FUNDAMENTA
MATHEMATICAE
229 (2015)

Countable tightness in the spaces of regular probability measures

by

Grzegorz Plebanek (Wrocław) **and Damian Sobota** (Warszawa)

To the memory of Jaś, *Kamil Duszenko (1986–2014)*

Abstract. We prove that if K is a compact space and the space $P(K \times K)$ of regular probability measures on $K \times K$ has countable tightness in its weak* topology, then $L_1(\mu)$ is separable for every $\mu \in P(K)$. It has been known that such a result is a consequence of Martin's axiom $\mathrm{MA}(\omega_1)$. Our theorem has several consequences; in particular, it generalizes a theorem due to Bourgain and Todorčević on measures on Rosenthal compacta.

1. Introduction. The *tightness* of a topological space X, denoted by $\tau(X)$, is the least cardinal number such that for every $A \subseteq X$ and $x \in \overline{A}$ there is a set $A_0 \subseteq A$ with $|A_0| \le \tau(X)$ and such that $x \in \overline{A}_0$.

Throughout, K stands for a compact Hausdorff topological space. By $C(K)$ we denote the Banach space of continuous functions on K equipped with the supremum norm. As usual, the conjugate space $C(K)^*$ is identified with $M(K)$, the space of signed Radon measures on K of finite variation. We denote by $P(K)$ the space of probability Radon measures on K and consider $P(K)$ endowed with the weak* topology inherited from $C(K)^*$.

In the present paper we focus on the following problem.

Problem 1.1. *Suppose that $P(K)$ has countable tightness. Does this imply that every $\mu \in P(K)$ has countable Maharam type* (*that is, $L_1(\mu)$ is separable*)*?*

There are several reasons why this problem seems to be quite interesting and delicate. We now briefly outline some aspects of Problem 1.1 and postpone a more detailed discussion to Section 5.

2010 *Mathematics Subject Classification*: Primary 46E15, 46E27, 28A33; Secondary 54C35.
Key words and phrases: weak-star topology, countable tightness, Radon measure, Maharam type.

DOI: 10.4064/fm229-2-4

Assuming Martin's axiom $\mathrm{MA}(\omega_1)$, Fremlin [9] showed that if a compact space K admits a measure of uncountable type, then K can be continuously mapped onto $[0,1]^{\omega_1}$, so in particular K must have uncountable tightness. Since $P(K)$ contains a subspace homeomorphic to K, it follows that Problem 1.1 has an affirmative answer under $\mathrm{MA}(\omega_1)$.

Talagrand [22] showed that if K admits a measure of type ω_2, then $P(K)$ can be continuously mapped onto $[0,1]^{\omega_2}$. Thus the following analogue of Problem 1.1 holds true: if $\tau(P(K)) \le \omega_1$, then every measure $\mu \in P(K)$ is of type $\le \omega_1$.

Pol [21] investigated whether the following duality holds: $P(K)$ has countable tightness if and only if the Banach space $C(K)$ has property (C) of Corson. By the main result of [7], in order to verify such a duality it is sufficient to prove that Problem 1.1 has an affirmative answer.

If K is *Rosenthal compact* (i.e. K can be represented as a pointwise compact space of Baire-one functions on some Polish space), then every μ in $P(K)$ has countable type. This fact, announced in Bourgain [2], was proved by Todorčević [23] basing on properties of Rosenthal compacta in forcing extensions (see also Marciszewski and Plebanek [15]). An affirmative answer to Problem 1.1 would be a generalization of that result since for Rosenthal compact K, the space $P(K)$ is also Rosenthal compact and consequently has countable tightness.

We shall prove below that for every compact space K, if $P(K \times K)$ has countable tightness, then K carries only measures of countable type. This does not solve Problem 1.1 completely, but seems to be a substantial step forward. After recalling basic definitions and facts in Section 2, we prove in Section 3 some auxiliary results on measures on product spaces. In Section 4 we prove our main result—Theorem 4.1. In the final section we present some consequences of Theorem 4.1, and state some related open problems.

2. Preliminaries. For a given space K, $\mathrm{Bor}(K)$ stands for the σ-algebra of all Borel subsets of K. Every $\mu \in P(K)$ is treated as a Borel measure on K, which is inner regular with respect to compact sets. Recall that the *weak* topology* on $P(K)$ is the weakest one making the function $\mu \mapsto \int_K g\,d\mu$ continuous for every $g \in C(K)$.

REMARK 2.1. Take an open set $U \subseteq K$ and a closed set $F \subseteq K$. Note that $V_{U,a} = \{\nu \in P(K) : \nu(U) > a\}$ is weak* open in $P(K)$ for every $a \in \mathbb{R}$.
Let $M \subseteq P(K)$ and $\nu_0 \in \overline{M}$. It follows that

(a) $\nu_0(U) \le a$ provided $\nu(U) \le a$ for every $\nu \in M$;
(b) $\nu_0(F) \ge a$ provided $\nu(F) \ge a$ for every $\nu \in M$.

The *Maharam type* of a measure $\mu \in P(K)$ can be defined as the least cardinal κ for which there exists a family $\mathcal{C} \subseteq \mathrm{Bor}(K)$ of cardinality κ and such that $\inf\{\mu(B \triangle C) : C \in \mathcal{C}\} = 0$ for every $B \in \mathrm{Bor}(K)$. Equivalently, μ has Maharam type κ if the space $L_1(\mu)$ of all μ-integrable functions has density κ as a Banach space. A measure $\mu \in P(K)$ is *homogeneous* if its type is the same on every set $B \in \mathrm{Bor}(K)$ of positive measure.

The following fact is well-known (see e.g. Plebanek [18, Lemma 2] or Fremlin [9, Introduction]).

Lemma 2.2. *If a compact space K carries a regular measure of uncountable type, then there is $\mu \in P(K)$ which is homogeneous of type ω_1.*

Let $\mu \in P(K)$ and denote its measure algebra by $\mathrm{Bor}(K)/_{\mu=0}$. For B in $\mathrm{Bor}(K)$, let $B^\bullet$ stand for the corresponding element of $\mathrm{Bor}(K)/_{\mu=0}$. We shall use the following standard result.

Lemma 2.3. *Let $\mu \in P(K)$ be a homogeneous measure of type ω_1 and let $\mathcal{C}$ be a countable family of Borel subsets of K. Then there is $B \in \mathrm{Bor}(K)$ such that $\mu(B) = 1/2$ and B is μ-*independent *of $\mathcal{C}$, i.e. $\mu(B \cap C) = \frac{1}{2}\mu(C)$ for every $C \in \mathcal{C}$.*

Proof. By the Maharam Theorem (see Maharam [14] or Fremlin [8]) there is a measure-preserving isomorphism of measure algebras $\varphi : \mathrm{Bor}(K)/_{\mu=0} \to \mathfrak{A}$, where $\mathfrak{A}$ is the measure algebra of the usual product measure λ on 2^{ω_1}. Let $\mathcal{C}^\bullet = \{C^\bullet : C \in \mathcal{C}\}$.

Recall that for every $a \in \mathfrak{A}$ there is a set $A \subseteq 2^{\omega_1}$ depending on coordinates in a countable set $I_A \subseteq \omega_1$ such that $A^\bullet = a$ (see Fremlin [10, Section 8]). Therefore, there is a countable set $I \subseteq \omega_1$ such that for every $C \in \mathcal{C}$ there is $A \subseteq 2^{\omega_1}$ such that $A = A' \times 2^{\omega_1 \setminus I}$ for some $A' \in \mathrm{Bor}(2^I)$ and $\varphi(C^\bullet) = A^\bullet$.

Take $\xi < \omega_1$ such that $\xi > \sup I$, and $B \in \mathrm{Bor}(K)$ for which $B^\bullet = \varphi^{-1}(c_\xi^\bullet)$, where $c_\xi = \{x \in 2^{\omega_1} : x(\xi) = 0\}$. Then B has the required property. ∎

The following corollary can be easily obtained using Lemma 2.3 and regularity of μ.

Corollary 2.4. *Let $\mu \in P(K)$ be a homogeneous measure of type ω_1. For every $\varepsilon > 0$ there exist sequences $\langle B_\xi \in \mathrm{Bor}(K) : \xi < \omega_1\rangle$ and $\langle U_\xi \in \mathrm{Open}(K) : \xi < \omega_1\rangle$ such that*

(i) $\mu(B_\xi) = 1/2$,
(ii) $B_\xi \subseteq U_\xi$ *and* $\mu(U_\xi \setminus B_\xi) < \varepsilon$,
(iii) B_ξ *is μ-independent of the algebra generated by* $\mathcal{C}_\xi = \{B_\eta, U_\eta : \eta < \xi\}$.

3. Measures on $K \times K$. In this section we consider a fixed measure $\mu \in P(K)$. Given two algebras $\mathcal{A}$ and $\mathcal{B}$, we write

$$\mathcal{A} \otimes \mathcal{B} = \mathrm{alg}(\{A \times B : A \in \mathcal{A},\ B \in \mathcal{B}\})$$

for their product algebra; here $\mathrm{alg}(\cdot)$ denotes the algebra of sets generated by a given family. Let $\mathcal{R}$ denote the Borel rectangle algebra in $K \times K$, i.e.

$$\mathcal{R} = \mathrm{Bor}(K) \otimes \mathrm{Bor}(K).$$

The following notation is crucial for our considerations: given an algebra $\mathcal{A} \subseteq \mathrm{Bor}(K)$, we write $P(\mathcal{A} \otimes \mathcal{A}, \mu)$ for the family of all finitely additive probability measures ν on $\mathcal{A} \otimes \mathcal{A}$ such that

$$\nu(A \times K) = \nu(K \times A) = \mu(A) \quad \text{for every } A \in \mathcal{A}.$$

By a result due to Marczewski and Ryll-Nardzewski [16] every ν in $P(\mathcal{R}, \mu)$ is automatically countably additive and can be extended to a (regular) measure on the product σ-algebra $\sigma(\mathrm{Bor}(K) \otimes \mathrm{Bor}(K))$. In turn, such a measure can be extended to a regular measure on $\mathrm{Bor}(K \times K)$. For completeness we outline a relatively short argument below (cf. Plebanek [17, Theorem 4]).

THEOREM 3.1. *Every $\nu \in P(\mathcal{R}, \mu)$ can be extended to a regular Borel measure on $K \times K$.*

Proof. Let $\mathcal{L}$ denote the family of finite unions of rectangles $F \times F'$ where $F, F' \subset K$ are closed. Using the fact that $\nu \in P(\mathcal{R}, \mu)$, it is easy to see that ν is *$\mathcal{L}$-regular*, i.e. for every $\varepsilon > 0$ and $A \in \mathcal{R}$ there exists $L \in \mathcal{L}$ contained in A and such that $\nu(A \setminus L) < \varepsilon$.

Let $\mathcal{F}$ be the lattice of all closed subsets of $K \times K$. By the main result of Bachman and Sultan [1], ν can be extended to an $\mathcal{F}$-regular finitely additive measure ν' on $\mathrm{alg}(\mathcal{R} \cup \mathcal{F})$. By $\mathcal{F}$-regularity and compactness, ν' is continuous from above at $\emptyset$ and the standard Carathéodory extension procedure gives an extension to a regular measure on $\sigma(\mathcal{R}) = \mathrm{Bor}(K \times K)$. ∎

For a subset $B \subseteq K$, we set $B^0 = B$ and $B^1 = B^c = K \setminus B$. We now prove two lemmas concerning extensions of measures on finite algebras with fixed marginal distributions.

LEMMA 3.2. *If $\mathcal{A}$ is a finite subalgebra of* $\mathrm{Bor}(K)$, *then every ν in $P(\mathcal{A} \otimes \mathcal{A}, \mu)$ can be extended to $\widehat{\nu} \in P(\mathcal{R}, \mu)$.*

Proof. Fix a finite algebra $\mathcal{A} \subseteq \mathrm{Bor}(K)$ and $\nu \in P(\mathcal{A} \otimes \mathcal{A}, \mu)$. Let $\mathcal{A}_1 = \mathrm{alg}(\mathcal{A} \cup \{B\})$ where $B \in \mathrm{Bor}(K) \setminus \mathcal{A}$. We shall check first that ν can be extended to $\nu_1 \in P(\mathcal{A}_1 \otimes \mathcal{A}_1, \mu)$.

Let $\{S_1, \ldots, S_l\}$ be the family of all atoms of $\mathcal{A}$ having positive measure. It is sufficient to define ν_1 on atoms of $\mathcal{A}_1 \times \mathcal{A}_1$. For $i, j \leq l$ let

$$\alpha_{i,j} = \nu(S_i \times S_j)/(\mu(S_i)\mu(S_j)).$$

For $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$ set

$$\nu_1((S_i \times S_j) \cap (B^{\varepsilon_1} \times B^{\varepsilon_2})) = \mu(S_i \cap B^{\varepsilon_1})\mu(S_j \cap B^{\varepsilon_2})\alpha_{i,j}.$$

It is easy to check that this uniquely defines the required $\nu_1 \in P(\mathcal{A}_1 \otimes \mathcal{A}_1, \mu)$ (cf. the proof of the next lemma).

It follows that ν admits an extension to $\nu_{\mathcal{D}} \in P(\mathcal{D} \otimes \mathcal{D}, \mu)$ for every finite algebra $\mathcal{D}$ such that $\mathcal{A} \subseteq \mathcal{D} \subseteq \mathrm{Bor}(K)$. Now the assertion follows by a compactness argument as follows (cf. the proof of Lemma 3.4). Let $P(\mathcal{R})$ denote the set of all finitely additive probability measures on $\mathcal{R}$; clearly, $P(\mathcal{R})$ is a closed subset of $[0,1]^{\mathcal{R}}$, hence it is compact. Denote by $P_{\mathcal{D}}$ the set of all measures $\widetilde{\nu} \in P(\mathcal{R})$ extending ν and such that $\widetilde{\nu}|_{\mathcal{D}\otimes\mathcal{D}} \in P(\mathcal{D} \otimes \mathcal{D}, \mu)$. Then $P_{\mathcal{D}}$ is a closed subset of $P(\mathcal{R})$. Let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be a sequence of finite extensions of $\mathcal{A}$ in $\mathrm{Bor}(K)$. Then $P_{\mathcal{D}_1} \cap \cdots \cap P_{\mathcal{D}_n}$ is nonempty and hence, by the finite intersection property, there exists

$$\widehat{\nu} \in \bigcap \{P_{\mathcal{D}} : \mathcal{D} \text{ is a finite extension of } \mathcal{A} \text{ in } \mathrm{Bor}(K)\}.$$

Clearly, $\widehat{\nu}$ is an element of $P(\mathcal{R}, \mu)$ extending ν. ∎

LEMMA 3.3. *Let $\mathcal{A}$ be a finite subalgebra of* $\mathrm{Bor}(K)$, $\mathcal{A}_1 = \mathrm{alg}(\mathcal{A} \cup \{B\})$, *where $B \in \mathrm{Bor}(K) \setminus \mathcal{A}$ is μ-independent of $\mathcal{A}$ and $\mu(B) = 1/2$. Then for every $\nu \in P(\mathcal{A} \otimes \mathcal{A}, \mu)$ there exists an extension $\nu_1 \in P(\mathcal{A}_1 \otimes \mathcal{A}_1, \mu)$ of ν such that $\nu_1(B \times B) = 1/2$.*

Proof. We extend ν to $\nu_1 \in P(\mathcal{A}_1 \times \mathcal{A}_1, \mu)$ in a way similar to the one in the proof of Lemma 3.2.

Let $T_1, \ldots, T_k$ be the list of all atoms of $\mathcal{A}$. For all $i, j \le k$ and $\varepsilon_1, \varepsilon_2 \in \{0,1\}$ set

$$\nu_1((T_i \times T_j) \cap (B^{\varepsilon_1} \times B^{\varepsilon_2})) = \tfrac{1}{2}\nu(T_i \times T_j)$$

if $\varepsilon_1 = \varepsilon_2$ and 0 otherwise. Then

$$\nu_1(B \times B) = \sum_i \sum_j \nu_1((T_i \times T_j) \cap (B \times B)) = \tfrac{1}{2} \sum_i \sum_j \nu(T_i \times T_j) = \tfrac{1}{2}.$$

We now prove that $\nu_1 \in P(\mathcal{A}_1 \otimes \mathcal{A}_1, \mu)$. It is sufficient to check $\nu_1(S \times K) = \nu_1(K \times S) = \mu(S)$ for every atom S of $\mathcal{A}_1$. We have

$$\begin{aligned}
\nu_1((T_i \cap B) \times K) &= \sum_j \nu_1((T_i \cap B) \times T_j) \\
&= \sum_j \big(\nu_1((T_i \cap B) \times (T_j \cap B)) + \nu_1((T_i \cap B) \times (T_j \cap B^c))\big) \\
&= \sum_j \nu_1((T_i \cap B) \times (T_j \cap B)) = \tfrac{1}{2} \sum_j \nu(T_i \times T_j) \\
&= \tfrac{1}{2}\nu(T_i \times K) = \tfrac{1}{2}\mu(T_i) = \mu(T_i \cap B),
\end{aligned}$$

where the last identity follows from the μ-independence of B and $\mathcal{A}$. Similarly one checks the remaining possibilities. ∎

LEMMA 3.4. *Let $\mu \in P(K)$ be a homogeneous measure of type ω_1, and suppose that $\langle B_\xi \in \mathrm{Bor}(K) : \xi < \omega_1\rangle$, $\langle U_\xi \in \mathrm{Open}(K) : \xi < \omega_1\rangle$ and $\mathcal{C}_\xi$ are as in Corollary 2.4. For every $\xi < \omega_1$ there is $\nu_\xi \in P(\mathcal{R}, \mu)$ such that*

- *$\nu_\xi(B_\eta \times B_\eta) = 1/2$ for $\eta \geq \xi$,*
- *$\nu_\xi(A \times A) = (\mu \otimes \mu)(A \times A)$ for every $A \in \mathcal{C}_\xi$.*

Proof. Fix $\xi < \omega_1$. Let $\mathcal{A}$ be a finite algebra generated by some elements of $\mathcal{C}_\xi$, and I be a finite subset of $\omega_1 \setminus \xi$. Then there is $\nu_{\mathcal{A},I} \in P(\mathcal{R}, \mu)$ such that

- $\nu_{\mathcal{A},I}(B_\eta \times B_\eta) = 1/2$ for $\eta \in I$,
- $\nu_{\mathcal{A},I}(A \times A) = (\mu \otimes \mu)(A \times A)$ for every $A \in \mathcal{A}$.

Indeed, such a $\nu_{\mathcal{A},I}$ can be first defined on $\mathrm{alg}(\mathcal{A} \cup \{B_\eta : \eta \in I\})$ using Lemma 3.3 and induction on $|I|$, and then extended to a member of $P(\mathcal{R}, \mu)$ using Lemma 3.2.

Now the existence of ν_ξ with the required properties follows again by a compactness argument: $P(\mathcal{R}, \mu)$ is clearly a closed subset of $[0,1]^{\mathcal{R}}$, so it is compact in the topology of convergence on all elements of $\mathcal{R}$. Hence the required measure ν_ξ can be defined as a cluster point of the net $\nu_{\mathcal{A},I}$ indexed by the pairs $(\mathcal{A}, I)$, where $\mathcal{A}$ is an algebra generated by a finite subset of $\mathcal{C}_\xi$ and I is a finite subset of $\omega_1 \setminus \xi$. ∎

4. Main result.

We are now ready to prove our main result.

THEOREM 4.1. *Let $P(K \times K)$ have countable tightness. Then every μ in $P(K)$ has countable type.*

Proof. Assume for contradiction that there exists $\mu \in P(K)$ of uncountable type. Without loss of generality we can assume that μ is a homogeneous measure of type ω_1 (see Lemma 2.2). Let $0 < \varepsilon < 1/16$.

Take the sequences $\langle B_\xi : \xi < \omega_1\rangle$ and $\langle U_\xi : \xi < \omega_1\rangle$ as in Corollary 2.4. For every $\xi < \omega_1$ take $\nu_\xi \in P(\mathcal{R}, \mu)$ as in Lemma 3.4 and extend it to $\widehat{\nu_\xi} \in P(K \times K)$ using Theorem 3.1. Let $\nu \in P(K \times K)$ be a cluster point of the sequence $\langle \widehat{\nu_\xi} : \xi < \omega_1\rangle$, i.e.

$$\nu \in \bigcap_{\xi < \omega_1} \overline{\{\widehat{\nu_\eta} : \eta \geq \xi\}}.$$

We shall show that $\nu \notin \overline{\{\widehat{\nu_\eta} : \eta \in I\}}$ for every countable $I \subseteq \omega_1$, which will contradict the countable tightness of $P(K \times K)$.

Let $I \subseteq \omega_1$ be countable. Take $\xi > \sup I$. By regularity of μ there exists closed $F \subseteq K$ such that $F \subseteq B_\xi$ and $\mu(B_\xi \setminus F) < \varepsilon$. For every $\eta \in I$ we have

$\widehat{\nu}_\eta(B_\xi \times B_\xi) = 1/2$ and

$$\begin{aligned}\widehat{\nu}_\eta((B_\xi \times B_\xi) \setminus (F \times F)) &\le \widehat{\nu}_\eta((B_\xi \setminus F) \times K) + \widehat{\nu}_\eta(K \times (B_\xi \setminus F)) \\ &= 2\mu(B_\xi \setminus F) < 2\varepsilon.\end{aligned}$$

Therefore $\widehat{\nu}_\eta(F \times F) > 1/2 - 2\varepsilon$ whenever $\eta \in I$.

On the other hand, if $\eta > \xi$ then

$$\nu_\eta(U_\xi \times U_\xi) = (\mu \otimes \mu)(U_\xi \times U_\xi) = \mu(U_\xi)^2 < (1/2 + \varepsilon)^2,$$

so by Remark 2.1(a), $\nu(F \times F) \le \nu(U_\xi \times U_\xi) \le (1/2 + \varepsilon)^2$.

As $\varepsilon < 1/16$, we have $(1/2 + \varepsilon)^2 < 1/2 - 2\varepsilon$. We conclude from Remark 2.1(b) that $\nu \notin \overline{\{\widehat{\nu}_\eta : \eta \in I\}}$, and the proof is complete. ■

Let us remark that modifying our proof of Theorem 4.1 one can obtain the following more general result.

THEOREM 4.2. *Suppose that K and L are compacta carrying measures of uncountable type. Then $\tau(P(K \times L)) \ge \omega_1$.*

5. Some consequences and open problems. In this final section we present several consequences of Theorem 4.1 as well as some open problems.

We do not know if one can give an affirmative answer to Problem 1.1 by an argument similar to the one presented above. To explain the main difficulty let us consider for a while a totally disconnected compact space L and a measure $\nu \in P(L)$ of uncountable type. The question is if we can find an uncountable family $\mathcal{C}$ of clopen subsets of L such that for any disjoint finite subfamilies $\mathcal{C}_0, \mathcal{C}_1 \subseteq \mathcal{C}$ there is a finitely additive measure ν' on $\mathrm{alg}(\mathcal{C}_0 \cup \mathcal{C}_1)$ such that $\nu'(C) = \nu(C)$ for $C \in \mathcal{C}_0$ while $|\nu'(C) - \nu(C)| \ge \varepsilon$ for $C \in \mathcal{C}_1$ (with some constant $\varepsilon > 0$). We were able to find such a family in the case when $L = K \times K$ and $\nu = \mu \otimes \mu$, thanks to special properties of rectangles.

D. Fremlin remarked that if Problem 1.1 cannot be settled in ZFC, then natural candidates to look at are hereditarily separable perfectly normal compacta carrying measures of uncountable type. Such spaces do exist under CH—see Džamonja and Kunen [5], and under some weaker axiom see Fremlin [11, 531Q].

It is not difficult to check that if every $\mu \in P(K)$ has countable type, then every $\nu \in P(K \times K)$ has countable type as well. In connection with Problem 1.1 and Theorem 4.1 it is natural to ask the following:

PROBLEM 5.1. *Suppose that $P(K)$ has countable tightness. Does $P(K \times K)$ have countable tightness?*

As far as we know, the problem is open. Note that $P(K) \times P(K)$ embeds into $P(K \times K)$, and if $\tau(P(K)) = \omega$ then $\tau(P(K) \times P(K)) = \omega$, since countable tightness is productive for compact spaces (see Engelking

[6, 3.12.8]). However, the space $P(K \times K)$ seems to be far more complicated than $P(K) \times P(K)$.

5.1. Rosenthal compacta. Recall that a compact space K is said to be *Rosenthal compact* if K embeds into $B_1(X)$, the space of Baire-one functions on a Polish space X equipped with the topology of pointwise convergence. The class of Rosenthal compacta is stable under taking countable products and, by a result of Godefroy [12], if K is Rosenthal compact, then so is $P(K)$. Moreover, Rosenthal compacta are Fréchet–Urysohn spaces (see Bourgain, Fremlin and Talagrand [3]), hence they have countable tightness. This, together with Theorem 4.1, implies the result of Bourgain and Todorčević mentioned in the introductory section.

COROLLARY 5.2. *If K is Rosenthal compact, then every $\mu \in P(K)$ has countable type.*

5.2. Property (C) of Corson. Let X be a Banach space. Corson [4] introduced the following convex analogue of the Lindelöf property: X is said to have *property* (C) if for every family $\mathcal{C}$ of convex closed subsets of X we have $\bigcap \mathcal{C} \neq \emptyset$ provided that every countable subfamily of $\mathcal{C}$ has nonempty intersection. For $C(K)$ spaces, Pol [21, Lemma 3.2] gave the following characterization of property (C).

THEOREM 5.3 (Pol). *For a compact space K the following are equivalent:*

(1) *the space $C(K)$ has property* (C)*;*
(2) *for every family $\mathcal{M} \subseteq P(K)$ and every $\mu \in \overline{\mathcal{M}}$ there exists a countable subfamily $\mathcal{N} \subseteq \mathcal{M}$ such that $\mu \in \overline{\mathrm{conv}}\, \mathcal{N}$.*

Let us say that $P(K)$ has *convex countable tightness* if $P(K)$ fulfils condition (2) of Theorem 5.3. Clearly, countable tightness implies convex countable tightness; Pol [21] asked if those properties are actually equivalent, which amounts to the following:

PROBLEM 5.4. *Assume $C(K)$ has property* (C)*. Does this imply the countable tightness of $P(K)$?*

Frankiewicz, Plebanek and Ryll-Nardzewski [7, Theorem 3.4] answered this question affirmatively assuming Martin's axiom $\mathrm{MA}(\omega_1)$. Without any additional set-theoretic assumptions they also obtained the following partial result [7, Theorem 3.2]:

THEOREM 5.5. *Assume that every $\mu \in P(K)$ has countable type. Then $\tau(P(K)) = \omega$ if and only if $C(K)$ has property* (C)*.*

Looking back at the proof of Theorem 4.1 it is easy to notice that we have in fact obtained the following (formally) stronger result.

THEOREM 5.6. *Let $P(K \times K)$ have convex countable tightness. Then every $\mu \in P(K)$ has countable type.*

Theorem 5.6 yields the following partial affirmative answer to Problem 5.4.

COROLLARY 5.7. *For any compact space K, $\tau(P(K \times K)) = \omega$ if and only if $C(K \times K)$ has property* (C).

Proof. Assume that $\tau(P(K\times K)) = \omega$. By Theorem 4.1, every $\mu \in P(K)$ has countable type, hence every $\mu \in P(K \times K)$ has countable type. By Theorem 5.5, $C(K \times K)$ has property (C).

For the converse, assume that $C(K \times K)$ has property (C). By Theorem 5.3, $P(K \times K)$ has convex countable tightness, which by Theorem 5.6 implies that every $\mu \in P(K \times K)$ has countable type. Using Theorem 5.5 again, we conclude that $\tau(P(K \times K)) = \omega$. ∎

In connection with Problem 5.1, one can ask the following question on property (C).

PROBLEM 5.8. *Let $C(K)$ have property* (C). *Does $C(K \times K)$ also have property* (C)*?*

Note that the converse holds true. Indeed, if X is a Banach space with property (C) and Y is its closed subspace, then Y also has property (C). Since $C(K)$ embeds isometrically into $C(K \times K)$ via the operator $C(K) \ni f \mapsto f \circ \pi \in C(K \times K)$, where $\pi : K \times K \to K$ is a projection, $C(K)$ is isometric to a closed subspace of $C(K \times K)$.

It is also worth noting that if $C(K)$ has property (C), then so does $C(K)\oplus C(K)$, since (C) is a three-space property (cf. Pol [20, Proposition 1]).

5.3. Topological dichotomy for $P(K \times K)$. The particular case of Theorem 2.2 of Krupski and Plebanek [13] states that given a compact space K, $P(K)$ contains either a $\mathbb{G}_\delta$ point (i.e. a point of countable character in $P(K)$) or a measure of uncountable type. Thus Theorem 4.1 immediately implies the following:

COROLLARY 5.9. *For every compact space K, either $P(K\times K)$ contains a $\mathbb{G}_\delta$ point or $P(K \times K)$ has uncountable tightness.*

Recall that a measure $\mu \in P(K)$ is *countably deteremined* (CD) if there is a countable family $\mathcal{F}$ of closed subsets of K such that $\mu(U) = \sup\{\mu(F) : F \subseteq U,\ F \in \mathcal{F}\}$ for every open $U \subseteq K$. Moreover, μ is *strongly countably determined* (SCD) if one can choose such a family $\mathcal{F}$ consisting of closed $\mathbb{G}_\delta$ sets; see [21] and [13] for basic properties of CD and SCD measures, and further references. For every $\mu \in P(K)$ we have the following implications:

$$\mu \text{ is SCD } \Rightarrow\ \mu \text{ is CD } \Rightarrow\ \mu \text{ has countable type.}$$

A measure $\mu \in P(K)$ is strongly countably determined if and only if μ is a $\mathbb{G}_\delta$ point of $P(K)$. Thus the statement 'every $\mu \in P(K)$ is strongly countably determined' is equivalent to $P(K)$ being first-countable. In the light of our main result, the following problem seems natural.

PROBLEM 5.10. *Suppose that $P(K)$ or $P(K^\omega)$ is a Fréchet–Urysohn space. Is every $\mu \in P(K)$ countably determined?*

It is not known whether every measure on a Rosenthal compactum is countably determined—see Marciszewski and Plebanek [15] for a partial positive solution.

Acknowledgments. G. Plebanek was partially supported by NCN grant 2013/11/B/ST1/03596 (2014–2017).

D. Sobota was partially supported by the Warsaw Center of Mathematics and Computer Science and grant S30028/I-18 from the Institute of Mathematics and Computer Science of the Wrocław University of Technology.

References

[1] G. Bachman and A. Sultan, *On regular extensions of measures*, Pacific J. Math. 86 (1980), 389–395.

[2] J. Bourgain, Thesis, Brussels, 1974.

[3] J. Bourgain, D. H. Fremlin and M. Talagrand, *Pointwise compact sets of Baire-measurable functions*, Amer. J. Math. 100 (1978), 845–886.

[4] H. H. Corson, *The weak topology of a Banach space*, Trans. Amer. Math. Soc. 101 (1961), 1–15.

[5] M. Džamonja and K. Kunen, *Measures on compact HS spaces*, Fund. Math. 143 (1993), 41–54.

[6] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[7] R. Frankiewicz, G. Plebanek and C. Ryll-Nardzewski, *Between Lindelöf property and countable tightness*, Proc. Amer. Math. Soc. 129 (2001), 97–103.

[8] D. H. Fremlin, *Measure algebras*, in: Handbook of Boolean Algebras, J. D. Monk and R. Bonnet (eds.), North-Holand, Amsterdam, 1989, Vol. 3, 877–980.

[9] D. H. Fremlin, *On compact spaces carrying Radon measures of uncountable Maharam type*, Fund. Math. 154 (1997), 295–304.

[10] D. H. Fremlin, *Sets determined by few coordinates*, Atti Sem. Mat. Fis. Univ. Modena 50 (2002), 23–36.

[11] D. H. Fremlin, *Measure Theory*, Vol. 5: Set-Theoretic Measure Theory, Torres Fremlin, Colchester, 2008.

[12] G. Godefroy, *Compacts de Rosenthal*, Pacific J. Math. 91 (1980), 293–306.

[13] M. Krupski and G. Plebanek, *A dichotomy for the convex spaces of probability measures*, Topology Appl. 158 (2011), 2184–2190.

[14] D. Maharam, *On homogeneous measure algebras*, Proc. Nat. Acad. Sci. U.S.A. 28 (1942), 108–111.

[15] W. Marciszewski and G. Plebanek, *On measures on Rosenthal compacta*, J. Math. Anal. Appl. 385 (2012), 185–193.

[16] E. Marczewski and C. Ryll-Nardzewski, *Remarks on the compactness and non direct products of measures*, Fund. Math. 40 (1953), 165–170.

[17] G. Plebanek, *Measures on two-dimensional products*, Mathematika 36 (1989), 253–258.

[18] G. Plebanek, *On Radon measures on first-countable spaces*, Fund. Math. 148 (1995), 159–164.

[19] G. Plebanek, *On compact spaces carrying Radon measures of large Maharam type*, Acta Univ. Carolin. Math. Phys. 43 (2002), 87–99.

[20] R. Pol, *On a question of H. H. Corson and some related problems*, Fund. Math. 109 (1980), 143–154.

[21] R. Pol, *Note on the spaces $P(S)$ of regular probability measures whose topology is determined by countable subsets*, Pacific J. Math. 100 (1982), 185–201.

[22] M. Talagrand, *Sur les espaces de Banach contenant $l^1(\tau)$*, Israel J. Math. 40 (1981), 324–330.

[23] S. Todorčević, *Compact subsets of the first Baire class*, J. Amer. Math. Soc. 12 (1999), 1179–1212.

Grzegorz Plebanek
Instytut Matematyczny
Uniwersytet Wrocławski
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: grzes@math.uni.wroc.pl

Damian Sobota
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-656 Warszawa, Poland
E-mail: damian.sobota@impan.pl

Received 16 July 2014;
in revised form 12 November 2014