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On topological groups with a small base and metrizability

by

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Abstract. A (Hausdorff) topological group is said to have a $\mathfrak{G}$-base if it admits a base of neighbourhoods of the unit, $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$, such that $U_\alpha \subset U_\beta$ whenever $\beta \le \alpha$ for all $\alpha, \beta \in \mathbb{N}^\mathbb{N}$. The class of all metrizable topological groups is a proper subclass of the class $\mathbf{TG}_\mathfrak{G}$ of all topological groups having a $\mathfrak{G}$-base. We prove that a topological group is metrizable iff it is Fréchet–Urysohn and has a $\mathfrak{G}$-base. We also show that any precompact set in a topological group $G \in \mathbf{TG}_\mathfrak{G}$ is metrizable, and hence G is strictly angelic. We deduce from this result that an almost metrizable group is metrizable iff it has a $\mathfrak{G}$-base. Characterizations of metrizability of topological vector spaces, in particular of $C_c(X)$, are given using $\mathfrak{G}$-bases. We prove that if X is a submetrizable k_ω-space, then the free abelian topological group $A(X)$ and the free locally convex topological space $L(X)$ have a $\mathfrak{G}$-base. Another class $\mathbf{TG}_{\mathcal{CR}}$ of topological groups with a compact resolution swallowing compact sets appears naturally. We show that $\mathbf{TG}_{\mathcal{CR}}$ and $\mathbf{TG}_\mathfrak{G}$ are in some sense dual to each other. We conclude with a dozen open questions and various (counter)examples.

1. Introduction. All topological spaces and groups in this paper are assumed to be Hausdorff.

The classical metrization theorem of Birkhoff and Kakutani states that a topological group G is metrizable if and only if G is *first-countable* (see [27]), i.e., there exists a *decreasing* sequence $\{U_n\}_{n\in\mathbb{N}}$ which forms a base of neighbourhoods at the unit e of G. Consider $\mathbb{N}^\mathbb{N}$ with the natural partial order, i.e., $\alpha \le \beta$ if $\alpha_i \le \beta_i$ for all $i \in \mathbb{N}$, where $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ and $\beta = (\beta_i)_{i\in\mathbb{N}}$. Then it is easily seen that G admits the following base indexed by $\mathbb{N}^\mathbb{N}$: $U_\alpha := U_{\alpha_1}$ for $\alpha = (\alpha_i) \in \mathbb{N}^\mathbb{N}$. Evidently, $U_\alpha \subseteq U_\beta$ whenever $\alpha \le \beta$ for $\alpha, \beta \in \mathbb{N}^\mathbb{N}$. This simple remark motivates introducing a new class of topological groups which contains all metrizable ones:

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Definition 1.1. Let G be a topological group. A family $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of neighbourhoods of the unit is called a $\mathfrak{G}$-*base* if $\mathcal{U}$ is a base of neighbourhoods at the unit and $U_\beta \subseteq U_\alpha$ whenever $\alpha \le \beta$ for all $\alpha, \beta \in \mathbb{N}^\mathbb{N}$.

Originally, the concept of a $\mathfrak{G}$-base has been formally introduced in [17] in the framework of locally convex spaces (LCS for short) for studying (DF)-spaces, $C(X)$-spaces, and spaces in the class $\mathfrak{G}$ in the sense of Cascales and Orihuela (see [10] and the monograph [29]). Recently a characterization of spaces $C_c(X)$ (of all continuous real-valued functions on a Tychonoff space endowed with the compact-open topology) having a $\mathfrak{G}$-base has been obtained in [16].

The main goal of the article is a thorough study of the class $\mathbf{TG}_\mathfrak{G}$ of all topological groups with a $\mathfrak{G}$-base.

The character of a topological group G will be denoted by $\chi(G)$. Denote by $\mathfrak{d}$ the *cofinality* of $\mathbb{N}^\mathbb{N}$. The cardinal $\mathfrak{d}$ was extensively studied, and it lies between $\aleph_1$ and the continuum $\mathfrak{c}$ (see [14]). So, if $G \in \mathbf{TG}_\mathfrak{G}$, then it has a small character in the sense that $\chi(G) \le \mathfrak{d} \le \mathfrak{c}$. We provide a few classes of topological groups in $\mathbf{TG}_\mathfrak{G}$ which are non-metrizable. Moreover, there exists a countable precompact abelian group G with $\chi(G) = \aleph_1$ which does not admit a $\mathfrak{G}$-base (see Example 6.4). Thus the class of metrizable groups is a proper subclass of $\mathbf{TG}_\mathfrak{G}$.

A natural generalization of metrizability is the Fréchet–Urysohn property. There are many Fréchet–Urysohn topological groups which do not have a $\mathfrak{G}$-base. On the other hand, one of the main results of our paper states the following:

Theorem 1.2. *A topological group G is metrizable if and only if it is Fréchet–Urysohn and has a $\mathfrak{G}$-base.*

So, Theorem 1.2 is closely related to the famous Malykhin problem (1978) (see for example [3]): Is there a separable Fréchet–Urysohn topological group that is not metrizable? See recent papers [28, 33] discussing this problem. Consequently, Malykhin's problem can be reformulated as follows: *Does every countable Fréchet–Urysohn group admit a $\mathfrak{G}$-base?* Note that the Fréchet–Urysohn property in Theorem 1.2 cannot be weakened to sequentiality. We show that the free abelian group $A(\mathbf{e})$ over the convergent sequence $\mathbf{e}$ is a countable sequential non-Fréchet–Urysohn group having a $\mathfrak{G}$-base (see Corollary 4.20). Recently the Pytkeev property which is strictly weaker than the Fréchet–Urysohn property has been investigated in spaces of continuous functions (see [43]). Example 4.11 of our paper shows that there exists a non-metrizable topological group $G \in \mathbf{TG}_\mathfrak{G}$ having the Pytkeev property. So, the Fréchet–Urysohn property in Theorem 1.2 cannot be weakened to the Pytkeev property either.

We provide a necessary condition for a topological group to have a $\mathfrak{G}$-base; this extends [10, Theorem 2].

THEOREM 1.3. *If $G \in \mathbf{TG}_{\mathfrak{G}}$, then every precompact subset K in G is metrizable.*

However, we show that there are topological groups whose precompact subsets are all metrizable but which do not have a $\mathfrak{G}$-base (see Examples 6.6 and 6.7). Applying Theorem 1.3, we prove that an almost metrizable group is metrizable if and only if it has a $\mathfrak{G}$-base (see Theorem 3.10).

The class **TVS** of topological vector spaces (TVS for short) is one of the most important subclasses of the class **TG** of all topological groups. We denote by $\mathbf{TVS}_{\mathfrak{G}}$ the class of TVS which admit a $\mathfrak{G}$-base. The next theorem is an analogue of Theorem 1.2 for TVS with the Baire property.

THEOREM 1.4. *Let $E \in \mathbf{TVS}_{\mathfrak{G}}$. If E is Baire, then E is metrizable.*

In the framework of locally convex spaces we also prove that a LCS E is metrizable if and only if E has a $\mathfrak{G}$-base and E is b-Baire-like.

The class of free abelian topological groups $A(X)$ over Tychonoff spaces X is one of the most important classes of topological groups. For submetrizable k_ω-spaces we prove the following result.

THEOREM 1.5. *If X is a submetrizable k_ω-space, then $A(X)$ has a $\mathfrak{G}$-base. If, additionally, X is non-discrete, then $\chi(A(X)) = \mathfrak{d}$.*

Note that this theorem provides an alternative and simpler proof of the equality $\chi(A(X)) = \mathfrak{d}$ for a non-discrete submetrizable k_ω-space X, which is one of the principal results of [34]. Similar results are obtained for the free locally convex spaces $L(X)$ (see Theorem 4.16).

Using the concept of a $\mathfrak{G}$-base we extend a number of results from [11] about the dual groups of abelian topological groups. Another class $\mathbf{TG}_{\mathcal{CR}}$ of topological groups with a compact resolution swallowing compact sets appears naturally in the article. We show that the classes $\mathbf{TG}_{\mathcal{CR}}$ and $\mathbf{TG}_{\mathfrak{G}}$ are in some sense dual to each other. The class $\mathbf{TG}_{\mathcal{CR}}$ contains all hemicompact groups.

We also provide many examples of topological groups with or without $\mathfrak{G}$-bases and pose a dozen open questions.

2. Topological groups with $\mathfrak{G}$-bases. We denote by $\mathcal{N}(G)$ the filter of all open neighbourhoods at the unit of a topological group G.

We define topological spaces with local $\mathfrak{G}$-bases.

DEFINITION 2.1. We say that a point x of a topological space X has a *local $\mathfrak{G}$-base* if there exists a base of neighbourhoods at x of the form $\mathcal{U}(x) = \{U_\alpha(x) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, where $U_\beta(x) \subseteq U_\alpha(x)$ whenever $\alpha \le \beta$ for all

$\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$. The space X is said to have a *local* $\mathfrak{G}$*-base* if it has a $\mathfrak{G}$-base at each point.

The class of all topological spaces having a local $\mathfrak{G}$-base is denoted by $\mathbf{T}_{\mathfrak{G}}$. Every metrizable space has a local $\mathfrak{G}$-base. Clearly, a topological group has a local $\mathfrak{G}$-base if and only if it has a local $\mathfrak{G}$-base at each point.

Remark 2.2. Note that Definition 2.1 can be generalized as follows. Let X be a topological space and I be a partially ordered set with an order $\leq$. We say that a family $\mathcal{U} = \{U_i\}_{i \in I}$ is a *local I-base at a point $x \in X$* if $\mathcal{U}$ is a base at x such that $U_i \subset U_j$ for all $i \geq j$ in I. The authors of numerous papers devoted to the study of LCS used the term "$\mathfrak{G}$-base" for a local $\mathbb{N}^{\mathbb{N}}$-base (see, for example, [10, 29]). We use this term for the more general setting of topological groups.

In the next propositions we establish some properties of the class $\mathbf{TG}_{\mathfrak{G}}$.

Proposition 2.3. *The class $\mathbf{TG}_{\mathfrak{G}}$ contains all metrizable groups.*

Proof. Let G be a metrizable group and let $\{V_n\}_{n \in \mathbb{N}}$ be a decreasing base for $\mathcal{N}(G)$. For every $\alpha \in \mathbb{N}^{\mathbb{N}}$, set $U_\alpha := V_{\alpha_1}$. Clearly, $\{U_\alpha\}_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ is a $\mathfrak{G}$-base in G. ∎

Proposition 2.3 shows that every topological group G of countable character (i.e., $\chi(G) \leq \aleph_0$) belongs to $\mathbf{TG}_{\mathfrak{G}}$.

It is clear that, if $G \in \mathbf{TG}_{\mathfrak{G}}$, then $\chi(G) \leq \mathfrak{d}$. The next strengthening of this remark was suggested to us by Taras Banakh. Recall that the small cardinal $\mathfrak{b}$ is the least cardinality of a subset of $\mathbb{N}^{\mathbb{N}}$ which cannot be covered by a σ-compact subset of $\mathbb{N}^{\mathbb{N}}$. Clearly, $\mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$. It is well known (see [14]) that the inequality $\aleph_1 < \mathfrak{b}$ is consistent with ZFC.

Proposition 2.4. *If $G \in \mathbf{TG}_{\mathfrak{G}}$, then $\chi(G) \in \{1, \aleph_0\} \cup [\mathfrak{b}, \mathfrak{d}]$.*

Proof. By the definition of $\mathfrak{d}$ we have $\chi(G) \leq \mathfrak{d}$. So it suffices to prove that $\chi(G) < \mathfrak{b}$ implies $\chi(G) \leq \aleph_0$. Since $\chi(G) < \mathfrak{b}$, we can find a subset $B \subset \mathbb{N}^{\mathbb{N}}$ of cardinality $|B| = \chi(G) < \mathfrak{b}$ such that $\{U_\beta\}_{\beta \in B}$ is a local base at the unit e. Since $|B| < \mathfrak{b}$, the set B can be covered by a σ-compact subset of $\mathbb{N}^{\mathbb{N}}$. The projection of each compact subset of $\mathbb{N}^{\mathbb{N}}$ onto any coordinate is finite. Therefore there is an increasing countable sequence $\{\alpha^n\}_{n \in \mathbb{N}}$ in $\mathbb{N}^{\mathbb{N}}$ such that for every $\beta \in B$ there is α^n for which $\beta \leq \alpha^n$. Then $\{U_{\alpha^n}\}_{n \in \mathbb{N}}$ is a countable local base at e. Indeed, let $e \in U_\beta$ for some $\beta \in B$. Then there is α^n for which $\beta \leq \alpha^n$, and by the definition of a local $\mathfrak{G}$-base this means that $e \in U_{\alpha^n} \subseteq U_\beta$. Thus G is first countable. ∎

Corollary 2.5. *A topological group G with a $\mathfrak{G}$-base is metrizable if and only if $\chi(G) < \mathfrak{b}$.*

Proposition 2.4 and Examples 6.3 and 6.4 (showing that there are abelian groups G with $\chi(G) = \aleph_1$ which do not have a $\mathfrak{G}$-base) suggest the following question.

Question 2.6. *Does there exist in ZFC without any additional set-theoretic assumptions a topological group G which has a $\mathfrak{G}$-base with $\chi(G) = \mathfrak{b}$?*

We show that the class $\mathbf{TG}_{\mathfrak{G}}$ is closed under natural operations.

Proposition 2.7. *The class $\mathbf{TG}_{\mathfrak{G}}$ is closed under taking subgroups, quotients, the* (*Raĭkov*) *completion and countable products.*

Proof. It is trivial that $\mathbf{TG}_{\mathfrak{G}}$ is closed under taking subgroups and quotients.

Assume that $G \in \mathbf{TG}_{\mathfrak{G}}$ with an open base $\{U_\alpha\}_{\alpha\in\mathbb{N}^\mathbb{N}}$. In what follows we use the notation and constructions of the Raĭkov completion of a topological group from [5, §3.6]. Let us show that the closure G^* of G also has a $\mathfrak{G}$-base. For every $\alpha \in \mathbb{N}^\mathbb{N}$, set $U^*_\alpha := \{\eta \in G^* : U_\alpha \in \eta\}$, where η is a canonical filter on G (see [5, §3.6]). Now Fact 13 of [5, §3.6] implies that $U^*_\alpha \cap G = U_\alpha$ and the family $\{U^*_\alpha\}_{\alpha\in\mathbb{N}^\mathbb{N}}$ is an open $\mathfrak{G}$-base in G^*.

We now show that $\mathbf{TG}_{\mathfrak{G}}$ is countably productive. Let $\{U_\alpha(e_i) : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a $\mathfrak{G}$-base at the unit e_i of G_i for every $i \in \mathbb{N}$. For each $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$, we set $\alpha^* = (\alpha_{i+1})_{i\in\mathbb{N}}$ and

$$U_\alpha := \prod_{i=1}^{\alpha_1} U_{\alpha^*}(e_i) \times \prod_{i>\alpha_1} G_i.$$

Clearly, $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a $\mathfrak{G}$-base in the product $\prod_{i\in\mathbb{N}} G_i$. ∎

Recall that a property $\mathcal{P}$ is said to be a *three-space property* if for every topological group G and a closed normal subgroup H of G, the fact that both H and G/H have $\mathcal{P}$ implies that G also enjoys $\mathcal{P}$. A classical result says that metrizability is a three-space property (see [27, 5.38]). Several topological properties have been investigated in this respect in [9]. In particular, it was shown that $\mathcal{P} = \{$each compact subset is metrizable$\}$ is a three-space property.

Question 2.8. *Is the property of having a $\mathfrak{G}$-base a three-space property?*

We obtain a partial answer.

Proposition 2.9. *Let G be a topological group. If G has a normal metrizable closed subgroup H such that G/H has a $\mathfrak{G}$-base, then G has a $\mathfrak{G}$-base.*

Proof. Let $\{W_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of open symmetric neighbourhoods of the unit in G such that $W_{n+1} \cdot W_{n+1} \subset W_n$ for all $n \in \mathbb{N}$, and $\{W_n \cap H\}_n$ is a base of open (symmetric) neighbourhoods of the unit in H. By assumption, G/H has a symmetric $\mathfrak{G}$-base, say $\{V_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$.

Set $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$, where $U_\alpha := q^{-1}(V_\alpha)$ and $q : G \to G/H$ is the quotient map. For each $n \in \mathbb{N}$ and $\alpha = (\alpha_i) \in \mathbb{N}^\mathbb{N}$ set $R_\beta := W_n \cap U_\alpha$, where $\beta = (n, \alpha_1, \alpha_2, \dots)$. Clearly, $\mathcal{R} := \{R_\beta : \beta \in \mathbb{N} \times \mathbb{N}^\mathbb{N}\}$ is a family of open symmetric neighbourhoods of the unit in G such that $R_\beta \subseteq R_\gamma$ for $\beta \geq \gamma$.

We show that $\mathcal{R}$ is a base for G. Let $U \in \mathcal{U}(G)$. Take a symmetric $V \in \mathcal{U}(G)$ such that $VV \subset U$. Choose $n \in \mathbb{N}$ such that $W_n \cap H \subset V$. Choose $\alpha \in \mathbb{N}^\mathbb{N}$ such that $q(U_\alpha) \subset q(V \cap W_{n+1})$. Set $R_\beta = W_{n+1} \cap U_\alpha$. We prove that $R_\beta \subset U$. Indeed, if $g \in R_\beta$, then $g \in W_{n+1}$ and $g \in U_\alpha \subset (V \cap W_{n+1}) \cdot H$. Clearly, $g = ab$ for some $a \in V \cap W_{n+1}$ and $b \in H$. Since $b = a^{-1}g \in (W_{n+1})^{-1}W_{n+1} \subset W_n$, we have $g \in (V \cap W_{n+1})(H \cap W_n) \subset VV \subset U$. Hence $\mathcal{R}$ is a base. By the first paragraph of the proof, it is a $\mathfrak{G}$-base in G. ∎

Let $(G_n, \tau_n)_{n\in\mathbb{N}}$ be a sequence of topological groups. The family of subsets of the form $\prod_{n\in\mathbb{N}} U_n$, where $U_n \in \mathcal{N}(G_n)$, forms a base for a group topology in $\prod_{n\in\mathbb{N}} G_n$. This topology is called the *box topology* and denoted by τ_b. The *restricted direct product* G of $\{(G_n, \tau_n)\}_{n\in\mathbb{N}}$ is the subgroup of $\prod_{n\in\mathbb{N}} G_n$ consisting all sequences with finite support. We will identify subsets A of $\prod_{k=1}^n G_k$ with $A \times (e_{G_{n+1}}, e_{G_{n+2}}, \dots) \subset G$. The restriction of τ_b to G will also be denoted by τ_b.

We will need the following natural construction. Divide $\mathbb{N}$ into a disjoint union $\bigsqcup_{n\in\mathbb{N}} I_n$ of infinite subsets. So $I_n = \{k_i^n\}_{i\in\mathbb{N}}$, where $k_1^n < k_2^n < \cdots$. Define:

- $s : \mathbb{N} \to \mathbb{N}$ by $s(i) := n$ if $i \in I_n$;
- the bijection $t_n : \mathbb{N} \to I_n$ by $t_n(i) := k_i^n$ for all $i \in \mathbb{N}$;
- $\xi : \mathbb{N} \to \mathbb{N}$ by $\xi(i) := t_{s(i)}^{-1}(i)$ for all $i \in \mathbb{N}$.

Then, for all $n, i \in \mathbb{N}$, we have

$$s(t_n(i)) = n \quad \text{and} \quad \xi(t_n(i)) = t_n^{-1}(t_n(i)) = i.$$

For every $n \in \mathbb{N}$, define $p_n : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $p_n(\alpha) := (\alpha_{t_n(i)})_{i\in\mathbb{N}}$. Clearly, $p_n(\alpha) \leq p_n(\beta)$ for all $\alpha, \beta \in \mathbb{N}^\mathbb{N}$ such that $\alpha \leq \beta$. Finally, we define $R : (\mathbb{N}^\mathbb{N})^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by

$$R((\alpha^n)_{n\in\mathbb{N}}) := (\alpha_{\xi(i)}^{s(i)})_{i\in\mathbb{N}}, \qquad \text{where } \alpha^n = (\alpha_i^n)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N},$$

and note that for every natural number n,

$$p_n \circ R((\alpha^n)_{n\in\mathbb{N}}) = p_n((\alpha_{\xi(i)}^{s(i)})_{i\in\mathbb{N}}) = (\alpha_{\xi(t_n(i))}^{s(t_n(i))})_{i\in\mathbb{N}} = (\alpha_i^n)_{i\in\mathbb{N}} = \alpha^n. \tag{2.1}$$

Proposition 2.10. *Every countable product of topological groups with a $\mathfrak{G}$-base, endowed with the box topology, belongs to $\mathbf{TG}_\mathfrak{G}$.*

Proof. Let $\{G_n\}_{n\in\mathbb{N}}$ be a countable family of topological groups with respective $\mathfrak{G}$-bases $\{U_{\alpha,n}\}_{\alpha\in\mathbb{N}^\mathbb{N}}$. We claim that $G := (\prod_{n\in\mathbb{N}} G_n, \tau_b)$ has a $\mathfrak{G}$-base. For every $\alpha \in \mathbb{N}^\mathbb{N}$, set $W_\alpha = \prod_{n\in\mathbb{N}} U_{p_n(\alpha),n}$. Clearly, $W_\beta \subseteq W_\alpha$ for all $\alpha, \beta \in \mathbb{N}^\mathbb{N}$ such that $\alpha \leq \beta$. To show that $\mathcal{W} := \{W_\alpha\}_{\alpha\in\mathbb{N}^\mathbb{N}}$ is an open $\mathfrak{G}$-base in G, let $U = \prod_{n\in\mathbb{N}} U_{\alpha^n,n} \in \mathcal{N}(G)$. Set $\alpha = R((\alpha^n)_{n\in\mathbb{N}})$. Then $W_\alpha = U$ by (2.1). Thus $\mathcal{W}$ is an open $\mathfrak{G}$-base in G. ∎

Let $G \in \mathbf{TG}_{\mathfrak{G}}$ and $\{U_\alpha\}_{\alpha\in\mathbb{N}^{\mathbb{N}}}$ be a $\mathfrak{G}$-base in G. For $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$, set

$$D_k(\alpha) := \bigcap_{\beta\in I_k(\alpha)} U_\beta, \quad \text{where} \quad I_k(\alpha) = \{\beta \in \mathbb{N}^{\mathbb{N}} : \beta_i = \alpha_i \text{ for } i = 1,\dots,k\}.$$

Clearly, $\{D_k(\alpha)\}_{k\in\mathbb{N}}$ is an increasing sequence of subsets of G containing the unit.

We will use the following technical lemma.

Lemma 2.11. *Let $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and $\beta_k = (\beta_i^k)_{i\in\mathbb{N}} \in I_k(\alpha)$ for every $k \in \mathbb{N}$. Then there is $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha \le \gamma$ and $\beta_k \le \gamma$ for every $k \in \mathbb{N}$.*

Proof. For every $i \in \mathbb{N}$, set

$$\gamma_i = \max\{\alpha_i, \beta_i^l : l = 1,\dots,i\} = \max\{\beta_i^l : l \in \mathbb{N}\}.$$

Clearly, $\gamma := (\gamma_i)_{i\in\mathbb{N}}$ is as desired. ∎

We now turn to Fréchet–Urysohn topological groups. Nyikos [36] found several necessary and sufficient conditions for the Fréchet–Urysohn property in topological groups. Further results in this direction were obtained in [11]. It has been shown [11, Lemma 1.3] (see also [29, Lemma 14.1]) that every Fréchet–Urysohn topological group G satisfies the following condition:

(AS) For any family $\{x_{n,k} : (n,k) \in \mathbb{N}\times\mathbb{N}\} \subset G$ with $\lim_n x_{n,k} = x \in G$, $k = 1,2,\dots$, there are strictly increasing sequences $(n_i)_{i\in\mathbb{N}}$ and $(k_i)_{i\in\mathbb{N}}$ of natural numbers such that $\lim_i x_{n_i,k_i} = x$.

The next theorem characterizes metrizability in the class $\mathbf{TG}_{\mathfrak{G}}$:

Theorem 2.12. *If $G \in \mathbf{TG}_{\mathfrak{G}}$ with a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, then the following are equivalent:*

(i) *G is metrizable.*
(ii) *G is Fréchet–Urysohn.*
(iii) *For every $\alpha \in \mathbb{N}^{\mathbb{N}}$ there exists $k \in \mathbb{N}$ such that $D_k(\alpha)$ is a neighbourhood of the unit e.*

Proof. (i)⇒(ii) is trivial.

(ii)⇒(iii). Suppose for contradiction that there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $D_k(\alpha)$ is not a neighbourhood of e, for any $k \in \mathbb{N}$. So e belongs to the closure of $G \setminus D_k(\alpha)$. Since G is Fréchet–Urysohn, for every natural k we can choose a sequence $\{x_{n,k}\}_{n\in\mathbb{N}}$ in $G \setminus D_k(\alpha)$ converging to e. Applying the property (AS) we can choose strictly increasing sequences $(n_i)_{i\in\mathbb{N}}$ and $(k_i)_{i\in\mathbb{N}}$ of natural numbers such that $\lim_i x_{n_i,k_i} = e$.

For every $i \in \mathbb{N}$, choose $\beta_{k_i} \in I_{k_i}(\alpha)$ such that $x_{n_i,k_i} \notin U_{\beta_{k_i}}$. By Lemma 2.11 (with $\beta_k = \alpha$ if $k \neq k_i$ for all i), take $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $\beta_{k_i} \leq \gamma$ for every $i \in \mathbb{N}$. So $x_{n_i,k_i} \notin U_\gamma$ for every $i \in \mathbb{N}$. Thus $x_{n_i,k_i} \not\to e$, a contradiction.

(iii)⇒(i). For every $\alpha \in \mathbb{N}^{\mathbb{N}}$ choose the minimal natural k_α such that $D_{k_\alpha}(\alpha)$ is a neighbourhood of e. The family $\{D_{k_\alpha}(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is countable because it is contained in

$$\{D_k(\alpha) : k \in \mathbb{N}, \alpha \in \mathbb{N}^{\mathbb{N}}\} = \{D_k(\alpha) : k \in \mathbb{N}, \alpha \in \mathbb{N}^{(\mathbb{N})}\}.$$

Hence, by the construction of $D_k(\alpha)$, the family $\{\mathrm{int}(D_{k_\alpha}(\alpha)) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a countable base of open neighbourhoods at e. Thus X is metrizable. ∎

Proof of Theorem 1.2. Immediately follows from Proposition 2.3 and Theorem 2.12. ∎

EXAMPLE 2.13. The Σ-product G of uncountably many copies of $\mathbb{Z}(2)$ is a Fréchet–Urysohn topological group which is not metrizable. Hence G does not admit a $\mathfrak{G}$-base.

Note that Shakhmatov gave a survey of the metrization problem of topological groups with various convergence properties (see [42]).

It is known that, if G is a countably infinite topological group, then $\chi(G) \leq 2^{|G|} = \mathfrak{c}$.

QUESTION 2.14. *Is there a countable Fréchet–Urysohn group G with $\chi(G) = \mathfrak{c}$?*

Both the hypotheses $\mathfrak{d} < \mathfrak{c}$ and $\mathfrak{d} = \mathfrak{c}$ are consistent with ZFC (see [14, §5]). So, under $\mathfrak{d} < \mathfrak{c}$, the positive answer to Question 2.14 and Proposition 2.4 gives a negative answer to Malykhin's aforementioned problem.

Recall that a topological space is called *submetrizable* if it admits a weaker metrizable topology. In the light of Malykhin's problem and Question 2.14, it is interesting to note that every countable topological group is submetrizable, by a result of Guran (see [5, 3.4.25]). We do not know whether the same holds for every topological group with a $\mathfrak{G}$-base.

QUESTION 2.15. *Does every topological group with a $\mathfrak{G}$-base admit a weaker metrizable group topology?*

Recall that a topological group is *minimal* if it does not admit a strictly weaker Hausdorff group topology (see [44]). On the other hand, by a result of Prodanov and Stoyanov [39] every abelian minimal topological group is precompact. Hence a minimal abelian topological group with a $\mathfrak{G}$-base must be metrizable by Theorem 5.5 below. Therefore the above Question 2.15 is of interest for non-minimal abelian or non-abelian minimal topological groups.

The rest of the section deals with sequential properties of restricted direct products of metrizable groups. Recall that a topological space X is *sequential* if every sequentially closed subset of X is closed. Trivially every metrizable space is sequential.

Let $\{(X_n, \tau_n)\}_{n\in\mathbb{N}}$ be a sequence of topological spaces with $X_n \subseteq X_{n+1}$ and $\tau_{n+1}|_{X_n} = \tau_n$ for all $n \in \mathbb{N}$. The union $X = \bigcup_{n\in\mathbb{N}} X_n$ with the weak topology τ (i.e., $U \in \tau$ if and only if $U \cap X_n \in \tau_n$ for every $n \in \mathbb{N}$) is called the *inductive limit* of the sequence $\{(X_n, \tau_n)\}_{n\in\mathbb{N}}$ and it is denoted by $(X, \tau) = \varinjlim (X_n, \tau_n)$. Recall that a topological space X is called a *k_ω-space* if it is the inductive limit of an increasing sequence of its compact subsets K_n, and that $\bigcup_{n\in\mathbb{N}} K_n$ is called a *k_ω-decomposition* of X. A topological group is called a *k_ω-group* if its underlying topological space is a k_ω-space.

PROPOSITION 2.16. *Let G be the restricted direct product of a sequence $\{G_n\}_{n\in\mathbb{N}}$ of metrizable groups endowed with the box topology τ_b. If G is a k-space, then G is sequential.*

Proof. According to [11, Lemma 1.5], if X is a topological k-space whose compact subsets are all metrizable, then X is sequential. Let K be a compact subset of G. It is well known that K is contained in some finite product $\prod_{k=1}^{n} G_k$. So K is metrizable and the claim of the proposition follows. ∎

COROLLARY 2.17. *Let G be the restricted direct product of a sequence $\{G_n\}_{n\in\mathbb{N}}$ of infinite locally compact metrizable groups endowed with the box topology τ_b. Then:*

(i) *G is a sequential group;*
(ii) *G has a $\mathfrak{G}$-base;*
(iii) *G is not Fréchet–Urysohn.*

Proof. (i) is a consequence of [36, Problem 1]. Note also that the sequentiality of G can be proved directly by showing that G has an open k_ω-subgroup and applying Proposition 2.16. (ii) follows from Proposition 2.7. (iii) follows from Theorem 2.12 and the well known fact that τ_b is non-metrizable. ∎

REMARK 2.18. The assumption on G_n to be locally compact in Corollary 2.17 is essential. Nyikos [36, Footnote 2] mentioned that van Douwen has shown that if even one of the factors in the above sequence is not locally compact, and infinitely many of the G_n are not discrete, then the resulting space is not sequential. Therefore, (G, τ_b) is not even a k-space, by Proposition 2.16. On the other hand, every metrizable k_ω-group G is locally compact (indeed, G is metrizable and complete by [5, 7.4.10], and hence locally compact by [5, 4.3.b]). We prove a stronger claim in Proposition 3.18.

3. Topological groups with a compact resolution swallowing compact sets. We start with the following definition (see [12] and [47], where this concept has been studied under other names).

Definition 3.1. A family $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of compact sets of a topological space X is called a *compact resolution* if $X = \bigcup\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ and $K_\alpha \subseteq K_\beta$ for all $\alpha \leq \beta$. If additionally every compact set in X is contained in some K_α we will say that $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ *swallows the compact sets of* X.

Denote by $\mathbf{T}_{\mathcal{CR}}$ and $\mathbf{TG}_{\mathcal{CR}}$ the families of topological spaces and of groups, respectively, having a compact resolution swallowing the compact sets.

Remark 3.2. Note that Definition 3.1 can be generalized as follows. Let X be a topological space and I be a set with a partial order $\leq$. We say that a family $\mathcal{A} = \{A_i\}_{i\in I}$ of subsets of X is *I-increasing* if $A_i \subset A_j$ for all $i \leq j$ in I. A family $\mathcal{K} = \{K_i\}_{i\in I}$ of compact subsets of X is *compact I-dominated* if $\mathcal{K}$ is I-increasing and *compact dominated*, i.e., for each compact subset K of X there is $i \in I$ such that $K \subset K_i$. Aiming to be consistent with [10, 29] we call a compact $\mathbb{N}^\mathbb{N}$-dominated family a compact resolution swallowing compact sets.

Any Polish space X has a compact resolution swallowing the compact sets of X. Indeed, let $\{x_n : n \in \mathbb{N}\}$ be a countable dense subset in X. For every $\alpha = (\alpha_k) \in \mathbb{N}^\mathbb{N}$, set $K_\alpha := \bigcap_{k\in\mathbb{N}} \bigcup_{j=1}^{\alpha_k} B(x_j, 1/k)$, where $B(x_j, 1/k)$ is the closed ball in X with center at x_j and radius $1/k$. It is easy to prove that the family $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is as required. In fact, a stronger result is known:

Proposition 3.3 ([12, Theorem 3.3]). *If X is a metrizable topological space, then the following are equivalent:*

(i) *X is a Polish space.*
(ii) *X has a compact resolution swallowing the compact sets of X.*

The following facts were noticed in [47].

Proposition 3.4 ([47]). *The class $\mathbf{T}_{\mathcal{CR}}$ contains all hemicompact topological spaces. Thus $\mathbf{TG}_{\mathcal{CR}}$ contains all hemicompact groups.*

Proposition 3.5 ([47]). *The class $\mathbf{TG}_{\mathcal{CR}}$ is closed under taking closed subgroups and countable cartesian products.*

Recall that a continuous mapping $q : G \to H$ is called *compact-covering* if for every compact subset K of H there exists a compact subset C of G such that $q(C) = K$.

Remark 3.6. If K is a compact subgroup of a topological group G, then the quotient mapping $q : G \to G/K$ onto the left coset space is a compact covering map. This immediately follows from [5, Theorem 1.5.7] and [15, Theorem 3.7.2]. Note also that, if G is Čech-complete, then every quotient homomorphism of G is compact-covering by [1, Theorem 1.2].

PROPOSITION 3.7. *Let $G \in \mathbf{TG}_{\mathcal{CR}}$ and $q : G \to H$ be a quotient compact-covering map. Then $H \in \mathbf{TG}_{\mathcal{CR}}$.*

Proof. If $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact resolution swallowing the compact sets of G, then $\{q(K_\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact resolution swallowing the compact sets of H (since q is compact-covering). ∎

We do not know any example of $G \in \mathbf{TG}_{\mathcal{CR}}$ such that G has a quotient group H without a resolution swallowing the compact sets of H.

The proof of the next theorem uses the following:

PROPOSITION 3.8 ([10, Theorem 1]). *A compact space K is metrizable if and only if $(K \times K) \setminus \Delta$ has a compact resolution swallowing its compact sets, where $\Delta := \{(x, x) : x \in K\}$.*

Recall that a topological space X is *angelic* (see [20]) if every relatively countably compact set A in X is relatively compact and for each $x \in \overline{A}$ there exists a sequence in A converging to x. A topological space X is *strictly angelic* if X is angelic and each separable compact subset of X is first countable. Now Theorem 1.3 can be formulated more precisely.

THEOREM 3.9. *If $G \in \mathbf{TG}_{\mathfrak{G}}$, then every precompact subset K in G is metrizable. Consequently, G is strictly angelic.*

Proof. Having in mind Proposition 2.7 and [5, Theorem 3.7.10], we assume that G is complete and that K is compact. Let $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be an open $\mathfrak{G}$-base in G. We may assume that all sets U_α are symmetric. We have to show that K is metrizable. To prove this, by Proposition 3.8, it is enough to show that the set $W := (K \times K) \setminus \Delta$ has a compact resolution which swallows its compact sets.

For each $\alpha \in \mathbb{N}^\mathbb{N}$, set

$$W_\alpha := \{(x, y) \in W : xy^{-1} \notin U_\alpha\}.$$

Then W_α is closed in $K \times K$, and hence it is compact for every $\alpha \in \mathbb{N}^\mathbb{N}$. Let us show that $\mathcal{W} := \{W_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact resolution in W. Indeed, for each compact subset C of W, the set $q(C) = \{xy^{-1} : (x, y) \in C\}$ is compact and does not contain the unit e of G. Since $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a local base at e, for some $\alpha \in \mathbb{N}^\mathbb{N}$ we obtain $U_\alpha \cap q(C) = \emptyset$. Hence $C \subset W_\alpha$. Thus $\mathcal{W}$ swallows the compact sets in W. Therefore K is metrizable by Proposition 3.8.

Since in uniform spaces every relatively countably compact set is precompact (see [20, 1.1.2]), the above conclusion shows that G is angelic. Moreover, as we have proved that every compact set in G is metrizable, it follows that G is strictly angelic. ∎

However, Example 6.7 below shows that there is a topological group G whose precompact subsets are all metrizable but G does not have a $\mathfrak{G}$-base.

Recall that a topological group G is called *almost metrizable* (or *feathered*) if it contains a non-empty compact set K of countable character in G (see [5]). All Čech-complete groups, in particular locally compact groups, are almost metrizable (see [5, p. 235]). By Choban's theorem [5, 4.3.16], every almost metrizable group can be embedded as a subgroup into a Čech-complete group. Applying Theorem 3.9 we have the following result.

THEOREM 3.10. *Let G be an almost metrizable group. Then G has a $\mathfrak{G}$-base if and only if G is metrizable.*

Proof. Suppose G has a $\mathfrak{G}$-base. By Pasynkov's theorem (see [5, 4.3.20]), G contains a compact subgroup H such that the left coset space G/H is metrizable. By Proposition 2.7 and Theorem 3.9, the subgroup H is metrizable. Thus G is metrizable by [27, 5.38]. The converse is clear. ∎

COROLLARY 3.11. *A locally precompact group G has a $\mathfrak{G}$-base if and only if G is metrizable.*

Proof. Let G be a locally precompact group with a $\mathfrak{G}$-base. The completion $\overline{G}$ of G is locally compact and has a $\mathfrak{G}$-base (by Proposition 2.7). Now Theorem 3.10 implies that $\overline{G}$ is metrizable. Thus G is also metrizable. The converse is clear. ∎

Following [25], a family $\mathcal{N}$ of subsets of a topological space X is called a *cs^*-network at a point $x \in X$* if for each sequence $(x_n)_{n\in\mathbb{N}}$ in X convergent to x and for each neighborhood O_x of x there is a set $N \in \mathcal{N}$ such that $x \in N \subset O_x$ and the set $\{n \in \mathbb{N} : x_n \in N\}$ is infinite. The *cs^*-character* of a topological group G is the least cardinality of cs^*-networks at the unit e of G. Topological groups having countable cs^*-character are thoroughly studied in [8].

THEOREM 3.12. *Each topological group G with a $\mathfrak{G}$-base has countable cs^*-character.*

Proof. Let $\{U_\alpha\}_{\alpha\in\mathbb{N}^\mathbb{N}}$ be a $\mathfrak{G}$-base in G. It is enough to show that the countable family $\mathcal{D} = \{D_k(\alpha) : \alpha \in \mathbb{N}^\mathbb{N}, k \in \mathbb{N}\}$ is a cs^*-network at the unit e. Take a sequence $S := (g_n)_{n\in\mathbb{N}}$ in G converging to e and a neighborhood U_α of e. We have to show that there is $k \in \mathbb{N}$ such that $S \cap D_k(\alpha)$ is infinite.

Suppose for a contradiction that $S \cap D_k(\alpha)$ is finite for every $k \in \mathbb{N}$. By induction, for every $k \in \mathbb{N}$ we can choose $n_k \in \mathbb{N}$ and $\beta_k \in I_k(\alpha)$ such that $n_1 < n_2 < \cdots$ and $g_{n_k} \notin U_{\beta_k}$. Clearly, $g_{n_k} \to e$. By Lemma 2.11, there is $\gamma \in \mathbb{N}^\mathbb{N}$ such that $\alpha \le \gamma$ and $\beta_k \le \gamma$ for every $k \in \mathbb{N}$. By construction, $g_{n_k} \notin U_\gamma$ for every $k \in \mathbb{N}$. So $g_{n_k} \not\to e$, a contradiction. Hence $S \cap D_k(\alpha)$ is infinite for some $k \in \mathbb{N}$. Thus $\mathcal{D}$ is a cs^*-network at e. ∎

As a corollary of Theorems 3.12 and 3.9 and [8, Theorem 1] we obtain:

Corollary 3.13. *Let $G \in \mathbf{TG}_{\mathfrak{G}}$. Then the following are equivalent:*

(i) *G is a k-space.*
(ii) *G is sequential.*
(iii) *G is metrizable or contains a submetrizable open k_ω-subgroup.*

Proof. (i)⇒(ii). Let G be a k-space. By Theorem 3.9, each compact subset K of G is metrizable. Thus G is sequential by [11, Lemma 1.5].

(ii)⇒(iii). By Theorem 3.12, the group G has countable cs^*-character. Since G is sequential, [8, Theorem 1] implies (iii).

(iii)⇒(i) is clear. ■

It would be interesting to know whether the k-property and sequentiality are equivalent for topological groups with countable cs^*-character.

Theorem 3.12, Corollary 3.13 and [8, Theorems 2 and 3] immediately imply

Corollary 3.14. *Let $G \in \mathbf{TG}_{\mathfrak{G}}$. Then G is metrizable if and only if G is a k-space and one of the following conditions holds:*

(i) *$\chi(G) < \mathfrak{d}$;*
(ii) *G is Baire;*
(iii) *G is not Weil complete.*

Remark 3.15. Note that Theorems 2.12 and 3.10 follow also from Corollary 3.13 and [8, Theorem 3]. However, our methods are much simpler and straightforward.

For almost metrizable groups with compact resolutions swallowing compact sets we have the following:

Theorem 3.16. *If G is an almost metrizable topological group, then the following are equivalent:*

(i) *G belongs to $\mathbf{TG}_{\mathcal{CR}}$.*
(ii) *G has a compact subgroup K such that the left coset space G/K is Polish.*

Consequently, if (i) *holds, then G is Čech-complete.*

Proof. (i)⇒(ii). It is well known that G contains a compact subgroup K such that G/K is metrizable [5, 4.3.20]. Assume that G has a compact resolution swallowing the compact subsets of G, say $\mathcal{K} = \{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Set $\mathcal{K}' = \{q(K_\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, where $q : G \to G/K$ is the quotient map. We claim that $\mathcal{K}'$ swallows the compact subsets of G/K. Indeed, if K' is compact in G/K, then $q^{-1}(K')$ is compact in G by Remark 3.6. So there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $q^{-1}(K') \subseteq K_\alpha$, and hence $K' \subseteq q(K_\alpha)$. Now Proposition 3.3 implies that G/K is Polish.

(ii)⇒(i). By Proposition 3.3, G/K has a compact resolution swallowing the compact sets of G/K, say $\mathcal{K}' = \{K'_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. For every $\alpha \in \mathbb{N}^\mathbb{N}$, set $K_\alpha := q^{-1}(K'_\alpha)$ where $q : G \to G/K$ is the quotient map. Then K_α is a compact subset of G by Remark 3.6. Hence $\mathcal{K} = \{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact resolution. Let now C be a compact subset of G. Then there exists $\alpha \in \mathbb{N}^\mathbb{N}$ such that $q(C) \subseteq K'_\alpha$. So $C \subseteq K_\alpha$. Hence $\mathcal{K}$ swallows the compact sets of G. Thus $G \in \mathbf{TG}_{\mathcal{CR}}$.

If (i) or (ii) holds, then the group G is Čech-complete by [5, 4.3.20]. ∎

Tkachuk [47] showed that a locally compact space $X \in \mathbf{T}_{\mathcal{CR}}$ need not be Lindelöf. For locally compact groups the situation changes.

Corollary 3.17. *Let G be a locally compact group. Then $G \in \mathbf{TG}_{\mathcal{CR}}$ if and only if G is a hemicompact space. In particular, G is Lindelöf.*

Proof. Assume that $G \in \mathbf{TG}_{\mathcal{CR}}$. Since any locally compact group G is almost metrizable (see [5, p. 235]), Theorem 3.16 implies that G has a compact subgroup K such that the locally compact space G/K is second countable. Therefore G/K is hemicompact. Thus G is hemicompact as well by Remark 3.6.

If G is hemicompact, it belongs to $\mathbf{TG}_{\mathcal{CR}}$ by Proposition 3.4. ∎

We know that every hemicompact group G belongs to $\mathbf{TG}_{\mathcal{CR}}$ (see Proposition 3.4). If in addition G is Fréchet–Urysohn, we prove the following:

Proposition 3.18. *Every Fréchet–Urysohn hemicompact topological group G is a separable locally compact metrizable group.*

Proof. Let $G = \bigcup_{n\in\mathbb{N}} K_n$, where $\{K_n\}_n$ is an increasing sequence of compact subsets of K containing the unit e such that every compact set in G is contained in some K_n.

Step 1. *There is $n \in \mathbb{N}$ such that K_n is a neighbourhood of the unit.*

Suppose for contradiction that there is no n for which K_n is a neighbourhood of e. Then for each $n \in \mathbb{N}$ and each $U \in \mathcal{N}(X)$ there exists $x_{U,n} \in U \setminus K_n$. For each $n \in \mathbb{N}$ we set

$$B_n := \{x_{U,n} : U \in \mathcal{N}(G)\}.$$

Then $e \in \overline{B_n}$. Since G is Fréchet–Urysohn, for each $n \in \mathbb{N}$ there exists a sequence $\{U_n(k)\}_k$ in $\mathcal{N}(X)$ such that $x_{U_n(k),n} \to e$ at $k \to \infty$. On the other hand, every Fréchet–Urysohn group satisfies the condition (AS). Therefore there exist strictly increasing sequences $(k_p)_p$ and $(n_p)_p$ such that $x_{U_{n_p(k_p)},n_p} \to e$ as $p \to \infty$. The set $B := \{x_{U_{n_p(k_p)},n_p} : p \in \mathbb{N}\} \cup \{e\}$ is compact in G, so there exists $m \in \mathbb{N}$ such that $B \subset K_m$, a contradiction (since $\{K_n\}_n$ is increasing).

Step 2. *G is a separable locally compact metrizable group.*

Indeed, by Step 1, the group G is locally compact. It is also well known (see [2]) that every Fréchet–Urysohn locally compact group is metrizable. So G, being metrizable and hemicompact, must be separable. ∎

We do not know whether hemicompactness of G in Proposition 3.18 can be replaced by the existence in G of a compact resolution swallowing compact sets.

CONJECTURE 3.19. *Every Fréchet–Urysohn topological group with a compact resolution swallowing compact sets is metrizable.*

We complete this section with the following remark.

REMARK 3.20. Let X be a locally compact space and $\bar{X} := X \cup \{e\}$ a one-point compactification of X. Then $X \in \mathbf{T}_{\mathcal{CR}}$ if and only if e has a local $\mathfrak{G}$-base. Indeed, by construction, a family $\{U_\alpha(e) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a local $\mathfrak{G}$-base at e if and only if $\{X \setminus U_\alpha(e) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution swallowing the compact sets in X.

4. Topological vector spaces and free abelian groups with a $\mathfrak{G}$-base. Recall that a subset A of a TVS E is called *bounded* if each neighbourhood of zero absorbs A. If B is a subset of E and $k \in \mathbb{N}$, we set $kB := \{kx : x \in B\}$.

We need:

LEMMA 4.1. *Let E be a TVS and $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-base in E consisting of closed and symmetric subsets. For each $\alpha = (\alpha_i) \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$, set*

$$D_k(\alpha) := \bigcap_{\beta \in I_k(\alpha)} U_\beta, \quad \textit{where} \quad I_k(\alpha) = \{\beta \in \mathbb{N}^{\mathbb{N}} : \beta_i = \alpha_i \textit{ for } i = 1, \dots, k\}.$$

Then, for each bounded subset $B \subset E$ and every $\alpha \in \mathbb{N}^{\mathbb{N}}$, there exists $k \in \mathbb{N}$ such that $B \subseteq kD_k(\alpha)$. In particular, $E = \bigcup_k kD_k(\alpha)$ for every $\alpha \in \mathbb{N}^{\mathbb{N}}$.

Proof. Assume there exists a bounded set B in E such that $B \not\subseteq kD_k(\alpha)$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, choose $x_k \in B$ such that $k^{-1}x_k \notin D_k(\alpha)$. Since B is bounded, $k^{-1}x_k \to 0$. Now, for every $k \in \mathbb{N}$, take $\beta_k \in I_k(\alpha)$ such that $k^{-1}x_k \notin U_{\beta_k}$. By Lemma 2.11, we can choose $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha \le \gamma$ and $\beta_k \le \gamma$ for every $k \in \mathbb{N}$. Since $U_\gamma \subseteq U_{\beta_k}$ for each $k \in \mathbb{N}$, we have $k^{-1}x_k \notin U_\gamma$. Thus $k^{-1}x_k \not\to 0$, a contradiction. ∎

Now we are in a position to prove Theorem 1.4.

Proof of Theorem 1.4. Let $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-base in E. We may assume that all sets U_α are closed and symmetric.

We use the notation from Lemma 4.1. Since E is Baire, we apply Lemma 4.1 to show that for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ there exists $k(\alpha) \in \mathbb{N}$ such that for every $k \geq k(\alpha)$ the set $D_k(\alpha) = \frac{1}{k} \cdot kD_k(\alpha)$ has a non-empty interior. Set

$$\mathcal{D} := \{D_k(\alpha) - D_k(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}} \text{ and } k \geq k(\alpha)\}.$$

Then $\mathcal{D}$ is a countable family of neighbourhoods of zero in E. We have to show that $\mathcal{D}$ is a base. Take a neighbourhood of zero V in E. Choose U_α such that $U_\alpha - U_\alpha \subseteq V$. Then, for $k \geq k(\alpha)$,

$$D_k(\alpha) - D_k(\alpha) \subseteq U_\alpha - U_\alpha \subseteq V.$$

So $\mathcal{D}$ is a base in E. Thus E is metrizable. ∎

We do not know whether Theorem 1.4 remains true for topological groups which are not k-spaces (see Proposition 3.14).

QUESTION 4.2. *Let $X \in \mathbf{TG}_{\mathfrak{G}}$ be Baire and not a k-space. Is X metrizable?*

We denote by $C_p(X)$ the space $C(X)$ endowed with the pointwise convergence topology τ_p. For $C_p(X)$ we have the following:

PROPOSITION 4.3. *Let X be a Tychonoff space. Then the following are equivalent:*

(i) *$C_p(X)$ is metrizable.*
(ii) *$C_p(X)$ has a $\mathfrak{G}$-base.*
(iii) *X is countable.*

Proof. (i)⇒(ii) follows from Proposition 2.3.

(ii)⇒(iii). It is well known that $C_p(X)$ is dense in the Tychonoff product $\mathbb{R}^X$. Since $C_p(X)$ has a $\mathfrak{G}$-base, Proposition 2.7 implies that $\mathbb{R}^X$ also has a $\mathfrak{G}$-base. Hence every precompact subset of $\mathbb{R}^X$ is metrizable by Theorem 3.9. So, in particular, the compact space $[0,1]^X$ is metrizable. This implies that X is countable.

(iii)⇒(i). If X is countable, then the space $\mathbb{R}^X$ is metrizable, so $C_p(X)$ is metrizable. ∎

REMARK 4.4. Note that, if a Tychonoff space X has a countable network, then every compact subset K of $C_p(X)$ is metrizable. (Indeed, since $C_p(X)$ has a countable network, [3, I.1.3] implies that so does every compact subset K of $C_p(X)$. Thus K is metrizable by [15, 3.1.19].) So, each compact subset of $C_p[0,1]$ is metrizable, but the space $C_p[0,1]$ does not admit a $\mathfrak{G}$-base by Proposition 4.3.

REMARK 4.5. Note that $C_p(X) \in \mathbf{TG}_{\mathcal{CR}}$ if and only if X is countable and discrete [47].

Recall that a LCS E is called *Baire-like* (Saxon) if every increasing sequence $\{A_n\}_{n\in\mathbb{N}}$ of absolutely convex closed subsets covering E contains a member which is a neighbourhood of zero. The space E is called *b-Baire-like*

(Ruess) if every increasing sequence $\{A_n\}_{n\in\mathbb{N}}$ of absolutely convex and closed sets covering E which is *bornivorous* (i.e., every bounded set in E is contained in some A_m) contains a member which is a neighbourhood of zero. Clearly, Baire LCS $\Rightarrow$ Baire-like $\Rightarrow$ b-Baire-like, and the converses fail. Every barrelled metrizable LCS is Baire-like (Saxon).

Applying the argument of the proof of Theorem 1.4, we can also prove the following:

THEOREM 4.6. *Let E be a LCS. Then E is metrizable if and only if E has a $\mathfrak{G}$-base and it is b-Baire-like.*

Proof. Let E be metrizable. So E has a $\mathfrak{G}$-base (see Proposition 2.3). By [29, Proposition 2.11], E is b-Baire-like.

Assume now that E has a $\mathfrak{G}$-base and is b-Baire-like. Since E is a LCS, elements of its $\mathfrak{G}$-base can be chosen to be absolutely convex symmetric closed sets. Clearly every neighbourhood of zero in E absorbs bounded sets of E. We use the notation from Lemma 4.1. Note that the sets $kD_k(\alpha)$ are absolutely convex symmetric and closed for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{\mathbb{N}}$.

Lemma 4.1 implies that the sequence $\{kD_k(\alpha)\}_{k\in\mathbb{N}}$ is bornivorous. Since E is b-Baire-like, we derive that for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ there exists $k(\alpha) \in \mathbb{N}$ such that the set $D_k(\alpha) = \frac{1}{k} \cdot kD_k(\alpha)$ has a non-empty interior for every $k \geq k(\alpha)$. Now we repeat word for word the arguments in the proof of Theorem 1.4 in order to show that E is metrizable. ■

Since $C_c(X)$ is trivially a LCS, Theorems 4.6 and 2.12 immediately imply:

COROLLARY 4.7. *Let X be a Tychonoff space. Then the following are equivalent:*

(i) *$C_c(X)$ is metrizable.*
(ii) *$C_c(X)$ is b-Baire-like and has a $\mathfrak{G}$-base.*
(iii) *$C_c(X)$ is Fréchet–Urysohn and has a $\mathfrak{G}$-base.*

A characterization of the space $C_c(X)$ admitting a $\mathfrak{G}$-base has been obtained by Ferrando and Kąkol [16]. Modifying their proof and using our terminology from Remarks 2.2 and 3.2 we generalize this characterization as follows.

THEOREM 4.8. *Let X be a Tychonoff space and $\mathbf{M}$ a partially ordered set. Then:*

(i) *If $C_c(X)$ has a local $\mathbf{M}$-base, then X has a compact $\mathbf{M}$-dominated family.*
(ii) *If X has a compact $\mathbf{M}$-dominated family, then $C_c(X)$ has a local $\mathbb{N} \times \mathbf{M}$-base.*

Proof. For $A \subseteq X$, $\epsilon > 0$ and $B \subseteq C_c(X)$, we define

$$[A, \epsilon] = \Big\{ f \in C(X) : \sup_{x\in A} |f(x)| < \epsilon \Big\},$$
$$B^{\lozenge} = \{x \in X : |f(x)| \le 1 \ \forall f \in B\}.$$

Clearly, if A is compact in X, then $[A, \epsilon]$ is open in $C_c(X)$.

(i) Assume that $C_c(X)$ has a local $\mathbf{M}$-base $\{U_\alpha : \alpha \in \mathbf{M}\}$. For every $\alpha \in \mathbf{M}$ set $C_\alpha := U_\alpha^{\lozenge}$. Let us show that the family $\mathcal{C} := \{C_\alpha : \alpha \in \mathbf{M}\}$ is compact $\mathbf{M}$-dominated.

Clearly, C_α is closed in X and $\alpha \le \beta$ implies that $C_\alpha \subseteq C_\beta$. So $\mathcal{C}$ is $\mathbf{M}$-increasing.

Let us show that C_α is compact in X. Indeed, take a compact subset K of X and $0 < \epsilon < 1$ such that $[K, \epsilon] \subseteq U_\alpha$. Note that $[K, \epsilon]^{\lozenge} \subseteq K$ since, if $x \in X \setminus K$, there is $f \in C(X)$ with $f(x) = 2$ and $f(K) = 0$. Now we have

$$C_\alpha = U_\alpha^{\lozenge} \subseteq [K, \epsilon]^{\lozenge} \subseteq K.$$

Hence C_α, being closed, is compact for every $\alpha \in \mathbb{N}^{\mathbb{N}}$.

We claim that $\mathcal{C}$ is compact $\mathbf{M}$-dominated. Indeed, let K be a compact subset in X. Take $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $U_\alpha \subseteq [K, 1]$. Then $K \subseteq [K, 1]^{\lozenge} \subseteq U_\alpha^{\lozenge} = C_\alpha$.

(ii) Let $\mathcal{K} := \{K_\alpha : \alpha \in \mathbf{M}\}$ be a compact $\mathbf{M}$-dominated family in X. For each $\alpha = (k, \alpha^*) \in \mathbb{N} \times \mathbf{M}$, set

$$U_\alpha := [K_{\alpha^*}, k^{-1}] \quad \text{and} \quad \mathfrak{U} := \{U_\alpha : \alpha \in \mathbb{N} \times \mathbf{M}\}.$$

Clearly, $\mathfrak{U}$ is a family of symmetric absolutely convex and absorbing open sets in $C(X)$ such that $U_{k,\beta^*} \subseteq U_{k,\alpha^*}$ for all $k \in \mathbb{N}$ and $\alpha \le \beta$.

We claim that $\mathfrak{U}$ is a base of a locally convex topology τ on $C(X)$. To prove this we have to check three conditions (see [30, §15.2]):

(a) For each $U \in \mathfrak{U}$ there is a $V \in \mathfrak{U}$ with $V + V \subseteq U$.
(b) For each $U \in \mathfrak{U}$ there is a $V \in \mathfrak{U}$ for which $\lambda V \subseteq U$ for all λ with $|\lambda| \le 1$.
(c) For each $U \in \mathfrak{U}$ and each $f \in C_c(X)$ there is an $m \in \mathbb{N}$ for which $f \in kU$.

For $\alpha = (k, \alpha^*) \in \mathbb{N} \times \mathbf{M}$, we set $\beta = (2k, \alpha^*)$. Then $U_\beta + U_\beta \subseteq U_\alpha$, which gives (a). The conditions (b) and (c) are fulfilled trivially.

If we prove that $\tau = \tau_c$, we infer that $\mathfrak{U}$ is a $\mathfrak{G}$-base in $C_c(X)$, as desired. Clearly, $\tau \le \tau_c$. Conversely, let K be a compact set in X and $\epsilon > 0$. Since $\mathcal{K}$ is compact $\mathbf{M}$-dominated, we can choose $\alpha^* \in \mathbf{M}$ and $k \in \mathbb{N}$ such that $K \subseteq K_{\alpha^*}$ and $k^{-1} < \epsilon$. Clearly, $U_{(k,\alpha^*)} \subseteq [K, \epsilon]$. Thus $\tau \ge \tau_c$ and $\mathfrak{U}$ is a local $\mathbb{N} \times \mathbf{M}$-base at zero in $C_c(X)$. ■

Since $\mathbb{N}^{\mathbb{N}} \cong \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$, the next result immediately follows from Theorem 4.8.

Theorem 4.9 (Ferrando–Kąkol [16]). *For a Tychonoff space X, the space $C_c(X)$ has a $\mathfrak{G}$-base if and only if X has a compact resolution that swallows compact sets.*

Corollary 4.10. *Let X be a Polish non-locally compact space. Then $C_c(X)$ has a $\mathfrak{G}$-base and is barrelled, but $C_c(X)$ is not b-Baire-like* (*hence not Baire*).

Proof. By Proposition 3.3 and Theorem 4.9, $C_c(X)$ has a $\mathfrak{G}$-base. Since X is not hemicompact, $C_c(X)$ is not metrizable. Thus, by Corollary 4.7, $C_c(X)$ is not b-Baire-like. ∎

Recall (see [43]) that a topological space X is said to have the *Pytkeev property* if for each $A \subset X$ and each $x \in \overline{A} \setminus A$, there exist infinite subsets $A_1, A_2, \ldots$ of A such that each neighbourhood of x contains some A_n.

Example 4.11. There is a non-metrizable LCS $E \in \mathrm{TVS}_{\mathfrak{G}}$ having the Pytkeev property. So, the Fréchet–Urysohn property in Theorem 1.2 cannot be weakened to the Pytkeev property. Indeed, by [48, Theorem 2.1] the space $C_c(\mathbb{N}^{\mathbb{N}})$ has the Pytkeev property. Since $\mathbb{N}^{\mathbb{N}}$ is a Polish space, and hence has a compact resolution that swallows compact sets, Theorem 4.9 implies that $C_c(\mathbb{N}^{\mathbb{N}})$ has a $\mathfrak{G}$-base. As $\mathbb{N}^{\mathbb{N}}$ is not hemicompact, $C_c(\mathbb{N}^{\mathbb{N}})$ is not metrizable.

The last part of this section deals with free topological groups and free locally convex spaces. The following concept is due to Markov [31] (see also Graev [26]).

Definition 4.12. Let X be a Tychonoff space. A topological group $F(X)$ (respectively, $A(X)$) is called *the* (*Markov*) *free* (respectively, *abelian*) *topological group* over X if $F(X)$ (respectively, $A(X)$) satisfies the following conditions:

- (i) There is a continuous mapping $i : X \to F(X)$ (respectively, $i : X \to A(X)$) such that $i(X)$ algebraically generates $F(X)$ (respectively, $A(X)$).
- (ii) If $f : X \to G$ is a continuous mapping to a topological (respectively, abelian topological) group G, then there exists a continuous homomorphism $\bar{f} : F(X) \to G$ (respectively, $\bar{f} : A(X) \to G$) such that $f = \bar{f} \circ i$.

The topological groups $F(X)$ and $A(X)$ always exist and are essentially unique. Note that i is a topological embedding [31, 26]. Let us also mention that when X is a Hausdorff topological group (respectively, an Abelian Hausdorff topological group) and $f : X \to X$ is the identical mapping, then the canonical mapping $\bar{f} : F(X) \to X$ (respectively, $\bar{f} : A(X) \to X$) is open.

If X is locally compact and second countable we have the following:

Lemma 4.13. *Let X be a locally compact second countable space. Then $A(X)$ and $F(X)$ are sequential groups.*

Proof. By [4], $A(X)$ and $F(X)$ are k-spaces. Now let K be a compact subset of $A(X)$ or $F(X)$. By [45] there exist a compact subset C of X and $n \in \mathbb{N}$ such that K is a continuous image of a compact subspace in C^n. Hence K is metrizable [15, 3.1.22]. Thus $A(X)$ and $F(X)$ are sequential [11, Lemma 1.5]. ∎

Analogously we can define free LCS:

Definition 4.14 ([31, 41, 18, 19, 49]). Let X be a Tychonoff space. The *free LCS* $L(X)$ on X is a pair consisting of a LCS $L(X)$ and a continuous mapping $i : X \to L(X)$ such that for every continuous mapping f from X to a LCS E there is a unique continuous linear operator $\bar{f} : L(X) \to E$ with $f = \bar{f} \circ i$.

Also the free LCS $L(X)$ always exists and is unique. The set X forms a Hamel basis for $L(X)$, and i is a topological embedding [41, 18, 19, 49]. The identity map $\mathrm{id}_X : X \to X$ extends to a canonical homomorphism $\mathrm{id}_{A(X)} : A(X) \to L(X)$. It is known that $\mathrm{id}_{A(X)}$ is an embedding of topological groups [46, 50]. For example, if X is a countably infinite discrete space, then $L(X) = \phi$, where ϕ is the restricted direct product of the sequence $\{\mathbb{R}^n\}_{n\in\mathbb{N}}$. It is well known that ϕ is a sequential non-Fréchet–Urysohn k_ω-space.

The following question arises naturally.

Question 4.15. *For which X do the groups $A(X)$, $F(X)$ and $L(X)$ have a $\mathfrak{G}$-base?*

It is well known that $L(X)$ admits a canonical continuous monomorphism $L(X) \to C_c(C_c(X))$. If X is a k-space, this monomorphism is an embedding of LCS [18, 19, 49]. So, for k-spaces, we obtain a chain of topological embeddings

$$A(X) \hookrightarrow L(X) \hookrightarrow C_c(C_c(X)). \tag{4.1}$$

This argument helps us to provide a partial answer to Question 4.15. The next theorem generalizes Theorem 1.5

Theorem 4.16. *If X is a submetrizable k_ω-space, then $A(X)$ and $L(X)$ have a $\mathfrak{G}$-base. Moreover,*

(i) *if X is not discrete, then $\chi(A(X)) = \chi(L(X)) = \mathfrak{d}$;*
(ii) *if X is discrete, then $\chi(A(X)) = 1$, and $\chi(L(X)) = \aleph_0$ for X finite and $\chi(L(X)) = \mathfrak{d}$ for X infinite.*

Proof. Since X is a submetrizable k_ω-space, $C_c(X)$ is a Polish space by [32, 4.2.2 and 5.8.1]. Hence $C_c(X)$ has a compact resolution that swallows the compact sets of $C_c(X)$ by Proposition 3.3. Thus $C_c(C_c(X))$ has a $\mathfrak{G}$-base by Theorem 4.9. Now Proposition 2.7 and (4.1) imply that $A(X)$ and $L(X)$ each have a $\mathfrak{G}$-base.

(i) It is well known that, if X is not discrete, then $A(X)$ is not even Fréchet–Urysohn. Since $A(X)$ is a k-space, Proposition 2.4 and Corollary 3.14(i) imply that $\mathfrak{d} \leq \chi(A(X)) \leq \chi(L(X)) \leq \mathfrak{d}$.

(ii) If X is discrete, then $A(X)$ is discrete, and hence $\chi(A(X)) = 1$. If X is finite, then $L(X) = \mathbb{R}^{|X|}$ is metrizable, and hence $\chi(L(X)) = \aleph_0$. If X is infinite, then X is countably infinite as a submetrizable k_ω-space. So $L(X) = \phi$. Now Corollary 3.14 implies that $\chi(L(X)) = \mathfrak{d}$. ∎

Also the following question is well motivated.

QUESTION 4.17. *Let X be a submetrizable k_ω-space. Does $F(X)$ have a $\mathfrak{G}$-base?*

QUESTION 4.18. *Let X be a k-space and $L(X) \in \mathbf{TG}_{\mathfrak{G}}$. Does $C_c(C_c(X))$ have a $\mathfrak{G}$-base?*

If $L(X)$ has a $\mathfrak{G}$-base, then $A(X) \in \mathbf{TG}_{\mathfrak{G}}$ (see Proposition 2.7). It is not clear whether the converse is also true.

QUESTION 4.19. *Let $A(X) \in \mathbf{TG}_{\mathfrak{G}}$. Does $L(X)$ have a $\mathfrak{G}$-base?*

According to Theorem 2.12, if a topological group G is Fréchet–Urysohn and has a $\mathfrak{G}$-base, then G is metrizable. Proposition 2.17 shows that we cannot replace "Fréchet–Urysohn" by "sequential". The next corollary shows the existence of even a *countable* sequential abelian group with a $\mathfrak{G}$-base which is not Fréchet–Urysohn.

Denote by $\mathbf{e} = \{e_n\}_{\mathbb{N}}$ the sequence in the direct sum $\mathbb{Z}^{(\mathbb{N})}$ with $e_1 = (1,0,0,\dots), e_2 = (0,1,0,\dots),\dots$. Then $\mathbf{e}$ converges to zero in the topology induced by the product topology on $(\mathbb{Z}_d)^{\mathbb{N}}$. Denote by $\tau_{\mathbf{e}}$ the finest group topology on $\mathbb{Z}^{(\mathbb{N})}$ in which $e_n \to 0$ (the topology $\tau_{\mathbf{e}}$ is described in [21] explicitly). In [21] it is pointed out that $A(\mathbf{e}) = (\mathbb{Z}^{(\mathbb{N})}, \tau_{\mathbf{e}})$.

COROLLARY 4.20. *The free abelian group $A(\mathbf{e})$ is a countable sequential group with a $\mathfrak{G}$-base which is not Fréchet–Urysohn.*

Proof. By Theorem 4.16, $A(\mathbf{e})$ is a countable group with a $\mathfrak{G}$-base. This group is sequential and is not Fréchet–Urysohn by [40, Theorem 2.3.10]. ∎

It seems of interest to study the class $\mathbf{TG}(\mathcal{CRG})$ of all topological groups having both a $\mathfrak{G}$-base and a compact resolution swallowing compact sets, i.e., $\mathbf{TG}(\mathcal{CRG}) = \mathbf{TG}_{\mathfrak{G}} \cap \mathbf{TG}_{\mathcal{CR}}$. This class contains all Polish groups (see Proposition 3.3) and all dual groups of abelian Polish groups, in particular $A(\mathbf{e})$, by Theorem 5.1.

Proposition 4.21. *Let G be a topological group and a k-space. Then $G \in \mathbf{TG}(\mathcal{CRG})$ if and only if G is either Polish or a submetrizable k_ω-group.*

Proof. Assume that $G \in \mathbf{TG}(\mathcal{CRG})$. Then, by Corollary 3.13, G is either metrizable or has an open submetrizable k_ω-subgroup. Since G has a compact resolution swallowing the compact subsets of G, Propositions 3.3 and 3.7 imply that G is either Polish or a submetrizable k_ω-group.

Conversely, if G is Polish, then $G \in \mathbf{TG}(\mathcal{CRG})$ by Propositions 2.3 and 3.7. If G is a submetrizable k_ω-group, then $G \in \mathbf{TG}(\mathcal{CRG})$ by Propositions 3.4 and 3.7 and Theorem 4.16. ∎

5. Abelian topological groups with a $\mathfrak{G}$-base and duality. For an abelian topological group G we denote by $\widehat{G}$ the group of all continuous characters on G. The group $\widehat{G}$ endowed with the compact-open topology is denoted by $G^\wedge$. The homomorphism $\alpha_G : G \to G^{\wedge\wedge}$, $x \mapsto (\chi \mapsto (\chi, x))$, is called the *canonical homomorphism*. If α_G is a topological isomorphism, the group G is called *reflexive*. The Pontryagin–van Kampen duality theorem states that every locally compact abelian group is reflexive.

For a subset A of an abelian topological group G, the *polar* of A is $A^\triangleright := \{\chi \in G^\wedge : \chi(A) \subseteq \mathbb{T}_+\}$, where $\mathbb{T}_+ := \{z \in \mathbb{T} : \mathrm{Re}(z) \geq 0\}$. The set A is called *locally quasi-convex* if for every $x \in G \setminus A$, there is a $\chi \in A^\triangleright$ such that $\mathrm{Re}(\chi, x) < 0$. For $B \subseteq X^\wedge$, the *inverse polar* of B is $B^\triangleleft := \{x \in X : x(B) \subseteq \mathbb{T}_+\}$. Obviously, A is quasi-convex [10, Theorem 2] if and only if $(A^\triangleright)^\triangleleft = A$. The group G is called *locally quasi-convex* if it has a base at zero whose elements are quasi-convex. Every locally compact abelian group is reflexive and hence locally quasi-convex. Note also that the family of sets of the form

$$P(K, \varepsilon) := \{\chi \in \widehat{G} : |1 - (\chi, x)| < \varepsilon, \forall x \in K\},$$

where K is compact in G and $\varepsilon > 0$, forms a base of open neighbourhoods at zero of the compact-open topology on $\widehat{G}$. For a subset D of G, set $(1)D := D$ and $(n+1)D := (n)D + D$ for $n \in \mathbb{N}$.

The next theorem gives the duality between the classes $\mathbf{TG}_{\mathfrak{G}}$ and $\mathbf{TG}_{\mathcal{CR}}$ in the framework of abelian groups.

Theorem 5.1. *Let G be an abelian topological group.*

(1) *Suppose G has an open $\mathfrak{G}$-base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. Denote $\mathcal{W} = \{W_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$, where $W_\alpha := U_\alpha^\triangleright$, the dual family of compact sets in $G^\wedge$. Then $\mathcal{W}$ is a compact resolution in $G^\wedge$.*
(2) *The following are equivalent:*
   (a) *The dual compact resolution $\mathcal{W}$ swallows the compact sets in $G^\wedge$.*
   (b) *Every compact subset of $G^\wedge$ is equicontinuous.*
   (c) *The canonical homomorphism α_G is continuous.*

(3) *If G has a compact resolution $\mathcal{W} = \{W_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ swallowing the compact sets in G, then the dual family $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$, where $U_\alpha := W_\alpha^\triangleright$, is a $\mathfrak{G}$-base in $G^\wedge$.*

Proof. (1) Note that W_α is a compact subset of $G^\wedge$ by [6, 3.5]. Clearly, $G^\wedge = \bigcup\{W_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ and $W_\alpha \subset W_\beta$ whenever $\alpha \le \beta$. Hence $G^\wedge$ has a compact resolution.

(2) (a)⇒(b). Suppose $\mathcal{W}$ swallows the compact sets in $G^\wedge$ and K is a compact subset of $G^\wedge$. Take $\alpha \in \mathbb{N}^\mathbb{N}$ such that $K \subseteq U_\alpha^\triangleright$.

Fix $\varepsilon > 0$ and choose $n \in \mathbb{N}$ such that $1/n < \varepsilon/20$. Choose $\beta \in \mathbb{N}^\mathbb{N}$ such that $(n)U_\beta \subseteq U_\alpha$. Then, for every $x \in U_\beta$ and each $0 \le k \le n$, we have $\chi(kx) \in \mathbb{T}_+$ for all $\chi \in K$. Hence $\arg(\chi(x)) \in \left[-\frac{\pi}{2n}, \frac{\pi}{2n}\right]$. This means that

$$|1-\chi(x)| \le \left|1-\exp\left\{i\frac{\pi}{2n}\right\}\right| \le 2\pi \cdot \frac{\pi}{2n} < \varepsilon, \quad \forall x \in U_\beta.$$

Thus K is equicontinuous.

(b)⇒(a). Fix a compact subset K of $G^\wedge$. Since by the assumption K is equicontinuous, there is $\alpha \in \mathbb{N}^\mathbb{N}$ such that $\chi(U_\alpha) \subset \mathbb{T}_+$ for every $\chi \in K$. This means that $K \subseteq U_\alpha^\triangleright = W_\alpha$. Thus $\mathcal{W}$ swallows the compact sets in $G^\wedge$.

The equivalence of (b) and (c) is proved in [6, 5.10].

(3) For each $\alpha \in \mathbb{N}^\mathbb{N}$, set $U_\alpha := W_\alpha^\triangleright$. Then U_α is a neighbourhood of the unit element of $G^\wedge$ by the definition of the compact-open topology. Clearly, $U_\beta \subseteq U_\alpha$ whenever $\alpha \le \beta$. Since $\mathcal{W}$ swallows the compact subsets of X, for every compact subset K of G there is $\alpha \in \mathbb{N}^\mathbb{N}$ such that $K \subset W_\alpha$. This means that $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a $\mathfrak{G}$-base in G. ∎

REMARK 5.2. Note that, if G is a k-space, then α_G is continuous [35].

As an immediate corollary of Theorem 5.1 we obtain:

COROLLARY 5.3. *Let G be a reflexive abelian topological group. Then G has a compact resolution swallowing the compact sets in G if and only if $G^\wedge$ has a $\mathfrak{G}$-base.*

REMARK 5.4. By [21, Theorem 1], the free abelian group $A(\mathbf{e})$ is reflexive and its dual group is Polish. Now Proposition 3.3 and Corollary 5.3 give an alternative proof of Corollary 4.20.

REMARK 5.5. In our opinion, it would be useful to provide an alternative proof of Corollary 3.11 for abelian locally precompact groups using duality theory arguments.

Assume that a locally precompact abelian group G has a $\mathfrak{G}$-base. We have to show that G is metrizable. Proposition 2.7 implies that the completion $\overline{G}$ also has a $\mathfrak{G}$-base. It is well known that $\overline{G} \cong \mathbb{R}^n \times G_0$, where $n \in \omega$ and G_0 has an open compact subgroup H [27, 24.30]. So, by Proposition 2.7, it is enough to show that H is metrizable.

Proposition 2.7 also implies that H has a $\mathfrak{G}$-base. By Theorem 5.1 the space $H^\wedge$ has a compact resolution swallowing its compact sets and it is discrete and hence metrizable. Now Proposition 3.3 shows that the space $H^\wedge$ is Polish, hence $H^\wedge$ is countable. Thus H is metrizable by [27, 24.15].

Conversely, if G is metrizable, then it has a $\mathfrak{G}$-base by Proposition 2.3.

The group G is called *maximally almost periodic* (MAP) if $\widehat{G}$ separates the points of G. For a MAP abelian group G we denote by $\sigma(G,\widehat{G})$ or τ^+ the weak topology on G, i.e., the smallest topology in G for which the elements of $\widehat{G}$ are continuous. The topology τ^+ is called *the Bohr modification* of τ. Set $G^+ := (G,\tau^+)$. It is well known that the groups G and G^+ have the same set of continuous characters, and $G^+ = G$ if and only if G is precompact (see [5]).

Now we obtain a complete description of MAP abelian groups G for which the group G^+ has a $\mathfrak{G}$-base.

Corollary 5.6. *For a MAP abelian group G the following assertions are equivalent:*

(i) *G^+ has a $\mathfrak{G}$-base.*
(ii) *G^+ is metrizable.*
(iii) *$\widehat{G}$ is countable.*

Proof. (i)⇔(ii) follows from Corollary 3.11 since G^+ is a precompact group.

Note that G and G^+ have the same set of continuous characters. Now the equivalence of (ii) and (iii) follows from [13]. ∎

We provide a few topological properties of dual groups. We start with the following fact (compare also with [11, Theorem 2.2]).

Proposition 5.7. *Let G be an abelian topological group. If G has a compact resolution swallowing the compact sets in G, then the following are equivalent:*

(i) *$G^\wedge$ is Fréchet–Urysohn.*
(ii) *$G^\wedge$ is metrizable.*

Proof. (i)⇒(ii) follows from Theorems 2.12 and 5.1(3).

(ii)⇒(i) is trivial. ∎

The next proposition generalizes [11, Theorem 2.2].

Proposition 5.8. *Let G be an abelian topological group. If $G^\wedge$ is a Fréchet–Urysohn hemicompact group, then $G^\wedge$ is a separable and metrizable locally compact group. If in addition, α_G is continuous and G is locally quasi-convex, then G is a separable and metrizable locally precompact abelian group.*

Proof. The first assertion is an immediate corollary of Proposition 3.18. In particular, $G^{\wedge\wedge}$ is a separable metrizable LCA group. If α_G is continuous and G is locally quasi-convex, then α_G is an embedding [6, 6.10]. Thus G is a separable metrizable locally precompact group. ∎

The next proposition is a direct consequence of Theorems 3.9 and 5.1(3), and also partially extends [11, Theorem 1.7].

PROPOSITION 5.9. *Let G be an abelian topological group. If $G \in \mathbf{TG}_{\mathcal{CR}}$, then every precompact set of $G^{\wedge}$ is metrizable. Thus $G^{\wedge}$ is strictly angelic.*

As an application we extend Theorem 2.8 of [11]. Let E be a LCS. Denote by E^*_β the *strong dual* of E, i.e. the space of all continuous linear functionals on E endowed with the topology of uniform convergence on bounded subsets of E. Note that, if E is metrizable, then E^*_β is metrizable if and only if E is normable. By the definition of the strong dual we obtain: E^*_β has a $\mathfrak{G}$-base if and only if there exists a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of bounded subsets of E which swallows all bounded sets in E. We show that this holds, for example, if E is a locally complete (in particular, complete) Quasi-(LB)-space in the sense of Valdivia [51]; for example, E can be the space of distributions $D'(\Omega)$ over an open $\Omega \subset \mathbb{R}^n$. A LCS E is said to be a *Quasi-(LB)-space* if E admits a resolution consisting of Banach discs (a subset A of E is called a *Banach disk* if it is a bounded absolutely convex set in E such that $E_A := \bigcup_n nA$, endowed with the norm $\|\cdot\|_A$ given by the Minkowski gauge of A, is a Banach space). Every (LF)-space (in particular, every metrizable and complete LCS) is a Quasi-(LB)-space, as also is its strong dual [51, Propositions 5, 6].

PROPOSITION 5.10. *Let E be a locally complete Quasi-(LB)-space. Then E^*_β admits a $\mathfrak{G}$-base. Therefore, E^*_β is Fréchet–Urysohn if and only if E^*_β is metrizable.*

Proof. By [51, Proposition 22] there is a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ in E consisting of Banach discs such that every Banach disc B of E is contained in some A_α. Since E is locally complete, the closure of any bounded set in E is a Banach disc [37, 5.1.6]. Hence $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ swallows all bounded sets in E. Then the polars in E^*_β of the sets A_α form a $\mathfrak{G}$-base in E^*_β. Now we apply Theorem 2.12 to complete the proof. ∎

6. Concluding examples and open questions. Now we provide some (counter-)examples which clarify relations between topological properties in the classes $\mathbf{TG}(\mathcal{CRG})$, $\mathbf{TG}_{\mathfrak{G}}$ and $\mathbf{TG}_{\mathcal{CR}}$, respectively.

EXAMPLE 6.1. There exists a countable abelian hemicompact group X with a $\mathfrak{G}$-base (and hence $X \in \mathbf{TG}(\mathcal{CRG})$) which is not a k-space.

Our example uses [7, Theorem 6]. Consider the metrizable topology τ' on $\mathbb{Z}^{(\mathbb{N})}$ generated by the base $\{U_n\}_{n\in\omega}$, where

$$U_n = \{(n_i)_{i\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}} : \ n_i \in 2^n \cdot \mathbb{Z} \text{ for } i \geq 1\}, \quad n \in \omega.$$

Set $G = A(\mathbf{e}) \times (\mathbb{Z}^{(\mathbb{N})}, \tau')$. By Proposition 2.7 and Corollary 4.20, G has a $\mathfrak{G}$-base. Let X be the diagonal subgroup of G. Then X is a countable abelian non-discrete group [7]. Further, X has a $\mathfrak{G}$-base by Proposition 2.7. Since every compact subset of X is finite [22, Example 4.1], the group X is hemicompact. Being non-discrete, X is not a k-space by [22, Proposition 4.6].

Example 6.2. There exists a countable abelian reflexive hemicompact group G with a $\mathfrak{G}$-base which is a sequential non-Fréchet–Urysohn space. Let $G = A(\mathbf{e})$. Then $A(\mathbf{e})$ is a countable, reflexive [38, 21], k_ω (and hence hemicompact) [40, 4.1.5], sequential non-Fréchet–Urysohn group [40, 2.3.10]. By Corollary 4.20, the space $A(\mathbf{e})$ has a $\mathfrak{G}$-base. Thus $A(\mathbf{e}) \in \mathbf{TG}(\mathcal{CRG})$.

The next example shows that the converse in Proposition 2.4 fails in general.

Example 6.3. Let $G = \prod_{i\in I} G_i$, where $|I| = \aleph_1$ and G_i is a metrizable non-trivial compact group for every $i \in I$. Then G is a compact abelian group of character $\chi(G) = \aleph_1$ [27, 24.15]. The group G does not have a $\mathfrak{G}$-base by Theorem 3.9.

In the next example we show that there exists even a countable precompact abelian group (G, τ) with $\chi(G) = \aleph_1$ which does not admit a $\mathfrak{G}$-base.

Example 6.4. Let $G = \mathbb{Z}$. Take an independent subset E of $\mathbb{T}$ of cardinality $\aleph_1$ without torsion elements. Set $\tau := \sigma(G, \langle E\rangle)$, where $\langle E\rangle$ is a subgroup of $\mathbb{T}$ generated by E. Since $|\langle E\rangle| = \aleph_1$ we have $\chi(G, \tau) = \aleph_1$. Since (G, τ) is a precompact non-metrizable group, we conclude that it has no $\mathfrak{G}$-base by Remark 5.5.

The last two examples and Theorem 3.9 suggest the following question:

Question 6.5. *Let G be a topological group of character $\chi(G) \leq \mathfrak{d}$ and such that all precompact subsets in G are metrizable. Does G admit a $\mathfrak{G}$-base?*

In other words, does the metrizability of all precompact subsets and the condition $\chi(G) \leq \mathfrak{d}$ characterize the class $\mathbf{TG}_{\mathfrak{G}}$? Each compact subset of a countable group is metrizable. Therefore, in view of Example 6.4, the metrizability assumption of all precompact sets in Question 6.5 is essential.

The next two examples show that the converse in Theorem 3.9 fails in general.

Example 6.6. It is known [5, 7.9.6] that the free abelian group $A(X)$ is (Raĭkov) complete if and only if X is Dieudonné complete. If every compact subset of X is metrizable, then by the same arguments as in Lemma 4.13, every compact subset of $A(X)$ is also metrizable. For example, let $X = T \cup \{e\}$, where T is discrete and the complement of any neighbourhood of e is countable. Then X is Lindelöf and each compact subset of X is finite. Now, if $|X| > \mathfrak{c}$, each precompact subset of $A(X)$ is finite (and metrizable), but $A(X)$ does not have a $\mathfrak{G}$-base by Proposition 2.4.

EXAMPLE 6.7. There exists a complete reflexive abelian group G such that each of its precompact subsets is finite (and hence metrizable) but G does not have a $\mathfrak{G}$-base. Indeed, let H be a reflexive abelian P-group of character $\chi(H) > \mathfrak{c}$ (see [23, Theorem 4.8]). Then its completion G is also a reflexive abelian P-group of character $\chi(G) > \mathfrak{c}$ [24, Proposition 4.10]. By Proposition 2.4, G has no $\mathfrak{G}$-base. Since every precompact subset of G is finite, the group G is as desired. Note that the dual group $G^\wedge$ of G is a precompact non-compact non-metrizable reflexive abelian group (by [24]) which does not have a compact resolution swallowing compact sets by Corollary 5.3.

We conclude the paper with the following remark. Assume that G is a metrizable group and $\{U_n\}_{n\in\mathbb{N}}$ is a decreasing base of open symmetric neighbourhoods of the unit e. For every $k \in \mathbb{N}$, there is $f(k) \in \mathbb{N}$ such that $U^k_{f(k)\cdot k} \subseteq U_k$ (where $U^{k+1} = U^k \cdot U$). Being motivated by this fact we try to consider a $\mathfrak{G}$-base on G enjoying the following condition:

(∗) there exists $f : \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ such that $U^k_{f(k)\cdot\alpha} \subseteq U_\alpha$ for every $k \in \mathbb{N}$ and each $\alpha \in \mathbb{N}^{\mathbb{N}}$, where $f(k) \cdot \alpha := (f(k)_n \cdot \alpha_n)$.

However, condition (∗) does not in general imply metrizability of G, as the next example shows.

EXAMPLE 6.8. Let ϕ be the restricted direct product of the sequence $(\mathbb{R}^n)$. It is well known that ϕ is a sequential non-Fréchet–Urysohn space. Set $B_n := \{x \in \mathbb{R} : |x| < 1/n\}$. Then the family of sets of the form

$$U_\alpha = \phi \cap \prod_{i\in\mathbb{N}} B_{\alpha_i}, \quad \text{where } \alpha = (\alpha_i) \in \mathbb{N}^{\mathbb{N}},$$

is a $\mathfrak{G}$-base for ϕ. For every $k \in \mathbb{N}$, set $f(k) = (f^k_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, where $f^k_i = k$ for every $i \in \mathbb{N}$. Since $B^k_{k\cdot n} = B_n$ we obtain $U^k_{f(k)\cdot\alpha} = U_\alpha$ for every $k \in \mathbb{N}$ and each $\alpha \in \mathbb{N}^{\mathbb{N}}$, and hence condition (∗) holds.

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