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Packing spectra for Bernoulli measures supported on Bedford–McMullen carpets

by

Thomas Jordan (Bristol) and **Michał Rams** (Warszawa)

Abstract. We consider the packing spectra for the local dimension of Bernoulli measures supported on Bedford–McMullen carpets. We show that typically the packing dimension of the regular set is smaller than the packing dimension of the attractor. We also consider a specific class of measures for which we are able to calculate the packing spectrum exactly, and we show that the packing spectrum is discontinuous as a function on the space of Bernoulli measures.

1. Introduction. The aim of this paper is to develop the theory of multifractal analysis for a special class of self-affine measures. These measures are the Bernoulli measures supported on Bedford–McMullen carpets [McM], [B], and their multifractal properties have been studied in several papers: [K], [O], [N], [BM], [GL], [JR], [R] and [BF]. However most of these papers focus on the Hausdorff spectra and very little is known about the packing spectra. For self-similar measures satisfying the open set condition the packing and Hausdorff spectra are the same [CM], [AP] but they are in general different for subsets of the irregular set where the lim inf and lim sup are specified [BOS]. For Bernoulli measures on Bedford–McMullen carpets an upper bound is given in [O] in terms of the Legendre transform of a certain function; this is typically greater than the Hausdorff spectra. However in [R] it is shown that this upper bound may not be sharp and that there are cases when the Hausdorff spectra and packing spectra are the same even when the Hausdorff and packing dimensions of the attractor are different. Moreover, Reeve [R] calculated the packing spectra for a specific class of self-affine measures.

We extend this theory in two ways, firstly by showing that for a generic class of self-affine measures the packing dimension of the attractor is strictly

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greater than the maximum of the packing spectra, i.e. the packing dimension of the regular set. Secondly we consider a specific family of Bedford–McMullen carpets and Bernoulli measures supported on these carpets. We explicitly calculate the packing spectra for this family and show that they are not a continuous function of the parameters for the measures. This is in contrast to the case for the Hausdorff and packing spectra for self-similar measures [CM], [AP] and for the Hausdorff spectra of self-affine measures on Bedford–McMullen carpets [BM], [K], [O] and [JR].

The setting we consider is that, for a given Bernoulli measure μ supported on a Bedford–McMullen carpet, we look at the set of points where the local dimension of μ equals α (to be denoted by X_α) and the set of points where the symbolic local dimension of μ (i.e. the local dimension obtained by using approximate squares instead of geometric balls) equals α (to be denoted by X_α^{symb}). Then the Hausdorff dimensions of X_α and X_α^{symb} coincide, they also coincide with the Hausdorff dimension of a certain Bernoulli measure μ_α, for which a typical point belongs to X_α^{symb}. The function $\dim_H X_\alpha$ is concave, it is the Legendre transform of some well-defined multifractal function. The maximum value achieved by $\dim_H X_\alpha$ equals the Hausdorff dimension of the whole Bedford–McMullen carpet. Moreover, $\dim_H X_\alpha$ is real analytic both as a function of α and as a function of μ.

Seeing this, Olsen [O] conjectured that (most of) the same properties hold for the packing spectrum. In particular, he conjectured that $\dim_P X_\alpha^{\mathrm{symb}}$ is the Legendre spectrum of another multifractal function, and then wrote down some properties of such a function. However in [R] it was shown that for a certain family of examples the packing spectra are the same as the Hausdorff spectra even when the packing dimension and Hausdorff dimension of the attractors are different. In particular this shows that the conjecture in [O] cannot hold in general. This is related to work by Nielsen [N] where the dimension of sets is determined by the frequencies of occurrence of each map in the iterated function system. Again in this case Nielsen shows that the packing and Hausdorff spectra are the same even when the Hausdorff and packing dimensions of the attractor are different.

In this note we go on to show that under a condition which holds for typical carpets the conjecture in [O] cannot hold, and then by considering a specific class of examples that the situation is much more complicated than the situation with the Hausdorff spectra. We prove two theorems. The first one, Theorem 2.3, states that it is unlikely that the maximum of (symbolic) packing spectrum equals the packing dimension of the Bedford–McMullen carpet. More precisely, there is a codimension one condition necessary and a codimension two condition sufficient for this to happen. Both conditions are on coefficients of the Bernoulli measure μ.

Our Theorem 2.4 is more interesting. We consider a very special family of Bernoulli measures, but the same phenomenon holds in much greater generality. Assume the Bedford–McMullen carpet has exactly two rows. We also assume that the Bernoulli measure μ is equally distributed in each row. For a given carpet such measures form a one-parameter family: they are uniquely determined by the measure of the first row. For such measures we have $\dim_P X_\alpha = \dim_P X_\alpha^{\mathrm{symb}}$ for all α. Then, for all parameter values except one, $\dim_H X_\alpha = \dim_P X_\alpha$ for all α. However, there is one exceptional measure μ in our family for which $\dim_H X_\alpha < \dim_P X_\alpha$ for all α in the interior of the spectrum interval. The function $\dim_P X_\alpha$ is still well behaved as a function of α, but as a function of μ it is not even continuous.

Note here that this phenomenon is not an artifact created by the fact that the packing dimension is in some way not adequate to study the local dimension spectrum. On the contrary, while for all nonexceptional measures μ the set X_α is equal to the set of μ_α-typical points, for the exceptional measure it is strictly greater. The Hausdorff dimension of those additional points is equal to $\dim_H \mu_\alpha$ (which is why this set does not cause the Hausdorff dimension to grow), but their packing dimension is strictly greater than $\dim_H X_\alpha$ for all α in the interior of the spectrum.

We are not able to present any conjectures as to what the packing spectrum really is in general. But it certainly seems to be an object worthy of further study.

2. Notation and results. We start this section by defining the packing dimension and stating some basic results we use to calculate the packing dimension of sets. For a set $A \subseteq \mathbb{R}^d$ and $s \geq 0$ let

$$\tilde{\mathcal{P}}(A) = \lim_{\varepsilon \to 0} \sup\Big\{\sum_i B(x_i, r_i) : x_i \in A,\ r_i < \varepsilon \text{ and } |x_i - x_j| > r_i + r_j \text{ for } i \neq j\Big\},$$

and define the outer packing measure by

$$\mathcal{P}(A) = \inf\Big\{\sum \tilde{\mathcal{P}}(A_i) : \bigcup_i A_i \supseteq A\Big\}.$$

Observe that $\mathcal{P}$ gives a measure when restricted to measurable sets, and we can define the packing dimension analogously to the Hausdorff dimension by

$$\dim_P A = \inf\{s : \mathcal{P}(A) = 0\} = \sup\{s : \mathcal{P}(A) = \infty\}.$$

While the main focus of this paper is on packing dimension we will at time use Hausdorff dimension, denoted by $\dim_H$, and upper box counting dimension denoted $\overline{\dim}_B$; for the definitions we refer the reader to [Ma]. We will use the following standard results on packing dimension. The first result relates packing dimension to Hausdorff dimension and upper box dimension.

Lemma 2.1. *For any $A \subseteq \mathbb{R}^d$ we have*

$$\dim_H A \le \dim_P A \le \overline{\dim}_B A.$$

Proof. See [Ma, p. 82]. ∎

The second is a version of Frostman's lemma for packing dimension.

Lemma 2.2. *For a Borel set $A \subseteq \mathbb{R}^d$, if there exists a Borel probability measure μ such that $\mu(A) = 1$ and*

$$\limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \ge s$$

for all $x \in A$, then $\dim_P A \ge s$.

Proof. See [Ma, Theorem 6.11, p. 97]. ∎

We now formally introduce Bedford–McMullen carpets. Let $m, n \in \mathbb{N}$ with $2 \le m < n$ and

$$D \subseteq \{0, \dots, m-1\} \times \{0, \dots, n-1\}$$

where $|D| \ge 2$, and define $\sigma := \frac{\log m}{\log n}$. For $(i,j) \in D$ let $T_{i,j} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T_{i,j}((x,y)) = \left(\frac{x+i}{n}, \frac{y+j}{m}\right),$$

and Λ be the unique nonempty compact set satisfying

$$\Lambda = \bigcup_{(i,j)\in D} T_{i,j}(\Lambda).$$

Sets of the form Λ were first studied in [B] and [McM] and are usually known as *Bedford–McMullen carpets*. We will let $L_0 = |D|$, and for $0 \le i \le m-1$ we let

$$n_i = |\{(i,j) \in D : 0 \le j \le n-1\}|.$$

Finally let $L_1 = |\{i : n_i \ne 0\}|$.

The geometry of affine sets is different from the geometry of conformal fractals because cylinders are not approximate balls anymore. Hence, in order to work with the geometric properties of the fractal, we need to use not single cylinders but some unions of cylinders. Given two positive integers $n_1 \le n_2$ and a point $x \in \Lambda$ we define the *rectangle*

$$R^{n_1,n_2}(x) = \{y \in \Lambda : i_k(y) = i_k(x)\ \forall k \le n_2 \text{ and } j_k(y) = j_k(x)\ \forall k \le n_1\}.$$

Note that $R^{N,N}(x)$ is just the Nth level cylinder containing x. The rectangle $R^{n_1,n_2}(x)$ is the intersection of Λ with a geometrical rectangle with horizontal side n^{-n_1} and vertical side m^{-n_2}. The Nth level *approximate square* is defined as $C_N(x) = R^{\lceil \sigma N \rceil, N}(x)$. It is a set of diameter approximately m^{-N}. It will sometimes be convenient to define $C_N(x)$ for noninteger values of $N > 0$: we will denote $C_N(x) = C_{\lceil N \rceil}(x)$.

We now introduce the class of measures we will be considering which are projections of Bernoulli measures onto Bedford–McMullen carpets. We let $\{p_{i,j}\}_{(i,j)\in D}$ be a probability vector and μ be the unique Borel probability measure such that for all Borel sets $B \subseteq \mathbb{R}^2$ we have

$$\mu(B) = \sum_{(i,j)\in D} p_{i,j}\mu(T_{i,j}^{-1}(B)).$$

An alternative way of defining this measure is as the natural projection of the $\{p_{i,j}\}$-Bernoulli measure from the shift space $D^{\mathbb{N}}$ onto Λ.

In this paper we deal with local dimensions. The *geometric local dimension* (or simply local dimension; we use the name geometric local dimension to distinguish it from the symbolic local dimension) of a Borel measure μ supported in $\mathbb{R}^2$ at a point x is defined as

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B_r(x))}{\log r},$$

provided the limit exists. However it is often easier to work with approximate squares rather than geometric balls. Thus we define the *symbolic local dimension* as

$$\delta_\mu(x) = \lim_{N\to\infty} \frac{\log \mu(C_N(x))}{-N\log m},$$

provided the limit exists. For any $\alpha \in \mathbb{R}$ we denote

$$X_\alpha = \{x \in \Lambda : d_\mu(x) = \alpha\} \quad \text{and} \quad X_\alpha^{\text{symb}} = \{x \in \Lambda : \delta_\mu(x) = \alpha\}.$$

The functions $\alpha \mapsto \dim_P X_\alpha$ and $\alpha \mapsto \dim_P X_\alpha^{\text{symb}}$ are called the *geometric* and the *symbolic packing local dimension spectra* of μ.

In several cases it is possible to show that $\delta_\mu(x) = d_\mu(x)$, however this is not always true. We let

$$\alpha_m = \min_{(i,j)\in D} \frac{-\sigma \log p_{ij} + (\sigma - 1)\log q_i}{\log m},$$
$$\alpha_M = \max_{(i,j)\in D} \frac{-\sigma \log p_{ij} + (\sigma - 1)\log q_i}{\log m},$$

where

$$q_i = \sum_j p_{i,j}.$$

Note that $\alpha \notin [\alpha_m, \alpha_M]$ is equivalent to $X_\alpha = X_\alpha^{\text{symb}} = \emptyset$.

Clearly, the relation between geometric and symbolic local dimensions at any given point is given by the geometric interplay between balls and approximate squares. On the one hand, for all $x \in \Lambda$ we have

$$C_N(x) \subseteq B_{m^{-N}}(x). \tag{2.1}$$

On the other hand, the point x can be very close to the boundary of an approximate square it lies in, and then the ball $B_{cm^{-N}}(x)$ will only be contained in $C_N(x)$ for very small c. We can describe the distance from x to the boundary of $C_N(x)$ using the symbolic expansion of x. Denote

$$I_N(x) = \frac{1}{N}\sup\{k \in \mathbb{N} : i_{N+1}(x) = \cdots = i_{N+k}(x) \in \{0, m-1\}\},$$
$$J_N(x) = \frac{1}{\lceil \sigma N\rceil}\sup\{k \in \mathbb{N} : j_{\lceil \sigma N\rceil+1}(x) = \cdots = j_{\lceil \sigma N\rceil+k}(x) \in \{0, n-1\}\}.$$

If some positive δ is greater than $I_N(x)$ and $J_N(x)$ then the distance from x to the boundary of $C_N(x)$ is at least $m^{-N(1+\delta)}$, and hence

$$B_{m^{-N(1+\delta)}}(x) \subseteq C_N(x). \tag{2.2}$$

An easy consequence of (2.1), (2.2) is that if both $I_N(x)$ and $J_N(x)$ converge to 0 as N goes to infinity (in particular, almost all points for any Bernoulli measure with nontrivial horizontal and vertical projections have this property) then the symbolic and local dimensions of any Bernoulli measure coincide at x.

The symbolic local dimension can thus be used to calculate the geometric local dimension at many points. At the same time, for any Bernoulli measure μ its symbolic local dimension at a point x is easy to calculate from the symbolic expansion of x. We have

$$\mu(C_N(x)) = \prod_{k=1}^{\lceil \sigma N\rceil} p_{i_k(x),j_k(x)} \prod_{k=\lceil \sigma N\rceil+1}^{N} q_{i_k(x)}, \tag{2.3}$$

and we can calculate $\delta_\mu(x)$ directly. Note also another important consequence of (2.3): there exists a constant $K > 0$ such that for any $x \in \Lambda$ and $N > 0$ we have

$$K < \frac{\mu(C_{N+1}(x))}{\mu(C_N(x))} \le 1. \tag{2.4}$$

Our results are as follows. We remind that the Bernoulli measure living on a Bedford–McMullen carpet is defined by the first iteration image: $n \times m$ rectangular grid in the unit square with probabilities corresponding to the rectangles, and that L_0 is the number of rectangles with nonzero probabilities, n_i is the number of rectangles in the ith horizontal row, L_1 is the number of nonempty rows, $p_{i,j}$ is the probability of the rectangle (i, j) and q_i is the probability of the ith horizontal row.

THEOREM 2.3. *The symbolic and geometric packing local dimension spectra are both strictly smaller than* $\dim_P \Lambda$ *for systems not satisfying*

$$\sum_i \frac{1}{L_1} \log q_i = \sum_i \frac{n_i}{L_0} \log q_i = A. \tag{2.5}$$

For systems satisfying both (2.5) *and*

$$\sum_{(i,j)\in D} \frac{1}{L_0} \log p_{ij} = \sum_{(i,j)\in D} \frac{1}{n_i L_1} \log p_{ij} = B \tag{2.6}$$

both the geometric and symbolic packing spectra are equal to $\dim_P \Lambda$ *at*

$$\alpha_0 = -\frac{1}{\log m}(\sigma B + (1-\sigma)A).$$

Consider now a special class of systems. Assume the Bedford–McMullen carpet has only two rows, with n_0 and n_1 rectangles respectively. Let $\mathcal{M}$ be the class of Bernoulli measures with probabilities equidistributed in each row. Denote by p_0 the probability of each rectangle in the first row and by p_1 the probability of each rectangle in the second row; the condition $n_0 p_0 + n_1 p_1 = 1$ must hold with $q_0 = n_0 p_0$ and $q_1 = n_1 p_1$.

THEOREM 2.4. *For* $\mu \in \mathcal{M}$ *and* $\alpha \in [\alpha_m, \alpha_M]$ *we have:*

(1) *If* $\mu \in \mathcal{M}$ *and* $\frac{\log(q_0/q_1)}{\sigma \log(n_0/n_1)} \neq -1$ *then*

$$\dim_P X_\alpha = \dim_P X_\alpha^{\mathrm{symb}} = \dim_H X_\alpha.$$

(2) *If* $\mu \in \mathcal{M}$ *is the unique measure for which* $\frac{\log(q_0/q_1)}{\sigma \log(n_0/n_1)} = -1$ *then for* $\alpha \in (\alpha_m, \alpha_M)$,

$$\dim_H X_\alpha < \dim_P X_\alpha = \dim_P X_\alpha^{\mathrm{symb}},$$

and for $\alpha \in \{\alpha_m, \alpha_M\}$,

$$\dim_H X_\alpha = \dim_P X_\alpha = \dim_P X_\alpha^{\mathrm{symb}}.$$

3. Proof of Theorem 2.3. The main part of the proof is to show that (2.5) is necessary for the geometric and symbolic packing spectra to achieve $\dim_P \Lambda$. To prove this for a fixed Bernoulli measure we construct two sets of dimension strictly smaller than $\dim_P \Lambda$, and then prove that the first set contains all symbolically regular points and that the second set contains all regular points. The second part of the theorem, that satisfying both conditions (2.5) and (2.6) is sufficient for the maximum of the geometric and symbolic packing spectra to achieve $\dim_P \Lambda$, is easy to show.

Let μ be the Bernoulli measure defined by the probability vector $\{p_{ij}\}$ given on the digit set $D \subseteq \{0,\dots,m-1\} \times \{0,\dots,n-1\}$, $m < n$. We assume that D contains at least two different i's and at least two different j's, otherwise the system would be a self-similar IFS on the line. We have

$$\dim_B \Lambda = \dim_P \Lambda = s := \frac{1}{\log m}(\sigma \log L_0 + (1-\sigma)\log L_1),$$

where $\sigma = \log m/\log n$.

Let us start with a simple geometric lemma. For $k_1, k_2 \in \mathbb{N}$ with $K_1 < k_2$ we denote by $F_{k_1}^{k_2}(i,j)(x)$ the frequency of the symbol (a,b) in the sequence $(i,j)_{k_1+1}(x), \ldots, (i,j)_{k_2}(x)$. More precisely,

$$F_{k_1}^{k_2}(i,j)(x) := \frac{\#\{(i_l(x), j_l(x)) = (i,j) : k_2 < l \le k_1\}}{k_2 - k_1}.$$

We extend this notation to the situation when k_1 and k_2 are not integers and $k_2 - k_1 \ge 1$, in which case $F_{k_1}^{k_2}(i,j)(x)$ will denote the frequency of the symbol (i,j) in the sequence $(i,j)_{\lceil k_1\rceil+1}(x), \ldots, (i,j)_{\lceil k_2\rceil}(x)$. For $a > 0$ we let $Z(N,a)$ be the set of points $x \in \Lambda$ such that for any $M > N$ one of the following seven *nongenericity* conditions holds:

(i) $F_0^M(i,j)(x) \notin [1/L_0 - a, 1/L_0 + a]$ for some $(i,j) \in D$,
(ii) $F_M^{\lceil M\sigma^{-1}\rceil}(i,j)(x) \notin [1/L_0 - a, 1/L_0 + a]$ for some $(i,j) \in D$,
(iii) $\sum_j F_{\lceil M\sigma^{-1}\rceil}^{\lceil M\sigma^{-2}\rceil}(x) \notin [1/(n_iL_1) - a, 1/(n_iL_1) + a]$ for some i,
(iv) all the symbols $j_{M+1}(x), \ldots, j_{\lceil M(1+a)\rceil}(x)$ are equal,
(v) all the symbols $i_{\lceil M\sigma^{-1}\rceil+1}(x), \ldots, i_{\lceil M\sigma^{-1}(1+a)\rceil}(x)$ are equal,
(vi) all the symbols $j_{\lceil M\sigma^{-1}\rceil+1}(x), \ldots, j_{\lceil M\sigma^{-1}(1+a)\rceil}(x)$ are equal,
(vii) all the symbols $i_{\lceil M\sigma^{-2}\rceil+1}(x), \ldots, i_{\lceil M\sigma^{-2}(1+a)\rceil}(x)$ are equal.

Let $\widetilde{Z}(N,a)$ be the subset of $Z(N,a)$ consisting of points $x \in \Lambda$ such that for any $M > N$ one of (i), (ii) and (iii) holds.

Lemma 3.1. *For any* $a > 0$,

$$\sup_N \overline{\dim}_B Z(N,a) < s.$$

Proof. Let us begin by calculating the upper box counting dimension of $\widetilde{Z}(N,a)$. The general idea of the proof is that we have approximately $z_M = L_0^{M\sigma^{-1}} L_1^{M(\sigma^{-2}-\sigma^{-1})}$ approximate squares of level $\lceil M\sigma^{-2}\rceil$ but for $N < M$ the subset $\widetilde{Z}(N,a)$ intersects at most $z_M e^{-cMa^2}$ of them.

There are three types of points $x \in \widetilde{Z}(N,a)$. If $F_0^M(i,j)(x) \notin [1/L_0 - a, 1/L_0 + a]$ for some $(i,j) \in D$ then there are only $L_0^{M(1-c_1a^2)}$ possible values of the initial M symbols $(i,j)_k(x)$. Indeed, the entropy of Bernoulli measure with probabilities $p_1, \ldots, p_{L_0}$ is $\log L_0$, achieved when all those probabilities are equal to $1/L_0$, and this maximum is nonflat. Hence, all points of this type can be covered by at most $L_0^{M(1-c_1a^2)} L_0^{\lceil M\sigma^{-1}\rceil - M} L_1^{\lceil M\sigma^{-2}\rceil - \lceil M\sigma^{-1}\rceil} \approx z_M L_0^{-c_1Ma^2}$ approximate squares of level $\lceil M\sigma^{-2}\rceil$.

If $F_M^{\lceil M\sigma^{-1}\rceil}(i,j)(x) \notin [1/L_0 - a, 1/L_0 + a]$ for some $(i,j) \in D$ then there are only $L_0^{(\lceil M\sigma^{-1}\rceil - M)(1-c_2a^2)}$ possible values of the symbols $(i,j)_k(x)$ for $k = M+1, \ldots, \lceil M\sigma^{-1}\rceil$. Hence, by a reasoning similar to the previous one, all points of this type can be covered by at most $z_M L_0^{-c_2a^2M(\sigma^{-1}-1)}$ approximate squares of level $\lceil M\sigma^{-2}\rceil$.

If $\sum_j F_{\lceil M\sigma^{-1}\rceil}^{\lceil M\sigma^{-2}\rceil}(i,j)(x) \notin [1/(n_iL_1) - a, 1/(n_iL_1) + a]$ for some i then there are only $l_1^{(\lceil M\sigma^{-2}\rceil - \lceil M\sigma^{-1}\rceil)(1-c_3a^2)}$ possible values of the symbols i_k for $k = \lceil M\sigma^{-1}\rceil + 1, \ldots, \lceil M\sigma^{-2}\rceil$. Hence, all points of this type can be covered by at most $z_M L_1^{-c_3a^2M(\sigma^{-2}-\sigma^{-1})}$ approximate squares of level $\lceil M\sigma^{-2}\rceil$.

The points in $\widetilde{Z}(N,a)$ might satisfy different nongenericity conditions for different M. However, for each $M > N$ all the points in $\widetilde{Z}(N,a)$ can be covered by at most $z_M e^{-cMa^2}$ approximate squares of level $\lceil M\sigma^{-2}\rceil$. Hence,

$$\overline{\dim}_B \widetilde{Z}(N,a) \le s - \tilde{c}a^2.$$

Consider now the set $Z(N,a)$. Instead of using approximate squares of level $\lceil M\sigma^{-2}\rceil$, we will use squares of level $\lceil M\sigma^{-2}(1+a/4)\rceil$. Note first that if $x \in \Lambda$ satisfies one of the nongenericity conditions (i), (ii), (iii) for M and a, it will also satisfy it for $\lceil M(1+a/4)\rceil$ and $a/2$.

Indeed, if x satisfies (i) then the symbols $((i,j)_k(x))_{k=0}^{\lceil M(1+a/4)\rceil}$ are the same as $((i,j)_k(x))_{k=0}^{M}$ (where frequencies differ from $(1/L_0, \ldots, 1/L_0)$ by at least a) plus approximately $Ma/4$ new symbols $((i,j)_k(x))_{k=M+1}^{\lceil M(1+a/4)\rceil}$ which can change the frequencies at most by $a/4$. If x satisfies either (ii) or (iii) then passing from M to $\lceil M(1+a/4)\rceil$ changes the ranges of k on both ends: some symbols at the beginning drop out, some symbols at the end are added. Altogether the change of frequencies cannot top $a(2+\sigma)/(4(1+\sigma)) < a/2$.

Hence, all the points $x \in \Lambda$ satisfying (i), (ii) or (iii) for given M and a can be covered with at most $z_{M(1+a/4)} m^{-\tilde{c}M\sigma^{-2}(1+a/4)a^2/4}$ sets of diameter $m^{-\lceil M\sigma^{-2}(1+a/4)\rceil}$, as in the first part of proof.

If $x \in \Lambda$ satisfies (iv) then the sequence $((i,j)_k(x))_{k=M+1}^{\lceil M(1+a/4)\rceil}$ can take only $(L_0-1)^{\lceil aM\rceil/4}$ possible values. Hence, the points of this type can be covered by at most $z_{M(1+a/4)}((L_0-1)/L_0)^{Ma/4}$ approximate squares of level $\lceil M\sigma^{-2}(1+a/4)\rceil$.

The cases of (v) and (vi) are almost identical: $((i,j)_k(x))_{\lceil M\sigma^{-1}+1\rceil}^{\lceil M\sigma^{-1}(1+a/4)\rceil}$ can take only $(L_0-1)^{\lceil a\sigma^{-1}M/4\rceil}$ possible values and we can cover those points with at most $z_{M(1+a/4)}((L_0-1)/L_0)^{M\sigma^{-1}a/4}$ approximate squares of level $\lceil M\sigma^{-2}(1+a/4)\rceil$.

Finally, if $x \in \Lambda$ satisfies (vii) then the sequence $\{i_k(x)\}_{k=\lceil M\sigma^{-2}\rceil+1}^{\lceil M\sigma^{-2}(1+a/4)\rceil}$ can take only $(L_1-1)^{\lceil aM\sigma^{-2}/4\rceil}$ possible values. Hence, the points of this type can be covered by at most $z_{M(1+a/4)}((L_1-1)/L_1)^{M\sigma^{-2}a/4}$ approximate squares of level $\lceil M\sigma^{-2}(1+a/4)\rceil$.

As a is small, a^2 is small in comparison to a. Hence, again we get a quadratic bound (independent of N) for the upper box counting dimension of $Z(N,a)$:

$$\overline{\dim}_B Z(N,a) \le s - ca^2. \ \blacksquare$$

We now proceed with the proof of the theorem. Let us consider the symbolic local dimensions first. Assume that (2.5) does not hold and let

$$\delta = \left| \frac{1-\sigma}{3\log m} \sum_i \left(\frac{1}{L_1} - \frac{n_i}{L_0} \right) \log q_i \right|.$$

Let

$$X_{\alpha,N,\delta} = \left\{ x \in \Lambda : \alpha - \delta < \frac{\log \mu(C_{N_0}(x))}{-N_0 \log m} < \alpha + \delta \ \forall N_0 > N \right\}.$$

We have

$$X_{\mathrm{regsymb}} \subseteq \bigcup_\alpha \bigcup_N X_{\alpha,N,\delta}.$$

Let $x \in X_{\alpha,N,\delta}$. Let $a_0 \ge 0$ be the smallest number for which the following are true:

- $1/L_0 - a_0 \le F_0^M(i,j)(x) \le 1/L_0 + a_0$ for all $(i,j) \in D$,
- $1/L_0 - a_0 \le F_M^{M\sigma^{-1}}(i,j)(x) \le 1/L_0 + a_0$ for all $(i,j) \in D$,
- $1/(n_i L_1) - a_0 \le \sum_j F_{M\sigma^{-1}}^{M\sigma^{-2}}(i,j)(x) \le 1/(n_i L_1) + a_0$ for all i.

By (2.3), we have

(3.1)
$$-\frac{\log m}{M\sigma^{-1}} \log \mu(C_{M\sigma^{-1}}(x)) = \sigma \sum_{(i,j)\in D} \frac{1}{L_0} \log p_{i,j} + (1-\sigma) \sum_i \frac{n_i}{L_0} \log q_i + Z_1$$

and

(3.2)
$$-\frac{\log m}{M\sigma^{-2}} \log \mu(C_{M\sigma^{-2}}(x)) = \sigma \sum_{(i,j)\in D} \frac{1}{L_0} \log p_{i,j} + (1-\sigma) \sum_i \frac{1}{L_1} \log q_i + Z_2,$$

where

$$|Z_1|, |Z_2| < a_0 \max\Big(\max_{i,j} |\log p_{i,j}|, n \max_i |\log q_i| \Big) + O(1/M).$$

Denoting $T = \max(\max_{i,j} |\log p_{i,j}|, n \max_i |\log q_i|)$, we see that, as the left hand sides of (3.1) and (3.2) can differ at most by $2\delta \log m$ and the right hand sides differ at least by $3\delta \log m - |Z_1| - |Z_2|$, we must have

$$a_0 > a = \frac{\log m}{2T}\,\delta.$$

Hence,

$$X_{\mathrm{regsymb}} \subseteq \bigcup_N \widetilde{Z}(N, a)$$

and the symbolic part of the assertion follows by Lemma 3.1.

Now consider the regular points for geometric local dimension. Let $x \in X_{\mathrm{reg}}$. There exist α (which can be chosen from a finite set) and $N > 0$ such that for all $M > N$,

$$\frac{\log \mu(B_{m^{-M}}(x))}{-M \log m} \in [\alpha - \delta/2, \alpha + \delta/2].$$

There are two cases: either the relation

$$\frac{\log \mu(C_{\tilde{M}}(x))}{-\tilde{M} \log m} \in [\alpha - \delta, \alpha + \delta]$$

holds for both $\tilde{M} = \lceil M\sigma^{-1} \rceil$ and $\tilde{M} = \lceil M\sigma^{-2} \rceil$, or it does not hold for at least one of those. If it holds for both values of $\tilde{M}$ then x must satisfy the nongenericity condition (i), (ii) or (iii) for M and a. If it does not hold for one of the possible values of $\tilde{M}$ then, necessarily, the measures of $C_{\tilde{M}}(x)$ and $B_{m^{-\tilde{M}}}(x)$ must differ at least by the factor $m^{\tilde{M}\delta/2}$. However, by (2.1), (2.2) and (2.4) this is only possible if either $I_{\tilde{M}}(x)$ or $J_{\tilde{M}}(x)$ is greater than $\tilde{a} = \delta |\log K|/(2 \log m)$. Hence, in this case x must satisfy (iv), (v), (vi) or (vii) for M and $\tilde{a}$. We come to the conclusion that

$$X_{\mathrm{reg}} \subseteq \bigcup_N Z(N, \min(a, \tilde{a}))$$

and we are done by Lemma 3.1.

For the second statement of Theorem 2.3, if both (2.5) and (2.6) hold, all points $x \in \Lambda$ whose symbolic expansions can be divided into parts in which symbols $(i, j) \in D$ appear with frequencies $\{1/L_0, \ldots, 1/L_0\}$ and parts with frequencies $\{1/(n_i L_1)\}$ belong to $X_{\alpha_0}^{\mathrm{symb}}$. If in addition we demand that they do not have long stretches of identical i's or j's then the geometric local dimension at those points is also α_0. It is easy to check that those points have full packing dimension.

4. Proof of Theorem 2.4. The proof is divided into four parts. In the first part we relate the symbolic and geometric local dimensions at any point. We continue by looking at how the local dimensions can be determined by observing the frequency of digits in initial parts of symbolic expansions.

This now naturally splits the argument into two cases which are the final two parts of the proof. The first case corresponds to part (1) of the theorem, and the second to part (2). In fact (2) is about only one measure but it will turn out this is the only case when the Hausdorff and packing spectra are different, and it is by far the most difficult case.

4.1. Geometric and symbolic local dimensions. We start by studying the relationship between the symbolic and geometric local dimensions.

LEMMA 4.1. *If* $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ *and* $d_\mu(x) = \alpha$ *but* $d_\mu^{\mathrm{symb}}(x) \neq \alpha$ *then for the symbolic expansion* (i_k, j_k) *for* x *there exist* $\eta > 0$ *and infinitely many integers* n_j *such that* $i_{n_j} = i_{n_j+k}$ *for all* $0 \le k \le [\eta n_j]$.

Proof. Fix a positive integer N and let $k(N) = \inf\{z : i_N \neq i_{N+z}\}$. Recall that the ball $B_{cm^{-N}}(x)$ contains $C_N(x)$ for $c = n+1$, and if $c = m^{-(k+1)}$ then it is contained in $C_N(x) \cup C_N(x')$ for some x' for which $i_k(x') = i_k(x)$ for all k. Note that either $\mu(C_N(x')) = 0$ or $\mu(C_N(x')) = \mu(C_N(x))$. Hence

$$\mu(B_{(n+1)m^{-N}}(x) \le \mu(C_N(x)) \le 2\mu(B_{(m^{-k+1}m^{-N}}(x)),$$

and if the symbolic and geometric local dimensions are not equal then we cannot have $K(N) = o(N)$, and the result follows. ■

LEMMA 4.2. $X_\alpha \subseteq X_\alpha^{\mathrm{symb}}$.

Proof. Let $x \in X_\alpha$. For any $\varepsilon > 0$ there exists N_0 such that

$$x \in \{x \in \Lambda : \mu(B(x, m^{-N})) \in (m^{-N(\alpha-\varepsilon)}, m^{-N(\alpha-\varepsilon)}) \text{ for all } N \ge N_0\}.$$

We will denote this set by $Y_{\alpha,N_0,\varepsilon}$. By Lemma 4.1 there exist $\delta > 0$ and infinitely many $N \ge N_0$ such that $K(N)/N \ge \delta$. We will fix such an N and assume without loss of generality that $i_N = 1$. For $c > n$ and $M \in [N, N(1+\delta)]$ the ball $B_{cm^{-M}}(x)$ will contain both the approximate square $C_M(x)$ and the approximate square $C_M(y_N)$, where y_N is the point with the same symbolic expansion as x except $i_k(y) = 1 - i_k(x)$ for $k \in [N, N(1+\delta)]$. Similarly, for c small the ball $B_{cm^{-M}}(x)$ will be contained in $C_M(x) \cup C_M(y_N) \cup C_M(x') \cup C_M(y'_N)$. Hence, as $x \in Y_{\alpha,N_0,\varepsilon}$ for all $M \in [N, N(1+\delta)]$ we have

$$\tfrac{1}{4} m^{-M(\alpha+2\varepsilon)} \le \max\big((\mu(C_M(x)), \mu(C_M(y_N))\big) \le m^{-M(\alpha-2\varepsilon)}.$$

We can also assume that for at least some N and M this maximum is not $\mu(C_M(x))$, otherwise the symbolic and geometric local dimensions at x would be equal. There are now several cases to consider:

CASE I: $q_0 = q_1$, $\delta \le \sigma^{-1}$. In this situation $\mu(C_M(x)) = \mu(C_M(y_N))$ for all $M \in [N, N(1+\delta)]$, because the measure of an approximate square of level M only depends on the initial $M\sigma$ symbols from the symbolic expansion, which are the same for x and for y_N.

CASE II: $q_0 = q_1$, $\delta > \sigma^{-1}$. Then $\mu(C_M(x)) = \mu(C_M(y_N))$ for all $M \in [N, N\sigma^{-1}]$ as in Case I, but for $M \in [N\sigma^{-1}, N(1+\delta)]$ we have

$$\frac{\mu(C_M(y_N))}{\mu(C_M(x))} = \left(\frac{p_1}{p_0}\right)^{\sigma M - N - 1}.$$

As $\mu(C_M(y_N)) > \mu(C_M(x))$ for at least some M, we must have $p_1 > p_0$. Hence, $-(1/M)\log(\mu(C_M(y_N))/\mu(C_M(x)))$ is an increasing function. By our assumptions about x we may assume that for all $M \in [N, N(1+\delta)]$,

$$\mu(C_M(y_N)) \in \left(\tfrac{1}{4}m^{-M(\alpha+2\varepsilon)}, m^{-M(\alpha-2\varepsilon)}\right)$$

and there exists $M \in [N, N(1+\delta)]$ such that

$$\mu(C_M(x)) < \tfrac{1}{4}m^{-M(\alpha+2\varepsilon)}.$$

Since $-(1/M)\log(\mu(C_M(y_N))/\mu(C_M(x)))$ is increasing in M, we have

$$\mu(C_{N(1+\delta)}(x)) < m^{-N(1+\delta)(\alpha+2\varepsilon)}.$$

At the same time, our assumptions give us

$$\mu(C_{N(1+\delta)}(x)) \geq m^{-N(1+\delta)(\alpha+2\varepsilon)},$$

which is a contradiction.

CASE III: $q_0 \neq q_1$, $\delta \leq \sigma^{-1}$. For all $M \in [N, N(1+\delta)]$ we have

$$\frac{\mu(C_M(y_N))}{\mu(C_M(x))} = \left(\frac{q_1}{q_0}\right)^{M - N - 1}.$$

As in Case II, $-(1/M)\log(\mu(C_M(y_N))/\mu(C_M(x)))$ is an increasing function, so for all $M \in [N, N(1+\delta)]$ we must have

$$\mu(C_M(y_N)) \in \left(\tfrac{1}{4}m^{-M(\alpha+2\varepsilon)}, m^{-M(\alpha-2\varepsilon)}\right);$$

but also

$$\mu(C_{N(1+\delta)}(x)) \approx m^{-N(1+\delta)\alpha},$$

which leads to a similar contradiction.

CASE IV: $q_0 \neq q_1$, $\delta > \sigma^{-1}$. Now $-(1/M)\log(\mu(C_M(y_N))/\mu(C_M(x)))$ is an increasing function for $M \in [N, N\sigma^{-1}]$ but might be decreasing for $M \in [N\sigma^{-1}, N(1+\delta)]$. Hence, if δ satisfies

$$m^{-2N(1+\delta)\varepsilon} \leq \left(\frac{q_1}{q_0}\right)^{N(1+\delta)(1-\sigma)} \left(\frac{p_1}{p_0}\right)^{N((\delta+1)\sigma-1)} \leq m^{2N(1+\delta)\varepsilon} \tag{4.1}$$

then it is possible that

$$\frac{1}{N(1+\delta)\log m}|\log \mu(C_{N(1+\delta)}(x)) - \log \mu(C_{N(1+\delta)}(y_N))| < 2\varepsilon$$

and the contradiction as in Cases II, III does not happen. However, note that

$$\mu(C_{N(1+\delta)}(y_N)) = \mu(C_{N\sigma^{-1}}(y_N))q_1^{N(1+\delta-\sigma^{-1})}p_1^{N((1+\delta)\sigma-1)}.$$

As

$$\frac{1}{N(1+\delta)} \log \mu(C_{N(1+\delta)}(y_N)) \in \big((-\alpha - 2\varepsilon) \log m, (-\alpha + 2\varepsilon) \log m\big)$$

and

$$\frac{1}{N\sigma^{-1}} \log \mu(C_{N\sigma^{-1}}(y_N)) \in \big((-\alpha - 2\varepsilon) \log m, (-\alpha + 2\varepsilon) \log m\big),$$

we must have

$$-\alpha \log m = \sigma \log p_1 + (1-\sigma) \log q_1,$$

which implies $\alpha = \alpha_{\min}$.

Consider now $M = N(1+\delta)\sigma^{-1}$. In the symbolic expansion of x there are at least $N\delta$ zeros on the first $N(1+\delta)$ places. Hence, the measure of any approximate square of level $N(1+\delta)\sigma^{-1}$ that can intersect the ball $B_{m^{-N\delta\sigma^{-1}}}(x)$ (the symbolic descriptions of those squares share the first $N(1+\delta)$ symbols with x) is at most

$$p_1^N p_0^{N\delta} q_0^{N(1+\delta)(\sigma^{-1}-1)} = m^{-\alpha N(1+\delta)\sigma^{-1}} \left(\frac{p_0}{p_1}\right)^{N\delta} \left(\frac{q_0}{q_1}\right)^{N(1+\delta)(\sigma^{-1}-1)}$$

(remember that $q_1 < q_0$ and $p_1 > p_0$ by (4.1)). By (4.1),

$$\left(\frac{p_0}{p_1}\right)^{N\delta} \left(\frac{q_0}{q_1}\right)^{N(1+\delta)(\sigma^{-1}-1)} \le m^{2N(1+\delta)\sigma^{-1}\varepsilon} \left(\frac{p_0}{p_1}\right)^{N(\sigma^{-1}-1)} < m^{-N(1+\delta)\sigma^{-1}\varepsilon}.$$

Hence, x cannot belong to $Y_{\alpha,N_0,\varepsilon}$. ■

4.2. Local dimensions and frequencies. We now look at the relationship between frequencies of digits and local dimensions. To do so, we need to introduce some notation adapted to this setting. Recall that in this case we have two rows, and μ is a Bernoulli measure giving weight q_0/n_0 to each rectangle in the first row and q_1/n_1 to each rectangle in the second row. Using this for $P \in (0,1)$ we define the following quantities:

$$\begin{aligned} H(P) &= -P \log P - (1-P) \log(1-P), \\ H_q(P) &= -P \log q_0 - (1-P) \log q_1, \\ CH(P) &= P \log(n_0^{-1} P) - (1-P) \log(n_1^{-1} P), \\ CH_q(P) &= P \log(n_0^{-1} q_0) - P \log(n_1^{-1} q_1). \end{aligned}$$

It will also be convenient to let $G_k = \sum_{j=0}^{n-1} F_0^{k-1}(0,j)(x)$.

Let $\mu \in \mathcal{M}$, $\alpha \in [\alpha_m, \alpha_M]$ and $x \in X_\alpha^{\mathrm{symb}}$. For any positive ε we have

$$X_\alpha^{\mathrm{symb}} \subseteq \bigcup_{N_0} \bigcap_{N > N_0} X_{\alpha,N,\varepsilon}. \tag{4.2}$$

We fix some small positive ε (in the future we will determine how small it should be). We assume that $x \in \bigcap_{N>N_0} X_{\alpha,N,\varepsilon}$ for some fixed N_0. Let

$$\beta(x) = \lim_{k\to\infty} G_k(x)$$

be the frequency of appearance of the 0 row in the symbolic expansion of x (if it exists). We will also use the finite approximations

$$P_k = G_{M\sigma^{2-k}-1}(x) \quad \text{and} \quad P'_k = \sum_{j=0}^{n-1} F_{M\sigma^{2-k}}^{M\sigma^{1-k}-1}(0,j)(x),$$

where $M > N_0$ is fixed, to be defined later.

Our first step is the standard calculation:

LEMMA 4.3. *If $\beta(x) = \beta$ then*

$$d_\mu^{\mathrm{symb}}(x) = \alpha(\beta) := \frac{1}{\log m}\big(\sigma C H_q(\beta) + (1-\sigma)H_q(\beta)\big).$$

Our approach is to relate the local dimension at a point x to $\beta(x)$ and use the following lemma.

LEMMA 4.4 (Gui–Li). *For all $\beta \in [0,1]$ we have*

$$\begin{aligned}\dim_P\{x : \beta(x) = \beta\} &= \dim_H\{x : \beta(x) = \beta\} \\ &= \frac{1}{\log m}\big(\sigma H_q(\beta) + (1-\sigma)H_q(\beta)\big).\end{aligned}$$

Proof. For $\beta \in (0,1)$ this follows from [GL, Theorem 1.1]. For $\beta = 0$ or 1 it is a simple exercise. ∎

In certain special cases it is straightforward to show that $\{x : \beta(x) = \beta\} = X_{\alpha_\beta}^{\mathrm{symb}}$.

LEMMA 4.5.

(1) *If $n_0 = n_1$ and $q_0 \neq q_1$ then $\{x : \beta(x) = \beta\} = X_{\alpha(\beta)}^{\mathrm{symb}}$.*
(2) *If $q_0 = q_1$ and $n_0 \neq n_1$ then $\{x : \beta(x) = \beta\} = X_{\alpha(\beta)}^{\mathrm{symb}}$.*
(3) *If $q_0 = q_1$ and $n_0 = n_1$ then $\alpha_m = \alpha_M$ and $X_{\alpha_m}^{\mathrm{symb}} = \Lambda$.*

Proof. For the first part, fix $x \in \Lambda$ and $k \in \mathbb{N}$. We calculate

$$-\log \mu(C_k(x)) = kH_q(G_k(x)) + \sigma_k \log n_0 + o(k),$$

and we can see that $d_\mu^{\mathrm{symb}} = \alpha(\beta)$ if and only if $\lim_{k\to\infty} G_k(x) = \beta$.

For the second part, if $q_0 = q_1 = 1/2$ then

$$-\log \mu(C_k(x)) = k\log 2 + k\sigma\big(G_{\sigma_k}\log n_0 + (1-G_{\sigma k})\log n_1\big) + o(k).$$

Again we clearly have $d_\mu^{\mathrm{symb}}(x) = \alpha(\beta)$ if and only if $\lim_{k\to\infty} G_k(x) = \beta$.

Finally for the third part, if $n_0 = n+1$ then

$$-\log \mu(C_k(x)) = k\log 2 + k\sigma \log n_0 + o(k)$$

and
$$d_\mu^{\mathrm{symb}}(x) = \frac{\log 2 + \sigma \log n_0}{\log m} = \dim \Lambda = \alpha_m = \alpha_M. \ \blacksquare$$

Thus in the rest of the proof of Theorem 2.4 we assume that $n_0 \neq n_1$ and $q_0 \neq q_1$, and define
$$A = \frac{\log(q_0/q_1)}{\sigma \log(n_0/n_1)}.$$
In the case of $|A| \neq 1$ we will again be able to show that $x \in X_\alpha^{\mathrm{symb}}$ uniquely determines $\beta(x)$. If $A = 1$ then the set X_α^{symb} contains not only points with a fixed (nonunique) $\beta(x)$ but also some additional points for which $\beta(x)$ does not exist. However, we will prove that this does not lead to an increase of either Hausdorff or packing dimension. Finally, if $A = -1$ then the set X_α^{symb} also contains some additional points with no $\beta(x)$. In this case, which is covered in part (2) of Theorem 2.4, this leads to an increase of packing dimension, though not of Hausdorff dimension.

LEMMA 4.6. *If $n_0 \neq n_1$ and $q_0 \neq q_1$ then*
$$P_{k+1} = F(P_k) + O(\varepsilon) + O(\sigma^k/M), \tag{4.3}$$
where
$$F(P_k) = A^{-1} P_k + B$$
and
$$B = -\frac{1}{\log(q_0/q_1)}(\alpha \log m + \sigma \log p_1 + (1-\sigma)\log q_1).$$

Proof. Given $x \in \Lambda$ and $M \in \mathbb{N}$, the measure $\mu(C_{\sigma^{1-k}M}(x))$ is precisely determined by $\{i_k(x)\}_{k=0}^{\lfloor \sigma^{1-k}M \rfloor - 1}$.

We have
$$-\frac{1}{\sigma^{1-k}M} \log \mu(C_{\sigma^{1-k}M}(x)) = \sigma(CH_q(P_k)) + (1-\sigma)H_q(P_k') + o(1). \tag{4.4}$$

For $x \in X_{\alpha, M\sigma^{1-k}, \varepsilon}$ we have
$$\frac{1}{\sigma^{1-k}M} \log \mu(C_{\sigma^{1-k}M}(x)) \in [-(\alpha+\varepsilon)\log m, -(\alpha-\varepsilon)\log m],$$
hence (4.4) lets us obtain a relation between P_k and P_k'. Applying the obvious relation
$$P_{k+1} = \sigma P_k + (1-\sigma)P_k' + O(\sigma^k/M)$$
we get

(4.5)
$$\log \frac{q_0}{q_1} P_{k+1} = \log \frac{n_0}{n_1} P_k - \alpha \log m - \sigma \log p_1 - (1-\sigma)\log q_1 + O(\varepsilon) + O(\sigma^k/M),$$
and the assertion follows. $\blacksquare$

We will denote by P the fixed point of F and by μ_P the Bernoulli measure given by the probability vector $\tilde{p}_{0j} = P/n_0$, $\tilde{p}_{1j} = (1-P)/n_1$. We note that μ_P is the Bernoulli measure constructed by King [K] and

$$\dim_H X_\alpha = \dim_H X_\alpha^{\mathrm{symb}} = \dim_H \mu_P.$$

The map F is very simple, we only need to consider several cases depending on the value of A. Note that A only depends on μ and α, not on M, N or ε. Hence, we can make our choice of ε only at this moment (this will matter in the proof of the following lemma).

4.3. Proof of part (1) of Theorem 2.4. Part (1) of Theorem 2.4 corresponds to the case when $A \neq -1$. We will start with the simple case $|A| \neq 1$.

LEMMA 4.7. *If* $|A| \neq 1$ *then*

$$X_\alpha^{\mathrm{symb}} = \{x : \beta(x) = P\}.$$

Proof. It follows from Lemma 4.3 that $X_\alpha^{\mathrm{symb}} \supseteq \{x : \beta(x) = P\}$. To obtain the other direction, we will consider two cases.

CASE I: $|A| > 1$. In this situation we choose $\varepsilon > 0$ to be small relative to $|A^{-1}|$ and note that the map F is contracting to the fixed point P. This also means that the real frequencies P_k will converge to the region $[P-c\varepsilon, P+c\varepsilon]$ and then stay there. As ε can be chosen arbitrarily small, the assertion follows.

CASE II: $|A| < 1$. Then we choose a small ε and the map $P_k \to P_{k+1}$ is diverging (except in some $c\varepsilon$-neighbourhood of P, where the error term can dominate the divergence of F). If P_k does not belong to $[P-c\varepsilon, P+c\varepsilon]$ then P_{k+1} will be even further from P and so on. However, for any point x all the frequencies P_k must belong to $[0,1]$, hence they must indeed be in some $c\varepsilon$-neighborhood of P for all $k > 0$. As in the first case, ε can be chosen arbitrarily small, and the assertion follows. ∎

We now see that by Lemmas 4.4 and 4.2,

$$\dim_H X_\alpha \le \dim_P X_\alpha \le \dim_P X_\alpha^{\mathrm{symb}} = \dim_H X_\alpha^{\mathrm{symb}}.$$

The assertion of the theorem when $|A| \neq 1$ now follows since by [JR] we know that $\dim_H X_\alpha = \dim_H X_\alpha^{\mathrm{symb}}$.

We are left with the case when $A = 1$. In this case (and in the case $A = -1$, treated in the following subsection), the symbolic local dimension of x does not determine $\beta(x)$. However when $A = 1$ we have the following statement:

LEMMA 4.8. *If* $A = 1$ *then* $\alpha_m = \alpha_M$ *and*

$$\dim_P X_{\alpha_m}^{\mathrm{symb}} = \dim_H X_{\alpha_m}^{\mathrm{symb}} = \dim_H \Lambda. \tag{4.6}$$

Proof. It is clear that $\alpha_m = \alpha_M$ and that μ is the measure of maximal dimension. We let $\alpha = \alpha_m$. Then for all $x \in X_{\alpha,N,\varepsilon}$ we have

$$|P_{k+1} - P_k| \le O(\varepsilon) + O(\sigma^k/M).$$

Hence, a drift is possible: for any $\tilde{M} > N$ we might not know the frequency $Q = \sum_j F_0^{\tilde{M}\sigma^{-1}-1}(0,j)(x)$. Still, the set $X_{\alpha,N,\varepsilon}$ can be covered by

$$\sup_{Q\in[0,1]} \exp(\tilde{M}CH(Q))\exp((\sigma^{-1}-1)\tilde{M}H(Q))\exp(\tilde{M}O(\varepsilon))$$

approximate squares of level $\tilde{M}\sigma^{-1}$, and hence

$$\begin{aligned}\overline{\dim}_B X_{\alpha,N,\varepsilon} &\le \sup_{Q\in[0,1]} \frac{1}{\log m}\big(\sigma CH(Q) + (1-\sigma)H(Q)\big) + O(\varepsilon)\\ &= \dim_H \Lambda + O(\varepsilon).\end{aligned}$$

As $\dim_H X_\alpha^{\mathrm{symb}} = \dim_H \Lambda$, formula (4.6) follows. ∎

The proof of part (1) of Theorem 2.4 can now be completed since by Lemma 4.2 we have

$$\dim_H X_{\alpha_m} \le \dim_P X_{\alpha_m} = \dim_H X_{\alpha_m} = \dim_H \Lambda$$

and we know that $\dim_H X_{\alpha_m} = \dim_H \Lambda$.

4.4. Proof of part (2) of Theorem 2.4. Finally we consider the most interesting case when $A = -1$ which corresponds to Theorem 2.4(2). Our goal is to prove the following theorem from which Theorem 2.4(2) follows.

THEOREM 4.9. *Assume $A = -1$. If $\alpha \in (\alpha_m, \alpha_M)$ then*

$$\dim_H X_\alpha^{\mathrm{symb}} < \dim_P X_\alpha^{\mathrm{symb}} = \dim_P X_\alpha.$$

If $\alpha \in \{\alpha_m, \alpha_M\}$ then

$$\dim_H X_\alpha^{\mathrm{symb}} = \dim_P X_\alpha^{\mathrm{symb}} \le \dim_H X_\alpha.$$

The proof will be split into several parts. The map F has only one fixed point P, but $F^2 \equiv \mathrm{id}$. As in the case $A = 1$, a drift is possible, so we do not know the frequencies $Q_1 = G_{\tilde{M}}(x)$ or $Q_2 = G_{\tilde{M}\sigma^{-1}}(x)$, but we know that $Q_1 + Q_2 = 2P + O(\varepsilon)$. Once again one can calculate

$$\overline{\dim}_B X_{\alpha,N,\varepsilon} \le \sup_\rho \frac{1}{\log m}\big(\sigma CH(P+\rho) + (1-\sigma)H(P-\rho)\big) + O(\varepsilon) \tag{4.7}$$

where $\delta = \rho(1+\sigma)/(1-\sigma)$ and the supremum is taken over ρ such that $P \pm \delta \in [0,1]$.

The first thing to note is that if $P \in \{0,1\}$ (which corresponds to the local dimensions $\alpha_{\min}$ and $\alpha_{\max}$) then $\rho = 0$ is the only admissible choice. In these cases we have equality in (4.7) and we have the following simple result.

LEMMA 4.10. *For $A = -1$ and $\alpha \in \{\alpha_m, \alpha_M\}$ we have*

$$\dim_P X_\alpha^{\mathrm{symb}} = \dim_H X_\alpha^{\mathrm{symb}} \le \dim_H X_\alpha.$$

Proof. Let $\alpha \in \{\alpha_m, \alpha_M\}$ and thus $P \in \{0,1\}$. Hence in inequality (4.7) the only choice is $\rho = 0$, and so

$$\overline{\dim}_B X_{\alpha,N,\varepsilon} \le \sup_\rho \frac{\sigma CH(P)}{\log m} + O(\varepsilon),$$

which means that

$$\dim_P X_{\alpha^{\mathrm{symb}}} \le \frac{\sigma CH(P)}{\log m}.$$

To complete the proof we need to show that $\dim_H X_{\alpha^{\mathrm{symb}}} \ge \frac{\sigma CH(P)}{\log m}$ and $\dim_H X_\alpha \ge \frac{\sigma CH(P)}{\log m}$. To do this we observe that either $\Lambda \cap \{(0,y) : y \in \mathbb{R}\} \subseteq X_{\alpha^{\mathrm{symb}}} \cup X_\alpha$ or $\Lambda \cap \{(1,y) : y \in \mathbb{R}\} \subseteq X_{\alpha^{\mathrm{symb}}} \cup X_\alpha$. We can then easily calculate

$$\dim_H \Lambda \cap \{(0,y) : y \in \mathbb{R}\} = \frac{\log n_0}{\log n} = \frac{\sigma CH(1)}{\log m},$$

$$\dim_H \Lambda \cap \{(1,y) : y \in \mathbb{R}\} = \frac{\log n_1}{\log n} = \frac{\sigma CH(0)}{\log m}. \blacksquare$$

However, in the interior of the spectrum the packing and the Hausdorff symbolic spectra are different.

DEFINITION 4.11. Given $\delta, K_0 > 0$ let W_{α,δ,K_0} be the set of points with the following properties:

- for K even and for any $a, b \in [\sigma^{-K}, \sigma^{-K+1} - 1]$ we have

$$\Big|\sum_{j=0}^{n-1} F_a^{b-1}(0,j)(x) - P - \delta\Big| < \frac{K_0}{b-a},$$

- for K odd and for any $a, b \in [\sigma^{-K}, \sigma^{-K+1} - 1]$ we have

$$\Big|\sum_{j=0}^{n-1} F_a^{b-1}(0,j)(x) - P + \delta\Big| < \frac{K_0}{b-a}.$$

If $x \in W_{\alpha,\delta,K_0}$ then whenever for some large M,

$$G_{M-1}(x) = P + \delta',$$

there exists $\delta'' = \delta' + O(1/M)$ such that for all $K \in \mathbb{N}$,

$$G_{M\sigma^{-K}-1}(x) = P + (-1)^K \delta'' + O(\sigma^K/M).$$

Obviously, $|\delta''| \le \delta/(1+\sigma)$. It follows that each of the sets W_{α,δ,K_0} is contained in X_α^{symb}.

Our nearest goal is to calculate the upper box counting dimension of W_{α,δ,K_0}. We will need the following simple lemma.

LEMMA 4.12. *Let M be large. Let $W \subseteq \{0,1\}^M$ be the set of words ω such that for all subwords $(\omega_a, \dots, \omega_{b-1})$,*

$$|F_a^{b-1}(0) - P| < \frac{K_0}{b-a}.$$

Then

$$\frac{1}{M}\log|W| \geq H(P) - O\left(\frac{\log K_0}{K_0}\right) - O\left(\frac{K_0}{M}\right).$$

Proof. Consider the set of sequences such that for all $0 \leq i < 2M/K_0$,

$$|F_{iK_0/2}^{(i+1)K_0/2-1}(0) - PK_0/2| \leq 1.$$

Moreover, if $PK_0/2$ is not an integer, choose the blocks for which

$$F_{iK_0/2}^{(i+1)K_0/2-1}(0) - PK_0/2 < 0$$

and blocks for which

$$F_{iK_0/2}^{(i+1)K_0/2-1}(0) - PK_0/2 > 0$$

in such a way that for any i, j,

$$|F_{iK_0/2}^{jK_0/2-1}(0) - PK_0/2| \leq 1$$

(the order in which the blocks with frequency higher than the target and blocks with frequency lower than the target are positioned forms a Rauzy sequence).

Let $\tilde{W} \subseteq \{0,1\}^M$ be the set we just defined. Clearly, $\tilde{W} \subseteq W$. By Stirling's formula we can estimate

$$\log|\tilde{W}| \geq \left\lfloor \frac{2M}{K_0} \right\rfloor \log\binom{K_0/2}{PK_0/2} = 2MH(P) + O\left(\frac{M\log K_0}{K_0}\right) + O(K_0). \ ■$$

Let $Y(\delta) = \sup_{0\leq\gamma\leq 2} \tilde{Y}(\gamma,\delta)$ where

$$\begin{aligned}(4.8)\qquad \tilde{Y}(\gamma,\delta) = {} & H(P) + \sigma\big(P\log n_0 + (1-P)\log n_1\big) \\ & + (1-\sigma^{\gamma-\lfloor\gamma\rfloor})\Delta((-1)^{\lfloor\gamma\rfloor}) \\ & + \frac{1}{1+\sigma}\big(\sigma^{-2\lfloor\gamma/2\rfloor}\Delta(-1) + \sigma^{-2\lfloor(\gamma-1)/2\rfloor+1}\Delta(+1)\big) \\ & + (-1)^{\lfloor\gamma\rfloor}\left(\frac{2\sigma^{\gamma-\lfloor\gamma\rfloor}}{1+\sigma} - 1\right)\delta\log\frac{n_0}{n_1}\end{aligned}$$

and

$$\Delta(s) := H(P+s\delta) - H(P).$$

Note that the first line of (4.8) has an important geometric meaning:

$$H(P) + \sigma(P\log n_0 + (1-P)\log n_1) = \log m \cdot \dim_H X_\alpha.$$

LEMMA 4.13.

$$\lim_{K_0\to\infty} \overline{\dim}_B W_{\alpha,\delta,K_0} = Y(\delta)/\log m.$$

Proof. We will estimate the number Z_r of approximate squares of level σ^{-r} necessary to cover W_{α,δ,K_0}. Here r is not necessarily an integer.

Let

$$Q(r) := \sum_{j=0}^{n-1} F_0^{\sigma^{-r+1}-1}(0,j).$$

We calculate

$$\begin{aligned} Q(r) &= (\sigma^{1-r} - \sigma^{1-\lfloor r\rfloor})(P + (-1)^{\lfloor r\rfloor -1}\delta) \\ &\quad + \sum_{\ell=1}^{\lfloor r\rfloor -1} (\sigma^{-\ell} - \sigma^{1-\ell})(P + (-1)^{1-\ell}\delta) + O(K_0 r). \end{aligned}$$

The geometric series is easy to sum, and we get

$$\sigma^{r-1}Q(r) = P + (-1)^{\lfloor r\rfloor}\left(\frac{2\sigma^{r-\lfloor r\rfloor}}{1+\sigma} - 1\right)\delta + O(\sigma^r K_0). \tag{4.9}$$

We can now write a formula for Z_r, using Lemma 4.12:

$$\begin{aligned} \log Z_r &= (\sigma^{-r} - \sigma^{-\lfloor r\rfloor})H(P + (-1)^{\lfloor r\rfloor}\delta) \\ &\quad + \sum_{\ell=1}^{\lfloor r\rfloor}(\sigma^{-\ell} - \sigma^{1-\ell})H(P + (-1)^{1-\ell}\delta) + Q(r)\log n_0 \\ &\quad + (\sigma^{1-r} - Q(r))\log n_1 + O\left(\frac{r\log K_0}{K_0}\right) + O(K_0). \end{aligned} \tag{4.10}$$

We sum the geometric series and substitute (4.9), obtaining the simple formula

$$\begin{aligned} \log Z_r &= \sigma^{-r}H(P) + \sigma^{1-r}\big(P\log n_0 + (1-P)\log n_1\big) \\ &\quad + (\sigma^{-r} - \sigma^{-\lfloor r\rfloor})\Delta((-1)^{\lfloor r\rfloor}) \\ &\quad + \frac{1}{1+\sigma}\big(\sigma^{-2\lfloor r/2\rfloor}\Delta(-1) + \sigma^{-2\lfloor (r-1)/2\rfloor+1}\Delta(+1)\big) \\ &\quad + (-1)^{\lfloor r\rfloor}\left(\frac{2\sigma^{r-\lfloor r\rfloor}}{1+\sigma} - 1\right)\delta\log\frac{n_0}{n_1} + O\left(\frac{r\log K_0}{K_0}\right) + O(K_0). \end{aligned}$$

We want to calculate the upper limit

$$\overline{\dim}_B W_{\alpha,\delta,K_0} = \limsup \sigma^r \frac{\log Z_r}{\log m}.$$

Note that the sequence $\lim_{k\to\infty} \sigma^{\gamma+2k} \log Z_{\gamma+2k}$ converges:

$$
\begin{aligned}
(4.11)\qquad &\lim_{k\to\infty} \sigma^{\gamma+2k} \log Z_{\gamma+2k} \\
&= H(P) + \sigma\big(P\log n_0 + (1-P)\log n_1\big) + (1-\sigma^{\gamma-\lfloor\gamma\rfloor})\Delta((-1)^{\lfloor\gamma\rfloor}) \\
&\quad + \frac{1}{1+\sigma}\big(\sigma^{-2\lfloor\gamma/2\rfloor}\Delta(-1) + \sigma^{-2\lfloor(\gamma-1)/2\rfloor+1}\Delta(+1)\big) \\
&\quad + (-1)^{\lfloor\gamma\rfloor}\left(\frac{2\sigma^{\gamma-\lfloor\gamma\rfloor}}{1+\sigma} - 1\right)\delta\log\frac{n_0}{n_1}.
\end{aligned}
$$

As the right hand side of (4.11) is exactly the function we denoted as $\tilde{Y}(\gamma,\delta)$, the assertion follows. ■

We do not need the exact formulation of Lemma 4.13, but only the following corollary:

COROLLARY 4.14. *For $\alpha \notin \{\alpha_m, \alpha_M\}$ and sufficiently small $\delta > 0$,*

$$\frac{1}{\log m} Y(\delta) > \dim_H X_\alpha.$$

Proof. Define

$$\tilde{Z}(\gamma,\delta) = \tilde{Y}(\gamma,\delta) - \log m \cdot \dim_H X_\alpha.$$

We can write the approximate form for $(\partial/\partial\delta)\tilde{Z}(\gamma,\delta)$:

$$
\begin{aligned}
(4.12)\qquad \frac{\partial}{\partial\delta}\tilde{Z}(\gamma,\delta) &= O(\delta) + (-1)^{\lfloor\gamma\rfloor}\Bigg(H'(P) - \log\frac{n_0}{n_1} \\
&\quad + \sigma^{\gamma-\lfloor\gamma\rfloor}\left(-H'(P) + \frac{2}{1+\sigma}\log\frac{n_0}{n_1}\right) + \sigma^{\lfloor\gamma\rfloor-\gamma}\frac{1-\sigma}{1+\sigma}H'(P)\Bigg).
\end{aligned}
$$

As $\tilde{Z}(\gamma,0) \equiv 0$ and

$$\frac{\partial}{\partial\delta}\tilde{Z}(\gamma,\delta) + \frac{\partial}{\partial\delta}\tilde{Z}(\gamma+1,\delta) = O(\delta),$$

the only possibility for $\sup_\gamma \tilde{Z}(\gamma,\delta)$ to stay nonpositive for small δ is that the function

$$H'(P) - \log\frac{n_0}{n_1} + \sigma^{\gamma}\left(-H'(P) + \frac{2}{1+\sigma}\log\frac{n_0}{n_1}\right) + \sigma^{-\gamma}\frac{1-\sigma}{1+\sigma}H'(P)$$

is equal to 0 for all $\gamma \in [0,1]$. However, this function is a linear combination of the functions 1, σ^γ and $\sigma^{-\gamma}$, which are linearly independent. Hence, all the coefficients must be equal to 0. In particular, $H'(P) = 0$ and $H'(P) - \log(n_0/n_1) = 0$. However, this implies $n_0 = n_1$, which is a contradiction. ■

The last task in proving Theorem 4.9 is to combine the following proposition with Lemma 4.2.

PROPOSITION 4.15.

$$\dim_P X_\alpha^{\mathrm{symb}} \le \max_{\delta\le\min(P,1-P)} \frac{Y(\delta)}{\log m}, \qquad \dim_P X_\alpha \ge \max_{\delta\le\min(P,1-P)} \frac{Y(\delta)}{\log m}.$$

Proof. Let us start with the lower bound.

Consider any $\delta < \min(P, 1-P)$. Let $\gamma_0(\delta)$ be such that

$$Y(\delta) = \tilde{Y}(\gamma_0, \delta).$$

Consider the measure ν_δ on $D^{\mathbb{N}}$ defined as follows. For any even K, for all $\ell \in [\sigma^{-K}, \sigma^{-K-1})$ we choose $i_\ell = 0$ with probability $P+\delta$ and $i_\ell = 1$ with probability $1-P-\delta_0$, independently. For any odd K, for all $\ell \in [\sigma^{-K}, \sigma^{-K-1})$ we choose $i_\ell = 0$ with probability $P-\delta_0$ and $i_\ell = 1$ with probability $1-P+\delta_0$. Whichever the choice of i_ℓ, we choose all the possible j_ℓ, $(i_\ell, j_\ell) \in D$, with the same probability $1/n_{i_\ell}$.

We will also use the projection of ν_δ onto Λ, still denoted by ν_δ.

Let us begin with the observation that for every $\varepsilon > 0$ and for ν_δ-almost every x there are only finitely many N such that all i_ℓ, $\ell = N, N+1, \ldots, N(1+\varepsilon)$, are equal. Hence,

$$\overline{d}_{\nu_\delta}(x) = \limsup_{\ell\to\infty} \frac{\log \nu_\delta(C_\ell(x))}{-\ell \log m} = \overline{\delta}_{\nu_\delta}(x) \tag{4.13}$$

ν_δ-almost everywhere.

To obtain the lower bound, we need the following two lemmas.

LEMMA 4.16. $\nu_\delta(X_\alpha) = 1$.

Proof. By (4.13), a ν_δ-typical point is in X_α if and only if it is in X_α^{symb}. Recall that a sufficient condition for $x \in X_\alpha^{\mathrm{symb}}$ is that there exists a function $\varepsilon_N \to 0$ such that for all except finitely many N,

$$\left| \frac{1}{\sigma N} \sum_{j=0}^{n-1} F_0^{\sigma N-1}(0,j)(x) + \frac{1}{N} \sum_{j=0}^{n-1} F_0^{N-1}(0,j)(x) - 2P \right| < \varepsilon_N. \tag{4.14}$$

Standard large deviation estimates show that for ν_δ-typical x, (4.14) is satisfied for all except finitely many N if we choose $\varepsilon_N = N^{-1/3}$. We skip the details. ∎

LEMMA 4.17. *For ν_δ-almost every x,*

$$\overline{d}_{\nu_\delta}(x) \ge \frac{1}{\log m} Y(\delta).$$

Proof. We just need to find a sequence $\ell \to \infty$ for which the limit in (4.13) would be no smaller than $Y(\delta)/\log m$. The right sequence is

$$\ell_K = \sigma^{a_K} \quad \text{where} \quad a_K = 2K + \gamma_0(\delta).$$

For ν_δ-typical x, for K large enough we have

$$\begin{aligned}\log \nu_\delta(C_{\ell_K}(x)) = &-(\sigma^{-a_K} - \sigma^{-\lfloor a_K\rfloor})H(P+(-1)^{\lfloor a_K\rfloor}\delta)\\ &-\sum_{k=1}^{\lfloor a_K\rfloor}(\sigma^{-k}-\sigma^{1-k})H(P+(-1)^{1-k}\delta)\\ &-(\sigma^{1-a_K}-\sigma^{1-\lfloor a_K\rfloor})(P+(-1)^{\lfloor a_K\rfloor-1}\delta)\log n_0\\ &-\Big(\sum_{k=1}^{\lfloor a_K\rfloor-1}(\sigma^{-k}-\sigma^{1-k})(P+(-1)^{1-k}\delta)\Big)\log n_0\\ &-\big(\sigma^{1-a_K}-(\sigma^{1-a_K}-\sigma^{1-\lfloor a_K\rfloor})(P+(-1)^{\lfloor a_K\rfloor-1}\delta)\big)\log n_1\\ &+\Big(\sum_{k=1}^{\lfloor a_K\rfloor-1}(\sigma^{-k}-\sigma^{1-k})(P+(-1)^{1-\ell}\delta)\Big)\log n_1 + O(\ell_K^{1/2}).\end{aligned}$$

Comparing with (4.10), we get

$$\log \nu_\delta(C_{\ell_K}(x)) = -Z_{\ell_K} + o(\ell_K)$$

and

$$\lim_{K\to\infty}\frac{1}{\ell_K}\log\nu_\delta(C_{\ell_K}(x)) = -\lim_{K_0\to\infty}\lim_{K\to\infty}\frac{1}{\ell_K}Z_{\ell_K}.$$

The last limit was calculated in the course of proof of Lemma 4.13:

$$\lim_{K_0\to\infty}\lim_{K\to\infty}\frac{1}{\ell_K}Z_{\ell_K} = Y(\delta).$$

The assertion follows. ■

To finish the proof of the lower bound in Proposition 4.15 we need only observe that Y is a continuous function of δ, hence

$$\max\{Y(\delta),\delta\le\min(P,1-P)\} = \sup\{Y(\delta),\delta<\min(P,1-P)\},$$

and apply Frostman's lemma.

To prove the upper bound, fix small ε_1 and $\varepsilon_2 < \varepsilon_1/(2U)$. Consider a finite family $\{I_k\}_{k=1}^V$ of intervals of size ε_1 covering $[0,1]$. Let $J_k = 3I_k$, that is, an interval with the same center as I_k but three times longer.

For every point $x\in X_\alpha^{\text{symb}}$ there is some minimal $N(x)$ such that for all $N > N(x)$,

$$|G_{\sigma N-1}(x) + G_{N-1}(x) - 2P| < \varepsilon_2. \tag{4.15}$$

We divide X_α^{symb} into subsets

$$X_{\alpha,N} = \{x\in X_\alpha^{\text{symb}} : N(x) = N\}.$$

Our goal is to prove

LEMMA 4.18. *For each N,*

$$\overline{\dim}_B X_{\alpha,N} \leq \frac{1}{\log m} \max_{\delta\leq\min(P,1-P)} Y(\delta).$$

Proof. Let $x \in X_{N,\alpha}$. Let k be such that

$$G_{N-1}(x) \in I_k.$$

By (4.15), this means that for all $\ell \leq U$ we have

$$G_{\sigma^{-2\ell}N-1}(x) \in J_k \quad \text{and} \quad G_{\sigma^{-2\ell+1}N-1}(x) \in 2P - J_k.$$

We estimate the number $A(k,N)$ of possible sequences $(i_N(x),\dots,i_{\lfloor\sigma^{-2U}N\rfloor})$. Just as for the lower bound, the estimation will be almost the same as in the proof of Lemma 4.13:

$$\frac{1}{(\sigma^{-2U}-1)N}\log A(k,N) \leq \sup_{\delta\colon \{P+\delta,P-\delta\}\cap J_k\neq\emptyset} Y(\delta) + O(\varepsilon_1).$$

There are only V possible k's, hence the number $B(N)$ of possible sequences $(i_N(x),\dots,i_{\lfloor\sigma^{-2U}N\rfloor})$ for all $x \in X_{\alpha,N}$ satisfies

$$\frac{1}{(\sigma^{-2U}-1)N}\log B(N) \leq \sup_{\delta\leq\min(P,1-P)} Y(\delta) + O(\varepsilon_1) + O\bigg(\frac{1}{\sigma^{-2U}N}\bigg).$$

Repeating the argument for $B(\sigma^{-2U}N)$ and so on and passing to the limit, we get

$$\overline{\dim}_B X_{\alpha,N} \leq \frac{1}{\log m}\min(P,1-P)Y(\delta) + O(\varepsilon_1),$$

and the assertion follows. ∎

As the packing dimension is not greater than the upper box counting dimension, this gives the upper bound in Proposition 4.15, and so the proof of the proposition is complete. ∎

Theorem 4.9 now follows by combining Proposition 4.15, Lemma 4.2 Lemma 4.10 and Corollary 4.14. This completes the proof of Theorem 2.4.

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Thomas Jordan
School of Mathematics
University of Bristol
University Walk
Bristol, BS8 1TW, UK
E-mail: thomas.jordan@bris.ac.uk

Michał Rams
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-656 Warszawa, Poland
E-mail: rams@impan.pl

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