is finitely satisfiable in $\mathbb{R}$) $\mathrm{tp}(t_1, h_1, t/\mathbb{R}, h)$ is finitely satisfiable in $\mathbb{R}$, so by Lemma 1.3(iii), $\mathrm{tp}(h/\mathbb{R}, t_1, h_1, t) = p_0|(\mathbb{R}, t_1, h_1, t)$. Now by Corollary 3.6, $h_1t = t_2h_2$ for t_2 realizing q_0 and $h_2 \in H$. We still have $\mathrm{tp}(h/\mathbb{R}, t_1, t_2, h_2) = p_0|(\mathbb{R}, t_1, t_2, h_2)$, because (h_2, t_2) is interdefinable with (h_1, t). Now p_0 is a (definable) left $H(\mathbb{R})$-invariant type of H, so for any model $K' \supset \mathbb{R}$, $p_0|K'$ is also a left $H(K')$-invariant type of H. Hence $h_3 = h_2h$ realizes $p_0|(\mathbb{R}, t_1, t_2, h_2)$.

Now $t_1h_1th = t_1t_2h_3$. As t_1, t_2 both realize q_0, or t_1 is the identity, their product t_1t_2 realizes q_0, and we have just seen that $\mathrm{tp}(t_1t_2/\mathbb{R}, h_3)$ is the unique coheir over $(\mathbb{R}, h_3)$ of q_0. So $t_1t_2h_3$ realizes r_0 as required. ∎

From Lemmas 3.5 and 3.7 we conclude easily:

Corollary 3.8. *The restriction of* $\pi\colon S_G(\mathbb{R}) \to S_V(\mathbb{R})$ *to* $\mathrm{cl}(G(\mathbb{R})r_0)$ *is a homeomorphism betweeen* $\mathrm{cl}(G(\mathbb{R})r_0)$ *and* $\mathrm{cl}(G(\mathbb{R}) \cdot \pi(r_0))$

Lemma 3.9. *The set* $S_{V,\mathrm{na}}(\mathbb{R})$ *of nonalgebraic types in* $S_V(\mathbb{R})$ *is the unique minimal closed* $G(\mathbb{R})$*-invariant subset of* $S_V(\mathbb{R})$.

Proof. Let for now S denote the set of nonalgebraic types in $S_V(\mathbb{R})$, a closed subspace. It is obviously $G(\mathbb{R})$-invariant. To show minimality it is enough to note that, identifying (via π) S with the space of nonalgebraic types in $S_T(\mathbb{R})$, it is minimal closed $T(\mathbb{R})$-invariant. If U is a basic open subset of $S_T(\mathbb{R})$ which is not a finite set (of isolated points), then by o-minimality, U contains an "interval" in $S_T(\mathbb{R})$, namely the set of types containing a formula defining an interval, with endpoints in $T(\mathbb{R})$, in T with respect to the circular ordering on T. But clearly if $s \in S$ then for some $g \in T(\mathbb{R})$, $gs \in U$. So S contains no proper $T(\mathbb{R})$-invariant closed subset, whereby S is minimal closed $T(\mathbb{R})$-invariant, as required. Uniqueness is clear. ∎

From 3.8 and 3.9 we deduce:

Proposition 3.10. $I = \mathrm{cl}(G(\mathbb{R})r_0)$ *is a minimal* $G(\mathbb{R})$*-invariant closed subspace of* $S_G(\mathbb{R})$ *and is homeomorphic as a* $G(\mathbb{R})$*-flow to* $S_{V,\mathrm{na}}(\mathbb{R})$ *under* π.

Lemma 3.11. $r_0 * r_0 = r_0$, *that is,* r_0 *is an idempotent in* I.

Proof. This is a special case of Lemma 3.7. ∎

We have so far accomplished the first aim: description of a minimal (closed) subflow I of $S_G(\mathbb{R})$ and an idempotent $r_0 \in I$. We now want to describe the "Ellis group" $r_0 * I$. Note first:

Lemma 3.12. *The restriction of* π *to* $r_0 * I$ *is a bijection with* $r_0 * S_{V,\mathrm{na}}(\mathbb{R})$.

Proof. By 3.3 and 3.10. ∎

We first consider the action of $H < G$ on $\mathbb{P}^1$ from Section 2. Note that H fixes $\binom{1}{0}$. We identify any other element $\binom{x}{1}$ of $\mathbb{P}^1$ with $x \in K$. With this notation:

Lemma 3.13. *Let* $x \in \mathbb{P}^1$, *and let* h *realize* p_0 *such that* $\mathrm{tp}(h/\mathbb{R}, x)$ *is finitely satisfiable in* $\mathbb{R}$. *Then* $h \cdot x$ *is positive infinite or negative infinite* (*in particular infinitesimally close to* $\binom{1}{0}$ *in* $\mathbb{P}^1$).

Proof. We have $h = (b, c)$ with b (positive) infinite, and c (positive) infinite over b. Moreover, $h \cdot x = b^2x + bc$. There are three cases to consider.

If x is finite (positive or negative), then clearly $b^2x + bc$ is positive infinite (as bc is infinite over $|b^2x|$).

If x is positive infinite, then clearly $b^2x + bc$ is positive infinite.

Finally, suppose x is negative infinite. Now as $\mathrm{tp}(x/b,c)$ is definable over $\mathbb{R}$, x is negative infinite *over* $\{b,c\}$, i.e. $x < \mathrm{dcl}(b,c)$. Hence $b^2x < \mathrm{dcl}(b,c)$, whereby $b^2 + bc$ is still negative infinite. ∎

Now we consider the homeomorphism (induced by π_1) between $T/\{I,-I\}$ and $\mathbb{P}^1$ given in 2.3 and the corresponding action $\cdot_1$ of G on $T/\{I,-I\}$. As the identity of $T/\{I,-I\}$ goes to $\binom{1}{0}$ under π_1, we deduce from Lemma 3.13:

Corollary 3.14. *Let $t/\{I,-I\} \in T/\{I,-I\}$ and let h realize p_0 such that $\mathrm{tp}(h/\mathbb{R},t)$ is finitely satisfiable in $\mathbb{R}$. Then $h\cdot_1(t/\{I,-I\})$ is infinitesimal in $T/\{I,-I\}$, namely infinitesimally close to the identity or equal to the identity.*

We now consider the action $\cdot$ of G on T induced by the action of G on V and the homeomorphism between T and V induced by π. We use Remark 2.4 to conclude:

Lemma 3.15. *Let $t \in T$ and let h realize p_0 such that $\mathrm{tp}(h/\mathbb{R},t)$ is finitely satisfiable in $\mathbb{R}$. Then $h\cdot t$ is infinitesimally close to the identity I (i.e. $(1,0)$) or to $-I$ (i.e. $(-1,0)$). Moreover both possibilities happen. Namely if t is infinitesimally close to $-I$ then so is $h\cdot t$, and if t is infinitesimally close to I then so is $h\cdot t$.*

Proof. The first part follows from Corollary 3.14. The rest follows by continuity and the fact that $h\cdot I = I$ and $h\cdot -I = -I$ (as $-I$ commutes with h). ∎

Remember q_0 is the type of a "positive infinitesimal" in T. We let q_1 denote the type of an element of T infinitesimally close to $-I$ and on the "positive side". Now we can conclude:

Proposition 3.16. *$r_0 * S_{V,\mathrm{na}}(\mathbb{R})$ has two elements, $\pi(q_0)$ and $\pi(q_1)$.*

Proof. We will work instead with the action $\cdot$ of G on $S_T(\mathbb{R})$ induced by the homeomorphism induced by π between $S_T(\mathbb{R})$ and $S_V(\mathbb{R})$. So for a type q of an element of T, by $r_0 * q$ we mean $\mathrm{tp}(g\cdot t/\mathbb{R})$ where t realizes q and g realizes r_0 such that $\mathrm{tp}(g/\mathbb{R},t)$ is finitely satisfiable in $\mathbb{R}$.

So let $t_1 \in T\setminus T(\mathbb{R})$ (i.e. t_1 realizes a nonalgebraic type in $S_T(\mathbb{R})$). And let th realize r_0 such that $\mathrm{tp}(th/\mathbb{R},t_1)$ is finitely satisfiable in $\mathbb{R}$. Then $\mathrm{tp}(h/\mathbb{R},t_1)$ is finitely satisfiable in $\mathbb{R}$, and we may assume that $\mathrm{tp}(t/\mathbb{R},h,t_1)$ is finitely satisfiable in $\mathbb{R}$. (And remember that t realizes q_0 and h realizes p_0). By Lemma 3.15, $h\cdot t_1 = t_2$, say, is infinitesimally close to either I or $-I$ (and each can happen for suitable choice of t_1). Note also that $t\cdot t_2$ is just tt_2 (product in T). Now as t realizes q_0 and its type over $(\mathbb{R},t_2)$ is finitely satisfiable in $\mathbb{R}$, it is easy to see that tt_2 realizes q_0 if t_2 is infinitesimally close to I, and realizes q_1 if t_2 is infinitesimally close to $-I$. This concludes the proof. ∎

Putting this together with earlier results we summarize (where r_0 is as in Definition 3.4):

THEOREM 3.17.

(i) $I = \mathrm{cl}(G(\mathbb{R})r_0)$ *is a minimal closed* $G(\mathbb{R})$*-invariant subset of* $S_G(M)$.
(ii) r_0 *is an idempotent with respect to the Ellis semigroup structure* $*$ *on* $S_G(\mathbb{R})$.
(iii) *The Ellis group* $(r_0 * I, *)$ *has two elements.*

Proof. (i) is Proposition 3.10. (ii) is Lemma 3.11. And (iii) follows from Proposition 3.16 and Lemma 3.12. ∎

We finish the paper with some remarks on routine extensions of our results and comparisons with the literature. See [3] for additional notions from topological dynamics.

Firstly, it is natural to also ask about the case where $G = \widetilde{\mathrm{SL}(2,\mathbb{R})}$, the universal cover of SL$(2,\mathbb{R})$. Now G can be naturally interpreted (defined) in the two-sorted structure $M = ((\mathbb{Z},+),(\mathbb{R},+,\times))$. Again all types over this standard model are definable, the $\mathbb{Z}$-sort being stable. H will be as before, and the role of the maximal compact T is now played by the universal cover of SO$(2,\mathbb{R})$, interpreted naturally on the set $\mathbb{Z}\times \mathrm{SO}(2,\mathbb{R})$. We leave it as an exercise to check that the above analysis goes through to show, among other things, that the Ellis group is $\widehat{\mathbb{Z}}$, the profinite completion of $(\mathbb{Z},+)$, which is in fact precisely the set of generic types of $\mathbb{Z}$.

Secondly, we can compare Theorem 3.17(iii) with Example 3.7 in Chapter VIII of [3] which says that the generalized strong Bohr compactification of SL$(2,\mathbb{R})$ considered as a topological group is the product of the Bohr compactification of the real line with $\mathbb{Z}/2\mathbb{Z}$.

Thirdly, the proof of Proposition 3.16 shows that $r_0 * S_{\mathbb{P}^1,\mathrm{na}}(\mathbb{R})$ has a unique element, implying that the $G(\mathbb{R})$-flow $S_{\mathbb{P}^1,\mathrm{na}}(\mathbb{R})$ is proximal. In fact according to our theory of definable topological dynamics, $S_{\mathbb{P}^1,\mathrm{na}}(\mathbb{R})$ will be the universal definable minimal proximal $G(\mathbb{R})$-flow. Again compare this to IV.4.1 of [3] where the universal minimal strongly proximal flow of SL$(2,\mathbb{R})$ as a topological group is given as $\mathbb{P}^1(\mathbb{R})$.

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