FUNDAMENTA
MATHEMATICAE
221 (2013)

Weak square sequences and special Aronszajn trees

by

John Krueger (Denton, TX)

Abstract. A classical theorem of set theory is the equivalence of the weak square principle $\square^*_\mu$ with the existence of a special Aronszajn tree on μ^+. We introduce the notion of a weak square sequence on any regular uncountable cardinal, and prove that the equivalence between weak square sequences and special Aronszajn trees holds in general.

Recall the *weak square principle* $\square^*_\mu$ for an infinite cardinal μ, which asserts the existence of a sequence $\langle \mathcal{C}_\alpha : \alpha \in \mu^+ \cap \mathrm{Lim}\rangle$ satisfying:

(1) for all $c \in \mathcal{C}_\alpha$, c is a club subset of α with order type at most μ;
(2) $|\mathcal{C}_\alpha| \leq \mu$;
(3) for all $c \in \mathcal{C}_\alpha$, if $\beta \in \lim(c)$ then $c \cap \beta \in \mathcal{C}_\beta$.

For a regular uncountable cardinal κ, a tree $(T, <_T)$ is a *κ-tree* if it has height κ and all its levels are of size less than κ. For a successor cardinal $\kappa = \mu^+$, a κ-tree $(T, <_T)$ is a *special Aronszajn tree* if T is the union of μ many antichains. Equivalently, T is special if there exists a function $f : T \to \mu$ such that $t <_T u$ implies $f(t) \neq f(u)$.

The following classical theorem was originally noted by Jensen [2]. Let μ be an infinite cardinal. Then $\square^*_\mu$ is equivalent to the existence of a special Aronszajn tree on μ^+.

Todorčević [3] introduced a more general definition of a special Aronszajn tree. For a regular uncountable cardinal κ, a tree $(T, <_T)$ of height κ is said to be a *special Aronszajn tree* if there exists a function $g : T \to T$ satisfying:

(1) $g(t) <_T t$ for all nonminimal $t \in T$;
(2) for all $u \in T$, $g^{-1}(\{u\})$ is the union of fewer than κ many antichains.

This definition coincides with the classical definition of a special Aronszajn tree when κ is a successor cardinal.

2010 *Mathematics Subject Classification*: Primary 03E05.
Key words and phrases: weak square sequence, special Aronszajn tree.

DOI: 10.4064/fm221-3-4

In this paper we introduce a definition of a weak square sequence which makes sense on any regular uncountable cardinal. We prove that the existence of such a sequence on a regular uncountable cardinal κ is equivalent to the existence of a special Aronszajn tree on κ in the sense of Todorčević.

Notation. Let Lim and Succ denote the classes of limit ordinals and successor ordinals respectively. Let $\mathrm{cof}(\omega)$ denote the class of limit ordinals of countable cofinality, and let $\mathrm{cof}(>\omega)$ denote the class of limit ordinals of uncountable cofinality. For a set a of ordinals, $\mathrm{ot}(a)$ is the order type of a, and $\lim(a)$ is the set of ordinals β such that $\sup(a \cap \beta) = \beta$.

A *tree* is a strict partial order $(T, <_T)$ such that for every node $x \in T$, the set $\{y \in T : y <_T x\}$ is well ordered by $<_T$. The *height* of a node $x \in T$, denoted by $\mathrm{ht}(x)$, is the order type of $\{y \in T : y <_T x\}$. Let $T_\alpha = \{x \in T : \mathrm{ht}(x) = \alpha\}$ denote level α of T, for any ordinal α. The *height* of the tree T is the least α such that T_α is empty. For finite sequences u and v, $u \sqsubseteq v$ means that u is an initial segment of v, and $u \sqsubset v$ means that u is a proper initial segment of v.

1. Weak square sequences. The next definition generalizes the idea of a weak square sequence to any regular uncountable cardinal.

Definition 1.1. Let κ be a regular uncountable cardinal. A sequence $\langle c_\alpha : \alpha \in C\rangle$ is a *weak square sequence* on κ if:

(1) $C \subseteq \kappa \cap \mathrm{Lim}$ is a club;
(2) for all $\alpha \in C$, c_α is a club subset of α with order type less than α;
(3) for every $\xi < \kappa$, $|\{c_\alpha \cap \xi : \alpha \in C\}| < \kappa$.

Note that if there exists a weak square sequence $\langle c_\alpha : \alpha \in C\rangle$ on κ, then κ is non-Mahlo. Indeed, (2) implies that every ordinal in the club C is singular.

The goal of this section is to show that for an infinite cardinal μ, the existence of a weak square sequence on μ^+ in the sense above is equivalent to the classical weak square principle $\square^*_\mu$. The main challenge lies in reducing the order type of the clubs on the sequence.

Let us note that for an infinite cardinal μ, $\square^*_\mu$ is equivalent to the existence of a sequence $\langle c_\alpha : \alpha \in \mu^+ \cap \mathrm{Lim}\rangle$, where each c_α is a club subset of α with order type at most μ, and for every $\xi < \mu^+$, $|\{c_\alpha \cap \xi : \alpha \in \mu^+ \cap \mathrm{Lim}\}| \le \mu$. For if we have such a sequence, we can define for each limit ordinal α the set $\mathcal{C}_\alpha$ to be the collection of sets of the form $c_\beta \cap \alpha$, where $\beta \in \mu^+ \cap \mathrm{Lim}$ and $\alpha \in \lim(c_\beta)$. Conversely, given $\langle \mathcal{C}_\alpha : \alpha \in \mu^+ \cap \mathrm{Lim}\rangle$, a sequence $\langle c_\alpha : \alpha \in \mu^+ \cap \mathrm{Lim}\rangle$ is obtained as required by choosing c_α to be any member of $\mathcal{C}_\alpha$.

Lemma 1.2. *Let κ be a regular uncountable cardinal. Suppose there exists a weak square sequence on κ. Then there exists a sequence $\langle c_\alpha : \alpha \in \kappa \cap \mathrm{Lim}\rangle$ satisfying:*

(1) *each c_α is a club subset of α;*
(2) *if α is singular then $\mathrm{ot}(c_\alpha) < \alpha$;*
(3) *there is a club $C \subseteq \kappa$ such that for all $\alpha \in C$, $\mathrm{ot}(c_\alpha) < \min(c_\alpha)$;*
(4) *for all $\alpha \in (\kappa \cap \mathrm{Lim}) \setminus C$, $\min(c_\alpha) > \sup(C \cap \alpha)$;*
(5) *for every $\xi < \kappa$, $|\{c_\alpha \cap \xi : \alpha \in \kappa \cap \mathrm{Lim}\}| < \kappa$.*

Proof. Fix a sequence $\langle d_\alpha : \alpha \in C\rangle$ satisfying Definition 1.1. We define a sequence $\langle c_\alpha : \alpha \in \kappa \cap \mathrm{Lim}\rangle$ as follows. If $\alpha \in C$, then $\mathrm{ot}(d_\alpha) < \alpha$. So let $c_\alpha = d_\alpha \setminus (\mathrm{ot}(d_\alpha) + 1)$. If $\alpha < \kappa$ is a limit ordinal not in C, then since C is a club, $\sup(C \cap \alpha) < \alpha$. Let c_α be any club subset of α with order type $\mathrm{cf}(\alpha)$ such that $\min(c_\alpha) > \sup(C \cap \alpha)$. Clearly (1)–(4) are satisfied.

We claim that for every $\xi < \kappa$, $|\{c_\alpha \cap \xi : \alpha \in \kappa \cap \mathrm{Lim}\}| < \kappa$. Let $\gamma = \min(C \setminus \xi)$. Then for every limit ordinal $\beta \in \kappa \setminus C$ which is larger than γ, $\min(c_\beta) > \gamma$, so $c_\beta \cap \xi = \emptyset$. It follows that the nonempty members of the set $\{c_\alpha \cap \xi : \alpha \in \kappa \cap \mathrm{Lim}\}$ are in the set

$$\bigcup_{\delta<\xi} \{d_\alpha \cap [\delta, \xi) : \alpha \in C\} \cup \{c_\beta \cap \xi : \beta \in \gamma \setminus C\}.$$

There are fewer than κ many elements in the set on the left by assumption, and clearly there are no more than $|\gamma| < \kappa$ many elements in the set on the right. ■

Lemma 1.3. *Let κ be a regular uncountable cardinal. Suppose $\langle c_\alpha : \alpha \in \kappa \cap \mathrm{Lim}\rangle$ is a sequence satisfying:*

(1) *each c_α is a club subset of α;*
(2) *if α is singular then $\mathrm{ot}(c_\alpha) < \alpha$;*
(3) *for every $\xi < \kappa$, $|\{c_\alpha \cap \xi : \alpha \in \kappa \cap \mathrm{Lim}\}| < \kappa$.*

For each limit ordinal $\alpha < \kappa$, let $f_\alpha : \mathrm{ot}(c_\alpha) \to c_\alpha$ be the increasing enumeration of c_α. Define a sequence $\langle d_\alpha : \alpha \in \kappa \cap \mathrm{Lim}\rangle$ by letting

$$d_\alpha = \begin{cases} c_\alpha & \text{if } \mathrm{ot}(c_\alpha) = \mathrm{cf}(\alpha), \\ f_\alpha[c_{\mathrm{ot}(c_\alpha)}] & \text{if } \mathrm{ot}(c_\alpha) > \mathrm{cf}(\alpha). \end{cases}$$

Then $\langle d_\alpha : \alpha \in \kappa \cap \mathrm{Lim}\rangle$ also satisfies conditions (1)–(3) *above; moreover, in the case that $\mathrm{ot}(c_\alpha) > \mathrm{cf}(\alpha)$, we have $\mathrm{ot}(d_\alpha) < \mathrm{ot}(c_\alpha)$.*

Proof. Consider a limit ordinal $\alpha < \kappa$. If $\mathrm{ot}(c_\alpha) = \mathrm{cf}(\alpha)$, then $d_\alpha = c_\alpha$ so (1) and (2) hold for d_α. Suppose $\mathrm{ot}(c_\alpha) > \mathrm{cf}(\alpha)$. Then, in particular, α is singular. Since $f_\alpha : \mathrm{ot}(c_\alpha) \to \alpha$ is normal and cofinal in α, $d_\alpha = f_\alpha[c_{\mathrm{ot}(c_\alpha)}]$ is a club subset of α with order type equal to $\mathrm{ot}(c_{\mathrm{ot}(c_\alpha)})$; but $\mathrm{ot}(c_{\mathrm{ot}(c_\alpha)}) \leq \mathrm{ot}(c_\alpha) < \alpha$. So (1) and (2) hold. For the final comment, assume

$\mathrm{ot}(c_\alpha) > \mathrm{cf}(\alpha)$. Note that $\mathrm{cf}(\mathrm{ot}(c_\alpha)) = \mathrm{cf}(\alpha) < \mathrm{ot}(c_\alpha)$, so $\mathrm{ot}(c_\alpha)$ is singular. Therefore $\mathrm{ot}(c_{\mathrm{ot}(c_\alpha)}) < \mathrm{ot}(c_\alpha)$ by (2). So $\mathrm{ot}(d_\alpha) = \mathrm{ot}(c_{\mathrm{ot}(c_\alpha)}) < \mathrm{ot}(c_\alpha)$.

Let $\xi < \kappa$ be given; we prove $|\{d_\alpha \cap \xi : \alpha \in \kappa \cap \mathrm{Lim}\}| < \kappa$. Note that

$$\{d_\alpha \cap \xi : \alpha \in \kappa \cap \mathrm{Lim},\ \mathrm{ot}(c_\alpha) = \mathrm{cf}(\alpha)\} \subseteq \{c_\alpha \cap \xi : \alpha \in \kappa \cap \mathrm{Lim}\},$$

so the set on the left has size less than κ. It remains to show that the set

$$\{d_\alpha \cap \xi : \alpha \in \kappa \cap \mathrm{Lim},\ \mathrm{ot}(c_\alpha) > \mathrm{cf}(\alpha)\}$$

has size less than κ.

Consider a limit ordinal α such that $\mathrm{ot}(c_\alpha) > \mathrm{cf}(\alpha)$. Then $d_\alpha = f_\alpha[c_{\mathrm{ot}(c_\alpha)}]$. Since f_α is the increasing enumeration of c_α, clearly $c_\alpha \cap \xi = f_\alpha[\mathrm{ot}(c_\alpha \cap \xi)]$. As $d_\alpha \subseteq c_\alpha$ and f_α is injective, we have $d_\alpha \cap \xi = d_\alpha \cap c_\alpha \cap \xi = f_\alpha[c_{\mathrm{ot}(c_\alpha)}] \cap f_\alpha[\mathrm{ot}(c_\alpha \cap \xi)] = f_\alpha[c_{\mathrm{ot}(c_\alpha)} \cap \mathrm{ot}(c_\alpha \cap \xi)]$. Let $g_\alpha : \mathrm{ot}(c_\alpha \cap \xi) \to c_\alpha \cap \xi$ be the increasing enumeration of $c_\alpha \cap \xi$. Then $g_\alpha = f_\alpha \restriction \mathrm{ot}(c_\alpha \cap \xi)$. So we have

$$d_\alpha \cap \xi = g_\alpha[c_{\mathrm{ot}(c_\alpha)} \cap \mathrm{ot}(c_\alpha \cap \xi)].$$

Now the function g_α is determined by $c_\alpha \cap \xi$, and there are fewer than κ many possibilities for $c_\alpha \cap \xi$. Once $c_\alpha \cap \xi$ is known, $d_\alpha \cap \xi$ is determined by $c_{\mathrm{ot}(c_\alpha)} \cap \mathrm{ot}(c_\alpha \cap \xi)$, and again there are fewer than κ many possibilities for this set. So there are fewer than κ many possibilities for $d_\alpha \cap \xi$. ∎

PROPOSITION 1.4. *Let κ be a regular uncountable cardinal. Suppose $\langle c_\alpha : \alpha \in \kappa \cap \mathrm{Lim}\rangle$ is a sequence satisfying:*

(1) *each c_α is a club subset of α;*
(2) *if α is singular then $\mathrm{ot}(c_\alpha) < \alpha$;*
(3) *for every $\xi < \kappa$, $|\{c_\alpha \cap \xi : \alpha \in \kappa \cap \mathrm{Lim}\}| < \kappa$.*

Then there exists a sequence $\langle d_\alpha : \alpha \in \kappa \cap \mathrm{Lim}\rangle$ satisfying (1)–(3), *and moreover, each d_α has order type equal to* $\mathrm{cf}(\alpha)$.

Proof. By induction we define for each $n < \omega$ a sequence

$$\langle c^n_\alpha : \alpha \in \kappa \cap \mathrm{Lim}\rangle.$$

The inductive hypotheses are that the sequence of c^n_α's satisfies (1)–(3), and moreover, if $\mathrm{ot}(c^n_\alpha) > \mathrm{cf}(\alpha)$, then $\mathrm{ot}(c^{n+1}_\alpha) < \mathrm{ot}(c^n_\alpha)$. Let $c^0_\alpha = c_\alpha$ for all limit ordinals $\alpha < \kappa$.

Fix $n < \omega$ and suppose that $\langle c^n_\alpha : \alpha \in \kappa \cap \mathrm{Lim}\rangle$ is defined as required. For each α let $f^n_\alpha : \mathrm{ot}(c^n_\alpha) \to c^n_\alpha$ be the increasing enumeration of c^n_α. Define c^{n+1}_α by

$$c^{n+1}_\alpha = \begin{cases} c^n_\alpha & \text{if } \mathrm{ot}(c^n_\alpha) = \mathrm{cf}(\alpha), \\ f^n_\alpha[c^n_{\mathrm{ot}(c^n_\alpha)}] & \text{if } \mathrm{ot}(c^n_\alpha) > \mathrm{cf}(\alpha). \end{cases}$$

Lemma 1.3 implies that $\langle c^{n+1}_\alpha : \alpha < \kappa \text{ limit}\rangle$ satisfies the inductive hypotheses. This completes the definition.

Now we define the sequence $\langle d_\alpha : \alpha \in \kappa \cap \mathrm{Lim}\rangle$. Consider a limit ordinal $\alpha < \kappa$. Since $\mathrm{ot}(c_\alpha^{n+1}) < \mathrm{ot}(c_\alpha^n)$ provided that $\mathrm{ot}(c_\alpha^n) > \mathrm{cf}(\alpha)$, there must exist a least k such that $\mathrm{ot}(c_\alpha^k) = \mathrm{cf}(\alpha)$. Then by definition, for all $m \geq k$, $c_\alpha^m = c_\alpha^k$. Let $d_\alpha = c_\alpha^k$, which is the eventual value of the club attached to α. Clearly d_α is a club subset of α with order type $\mathrm{cf}(\alpha)$, and in particular, if α is singular then $\mathrm{ot}(c_\alpha) < \alpha$.

To show (3), consider $\xi < \kappa$. Then for all $n < \omega$, $|\{c_\alpha^n \cap \xi : \alpha \in \kappa \cap \mathrm{Lim}\}| < \kappa$. But

$$\{d_\alpha \cap \xi : \alpha \in \kappa \cap \mathrm{Lim}\} \subseteq \bigcup_{n<\omega} \{c_\alpha^n \cap \xi : \alpha \in \kappa \cap \mathrm{Lim}\};$$

so the set on the left is a subset of a countable union of sets each having cardinality less than κ. ∎

THEOREM 1.5. *Let μ be an infinite cardinal. Then $\square_\mu^*$ holds iff there exists a weak square sequence on μ^+ in the sense of Definition* 1.1.

Proof. If $\square_\mu^*$ holds, then as noted above there exists a sequence $\langle c_\alpha : \alpha \in \mu^+ \cap \mathrm{Lim}\rangle$ such that each c_α is a club subset of α with order type at most μ, and for every $\xi < \mu^+$, $|\{c_\alpha \cap \xi : \alpha \in \mu^+ \cap \mathrm{Lim}\}| \leq \mu$. Let C be the club set of limit ordinals α with $\mu < \alpha < \mu^+$. Then $\langle c_\alpha : \alpha \in C\rangle$ satisfies Definition 1.1. Conversely, suppose there exists a weak square sequence on μ^+. Then by Lemma 1.2 and Proposition 1.4, there exists a sequence $\langle d_\alpha : \alpha \in \kappa \cap \mathrm{Lim}\rangle$ such that each d_α is a club subset of α with order type $\mathrm{cf}(\alpha) \leq \mu$, and for every $\xi < \kappa$, $|\{d_\alpha \cap \xi : \alpha \in \kappa \cap \mathrm{Lim}\}| < \kappa$. Therefore $\square_\mu^*$ holds. ∎

2. A special Aronszajn tree implies weak square. According to the classical definition, for an infinite cardinal μ, a tree $(T, <_T)$ of height μ^+ is a *special Aronszajn tree* if T is the union of μ many antichains, or equivalently, if there exists a function $f : T \to \mu$ such that for all $t, u \in T$, $t <_T u$ implies $f(t) \neq f(u)$.

Todorčević [3] introduced a more general definition of a special Aronszajn tree which makes sense for any regular uncountable cardinal. Recall that if $(T, <_T)$ is a tree, a function $g : T \to T$ is said to be *regressive* if $f(a) <_T a$ for all nonminimal $a \in T$.

DEFINITION 2.1. Let κ be a regular uncountable cardinal. A tree $(T, <_T)$ with height κ is a *special Aronszajn tree* if there exists a regressive function $g : T \to T$ such that for all $b \in T$, the set $g^{-1}(\{b\})$ is the union of fewer than κ many antichains.

We will sometimes abbreviate "special Aronszajn tree" to "special tree". A *special Aronszajn tree on κ* means a κ-tree which is special. Note that T is special iff there is a regressive function $g : T \to T$ such that for all $b \in T$, there is an ordinal $\lambda_b < \kappa$ and a function $f_b : g^{-1}(\{b\}) \to \lambda_b$ such that for all $t, u \in g^{-1}(\{b\})$, $t <_T u$ implies $f_b(t) \neq f_b(u)$.

The equivalence between the two definitions of "special" for successor cardinals was noted in [3] without proof.

Proposition 2.2 (Todorčević). *Let μ be an infinite cardinal and let $(T, <_T)$ be a tree of height μ^+. Then T is a special Aronszajn tree in the classical sense iff T satisfies Definition* 2.1.

Proof. The forward direction of the equivalence is trivial: just define a regressive function which maps every node to a minimal node. Now suppose there is a regressive function $g : T \to T$, and for each $b \in T$, some ordinal $\lambda_b < \mu^+$ and a function $f_b : g^{-1}(\{b\}) \to \lambda_b$ such that for all $t, u \in g^{-1}(\{b\})$, $t <_T u$ implies $f_b(t) \neq f_b(u)$. Without loss of generality, we can assume $\lambda_b = \mu$ for all b.

We define a function $f : T \to {}^{<\omega}\mu$ so that $c <_T d$ implies $f(c) \neq f(d)$ for all $c, d \in T$. Clearly this suffices since ${}^{<\omega}\mu$ has size μ. Consider a node $a \in T$. If a is minimal then let $f(a)$ be the empty sequence. Suppose a is not minimal. Define g^k for $k < \omega$ by recursion, letting $g^0(a) = a$, and $g^{k+1}(a) = g(g^k(a))$ if $g^k(a)$ is not minimal. Since g is regressive, we have $\mathrm{ht}(g^1(a)) > \mathrm{ht}(g^2(a)) > \cdots > \mathrm{ht}(g^k(a))$. Let m be least such that $g^m(a)$ is minimal. Define $f(a)$ by

$$f(a) = \langle f_{g(a)}(a), f_{g^2(a)}(g(a)), \ldots, f_{g^m(a)}(g^{m-1}(a)) \rangle.$$

Suppose for a contradiction $c <_T d$ but $f(c) = f(d)$. Since d is not minimal, $f(c) = f(d)$ is not empty, so c is not minimal either. Let $m > 0$ be least such that $g^m(c)$ is minimal. Since m is the length of the sequence $f(c) = f(d)$, m is also least such that $g^m(d)$ is minimal. As $g^m(c) <_T c <_T d$, $g^m(c) <_T d$, and hence $g^m(c) = g^m(d)$. Let $0 < k \leq m$ be least such that $g^k(c) = g^k(d)$.

Since $g^{k-1}(c) \leq_T c$, $g^{k-1}(d) \leq_T d$, and $c <_T d$, $g^{k-1}(c)$ and $g^{k-1}(d)$ are comparable and not equal. But $g(g^{k-1}(c)) = g^k(c) = g^k(d) = g(g^{k-1}(d))$. Therefore $f_{g^k(c)}(g^{k-1}(c)) \neq f_{g^k(d)}(g^{k-1}(d))$, which contradicts $f(c) = f(d)$. ∎

Recall the standard fact that for a strongly inaccessible cardinal κ, κ is weakly compact iff there does not exist an Aronszajn tree on κ. Todorčević [3] used his general definition of a special Aronszajn tree to provide an analogue of this result which characterizes Mahlo cardinals.

Theorem 2.3 (Todorčević). *Let κ be a strongly inaccessible cardinal. Then the following are equivalent:*

(1) *κ is a Mahlo cardinal;*
(2) *there does not exist a special Aronszajn tree on κ.*

We will prove that for a regular uncountable cardinal κ, the existence of a special Aronszajn tree on κ is equivalent to the existence of a weak square sequence on κ. We first show the forward direction; the proof follows the lines of Section 5.2 in [1], which handles the case when κ is a successor cardinal.

First let us give a simpler characterization of a special Aronszajn tree on κ.

LEMMA 2.4. *Let $(T, <_T)$ be a κ-tree, where κ is a regular uncountable cardinal. Then T is special iff there exists a function $g : T \to \kappa$ such that $g(t) < \mathrm{ht}(t)$ for all nonminimal t, and for all $\beta < \kappa$, $g^{-1}(\{\beta\})$ is the union of fewer than κ many antichains.*

Proof. For the forward direction, given a regressive $f : T \to T$ witnessing that T is special, define $g(t) = \mathrm{ht}(f(t))$. Then $g^{-1}(\{\beta\}) = \bigcup\{f^{-1}(\{b\}) : \mathrm{ht}(b) = \beta\}$. Each $f^{-1}(\{b\})$ is the union of fewer than κ many antichains, and there are fewer than κ many such b's since T is a κ-tree. Hence $g^{-1}(\{\beta\})$ is the union of fewer than κ many antichains. Conversely, given $g : T \to \kappa$ as described above, define $f(b) = b \restriction g(b)$ for nonminimal b. ∎

THEOREM 2.5. *Let κ be a regular uncountable cardinal. If there exists a special Aronszajn tree on κ, then there exists a weak square sequence on κ.*

Proof. Let $(T, <_T)$ be a κ-tree and suppose that T is special. Fix a function $g : T \to \kappa$, where $g(t) < \mathrm{ht}(t)$ for all nonminimal t, and for each $\beta < \kappa$ a function $f_\beta : g^{-1}(\{\beta\}) \to \lambda_\beta$, where $\lambda_\beta < \kappa$, such that for all $c, d \in g^{-1}(\{\beta\})$, $c <_T d$ implies $f_\beta(c) \neq f_\beta(d)$.

For each limit ordinal α we define a family $\mathcal{A}_\alpha$ of cofinal subsets of α. Fix a limit ordinal α. Consider the following property which a node x in T_α may or may not satisfy: there exists $\beta < \alpha$ such that the set

$$\{\mathrm{ht}(y) : y <_T x \wedge g(y) < \beta\}$$

is cofinal in α.

We claim that if α has uncountable cofinality, then this property is true for all $x \in T_\alpha$. Indeed, fix a sequence $\langle \alpha_i : i < \mathrm{cf}(\alpha) \rangle$ which is increasing, continuous, and cofinal in α. Since $g(t) < \mathrm{ht}(t)$ for all nonminimal t, there exists a regressive function $h : \mathrm{cf}(\alpha) \cap \mathrm{Lim} \to \mathrm{cf}(\alpha)$ so that for all limit ordinals $\gamma < \mathrm{cf}(\alpha)$, if $z <_T x$ has height α_γ, then $g(z) < \alpha_{h(\gamma)}$. Since $\mathrm{cf}(\alpha)$ is regular, there is some $\delta < \mathrm{cf}(\alpha)$ such that $h^{-1}(\{\delta\})$ is stationary in $\mathrm{cf}(\alpha)$. Let $X = \{\alpha_\gamma : \gamma \in h^{-1}(\{\delta\})\}$. Then X is cofinal in α and $X \subseteq \{\mathrm{ht}(y) : y <_T x \ \wedge \ g(y) < \alpha_\delta\}$.

For each limit ordinal $\alpha < \kappa$ and each $x \in T_\alpha$, we define a set d_x which is a club in α. Let β_x be the least ordinal such that the set $\{\mathrm{ht}(y) : y <_T x \ \wedge \ g(y) < \beta_x\}$ is cofinal in α. Note that $\beta_x \leq \alpha$, and if $\mathrm{cf}(\alpha) > \omega$ then $\beta_x < \alpha$.

The process of defining the club d_x involves defining a limit ordinal $\delta_x \leq \alpha$ and sequences

$$\langle \beta(x,i) : i \in \delta_x \cap \mathrm{Succ} \rangle, \quad \langle \alpha(x,i) : i < \delta_x \rangle, \quad \langle z(x,i) : i \in \delta_x \cap \mathrm{Succ} \rangle$$

which satisfy:

(1) $\beta(x,j) \leq \beta(x,i) < \beta_x$ for all successor ordinals $j < i < \delta_x$;
(2) $\langle \alpha(x,i) : i < \delta_x \rangle$ is an increasing and continuous sequence of ordinals cofinal in α;
(3) $z(x,i)$ is the unique node with height $\alpha(x,i)$ such that $z(x,i) <_T x$ for all $i \in \delta_x \cap \mathrm{Succ}$;
(4) $g(z(x,i)) = \beta(x,i)$ for all $i \in \delta_x \cap \mathrm{Succ}$;
(5) if $j < i < \delta_x$ are successor ordinals and $\beta(x,j) = \beta(x,i)$, then

$$f_{\beta(x,j)}(z(x,j)) < f_{\beta(x,j)}(z(x,i)).$$

After the construction is complete, we let $d_x = \{\alpha(x,i) : i < \delta_x\}$, which is a club subset of α with order type δ_x.

Let i be given and suppose that the objects above are defined as required for all $j < i$. If $\sup_{j<i} \alpha(x,j) = \alpha$, then let $i = \delta_x$ and we are done. Now assume $\sup_{j<i} \alpha(x,j) < \alpha$. If $i = 0$ then let $\alpha(x,i) = 0$, and if i is a limit ordinal then let $\alpha(x,i) = \sup_{j<i} \alpha(x,j)$. Suppose that i is a successor ordinal.

Consider the set

$$\{y <_T x : \mathrm{ht}(y) > \alpha(x,i-1)\}.$$

By the choice of β_x, there exists y in this set such that $g(y) < \beta_x$. Let $\beta(x,i)$ be the least ordinal such that there is $y <_T x$ with height greater than $\alpha(x,i-1)$ and $g(y) = \beta(x,i)$. Then $\beta(x,i) < \beta_x$. We claim that for all successor ordinals $j < i$, $\beta(x,j) \leq \beta(x,i)$. Since $\alpha(x,j-1) < \alpha(x,i-1)$, there exists z in the set $\{y <_T x : \mathrm{ht}(y) > \alpha(x,j-1)\}$ such that $g(z) = \beta(x,i)$. By the minimality of $\beta(x,j)$, $\beta(x,j) \leq \beta(x,i)$.

To define $\alpha(x,i)$, consider the set

$$\{y <_T x : \mathrm{ht}(y) > \alpha(x,i-1) \ \wedge \ g(y) = \beta(x,i)\}.$$

By the choice of $\beta(x,i)$, this set is nonempty. Moreover, since this set is a chain, $f_{\beta(x,i)}$ is injective on it. Let $z(x,i)$ be the unique element in this set with the minimal value under $f_{\beta(x,i)}$. Then let $\alpha(x,i) = \mathrm{ht}(z(x,i))$.

We claim that if $j < i$ is a successor ordinal and $\beta(x,i) = \beta(x,j)$, then

$$f_{\beta(x,j)}(z(x,j)) < f_{\beta(x,j)}(z(x,i)).$$

For since $\alpha(x,j-1) < \alpha(x,i-1)$ and $\beta(x,i) = \beta(x,j)$, the node $z(x,i)$ is in the set

$$\{y <_T x : \mathrm{ht}(y) > \alpha(x,j-1) \wedge g(y) = \beta(x,j)\}.$$

Since $z(x,j)$ has the minimal value in this set under $f_{\beta(x,j)}$, $f_{\beta(x,j)}(z(x,j)) < f_{\beta(x,j)}(z(x,i))$ as desired.

This completes the construction. Let us consider the order type δ_x of d_x for a node $x \in T$. For any ordinal $\beta < \kappa$, let $\theta(\beta)$ denote the order type of the well-order whose underlying set is

$$\bigcup_{\gamma<\beta} \gamma \times \lambda_\gamma$$

and ordered by lexicographical order $<_{\text{lex}}$. Note that $\theta(\beta) < \kappa$. For each $x \in T$, (1) and (5) imply that the function

$$i \mapsto \langle \beta(x,i), f_{\beta(x,i)}(z(x,i)) \rangle,$$

which maps from $\delta_x \cap \text{Succ}$ into the well-order $(\bigcup_{\gamma<\beta_x} \gamma \times \lambda_\gamma, <_{\text{lex}})$, is increasing. Since δ_x is a limit ordinal, δ_x and $\delta_x \cap \text{Succ}$ have the same order type. It follows that $\delta_x \leq \theta(\beta_x)$.

Let C be the club set of limit ordinals $\alpha < \kappa$ greater than ω such that for all $\beta < \alpha$, $\theta(\beta) < \alpha$. If $\alpha \in C$ has uncountable cofinality and $x \in T_\alpha$, then $\beta_x < \alpha$ and so $\theta(\beta_x) < \alpha$. Therefore $\text{ot}(d_x) = \delta_x \leq \theta(\beta_x) < \alpha$.

Now we prove the following statement: for every limit ordinal $\alpha < \kappa$ and for every node x with height α, if $\xi \in \lim(d_x)$, then letting $w <_T x$ have height ξ, $d_x \cap \xi = d_w$. So let such α, x, ξ, and w be given. Recall that β_w is the least ordinal such that the set $\{\text{ht}(y) : y <_T w \ \wedge \ g(y) < \beta_w\}$ is cofinal in ξ. Since $d_x \cap \xi$ is cofinal in ξ and for all $\gamma \in d_x$, $g(\gamma) < \beta_x$, clearly $\beta_w \leq \beta_x$.

Let δ'_w be the least ordinal such that $\{\alpha(x,i) : i < \delta'_w\}$ is cofinal in ξ. We will prove by induction that for all $i < \delta'_w$, $\alpha(x,i) = \alpha(w,i)$. It follows immediately that $\delta'_w = \delta_w$ and $d_x \cap \xi = d_w$.

So let $i < \delta'_w$ be given and suppose that for all $j < i$, $\alpha(x,j) = \alpha(w,j)$. If $i = 0$ then $\alpha(x,0) = 0 = \alpha(w,0)$, and if i is a limit ordinal then $\alpha(x,i) = \sup_{j<i} \alpha(x,j) = \sup_{j<i} \alpha(w,j) = \alpha(w,i)$. Suppose i is a successor ordinal.

Recall that $\beta(x,i)$ is the least ordinal such that there is $y <_T x$ with $\text{ht}(y) > \alpha(x,i-1)$ and $g(y) = \beta(x,i)$. And $z(x,i)$ is the element of the set

$$\{y <_T x : \text{ht}(y) > \alpha(x,i-1) \wedge g(y) = \beta(x,i)\}.$$

with the least $f_{\beta(x,i)}$ value. Let us show that $\beta(x,i) = \beta(w,i)$. We have $g(z(x,i)) = \beta(x,i) < \alpha(x,i) = \text{ht}(z(x,i)) < \xi$ and $z(x,i) <_T w$. So $z(x,i)$ is a witness to the statement that there is $y <_T w$ such that $\text{ht}(y) > \alpha(w,i-1)$ and $g(y) = \beta(x,i)$. By minimality it follows that $\beta(w,i) \leq \beta(x,i)$. If $\beta(w,i) < \beta(x,i)$, then there is $y <_T w$ with height greater than $\alpha(w,i-1) = \alpha(x,i-1)$ such that $g(w) < \beta(x,i)$. But then $y <_T x$ and we have a contradiction to the minimality of $\beta(x,i)$. So $\beta(x,i) = \beta(w,i)$.

Since $\text{ht}(z(x,i)) < \xi$, $z(x,i) <_T w$. So $z(x,i)$ is in the set

$$\{y <_T w : \text{ht}(y) > \alpha(w,i-1) \wedge g(y) = \beta(w,i)\}.$$

Since $z(w,i)$ is the element of this set with the least $f_{\beta(w,i)}$ value, $f_{\beta(w,i)}(z(w,i)) \leq f_{\beta(w,i)}(z(x,i))$. On the other hand, $z(w,i)$ is in the set

$$\{y <_T x : \mathrm{ht}(y) > \alpha(x,i-1) \wedge g(y) = \beta(x,i)\},$$

so for the same reason, $f_{\beta(w,i)}(z(x,i)) \leq f_{\beta(w,i)}(z(w,i))$. Therefore $f_{\beta(w,i)}(z(x,i)) = f_{\beta(w,i)}(z(w,i))$. Since $z(x,i)$ and $z(w,i)$ are both below x, they are comparable. But $f_{\beta(w,i)}$ is injective on chains, so $z(x,i) = z(w,i)$. This completes the proof that $d_x \cap \xi = d_w$.

Now we are ready to define a weak square sequence on κ. Recall that C is a club subset of κ such that for all $\alpha \in C$ with uncountable cofinality and all $x \in T_\alpha$, $\mathrm{ot}(d_x) < \alpha$. Define $\langle c_\alpha : \alpha \in C\rangle$ as follows. For α in C with uncountable cofinality, let $c_\alpha = d_x$ for some $x \in T_\alpha$. For α in C with cofinality ω, let c_α be a cofinal subset of α with order type ω.

It remains to show that for every $\xi < \kappa$, $|\{c_\alpha \cap \xi : \alpha \in C\}| < \kappa$. First note that if $\mathrm{cf}(\alpha) = \omega$, then $c_\alpha \cap \xi$ is either equal to c_α if $\alpha \leq \xi$, or is finite otherwise. Hence $|\{c_\alpha \cap \xi : \alpha \in C \cap \mathrm{cof}(\omega)\}| < \kappa$.

For each $\xi < \kappa$, let $\mathcal{D}_\xi = \{c_\alpha \cap \xi : \alpha \in C \cap \mathrm{cof}(>\omega)\}$. We prove by induction on ξ that $|\mathcal{D}_\xi| < \kappa$. The successor case is easy, so assume that ξ is a limit ordinal. The set $\mathcal{D}_\xi$ splits into two sets:

$$\{c_\alpha \cap \xi : \alpha \in C \cap \mathrm{cof}(>\omega), \sup(c_\alpha \cap \xi) < \xi\},$$
$$\{c_\alpha \cap \xi : \alpha \in C \cap \mathrm{cof}(>\omega), \sup(c_\alpha \cap \xi) = \xi\}.$$

The first set is contained in the union $\bigcup_{\xi'<\xi} \mathcal{D}_{\xi'}$, so has size less than κ by the inductive hypothesis. The second set is a subset of $\{d_w : w \in T_\xi\}$, which has size less than κ since $|T_\xi| < \kappa$. ∎

3. The full code of a C-sequence. Fix a regular uncountable cardinal κ. A *C-sequence* on κ is a sequence $\langle c_\alpha : \alpha < \kappa\rangle$ satisfying:

(1) $c_0 = \emptyset$;
(2) $c_{\alpha+1} = \{\alpha\}$;
(3) if α is a limit ordinal then c_α is a club subset of α.

We will review the *full code* ρ_0 of Todorčević [3], defined from a given C-sequence on κ. We propose that ρ_0 and its corresponding tree $T(\rho_0)$ can be developed most naturally in the context of weak square.

Fix a C-sequence $\langle c_\alpha : \alpha < \kappa\rangle$.

Definition 3.1. Let $\alpha \leq \beta < \kappa$.

(1) The *walk from β to α* is the unique sequence $\langle \beta_0, \ldots, \beta_n\rangle$ such that $\beta_0 = \beta$, $\beta_{k+1} = \min(c_{\beta_k} \setminus \alpha)$ for $k < n$, and $\beta_n = \alpha$.
(2) $\rho_0(\alpha,\beta) = \langle \mathrm{ot}(c_{\beta_0} \cap \alpha), \ldots, \mathrm{ot}(c_{\beta_{n-1}} \cap \alpha)\rangle$.

In (2) we mean $\rho_0(\alpha,\alpha) = \emptyset$ in the case $\alpha = \beta$. Note that the length of $\rho_0(\alpha,\beta)$ is 1 less than the length of the walk from β to α. If $\langle \beta_0, \ldots, \beta_n\rangle$ is the walk from β to α, then obviously for all $i = 0, \ldots, n$, $\langle \beta_i, \ldots, \beta_n \rangle$ is the walk from β_i to α. That $\langle \beta_0, \ldots, \beta_i \rangle$ is the walk from β to β_i follows from the next lemma.

LEMMA 3.2. ([1]) *Let $\alpha \leq \gamma \leq \beta$. Let $\langle \beta_0, \ldots, \beta_m \rangle$ be the walk from β to γ. Then the following are equivalent:*

(1) *the sequence $\langle \beta_0, \ldots, \beta_m \rangle$ is an initial segment of the walk from β to α;*
(2) *γ is in the walk from β to α;*
(3) *for all $i = 0, \ldots, m-1$, $c_{\beta_i} \cap [\alpha, \gamma) = \emptyset$.*

Proof. (1)⇒(2) is immediate since $\beta_m = \gamma$. For (3)⇒(1), it is easy to prove by induction on $i \leq m$ that β_i is the ith element in the walk from β to α; namely, $\beta_0 = \beta$, and if β_i is as required for a fixed $i < m$, then $\beta_{i+1} = \min(c_{\beta_i} \setminus \gamma) = \min(c_{\beta_i} \setminus \alpha)$, which is the $i+1$st element in the walk from β to α. To show (2)⇒(3), assume (2) holds and (3) fails. Let $i < m$ be least such that $c_{\beta_i} \cap [\alpha, \gamma) \neq \emptyset$. Then by the implication (3)⇒(1) just shown, $\langle \beta_0, \ldots, \beta_i \rangle$ is an initial segment of the walk from β to α, and the next step of this walk is $\min(c_{\beta_i} \setminus \alpha)$, which is less than γ by the choice of i. This contradicts that γ is in the walk from β to α. ∎

LEMMA 3.3. *Let $\alpha \leq \gamma \leq \beta$. Then the following are equivalent:*

(1) *$\rho_0(\alpha, \beta) = \rho_0(\gamma, \beta)^\frown \rho_0(\alpha, \gamma)$;*
(2) *$\rho_0(\gamma, \beta)$ is an initial segment of $\rho_0(\alpha, \beta)$;*
(3) *γ is in the walk from β to α.*

Proof. (1)⇒(2) is immediate. For (2)⇒(3), let $\langle \beta_0, \ldots, \beta_n \rangle$ and $\langle \beta'_0, \ldots, \beta'_m \rangle$ be the walks from β to α and from β to γ. If γ is not in the walk from β to α, let $0 < k \leq m$ be least such that $\beta_k \neq \beta'_k$. Then $\beta_k = \min(c_{\beta_{k-1}} \setminus \alpha) < \gamma$. So $c_{\beta_{k-1}} \cap \alpha$ is a proper initial segment of $c_{\beta_{k-1}} \cap \gamma$. Therefore $\rho_0(\alpha, \beta)(k-1) = \mathrm{ot}(c_{\beta_{k-1}} \cap \alpha) < \mathrm{ot}(c_{\beta_{k-1}} \cap \gamma) = \rho_0(\gamma, \beta)(k-1)$. So (2) fails.

Now assume (3). Let $\langle \beta_0, \ldots, \beta_n \rangle$ be the walk from β to α. By Lemma 3.2, fix $k < n$ such that $\langle \beta_0, \ldots, \beta_k \rangle$ is the walk from β to γ. Also by Lemma 3.2, for all $i \leq k-1$, $c_{\beta_i} \cap [\alpha, \gamma)$ is empty, and therefore $\rho_0(\gamma, \beta)(i) = \mathrm{ot}(c_{\beta_i} \cap \gamma) = \mathrm{ot}(c_{\beta_i} \cap \alpha) = \rho_0(\alpha, \beta)(i)$. So $\rho_0(\gamma, \beta) = \rho_0(\alpha, \beta) \restriction k$. By the definition of ρ_0 and the fact that $\langle \beta_k, \ldots, \beta_n \rangle$ is the walk from γ to α, for all $i < n-k$ we have $\rho_0(\alpha, \beta)(k+i) = \mathrm{ot}(c_{\beta_{k+i}} \cap \alpha) = \rho_0(\alpha, \gamma)(i)$. Thus $\rho_0(\alpha, \beta) = \rho_0(\gamma, \beta)^\frown \rho_0(\alpha, \gamma)$. ∎

([1]) Lemmas 3.2–3.4 are due to Todorčević; they are discussed in Lemmas 2.1.6 and 2.1.16 of [4] in the case $\kappa = \omega_1$.

Define the *right lexicographical order* $<_r$ on ${}^{<\omega}\kappa$ by letting $t <_r s$ if either s is a proper initial segment of t, or there is k such that $s(k) \neq t(k)$, and the least such k satisfies $t(k) < s(k)$.

LEMMA 3.4. *Let $\alpha < \gamma \leq \beta$. Then $\rho_0(\alpha,\beta) <_r \rho_0(\gamma,\beta)$.*

Proof. Let $\langle \beta_0,\ldots,\beta_n\rangle$ and $\langle \beta'_0,\ldots,\beta'_m\rangle$ be the walks from β to γ and from β to α respectively. If γ is in the walk from β to α, then by Lemma 3.3, $\rho_0(\gamma,\beta)$ is a proper initial segment of $\rho_0(\alpha,\beta)$, so $\rho_0(\alpha,\beta) <_r \rho_0(\gamma,\beta)$. Otherwise let $k > 0$ be least such that $\beta_k \neq \beta'_k$. Since β_{k-1} is in both walks, $\rho_0(\beta_{k-1},\beta)$ is an initial segment of both $\rho_0(\gamma,\beta)$ and $\rho_0(\alpha,\beta)$. In particular, the least place where $\rho_0(\gamma,\beta)$ and $\rho_0(\alpha,\beta)$ can differ is at $k-1$. Since $\beta'_k \in c_{\beta_{k-1}} \cap [\alpha,\gamma)$, we see that $c_{\beta_{k-1}} \cap \alpha$ is a proper initial segment of $c_{\beta_{k-1}} \cap \gamma$. Therefore $\rho_0(\alpha,\beta)(k-1) = \mathrm{ot}(c_{\beta_{k-1}} \cap \alpha) < \mathrm{ot}(c_{\beta_{k-1}} \cap \gamma) = \rho_0(\gamma,\beta)(k-1)$. Hence $\rho_0(\alpha,\beta) <_r \rho_0(\gamma,\beta)$. ■

In order to construct a special Aronszajn tree from a weak square sequence, we will need to analyze the following situation: suppose $\alpha \leq \beta,\gamma$, where α is a limit ordinal, and for all $\xi < \alpha$, $\rho_0(\xi,\beta) = \rho_0(\xi,\gamma)$. What can be said about the relationship between $\rho_0(\alpha,\beta)$ and $\rho_0(\alpha,\gamma)$? This relationship is described precisely in Proposition 3.6 below. If $\kappa = \omega_1$ then in the situation just described $\rho_0(\alpha,\beta) = \rho_0(\alpha,\gamma)$. But this is not true in general. For example, it is not true if $\alpha = \beta$ is a limit ordinal, $\alpha < \gamma$, and $c_\alpha = c_\gamma \cap \alpha$.

We make some additional observations about ρ_0 in preparation for Proposition 3.6. Let $\langle \beta_0,\ldots,\beta_n\rangle$ be the walk from β to α, where $\alpha \leq \beta$. Then for all $i = 0,\ldots,n-2$, $\sup(c_{\beta_i} \cap \alpha) < \alpha$. Namely, if $\sup(c_{\beta_i} \cap \alpha) = \alpha$, then $\alpha \in c_{\beta_i}$, and hence $\alpha = \min(c_{\beta_i} \setminus \alpha)$. This is only possible if $i = n-1$.

LEMMA 3.5. *Let $\langle \beta_0,\ldots,\beta_n\rangle$ be the walk from β to α, where α is a limit ordinal and $\alpha \leq \beta$. Assume that $\xi < \alpha$ is larger than $\sup(c_{\beta_i} \cap \alpha)$ for all $i = 0,\ldots,n-2$. Then $\langle \beta_0,\ldots,\beta_{n-1}\rangle$ is an initial segment of the walk from β to ξ, namely, the part of the walk consisting of ordinals above α.*

Proof. The proof is by induction on $k < n$. Assume $\langle \beta_0,\ldots,\beta_k\rangle$ is an initial segment of the walk from β to ξ, where $k < n-1$. By assumption, $\sup(c_{\beta_k} \cap \alpha) < \xi$, and hence $\beta_{k+1} = \min(c_{\beta_k} \setminus \alpha) = \min(c_{\beta_k} \setminus \xi)$, which is the next step of the walk from β to ξ. Finally, $\alpha = \min(c_{\beta_{n-1}} \setminus \alpha) \geq \min(c_{\beta_{n-1}} \setminus \xi)$, and $\min(c_{\beta_{n-1}} \setminus \xi)$ is the next step of the walk from β to ξ after β_{n-1}. ■

Let $\langle \beta_0,\ldots,\beta_n\rangle$ be the walk from β to α, where α is a limit ordinal and $\alpha \leq \beta$. Suppose $\sup(c_{\beta_{n-1}} \cap \alpha) < \alpha$. Let $\xi < \alpha$ be larger than $\sup(c_{\beta_i} \cap \alpha)$ for all $i = 0,\ldots,n-1$. Then for $i = 0,\ldots,n-1$, $\sup(c_{\beta_i} \cap \alpha) < \xi$ implies $c_{\beta_i} \cap [\xi,\alpha) = \emptyset$. By Lemma 3.2, α is in the walk from β to ξ. Therefore $\rho_0(\alpha,\beta)$ is an initial segment of $\rho_0(\xi,\beta)$.

On the other hand, suppose $\sup(c_{\beta_{n-1}} \cap \alpha) = \alpha$. Let $\xi < \alpha$ be larger than $\sup(c_{\beta_i} \cap \alpha)$ for all $i = 0,\ldots,n-2$. By Lemma 3.5, $\langle \beta_0,\ldots,\beta_{n-1}\rangle$ is an initial segment of the walk from β to ξ. But since $c_{\beta_{n-1}} \cap [\xi,\alpha)$ is nonempty, Lemma 3.2 implies that α is not in the walk from β to ξ. The next step of the walk from β to ξ after β_{n-1} is $\min(c_{\beta_{n-1}} \setminus \xi)$, which is less than α.

PROPOSITION 3.6. *Let $\alpha < \beta, \gamma$ be given, where α is a limit ordinal. Suppose that for all $\xi < \alpha$, $\rho_0(\xi,\beta) = \rho_0(\xi,\gamma)$. Let $\langle \beta_0,\ldots,\beta_n\rangle$ be the walk from β to α and let $\langle \gamma_0,\ldots,\gamma_m\rangle$ be the walk from γ to α. Let $\alpha_0 = \sup(c_{\beta_{n-1}} \cap \alpha)$ and $\alpha_1 = \sup(c_{\gamma_{m-1}} \cap \alpha)$.*

(1) *If $\alpha_0 < \alpha$ and $\alpha_1 < \alpha$, then $\rho_0(\alpha,\beta) = \rho_0(\alpha,\gamma)$.*
(2) *If $\alpha_0 = \alpha_1 = \alpha$, then $\rho_0(\alpha,\beta) = \rho_0(\alpha,\gamma)$.*
(3) *If $\alpha_0 < \alpha$ and $\alpha_1 = \alpha$, then $\rho_0(\alpha,\gamma) = \rho_0(\alpha,\beta)^\frown \mathrm{ot}(c_\alpha)$.*

Proof. Note that for all $\xi < \alpha$, $\rho_0(\xi,\beta) = \rho_0(\xi,\gamma)$ implies that the walks from β to ξ and from γ to ξ have the same length.

Suppose $\alpha_0 < \alpha$ and $\alpha_1 < \alpha$. Then for all large enough $\xi < \alpha$, α is in the walk from β to ξ and in the walk from γ to ξ. So for all large enough $\xi < \alpha$,

$$\rho_0(\xi,\beta) = \rho_0(\alpha,\beta)^\frown \rho_0(\xi,\alpha) \quad \text{and} \quad \rho_0(\xi,\gamma) = \rho_0(\alpha,\gamma)^\frown \rho_0(\xi,\alpha).$$

Since $\rho_0(\xi,\beta) = \rho_0(\xi,\gamma)$, equating the sequences above and removing the common tails yields $\rho_0(\alpha,\beta) = \rho_0(\alpha,\gamma)$.

Now suppose $\alpha_0 = \alpha_1 = \alpha$. First we show that $n = m$. For all large enough $\xi < \alpha$, $\langle \beta_0,\ldots,\beta_{n-1}\rangle$ is an initial segment of the walk from β to ξ, and $\langle \gamma_0,\ldots,\gamma_{m-1}\rangle$ is an initial segment of the walk from γ to ξ. Consider a large enough ordinal $\xi \in c_{\beta_{n-1}} \cap \alpha$. Then the walk from β to ξ equals $\langle \beta_0,\ldots,\beta_{n-1},\xi\rangle$, which has length $n+1$. Since $\rho_0(\xi,\beta) = \rho_0(\xi,\gamma)$, the walk from γ to ξ has length $n+1$ also. So the walk from γ to γ_{m-1}, namely $\langle \gamma_0,\ldots,\gamma_{m-1}\rangle$, has length less than $n+1$. Hence $m \le n$. A symmetric argument shows that $n \le m$.

For all large enough ξ, β_{n-1} is in the walk from β to ξ, and hence $\rho_0(\beta_{n-1},\beta) \sqsubset \rho_0(\xi,\beta)$ by Lemma 3.3. Similarly, for all large enough ξ, $\rho_0(\gamma_{n-1},\gamma) \sqsubset \rho_0(\xi,\gamma)$. Since $\rho_0(\xi,\beta) = \rho_0(\xi,\gamma)$ and $\rho_0(\beta_{n-1},\beta)$ and $\rho_0(\gamma_{n-1},\gamma)$ have the same length, $\rho_0(\beta_{n-1},\beta) = \rho_0(\gamma_{n-1},\gamma)$. Since $\rho_0(\alpha,\beta) = \rho_0(\beta_{n-1},\beta)^\frown \mathrm{ot}(c_{\beta_{n-1}} \cap \alpha)$ and $\rho_0(\alpha,\gamma) = \rho_0(\gamma_{n-1},\gamma)^\frown \mathrm{ot}(c_{\gamma_{n-1}} \cap \alpha)$, it suffices to show that $\mathrm{ot}(c_{\beta_{n-1}} \cap \alpha) = \mathrm{ot}(c_{\gamma_{n-1}} \cap \alpha)$.

Since α is a limit ordinal, it is enough to show that for all large enough $\xi < \alpha$, $\mathrm{ot}(c_{\beta_{n-1}} \cap \xi) = \mathrm{ot}(c_{\gamma_{n-1}} \cap \xi)$. But for all large enough ξ, $\rho_0(\xi,\beta)(n-1) = \mathrm{ot}(c_{\beta_{n-1}} \cap \xi)$ and $\rho_0(\xi,\gamma)(n-1) = \mathrm{ot}(c_{\gamma_{n-1}} \cap \xi)$. Since $\rho_0(\xi,\beta) = \rho_0(\xi,\gamma)$, $\mathrm{ot}(c_{\beta_{n-1}} \cap \xi) = \mathrm{ot}(c_{\gamma_{n-1}} \cap \xi)$.

Finally, suppose that $\alpha_0 < \alpha$ and $\alpha_1 = \alpha$. First we prove that $m = n+1$. If we take a large enough $\xi \in c_\alpha$, then the walk from β to ξ is equal to $\langle \beta_0,\ldots,\beta_n,\xi\rangle$, and $\langle \gamma_0,\ldots,\gamma_{m-1}\rangle$ is a proper initial segment of the walk from γ to ξ. Since $\rho_0(\xi,\beta) = \rho_0(\xi,\gamma)$, the walks from β to ξ and from γ to ξ

have the same length, namely $n+2$. Therefore the walk $\langle\gamma_0,\dots,\gamma_{m-1}\rangle$ has length at most $n+1$, that is, $m \le n+1$.

On the other hand, choosing a large enough ξ in $c_{\gamma_{m-1}}\cap\alpha$, $\langle\gamma_0,\dots,\gamma_{m-1},\xi\rangle$ is the walk from γ to ξ, and α is in the walk from β to ξ. So the walk from γ to ξ has length $m+1$. Since $\rho_0(\xi,\beta)=\rho_0(\xi,\gamma)$, the walk from β to ξ has length $m+1$. But the sequence $\langle\beta_0,\dots,\beta_n\rangle$ is a proper initial segment of the walk from β to ξ, so the length of this sequence is less than $m+1$, that is, $n+1\le m$. So $m=n+1$.

Now we show that $\rho_0(\alpha,\beta)=\rho_0(\gamma_{m-1},\gamma)$. Since $m=n+1$, the walks from β to α and from γ to γ_{m-1} have the same length, so $\rho_0(\alpha,\beta)$ and $\rho_0(\gamma_{m-1},\gamma)$ have the same length. To show they are equal, it suffices to show they are initial segments of the same sequence. Choose a large enough ξ so that α is in the walk from β to ξ and γ_{m-1} is in the walk from γ to ξ. Then $\rho_0(\alpha,\beta)\sqsubset\rho_0(\xi,\beta)$ and $\rho_0(\gamma_{m-1},\gamma)\sqsubset\rho_0(\xi,\gamma)$ by Lemma 3.3. Since $\rho_0(\xi,\beta)=\rho_0(\xi,\gamma)$, $\rho_0(\alpha,\beta)=\rho_0(\gamma_{m-1},\gamma)$.

Now $\rho_0(\alpha,\gamma)=\rho_0(\gamma_{m-1},\gamma)^\frown\mathrm{ot}(c_{\gamma_{m-1}}\cap\alpha)=\rho_0(\alpha,\beta)^\frown\mathrm{ot}(c_{\gamma_{m-1}}\cap\alpha)$. So to complete the proof, it suffices to show that $\mathrm{ot}(c_{\gamma_{m-1}}\cap\alpha)=\mathrm{ot}(c_\alpha)$. Since α is a limit ordinal, it suffices to show that for all large enough $\xi<\alpha$, $\mathrm{ot}(c_{\gamma_{m-1}}\cap\xi)=\mathrm{ot}(c_\alpha\cap\xi)$. Choose ξ large enough so that α is in the walk from β to ξ and γ_{m-1} is in the walk from γ to ξ. Then $\rho_0(\xi,\beta)(n)=\mathrm{ot}(c_\alpha\cap\xi)$ and $\rho_0(\xi,\gamma)(m-1)=\mathrm{ot}(c_{\gamma_{m-1}}\cap\xi)$. Since $\rho_0(\xi,\beta)=\rho_0(\xi,\gamma)$ and $n=m-1$, we are done. ∎

4. Weak square implies a special Aronszajn tree. We prove now that the existence of a weak square sequence on a regular uncountable cardinal κ implies the existence of a special Aronszajn tree on κ. Fix a C-sequence $\langle c_\alpha:\alpha<\kappa\rangle$, and let ρ_0 be the full code. For each $\beta<\kappa$, define $\rho_{0\beta}:\beta\to{}^{<\omega}\beta$ by letting $\rho_{0\beta}(\xi)=\rho_0(\xi,\beta)$ for $\xi<\beta$. Recall the tree $T(\rho_0)$ of Todorčević [3]: for each $\alpha<\kappa$, level α of $T(\rho_0)$ consists of functions of the form $\rho_{0\beta}\restriction\alpha$, where $\alpha\le\beta<\kappa$. For $u,v\in T(\rho_0)$, $u<_{T(\rho_0)}v$ if $v\restriction\mathrm{dom}(u)=u$.

Our goal is to prove that under some additional assumptions on the C-sequence, the tree $T(\rho_0)$ is a special Aronszajn tree. The existence of a C-sequence satisfying these assumptions follows from the existence of a weak square sequence. Our proof is based on the proof of Todorčević [4] that there exists a special Aronszajn tree on κ for any non-Mahlo strongly inaccessible cardinal κ (2).

It is clear that $T(\rho_0)$ is a tree of height κ. The next lemma will imply that if $|\{c_\beta\cap\xi:\beta<\kappa\}|<\kappa$ for every $\xi<\kappa$, then $T(\rho_0)$ is a κ-tree. The

(2) In that proof it is claimed that for a limit ordinal α and $\alpha\le\beta,\gamma$, if $\rho_0(\xi,\beta)=\rho_0(\xi,\gamma)$ for all $\xi<\alpha$, then $\rho_0(\alpha,\beta)=\rho_0(\alpha,\gamma)$. This claim appears to be incorrect even with the C-sequence used there. We replace this claim with Proposition 3.6.

proof is based on the argument in [1] that $\square^*_\mu$ implies the existence of a special Aronszajn tree on μ^+ for any infinite cardinal μ.

LEMMA 4.1. *Let $\alpha < \kappa$ be a limit ordinal, and let $\alpha \le \beta, \gamma$. Let $\langle \beta_0, \ldots, \beta_n \rangle$ be the walk from β to α and let $\langle \gamma_0, \ldots, \gamma_m \rangle$ be the walk from γ to α. Suppose that the sequences $\langle c_{\beta_0} \cap \alpha, \ldots, c_{\beta_n} \cap \alpha \rangle$ and $\langle c_{\gamma_0} \cap \alpha, \ldots, c_{\gamma_m} \cap \alpha \rangle$ are equal. Then $\rho_{0\beta} \restriction \alpha = \rho_{0\gamma} \restriction \alpha$.*

Proof. Note that $n = m$. Let $\xi < \alpha$ be given. Let $i \le n$ be least such that $c_{\beta_i} \cap [\xi, \alpha)$ is nonempty. By Lemma 3.2, β_i is in the walk from β to ξ. The next step of the walk from β to ξ after β_i is $\beta^* = \min(c_{\beta_i} \setminus \xi) < \alpha$. Due to the agreement described in the assumptions, i is also least such that $c_{\gamma_i} \cap [\xi, \alpha)$ is nonempty, γ_i is in the walk from γ to ξ, and $\gamma^* = \min(c_{\gamma_i} \setminus \xi) = \beta^*$ is the next step of the walk from γ to ξ after γ_i. By the agreement we have $\rho_0(\xi, \beta) = \langle \mathrm{ot}(c_{\beta_0} \cap \xi), \ldots, \mathrm{ot}(c_{\beta_i} \cap \xi) \rangle^\frown \rho_0(\xi, \beta^*) = \langle \mathrm{ot}(c_{\gamma_0} \cap \xi), \ldots$ $\ldots, \mathrm{ot}(c_{\gamma_i} \cap \xi) \rangle^\frown \rho_0(\xi, \gamma^*) = \rho_0(\xi, \gamma)$. ∎

PROPOSITION 4.2. *Suppose the C-sequence $\langle c_\alpha : \alpha < \kappa \rangle$ is such that for every $\xi < \kappa$, $|\{c_\beta \cap \xi : \beta < \kappa\}| < \kappa$. Then $T(\rho_0)$ is a κ-tree.*

Proof. Let $\xi < \kappa$ be given; we show that level ξ of the tree $T(\rho_0)$ has size less than κ. Note that it suffices to prove this statement for limit ordinals ξ. For in general, level γ of the tree is equal to $\{\rho_{0\gamma+n} \restriction \gamma : n < \omega\} \cup \{t \restriction \gamma : t \in T(\rho_0)_{\gamma+\omega}\}$.

So let ξ be a limit ordinal. By the previous lemma, for all $\beta \ge \xi$, the function $\rho_{0\beta} \restriction \xi$ is determined from the finite sequence $\langle c_{\beta_0} \cap \xi, \ldots, c_{\beta_n} \cap \xi \rangle$, where $\langle \beta_0, \ldots, \beta_n \rangle$ is the walk from β to ξ. By assumption, there are fewer than κ many possibilities for such a sequence. So there are fewer than κ many functions of the form $\rho_{0\beta} \restriction \xi$ for $\beta < \kappa$. ∎

Assume that there exists a weak square sequence on κ. Then by Lemma 1.2, we can fix a C-sequence $\langle c_\alpha : \alpha < \kappa \rangle$ satisfying the following conditions:

(1) there exists a club $C \subseteq \kappa \cap \mathrm{Lim}$ such that for all α in C, $\mathrm{ot}(c_\alpha) < \min(c_\alpha)$;
(2) for all $\alpha \in (\kappa \cap \mathrm{Lim}) \setminus C$, $\min(c_\alpha) > \sup(C \cap \alpha)$;
(3) for every $\xi < \kappa$, $|\{c_\alpha \cap \xi : \alpha < \kappa\}| < \kappa$.

Let ρ_0 be the full code defined from this C-sequence. We will prove that $T(\rho_0)$ is a special Aronszajn tree.

Let $\langle \alpha_0, \ldots, \alpha_n \rangle \mapsto \ulcorner \langle \alpha_0, \ldots, \alpha_n \rangle \urcorner$ be some coding of finite sequences of ordinals in κ by ordinals in κ. Let D be the club set of ordinals $\alpha \in C$ which are closed under this mapping.

LEMMA 4.3. *For all $\alpha \in C$ and $\beta \ge \alpha$, $\mathrm{ot}(c_\beta \cap \alpha) < \alpha$. Hence for all $\alpha \in D$ and $\gamma \ge \alpha$, $\ulcorner \rho_0(\alpha, \gamma) \urcorner < \alpha$.*

Proof. Fix $\alpha \in C$ and $\beta \geq \alpha$. If β is a successor ordinal then $c_\beta \cap \alpha = \emptyset$. Suppose β is a limit ordinal. If β is not in C, then $\alpha \leq \sup(C \cap \beta) < \min(c_\beta)$. Therefore $c_\beta \cap \alpha = \emptyset$. Now suppose that β is in C. If $c_\beta \cap \alpha = \emptyset$ then we are done. Otherwise $\mathrm{ot}(c_\beta \cap \alpha) \leq \mathrm{ot}(c_\beta) < \min(c_\beta) < \alpha$. ∎

THEOREM 4.4. *The tree $T(\rho_0)$ is a special Aronszajn tree.*

Proof. Let $U = \{t \in T(\rho_0) : \mathrm{ht}(t) \in D\}$. We will define a function $g : U \to \kappa$ satisfying:

(a) $g(t) < \mathrm{ht}(t)$ for all $t \in U$;
(b) $t \sqsubset u$ in U implies $g(t) \neq g(u)$.

Let us note that the existence of such a function g implies that $T(\rho_0)$ is special. For in that case, define $h : T(\rho_0) \to \kappa$ as follows. For $t \in U$, let $h(t) = g(t)$. For $t \in T(\rho_0) \setminus U$, let $h(t) = \sup(D \cap \mathrm{ht}(t))$. Then $h(t) < \mathrm{ht}(t)$ for all nonminimal t. Consider $\nu < \kappa$; we show that $h^{-1}(\{\nu\})$ is the union of fewer than κ many antichains. If $h(t) = \nu$ and $t \notin U$, then $\nu < \mathrm{ht}(t) < \min(D \setminus \nu+1)$. There are fewer than κ many such nodes t. Enumerate them as $\{t_i : i < \lambda\}$ where $\lambda < \kappa$. Define $f_\nu : h^{-1}(\{\nu\}) \to \lambda + 1$ by letting $f_\nu(t_i) = i$ for $i < \lambda$ and $f_\nu(t) = \lambda$ if $h(t) = \nu$ and $t \in U$. If $f_\nu(t) = f_\nu(u)$ then clearly $t, u \in U$. Hence $h(t) = g(t) = \nu$ and $h(u) = g(u) = \nu$, so $t \sqsubset u$ is not possible by the properties of g.

Now we define the function $g : U \to \kappa$. Consider $t \in T(\rho_0)$ with $\mathrm{ht}(t) \in D$. Let $\alpha = \mathrm{ht}(t)$. Define $A(t,0)$ as the set of $\beta \geq \alpha$ with $\rho_{0\beta} \restriction \alpha = t$ such that, letting $\langle \beta_0, \ldots, \beta_n \rangle$ be the walk from β to α, $\sup(c_{\beta_{n-1}} \cap \alpha) < \alpha$. Define $A(t,1)$ as the set of $\gamma \geq \alpha$ with $\rho_{0\gamma} \restriction \alpha = t$ such that, letting $\langle \gamma_0, \ldots, \gamma_m \rangle$ be the walk from γ to α, $\sup(c_{\gamma_{m-1}} \cap \alpha) = \alpha$. By Proposition 3.6 we have:

(1) for all $\beta, \beta' \in A(t,0)$, $\rho_0(\alpha, \beta) = \rho_0(\alpha, \beta')$;
(2) for all $\gamma, \gamma' \in A(t,1)$, $\rho_0(\alpha, \gamma) = \rho_0(\alpha, \gamma')$;
(3) for all $\beta \in A(t,0)$ and $\gamma \in A(t,1)$, $\rho_0(\alpha, \gamma) = \rho_0(\alpha, \beta)^\frown \mathrm{ot}(c_\alpha)$.

The definition of $g(t)$ splits into cases. First assume that one of $A(t,0)$ or $A(t,1)$ is empty. Fix any $\gamma \geq \alpha$ with $t = \rho_{0\gamma} \restriction \alpha$, and let

$$g(t) = \ulcorner \langle \ulcorner \rho_0(\alpha, \gamma) \urcorner, 0 \rangle \urcorner.$$

Note that by (1) and (2) and the case assumption, the definition of $g(t)$ is independent of γ. Secondly, assume that $A(t,0)$ and $A(t,1)$ are both nonempty. Fix any $\gamma \in A(t,1)$, and define

$$g(t) = \ulcorner \langle \ulcorner \rho_0(\alpha, \gamma) \urcorner, 1 \rangle \urcorner.$$

By (2), the definition of $g(t)$ is independent of γ. Note that $g(t) < \mathrm{ht}(t)$ by Lemma 4.3.

To complete the proof, we show that if $t, u \in U$, then $t \sqsubset u$ implies $g(t) \neq g(u)$. So let $t \sqsubset u$ be given, and let $\alpha = \mathrm{ht}(t)$ and $\delta = \mathrm{ht}(u)$. So $\alpha < \delta$. Assume for a contradiction that $g(t) = g(u)$. Note that $g(t)$ and $g(u)$ are defined by the same case, since the case is coded by a 0 or 1 in the definition of g.

First suppose $g(t)$ and $g(u)$ are defined as in the first case. Fix $\gamma \geq \delta$ such that $u = \rho_{0\gamma}\restriction\delta$. Since $t \sqsubset u$, $t = \rho_{0\gamma}\restriction\alpha$. So

$$\ulcorner\langle\ulcorner\rho_0(\alpha,\gamma)\urcorner, 0\rangle\urcorner = g(t) = g(u) = \ulcorner\langle\ulcorner\rho_0(\delta,\gamma)\urcorner, 0\rangle\urcorner.$$

Therefore $\rho_0(\alpha,\gamma) = \rho_0(\delta,\gamma)$. But by Lemma 3.4, $\alpha < \delta$ implies that $\rho_0(\alpha,\gamma) <_r \rho_0(\delta,\gamma)$, and in particular these sequences are different. So we have a contradiction.

Now suppose $g(t)$ and $g(u)$ are defined as in the second case. Fix $\gamma \in A(u,1)$. Then $u = \rho_{0\gamma}\restriction\delta$ and

$$g(u) = \ulcorner\langle\ulcorner\rho_0(\delta,\gamma)\urcorner, 1\rangle\urcorner.$$

Since $t \sqsubset u$, $t = \rho_{0\gamma}\restriction\alpha$. Now there are two cases, depending on whether γ is in $A(t,0)$ or $A(t,1)$. If $\gamma \in A(t,1)$, then

$$g(t) = \ulcorner\langle\ulcorner\rho_0(\alpha,\gamma)\urcorner, 1\rangle\urcorner.$$

But $g(t) = g(u)$ implies $\rho_0(\alpha,\gamma) = \rho_0(\delta,\gamma)$. This contradicts Lemma 3.4.

If $\gamma \in A(t,0)$, then fix some $\gamma' \in A(t,1)$. Then

$$g(t) = \ulcorner\langle\ulcorner\rho_0(\alpha,\gamma')\urcorner, 1\rangle\urcorner.$$

Since $g(t) = g(u)$, we have $\rho_0(\alpha,\gamma') = \rho_0(\delta,\gamma)$. But by Proposition 3.6(3),

$$\rho_0(\delta,\gamma) = \rho_0(\alpha,\gamma') = \rho_0(\alpha,\gamma)^\frown\mathrm{ot}(c_\alpha).$$

So $\rho_0(\alpha,\gamma)$ is a proper initial segment of $\rho_0(\delta,\gamma)$, which implies $\rho_0(\delta,\gamma) <_r \rho_0(\alpha,\gamma)$. But by Lemma 3.4, $\alpha < \delta$ implies $\rho_0(\alpha,\gamma) <_r \rho_0(\delta,\gamma)$, and we have a contradiction. ∎

Remark. If κ is a strongly inaccessible non-Mahlo cardinal, then there exists a weak square sequence on κ. Namely, let C be a club set of singular cardinals, and for each $\alpha \in C$, choose c_α as a club subset of α with order type $\mathrm{cf}(\alpha)$. Then for every $\xi < \kappa$, $|\{c_\alpha \cap \xi : \alpha < \kappa\}| \leq 2^{|\xi|} < \kappa$. We pose the following question: is it consistent that there is a weakly inaccessible non-Mahlo cardinal which does not carry a weak square sequence?

References

[1] J. Cummings, *Notes on singular cardinal combinatorics*, Notre Dame J. Formal Logic 46 (2005), 251–282.

[2] R. Jensen, *The fine structure of the constructible hierarchy*, Ann. Math. Logic 4 (1972), 229–308.

[3] S. Todorčević, *Partitioning pairs of countable ordinals*, Acta Math. 159 (1987), 261–294.

[4] S. Todorčević, *Walks on Ordinals and Their Characteristics*, Progr. Math. 263, Birkhäuser, Basel, 2007.

John Krueger
Department of Mathematics
University of North Texas
1155 Union Circle #311430
Denton, TX 76203, U.S.A.
E-mail: jkrueger@unt.edu

Received September 3, 2012;
in revised form March 17, 2013