FUNDAMENTA
MATHEMATICAE
189 (2006)

Refining thick subcategory theorems

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Abstract. We use a K-theory recipe of Thomason to obtain classifications of triangulated subcategories via refining some standard thick subcategory theorems. We apply this recipe to the full subcategories of finite objects in the derived categories of rings and the stable homotopy category of spectra. This gives, in the derived categories, a complete classification of the triangulated subcategories of perfect complexes over some commutative rings. In the stable homotopy category of spectra we obtain only a partial classification of the triangulated subcategories of the finite p-local spectra. We use this partial classification to study the lattice of triangulated subcategories. This study gives some new evidence for a conjecture of Adams that the thick subcategory $\mathcal{C}_2$ can be generated by iterated cofiberings of the Smith–Toda complex. We also discuss several consequences of these classification theorems.

1. INTRODUCTION

Classifying various subcategories of triangulated categories like the derived categories and the homotopy category of spectra has been an active area and has proved to be extremely useful in the study of global problems in (stable) homotopy theory. Several mathematicians brought to light many amazing a priori different theories by classifying various subcategories of triangulated categories. Following the seminal work of Devinatz, Hopkins, and Smith [DHS88] in stable homotopy theory, this line of research was initiated by Hopkins in the 80s. In his famous 1987 paper [Hop87], Hopkins classified the thick subcategories (triangulated subcategories that are closed under retractions) of the finite p-local spectra and those of perfect complexes over a noetherian ring. He showed that thick subcategories of the finite spectra are determined by the Morava K-theories and those of perfect complexes by the prime spectrum of the ring. These results have had a tremendous impact in their respective fields. The thick subcategory theorem for finite spectra played a vital role in the study of nilpotence and period-

2000 *Mathematics Subject Classification*: Primary: 55P42, 18E30, 18F30.

Key words and phrases: thick subcategory, triangulated subcategory, Grothendieck group, derived category, finite spectra, Smith–Toda complex.

icity. For example, using this theorem Hopkins and Smith [HS98] were able to settle the class-invariance conjecture of Ravenel [Rav84] which classified the Bousfield classes of finite spectra. Similarly the thick subcategory theorem for the derived category establishes a surprising connection between stable homotopy theory and algebraic geometry; using this theorem one is able to recover the spectrum of a ring from the homotopy structure of its derived category! These ideas were later pushed further into the world of derived categories of rings and schemes by Neeman [Nee92] and Thomason [Tho97], and into modular representation theory by Benson, Carlson and Rickard [BCR97]. Motivated by the work of Hopkins [Hop87], Neeman [Nee92] classified the Bousfield classes and localising subcategories in the derived category of a noetherian ring. In modular representation theory, the Benson–Carlson–Rickard classification of the thick subcategories of stable modules over group algebras has led to some deep structural information on the representation theory of finite groups. Finally, the birth of axiomatic stable homotopy theory [HPS97] in the mid 90s encompassed all these various theories and ideas and studied them all in a more general framework. With all these developments over the last 30 years, the importance of triangulated categories in modern mathematics is by now abundantly clear.

The goal of this paper is to classify the triangulated subcategories analogous to the aforementioned classifications of thick subcategories. To motivate this project further, let us consider the following question. Let $\mathcal{T}$ be a triangulated category and X and Y be two objects in $\mathcal{T}$.

Question A. When can Y be generated from X using *cofibrations* and *retractions*?

Knowledge of the thick subcategories of $\mathcal{T}$ will help us answer this question. For example, if $\mathcal{T}$ is the category of perfect complexes over a noetherian ring or the category of finite p-local spectra, then we know (from the Hopkins thick subcategory theorems) that Y can be generated from X using cofibrations and retractions if and only if $\mathrm{Supp}(Y) \subseteq \mathrm{Supp}(X)$. (When X is a perfect complex, $\mathrm{Supp}(X)$ is the set of primes p for which $X \otimes R_p \neq 0$; and when X is a finite p-local spectrum, by $\mathrm{Supp}(X)$ we mean the *chromatic support* of X, i.e., the set of non-negative integers n for which $K(n)_* X \neq 0$.) This is quite remarkable because often "support" is a computable invariant while cofibrations and retractions can be extremely hard to understand. Now let us ask an even more subtle and stringent question.

Question B. When can Y be generated from X using *cofibrations* alone?

We say that this is a stringent question because it is much harder, in general, to work with cofibrations alone. For example, take $X = M(p)$ (the

Moore spectrum) and $Y = M(p^2)$; clearly $\mathrm{Supp}(M(p)) = \mathrm{Supp}(M(p^2))$, therefore $M(p)$ can be generated from $M(p^2)$ using cofibrations and retractions. However, it is impossible, as we will see, to generate $M(p)$ from $M(p^2)$ just using cofibrations. On the other hand, $M(p^2)$ can be generated from $M(p)$ using cofibrations: There is a cofibre sequence

$$\Sigma^{-1}M(p) \to M(p) \to M(p^2).$$

So having a classification of triangulated subcategories will help us answer Question B.

In order to classify the triangulated subcategories, we use a K-theoretic approach of Thomason to refine Hopkins's thick subcategory theorems, both for finite spectra and perfect complexes. Note that, in both these categories, the support condition ($\mathrm{Supp}(Y) \subseteq \mathrm{Supp}(X)$) is a necessary condition for the generation in Question B. Thomason's K-theory technique helps us to come up with a sufficient condition! More precisely, we construct universal Euler characteristic functions χ on $\mathcal{T}$, and show that Y can be generated from X using cofibrations if and only if $\mathrm{Supp}(Y) \subseteq \mathrm{Supp}(X)$ and $\chi(X)$ divides $\chi(Y)$; see Corollaries 3.7, 3.13, 4.14, and 4.21.

We also discuss the following conjecture of Adams. In his last (unpublished) paper [Ada92], Adams conjectured that the Smith–Toda complex (which is known to exist at odd primes) generates the thick subcategory $\mathcal{C}_2$ by cofibrations. (Note that this is clearly possible if we allow retractions.) Using BP-based homology theories, we construct Euler characteristic functions χ_n on the thick subcategory $\mathcal{C}_n$:

$$\chi_n(X) := \sum_i (-1)^i \log_p |\mathrm{BP}\langle n-1\rangle_i X|.$$

These Euler characteristic functions give us triangulated subcategories $\mathcal{C}_n^k$ of $\mathcal{C}_n$:

$$\mathcal{C}_n^k = \{X \in \mathcal{C}_n : \chi_n(X) \equiv 0 \bmod l_n k\},$$

where l_n is the smallest positive value of χ_n on $\mathcal{C}_n$. We then show (using a Bockstein spectral sequence) that, for all n and k,

$$\mathcal{C}_{n+1} \subsetneq \mathcal{C}_n^k \subsetneq \mathcal{C}_n.$$

This gives only some evidence for Adams's conjecture (when $n = 1$ the above inclusions follow trivially from the Adams conjecture). The conjecture, however, remains open.

This paper is organised as follows. We set up our categorical stage in the next section. We begin with a quick recap of Grothendieck groups and some categorical definitions. We then explain a K-theoretic technique of Thomason which will be used in the later sections to obtain classifications of triangulated subcategories. We also revise Thomason's theorem for triangulated categories that are equipped with a nice product. We apply these techniques to spectra (Section 3) and derived categories (Section 4). As consequences of these classifications, we will study various structural results on triangulated subcategories. We also record some questions that come up along the way.

Acknowledgements. I would like to thank my advisor John Palmieri for the innumerable conversations that we had on this material. Among others, I thank Mark Hovey, for bringing to my attention the work of Adams on universal Euler characteristics, and Steve Mitchell and Hal Sadofsky for some interesting discussions on the subject. I am also grateful to an anonymous referee for many valuable suggestions.

2. THOMASON'S K-THEORY RECIPE

2.1. Grothendieck groups. We begin by recalling some definitions and results from [Tho97]. Let $\mathcal{T}$ denote a triangulated category that is essentially small (i.e., it has only a set of isomorphism classes of objects). Then the *Grothendieck group* $K_0(\mathcal{T})$ is defined to be the free abelian group on the isomorphism classes of $\mathcal{T}$ modulo the Euler relations $[B] = [A] + [C]$ whenever $A \to B \to C \to \Sigma A$ is an exact triangle in $\mathcal{T}$ (here $[X]$ denotes the element in the Grothendieck group that is represented by the isomorphism class of the object X). This is clearly an abelian group with $[0]$ as the identity element and $[\Sigma X]$ as the inverse of $[X]$. The identity $[A]+[B] = [A \amalg B]$ holds in the Grothendieck group. Also note that any element of $K_0(\mathcal{T})$ is of the form $[X]$ for some $X \in \mathcal{T}$. All these facts follow easily from the axioms for a triangulated category.

Grothendieck groups have the following universal property: Any map σ from the set of isomorphism classes of $\mathcal{T}$ to an abelian group G such that the Euler relations hold in G factors through a unique homomorphism $f : K_0(\mathcal{T}) \to G$; see the diagram below.

$$\begin{tikzcd}
\{\text{isomorphism classes of objects in}\,\mathcal{T}\} \arrow[d, "\pi"] \arrow[dr, "\sigma"] & \\
K_0(\mathcal{T}) \arrow[r, dotted, "f"] & G
\end{tikzcd}$$

Therefore the map π will be called the *universal Euler characteristic function.* $K_0(-)$ is clearly a covariant functor from the category of small triangulated categories to the category of abelian groups.

Unless stated otherwise, all subcategories in this paper are assumed to be *full* subcategories.

DEFINITION 2.1. An object X in a triangulated category $\mathcal{T}$ is *small* if the natural map

$$\bigoplus_{\alpha\in\Lambda} \mathrm{Hom}(X, A_\alpha) \to \mathrm{Hom}\Big(X, \coprod_{\alpha\in\Lambda} A_\alpha\Big)$$

is an isomorphism for all set-indexed collections of objects A_α in $\mathcal{T}$. (Some authors call such objects finite or compact objects.)

DEFINITION 2.2. A triangulated subcategory $\mathcal{C}$ of $\mathcal{T}$ is said to be *thick* if it is closed under retractions, i.e., given a commuting diagram

$$\begin{tikzcd} B \arrow[r, "i"] \arrow[rr, bend right, "="'] & A \arrow[r, "r"] & B \end{tikzcd}$$

in $\mathcal{T}$ such that A is an object of $\mathcal{C}$, then so is B. Since retractions split in a triangulated category, this property of $\mathcal{C}$ is equivalent to saying that $\mathcal{C}$ is closed under direct summands: $A \amalg B \in \mathcal{C} \Rightarrow A \in \mathcal{C}$ and $B \in \mathcal{C}$.

EXAMPLE 2.3. The full subcategory of small objects in any triangulated category is thick.

DEFINITION 2.4. We say that a triangulated subcategory $\mathcal{C}$ is *dense* in $\mathcal{T}$ if every object in $\mathcal{T}$ is a direct summand of some object in $\mathcal{C}$.

The following theorem due to Thomason [Tho97] is the foundational theorem that motivated this paper.

THEOREM 2.5 ([Tho97, Theorem 2.1]). *Let $\mathcal{T}$ be an essentially small triangulated category. Then there is a natural order preserving bijection between the posets*

$$\{\textit{dense triangulated subcategories } \mathcal{A} \textit{ of } \mathcal{T}\} \underset{g}{\overset{f}{\rightleftarrows}} \{\textit{subgroups } H \textit{ of } K_0(\mathcal{T})\}.$$

The map f sends $\mathcal{A}$ to the image of the map $K_0(\mathcal{A}) \to K_0(\mathcal{T})$, and the map g sends H to the full subcategory of all objects X in $\mathcal{T}$ such that $[X] \in H$.

2.2. Thomason's K-theory recipe. The importance of Thomason's Theorem 2.5 can be seen from the following simple observation. *Every triangulated subcategory $\mathcal{A}$ of $\mathcal{T}$ is dense in a unique thick subcategory of $\mathcal{T}$*—the one obtained by taking the intersection of all the thick subcategories of $\mathcal{T}$ that contain $\mathcal{A}$. This observation in conjunction with Theorem 2.5 gives the following brilliant recipe of Thomason for the problem of classifying the triangulated subcategories of $\mathcal{T}$:

(1) Classify the thick subcategories of $\mathcal{T}$.
(2) Compute the Grothendieck groups of all thick subcategories.
(3) Apply Thomason's Theorem 2.5 to each thick subcategory of $\mathcal{T}$.

We will apply this recipe to the categories of small objects in some stable homotopy categories like the stable homotopy category of spectra and the derived categories of rings. The stable homotopy structure on these categories will guide us while applying this recipe and consequently we will derive some structural information on these categories.

2.3. Grothendieck ring. Throughout this subsection $\mathcal{T}$ will denote a tensor triangulated category that is essentially small. Now making use of the available smash product, we want to define a ring structure on the Grothendieck group. This can be done in a very natural and obvious way.

Definition 2.6. If $[A]$ and $[B]$ are any two elements of $K_0(\mathcal{T})$, then define $[A][B] := [A \wedge B]$.

This can be easily shown to be a well defined operation and endows $K_0(\mathcal{T})$ with the structure of a commutative ring. The Grothendieck class of the unit object $[S]$ serves as the identity element in the ring.

The following three definitions are motivated by their analogues in commutative ring theory.

Definition 2.7. A full triangulated subcategory $\mathcal{C}$ of $\mathcal{T}$ is said to be $\otimes$-*closed* or a *triangulated ideal* if for all $A \in \mathcal{T}$ and all $B \in \mathcal{C}$, $B \wedge A \in \mathcal{C}$. A full triangulated ideal is said to be respectively thick or dense if it is so as a triangulated subcategory.

Definition 2.8. A full triangulated ideal $\mathcal{B}$ of $\mathcal{T}$ is said to be *prime* if for all $X, Y \in \mathcal{T}$ such that $X \wedge Y \in \mathcal{B}$, either $X \in \mathcal{B}$ or $Y \in \mathcal{B}$. Similarly $\mathcal{B}$ is said to be *maximal* in $\mathcal{T}$ if there is no triangulated ideal $\mathcal{A}$ such that $\mathcal{B} \subsetneq \mathcal{A} \subsetneq \mathcal{T}$.

Definition 2.9. We say that a triangulated category $\mathcal{A}$ is a *triangulated module* over a tensor triangulated category $\mathcal{T}$ if there is a triangulated functor

$$\phi : \mathcal{T} \times \mathcal{A} \to \mathcal{A}$$

that is covariant and exact in each variable and satisfies the obvious unital and associative conditions. A full triangulated subcategory $\mathcal{B}$ of $\mathcal{A}$ is a *triangulated submodule* of $\mathcal{A}$ if the functor ϕ maps $\mathcal{T} \times \mathcal{B} \to \mathcal{B}$. Note that in this situation, $K_0(A)$ becomes a $K_0(\mathcal{T})$-module, and $K_0(B)$ a $K_0(\mathcal{T})$-submodule.

We need the following lemma to upgrade Thomason's theorem to tensor triangulated categories.

Lemma 2.10 ([Tho97, Lemma 2.2]). *Let $\mathcal{A}$ be a dense triangulated subcategory of an essentially small triangulated category $\mathcal{T}$. Then for any object X in $\mathcal{T}$, $X \in \mathcal{A}$ if and only if $[X] = 0 \in K_0(\mathcal{T})/\mathrm{Im}(K_0(\mathcal{A}) \to K_0(\mathcal{T}))$.*

The following results are now expected.

THEOREM 2.11. *Let $\mathcal{T}$ be a tensor triangulated category that is essentially small. Then, under Thomason's bijection* (*Theorem* 2.5)

$$\{\textit{dense triangulated subcategories of } \mathcal{T}\} \leftrightarrow \{\textit{subgroups of } K_0(\mathcal{T})\},$$

we have the following correspondences.

(1) *The dense triangulated ideals correspond precisely to the ideals of the ring $K_0(\mathcal{T})$.*
(2) *The dense prime triangulated ideals correspond precisely to the prime ideals of $K_0(\mathcal{T})$.*
(3) *The dense maximal triangulated ideals correspond precisely to the maximal ideals of $K_0(\mathcal{T})$.*

Proof. Except possibly the second statement, everything else is straightforward.

1. Let I be any ideal in $K_0(\mathcal{T})$. Then the corresponding dense triangulated subcategory is $\mathcal{T}_I = \{X \in \mathcal{T} : [X] \in I\}$. Now for $A \in \mathcal{T}$ and $B \in A_I$, note that $[B \wedge A] = [B][A] \in I$ (I being an ideal) and therefore $B \wedge A \in \mathcal{T}_I$. This shows that $\mathcal{T}_I$ is a triangulated ideal. The other direction is equally easy.

2. Suppose H is a prime ideal in $K_0(\mathcal{T})$. The corresponding dense triangulated ideal is given by $\mathcal{B} = \{X : [X] \in H\}$. Now if $A \wedge B \in \mathcal{B}$, then by definition of $\mathcal{B}$, we have $[A \wedge B] \in H$, or equivalently $[A][B] \in H$. Now primality of H implies that either $[A] \in H$ or $[B] \in H$, which means either $A \in \mathcal{B}$ or $B \in \mathcal{B}$. For the other direction, suppose $\mathcal{B}$ is a dense prime triangulated ideal of $\mathcal{T}$. The corresponding subgroup H is the image of the map $K_0(\mathcal{B}) \to K_0(\mathcal{T})$. Now suppose the product of two elements $[A]$ and $[B]$ belongs to H. Then we have $[A \wedge B] \in H$ or equivalently $[A \wedge B] = 0$ in $K_0(\mathcal{T})/H$. By the above lemma, we then have $A \wedge B \in \mathcal{B}$. Since $\mathcal{B}$ is prime, this implies that either $A \in \mathcal{B}$ or $B \in \mathcal{B}$, or equivalently $[A] \in H$ or $[B] \in H$.

3. This follows directly from the fact that Thomason's bijection is a map of posets. ■

In the same spirit we get the following result for triangulated modules over tensor triangulated categories. We leave the proof, which is similar to that of the above theorem, as an easy exercise to the reader.

THEOREM 2.12. *Let $\mathcal{A}$ be a triangulated module over a tensor triangulated category $\mathcal{T}$. Then, under Thomason's bijection* (*Theorem* 2.5)

$$\{\textit{dense triangulated subcategories of } \mathcal{A}\} \leftrightarrow \{\textit{subgroups of } K_0(\mathcal{A})\},$$

we have the following correspondences.

(1) *The dense triangulated submodules of $\mathcal{A}$ correspond precisely to the $K_0(\mathcal{T})$-submodules of $K_0(\mathcal{A})$.*

(2) *The dense maximal triangulated submodules correspond precisely to the maximal $K_0(\mathcal{T})$-submodules of $K_0(\mathcal{A})$.*

3. TRIANGULATED SUBCATEGORIES OF FINITE SPECTRA

In this section we apply Thomason's Theorem 2.5 to the category $\mathcal{F}_p$ of finite p-local spectra. So we begin by recalling the celebrated thick subcategory theorem of Hopkins and Smith which is a key ingredient in Thomason's recipe for classifying the triangulated subcategories of $\mathcal{F}_p$. We then examine the Grothendieck groups of these thick subcategories. Our partial knowledge of these groups gives us a family of triangulated subcategories of $\mathcal{F}_p$. We then study the lattice of these subcategories in Subsection 3.5 using some spectral sequences. This study gives new evidence for a conjecture of Adams. We end the section with some questions.

For each non-negative integer n, there is a field spectrum called the nth Morava K-theory $K(n)$, whose coefficient ring is $\mathbb{F}_p[v_n, v_n^{-1}]$ with $|v_n| = 2(p^n - 1)$. According to the thick subcategory theorem of Hopkins–Smith these Morava K-theories determine the thick subcategories of $\mathcal{F}_p$. More precisely:

THEOREM 3.1 ([HS98]). *For each positive integer n, let $\mathcal{C}_n$ denote the full subcategory of all finite p-local spectra that are $K(n-1)$-acyclic. Then a non-zero subcategory $\mathcal{C}$ of $\mathcal{F}_p$ is thick if and only if $\mathcal{C} = \mathcal{C}_n$ for some n. Further these thick subcategories give a nested decreasing filtration of $\mathcal{F}_p$* ([Rav84, Mit85]):

$$\cdots \subsetneq \mathcal{C}_{n+1} \subsetneq \mathcal{C}_n \subsetneq \mathcal{C}_{n-1} \subsetneq \cdots \subsetneq \mathcal{C}_1 \subsetneq \mathcal{C}_0 \ (= \mathcal{F}_p).$$

A property P of finite spectra is said to be *generic* if the collection of all spectra in $\mathcal{F}_p$ which satisfy P is $\mathcal{C}_n$ for some n. A spectrum is said to be of *type n* if it belongs to $\mathcal{C}_n - \mathcal{C}_{n+1}$. For example, the sphere spectrum S is of type 0, the mod-p Moore spectrum $M(p)$ is of type 1, etc.

The problem of computing the Grothendieck groups of thick subcategories of the finite p-local spectra was first considered, to our knowledge, by Frank Adams. This appeared in an unpublished manuscript [Ada92, pp. 528–529] of Adams on the work of Hopkins. We begin with a recapitulation of Adams's work and then bring it to this new context of classifying triangulated subcategories. We should also point out that our Euler characteristic functions are simpler than the ones considered by Adams.

3.1. $\mathcal{C}_0$: finite p-local spectra. We begin with the fundamental notion of Euler characteristic of a finite spectrum. Recall that if X is any finite p-local spectrum, the Euler characteristic of X with rational coefficients is given by

$$(1) \qquad \chi_0(X) = \sum_{i=-\infty}^{\infty} (-1)^i \dim_{\mathbb{Q}} H\mathbb{Q}_i(X).$$

Since X is a finite spectrum it has homology concentrated only in some finite range and therefore this is a well defined function.

EXAMPLE 3.2. For every non-negative integer m, define a full subcategory $\mathcal{C}_0^m$ of $\mathcal{C}_0\,(=\mathcal{F}_p)$ as

$$\mathcal{C}_0^m = \{X \in \mathcal{C}_0 : \chi_0(X) \equiv 0 \bmod m\}.$$

It is an easy exercise to verify that these are all dense triangulated subcategories of $\mathcal{C}_0$.

PROPOSITION 3.3. *A triangulated subcategory $\mathcal{C}$ of $\mathcal{C}_0\,(=\mathcal{F}_p)$ is dense in $\mathcal{C}_0$ if and only if $\mathcal{C} = \mathcal{C}_0^m$ for some m.*

Proof. Let S denote the p-local sphere spectrum and recall that $\mathcal{C}_0$ can be generated by iterated cofiberings of the sphere spectrum S^0. Now using the Euler relations, it is clear that the Grothendieck group of $\mathcal{C}_0$ is a cyclic group generated by $[S]$. The map

$$\chi_0 : \{\text{isomorphism classes of } \mathcal{C}_0\} \to \mathbb{Z}$$

which sends an isomorphism class to its Euler characteristic shows that $K_0(\mathcal{C}_0)$ is isomorphic to $\mathbb{Z}$. The proposition now follows by invoking Theorem 2.5. ∎

PROPOSITION 3.4. *$K_0(\mathcal{C}_0) \cong \mathbb{Z}$ as commutative rings.*

Proof. First note that the smash product on $\mathcal{C}_0$ induces a ring structure on $K_0(\mathcal{C}_0)$. Now the isomorphism of abelian groups $\phi_0 : K_0(\mathcal{C}_0) \to \mathbb{Z}$ (from the proof of the previous proposition) maps the Grothendieck class of the wedge of n copies of the sphere spectrum to n. Since the smash product distributes over the wedge, we conclude that ϕ_0 is a ring isomorphism. ∎

We now give some nice consequences of this proposition.

COROLLARY 3.5. *Every triangulated subcategory of $\mathcal{C}_0$ is a triangulated ideal.*

Proof. Let $\mathcal{C}$ be a triangulated subcategory of $\mathcal{C}_0$, and let $\mathcal{C}_n$ denote the unique thick subcategory in which $\mathcal{C}$ is dense. Now note that $\mathcal{C}_n$ is a triangulated module over $\mathcal{C}_0$, therefore $K_0(\mathcal{C}_n)$ is a $K_0(\mathcal{C}_0)$-module. We have seen that $K_0(\mathcal{C}_0) \cong \mathbb{Z}$ as rings. So we apply Theorem 2.12 and observe the simple fact that subgroups of the abelian group $K_0(\mathcal{C}_n)$ are precisely the $\mathbb{Z}$-submodules of $K_0(\mathcal{C}_n)$. ∎

Corollary 3.6. *If $\mathcal{B}$ is a dense triangulated subcategory of $\mathcal{F}_p$, then $\mathcal{B}$ is prime if and only if $\mathcal{B} = \mathcal{C}_0^0$ or $\mathcal{C}_0^p$ for some prime number p. In particular, a dense triangulated subcategory of $\mathcal{F}_p$ is maximal if and only if it is $\mathcal{C}_0^p$ for some prime p.*

Proof. By Theorem 2.11, we have a correspondence between the prime (maximal) dense triangulated subcategories of $\mathcal{F}_p$ and the prime (maximal) ideals of $K_0(\mathcal{F}_p)$ ($\cong \mathbb{Z}$). The corollary now follows by comparing the prime ideals and maximal ideals of $\mathbb{Z}$. ■

Corollary 3.7. *Let $\Delta(X_1, \dots, X_k)$ denote the triangulated subcategory generated by spectra $X_1, \dots, X_k$. Then we have the following.*

(1) *If A is a type-0 spectrum, then $\Delta(A)$ consists of all spectra X in $\mathcal{C}_0$ for which $\chi_0(X)$ is divisible by $\chi_0(A)$.*
(2) *If A is a type-0 spectrum and B is a spectrum in $\mathcal{C}_0$, then B can be generated from A using cofibre sequences if and only if $\chi_0(B)$ is divisible by $\chi_0(A)$.*
(3) *If A is a type-0 spectrum and B is a spectrum in $\mathcal{C}_0$, then $\Delta(A, B)$ consists of all spectra X in $\mathcal{C}_0$ for which $\chi_0(X)$ is divisible by the highest common factor of $\chi_0(A)$ and $\chi_0(B)$.*

Remark 3.8. The above corollaries are interesting structural results on the subcategories of $\mathcal{F}_p$. It is not clear how one would establish such results without using this K-theory approach of Thomason.

3.2. $\mathcal{C}_1$: finite p-torsion spectra. Having classified the dense triangulated subcategories of $\mathcal{C}_0$, we now look at the thick subcategory $\mathcal{C}_1$ consisting of the finite p-torsion spectra. A potential candidate which might generate $\mathcal{C}_2$ as a triangulated category is the Moore spectrum $M(p)$. So before going any further, it is natural to ask if $M(p^2)$ can be generated from $M(p)$ by iterated cofiberings. That this is possible can be easily seen using the octahedral axiom which gives the following commutative diagram of cofibre sequences:

$$\begin{tikzcd}
S \arrow[r, "p"] \arrow[d, equal] & S \arrow[r] \arrow[d, "p"] & M(p) \arrow[d, dotted] \\
S \arrow[r, "p^2"] \arrow[d, "0"] & S \arrow[r] \arrow[d] & M(p^2) \arrow[d, dotted] \\
0 \arrow[r] & M(p) \arrow[r] & M(p)
\end{tikzcd}$$

The exact triangle in the far right is the desired cofibre sequence. Inductively, it is easy to see that $M(p^i)$ can be generated from $M(p)$ using cofibre sequences. This example motivates the next proposition.

PROPOSITION 3.9. *Every spectrum in $\mathcal{C}_1$ can be generated by iterated cofiberings of $M(p)$.*

Proof. First observe that the integral homology of any spectrum in $\mathcal{C}_1$ consists of finite abelian p-groups. Further these spectra have homology concentrated only in a finite range, so we induct on $|H\mathbb{Z}_*(-)|$. If $|H\mathbb{Z}_*(X)|=1$, that means X is a trivial spectrum and it is obviously generated by $M(p)$. So assume that $|H\mathbb{Z}_*(X)| > 1$ and let k be the smallest integer such that $H\mathbb{Z}_k(X)$ is non-zero. Then by the Hurewicz theorem, we know that $\pi_k(X) \cong H\mathbb{Z}_k(X)$. So pick an element of order p in $H\mathbb{Z}_k(X)$ and represent it by a map $\alpha : S^k \to X$. Since $p\alpha = 0$, the composite $S^k \xrightarrow{p} S^k \xrightarrow{\alpha} X$ is zero and hence the map α factors through $\Sigma^k M(p)$. This gives the following diagram where the vertical sequence is a cofibre sequence:

$$\begin{tikzcd} \Sigma^k S \arrow[r, "p"] \arrow[drr, "0"'] & \Sigma^k S \arrow[r] \arrow[dr, "\alpha"] & \Sigma^k M(p) \arrow[d, dotted, "\alpha'"] \\ & & X \arrow[d] \\ & & Y \end{tikzcd}$$

It is easily seen that $H\mathbb{Z}_i(\alpha')$ is non-zero (and hence injective) if $i = k$, and is zero otherwise. Therefore, from the long exact sequence in integral homology induced by the vertical cofibre sequence, it follows that $|H\mathbb{Z}_*(Y)| = |H\mathbb{Z}_*(X)| - p$. The induction hypothesis tells us that Y can be generated by cofibre sequences using $M(p)$ and then the above vertical cofibre sequence tells us that X can also be generated by cofibre sequences using $M(p)$. ∎

Note that the regular Euler characteristic is not good for spectra in $\mathcal{C}_1$ because these spectra are all rationally acyclic and therefore their Euler characteristic is always zero (no matter which field coefficients we use). The integral homology of these spectra consists of finite abelian p-groups and that motivates the following definition.

DEFINITION 3.10. For any spectrum X in $\mathcal{C}_1$, define

$$\chi_1(X) := \sum_{i=-\infty}^{\infty} (-1)^i \log_p |H\mathbb{Z}_i X|, \tag{2}$$

and for every non-negative integer m, define a full subcategory

$$\mathcal{C}_1^m := \{X \in \mathcal{C}_1 : \chi_1(X) \equiv 0 \bmod m\}.$$

THEOREM 3.11. *A triangulated subcategory $\mathcal{C}$ of $\mathcal{C}_1$ is dense in $\mathcal{C}_1$ if and only if $\mathcal{C} = \mathcal{C}_1^m$ for some non-negative integer m.*

Proof. The function $\log_p|-|$ is clearly an additive function on the abelian category of finite abelian p-groups, i.e., whenever $0 \to A \to B \to C \to 0$ is a short exact sequence of finite abelian p-groups, we have $\log_p|B| = \log_p|A| + \log_p|C|$. Using this it is elementary to show that if we have a bounded exact sequence of finite abelian p-groups, then the alternating sum of $\log_p|-|$'s is zero. This applies, in particular, to the long exact sequence in integral homology for a cofibre sequence in $\mathcal{C}_1$. In other words, $\chi_1(-)$ respects the Euler relations in $\mathcal{C}_1$. So we get an induced map $\phi_1 : K_0(\mathcal{C}_1) \to \mathbb{Z}$. From Proposition 3.9, we know that $K_0(\mathcal{C}_1)$ is a cyclic group generated by $[M(p)]$. Since $\phi_1([M(p)]) = 1$, it follows that ϕ_1 is an isomorphism. The given classification is now clear from Theorem 2.5. ∎

COROLLARY 3.12. *A triangulated subcategory $\mathcal{C}$ of $\mathcal{C}_1$ is a maximal triangulated subcategory if and only if $\mathcal{C} = \mathcal{C}_1^p$ for some prime number p.*

Proof. Since $K_0(\mathcal{C}_1) \cong \mathbb{Z}$, the corollary follows (as before) by observing that the maximal subgroups of $\mathbb{Z}$ are precisely the subgroups generated by prime numbers. ∎

Now we state the analogue of Corollary 3.7 for finite p-torsion spectra.

COROLLARY 3.13. *Let $\Delta(X_1, \ldots, X_k)$ denote the triangulated subcategory generated by spectra $X_1, \ldots, X_k$. Then we have the following.*

(1) *If A is a type-1 spectrum, then $\Delta(A)$ consists of all spectra X in $\mathcal{C}_1$ for which $\chi_1(X)$ is divisible by $\chi_1(A)$.*
(2) *If A is a type-1 spectrum and B is a spectrum in $\mathcal{C}_1$, then B can be generated from A using cofibre sequences if and only if $\chi_1(B)$ is divisible by $\chi_1(A)$.*
(3) *If A is a type-1 spectrum and B is a spectrum in $\mathcal{C}_1$, then $\Delta(A, B)$ consists of all spectra X in $\mathcal{C}_1$ for which $\chi_1(X)$ is divisible by the highest common factor of $\chi_1(A)$ and $\chi_1(B)$.*

We now examine the Grothendieck group of the Verdier quotient category $\mathcal{C}_0/\mathcal{C}_n$. We begin with some generalities. If $\mathcal{A}$ is any thick subcategory of a triangulated category $\mathcal{C}$, then we have natural functors fitting into an exact sequence

$$\mathcal{A} \to \mathcal{C} \to \mathcal{C}/\mathcal{A} \to 0,$$

where the first functor is the inclusion functor and the second one is the quotient functor into the Verdier quotient. Applying $K_0(-)$ to the above sequence induces an exact sequence [Gro77, p. 355, Proposition 3.1]

$$K_0(\mathcal{A}) \to K_0(\mathcal{C}) \to K_0(\mathcal{C}/\mathcal{A}) \to 0. \tag{3}$$

Moreover, if $\mathcal{C}$ is a tensor triangulated category and $\mathcal{A}$ is a thick ideal in $\mathcal{C}$, then $K_0(\mathcal{C}/\mathcal{A})$ is a $K_0(\mathcal{C})$-algebra. The first map in the above exact sequence is in general not injective. Here is an example. If $\mathcal{A} = \mathcal{C}_n$ $(n \geq 1)$ and $\mathcal{C} = \mathcal{C}_0$, then the inclusion functor $\mathcal{C}_n \hookrightarrow \mathcal{C}_0$ induces a map

$$K_0(\mathcal{C}_n) \to K_0(\mathcal{C}_0) \cong \mathbb{Z}.$$

This is evidently the zero map because the Euler characteristic χ_0 applied to any rational acyclic gives zero. Exactness of the sequence $K_0(\mathcal{C}_n) \to K_0(\mathcal{C}_0) \to K_0(\mathcal{C}_0/\mathcal{C}_1) \to 0$ gives the following corollary.

Corollary 3.14. *$K_0(\mathcal{C}_0/\mathcal{C}_n) \cong \mathbb{Z}$ as commutative rings, and is generated by the Grothendieck class of the image of the p-local sphere spectrum under the quotient functor $\mathcal{C}_0 \to \mathcal{C}_0/\mathcal{C}_n$.*

Since the thick subcategories of $\mathcal{F}_p$ are all nested $(\mathcal{C}_{n+1} \subseteq \mathcal{C}_n)$, we get an exact sequence of triangulated functors

$$\mathcal{C}_{n+1} \to \mathcal{C}_n \to \mathcal{C}_n/\mathcal{C}_{n+1} \to 0.$$

Applying the functor $K_0(-)$ gives an exact sequence of abelian groups

$$K_0(\mathcal{C}_{n+1}) \to K_0(\mathcal{C}_n) \to K_0(\mathcal{C}_n/\mathcal{C}_{n+1}) \to 0.$$

We do not know much about these groups beyond the fact that they are countably generated abelian groups. The hard and interesting thing here is to determine these groups and understand the map $K_0(\mathcal{C}_{n+1}) \to K_0(\mathcal{C}_n)$ in the above exact sequence. Note that if X is a type-n spectrum and v is a v_n-self map on X [HS98], then v being an even degree self map, the Grothendieck class of cofibre X/v is the zero element in $K_0(\mathcal{C}_n)$. Now heuristically we expect every finite type-$(n+1)$ spectrum to be the cofibre of a v_n-self map on a type-n spectrum. Therefore it is reasonable to conjecture that the map $K_0(\mathcal{C}_{n+1}) \to K_0(\mathcal{C}_n)$ is trivial. Also note that this conjecture is equivalent to the conjecture that $\mathcal{C}_{n+1}$ is contained in every dense triangulated subcategory of $\mathcal{C}_n$. We give some evidence for this conjecture in Subsection 3.5; see Proposition 3.25.

3.3. $\mathcal{C}_n$: higher thick subcategories. Now we want to study the triangulated subcategories of the thick subcategories $\mathcal{C}_n$ for $n > 1$. To this end, we make use of the spectra related to BP and their homology theories to construct some Euler characteristic functions. Recall that there is a ring spectrum called the Brown–Peterson spectrum (denoted by BP) whose coefficient ring is given by $\mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n, \dots]$, with $|v_i| = 2p^i - 2$. Associated to BP, we have, for each $n \geq 1$, the Johnson–Wilson spectrum $\mathrm{BP}\langle n\rangle$ whose coefficient ring is $\mathbb{Z}_{(p)}[v_1, \dots, v_n]$.

For each $n \geq 1$, recall that there is a generalised Moore spectrum of type n. For $n = 1$ this is just the Moore spectrum $M(p)$. For $n = 2$ this is the cofibre of a self map of the Moore spectrum

$$\Sigma^{|v_1^{i_1}|} M(p) \xrightarrow{v_1^{i_1}} M(p)$$

that induces multiplication by v_1^i (for some i) in BP_* homology and is denoted by $M(p, v_1^{i_1})$. For higher values this is defined inductively: A type-n generalised Moore spectrum is obtained by taking the cofibre of a self map

$$\Sigma^{|v_n^{i_n}|} M(p, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}) \xrightarrow{v_n^{i_n}} M(p, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$$

that induces multiplication by $v_n^{i_n}$ (for some i_n) in BP_* homology and is inductively denoted by $M(p, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}, v_n^{i_n})$. For sufficiently large powers of the v_i's these spectra are known to exist [HS98]. However the problem of existence of such spectra with specified exponents seems to be hard.

LEMMA 3.15. *Let X be any spectrum in $\mathcal{C}_n$ $(n \geq 1)$. Then* $\mathrm{BP}\langle n-1\rangle_i X$ *is always a finite abelian p-group, and is zero for all but finitely many i.*

Proof. The strategy here is a thick subcategory argument. Say that a finite p-local spectrum X has property P if $\mathrm{BP}\langle n-1\rangle_i X$ satisfies the conditions in the statement of the lemma. It is straightforward to verify that property P is generic. Now by the Hopkins–Smith thick subcategory theorem, we will be done if we can exhibit one generic type-n spectrum for which P holds. To this end, we consider a generalised type-n Moore spectrum $M(p, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$. We have, for each $n \geq 1$,

$$\mathrm{BP}\langle n-1\rangle_* M(p, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}) = \frac{\mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}]}{(p, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})}.$$

This homology is a finitely generated $\mathbb{F}_p$-algebra, and therefore the generalised Moore spectrum in question has property P. So we are done. In fact it is also clear that the full subcategory of all finite p-local spectra with property P is precisely the thick subcategory $\mathcal{C}_n$. ∎

With the above lemma at hand, we can define a function $\chi_n : \mathcal{C}_n \to \mathbb{Z}$ as

$$\chi_n X = \sum_{i=-\infty}^{\infty} (-1)^i \log_p |\mathrm{BP}\langle n-1\rangle_i(X)|. \tag{4}$$

The previous lemma shows that this is a well defined function. It is straightforward to verify that this function is an Euler characteristic function. Moreover the $\mathrm{BP}\langle n-1\rangle_*$ homology of the generalised type-n Moore spectrum $M(p, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$ is non-trivial and is concentrated in a finite range of even degrees and hence $\chi_n M(p, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$ is non-zero. So by the universal property of the Euler characteristic function χ_n, we have the following split short exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(\phi_n) \arrow[r] & K_0(\mathcal{C}_n) \arrow[r] \arrow[dr, "\phi_n"'] & \mathrm{Im}(\phi_n) \arrow[r] \arrow[d, hook] & 0 \\ & & & \mathbb{Z} & \end{tikzcd}$$

This discussion can be summarised in the following proposition.

PROPOSITION 3.16. *For each $n \geq 1$, $K_0(\mathcal{C}_n)$ has a direct summand isomorphic to $\mathbb{Z}$. This gives, for each $k \geq 0$, a dense triangulated subcategory of $\mathcal{C}_n$ defined by*

$$\mathcal{C}_n^k := \{X \in \mathcal{C}_n : \chi_n(X) \equiv 0 \bmod l_n k\}$$

where l_n is a generator for the cyclic group $\mathrm{Im}(\phi_n)$.

Note that these triangulated subcategories correspond to the subgroups $n\mathbb{Z} \oplus \mathrm{Ker}(\phi_n)$ and hence are dense in $\mathcal{C}_n$ by Theorem 2.5.

REMARK 3.17. The above proposition recovers the dense triangulated subcategories $\mathcal{C}_1^k$ of Theorem 3.11. In fact, the Euler characteristic function (4) agrees with (2) when $n = 1$. This is because $\mathrm{BP}\langle 0\rangle \cong H\mathbb{Z}_{(p)}$ (the Eilenberg–Mac Lane spectrum for the integers localised at p), and for p-torsion spectra, ${H\mathbb{Z}_{(p)}}_*(X) \cong H\mathbb{Z}_*(X)$. Thus, for X in $\mathcal{C}_1$, we have $\mathrm{BP}\langle 0\rangle_* X \cong H\mathbb{Z}_*(X)$ and consequently the two Euler characteristic functions agree.

3.4. A conjecture of Frank Adams. It is easily seen that if the Smith-Toda complex $V(n-1)$ $(:= M(p, v_1, \ldots, v_{n-1}))$ exists, then the map $\phi_n : K_0(\mathcal{C}_n) \to \mathbb{Z}$ is surjective. The natural thing to do now is to determine the kernel of ϕ_n. This turns out to be a very hard problem. It is known that $V(1)$ exists at odd primes. In view of this Frank Adams made the following conjecture.

CONJECTURE 3.18 ([Ada92, p. 529]). *The thick subcategory $\mathcal{C}_2$ (at odd primes) can be generated by iterated cofiberings of the Smith–Toda complex $V(1)$.*

This conjecture is equivalent to saying that $\mathrm{Ker}(\phi_2) = 0$, or equivalently that $K_0(\mathcal{C}_2) \cong \mathbb{Z}$. Adams [Ada92, p. 528] also asked the following weaker question.

QUESTION. What is a good set of generators for $\mathcal{C}_n$? (A set A *generates* $\mathcal{C}_n$ if the smallest triangulated category that contains A is $\mathcal{C}_n$.)

This is a very important question and one answer was given by Kai Xu [Xu95]. Before we can state his result we need to set up some terminology.

Recall that a spectrum X is *atomic* if it does not admit any non-trivial idempotents, i.e., whenever $f \in [X, X]$ is such that $f^2 = f$, then $f = 0$ or 1. Since idempotents split in the stable homotopy category, this is also equivalent to saying that X does not have any non-trivial summands. Now if X is a finite p-torsion atomic spectrum, then the finite non-commutative ring $[X, X]$ of degree zero self maps of X modulo its Jacobson radical (intersection of all left maximal ideals) is isomorphic to a finite field [AK89]:

$$[X, X]/\mathrm{rad} \cong \mathbb{F}_{p^k} \quad \text{for some } k.$$

So for every p-torsion atomic spectrum X, we define $e(X)$ to be the integer k given by the above isomorphism.

EXAMPLE 3.19. It can be easily verified that for all $i \geq 1$, $M(p^i)$ is an atomic spectrum with $e(M(p^i)) = 1$.

The main result of [Xu95] which uses the nilpotence results [DHS88, HS98] as the main tools then states:

THEOREM 3.20 ([Xu95]). *For every pair of natural numbers* (n, k), *there is an atomic spectrum* X *of type* n *such that* $e(X) = k$. *Further*, *if* $C(n, k)$ *denotes the triangulated subcategory of* $\mathcal{C}_n$ *generated by the type-*n *atomic spectra with* $e(-) \leq k$, *then* $C(n, k) = \mathcal{C}_n$.

So this gives an answer to Adams's question. One set of generators for $\mathcal{C}_n$ can be taken to be the collection of all type-n atomic spectra X with $e(X) = 1$. The next natural question is: how big is this set? Is it finite? If so, then we can infer that the Grothendieck groups $K_0(\mathcal{C}_n)$ are all finitely generated abelian groups. But we do not know the answer to this question.

Using this theorem of Xu, we can now revise Adams's conjecture as follows.

CONJECTURE 3.21. *At odd primes*, $V(1)$ *generates* (*by iterated cofiberings*) *all type-*2 *atomic spectra* X *such that* $e(X) = 1$.

3.5. The lattice of triangulated subcategories of $\mathcal{C}_0$. We will now study the lattice of triangulated subcategories of finite p-local spectra. First recall that the thick subcategories of finite p-local spectra are nested [Mit85]:

$$\cdots \subsetneq \mathcal{C}_{n+1} \subsetneq \mathcal{C}_n \subsetneq \mathcal{C}_{n-1} \subsetneq \cdots \subsetneq \mathcal{C}_1 \subsetneq \mathcal{C}_0.$$

Our goal now is to understand how the triangulated subcategories $\mathcal{C}_n^k$ fit in this chromatic chain. We begin with a simple proposition.

PROPOSITION 3.22. $\mathcal{C}_1$ *is contained in every dense triangulated subcategory of* $\mathcal{C}_0$, *i.e.*, $\mathcal{C}_1 \subsetneq \mathcal{C}_0^k$ *for all* $k \geq 0$.

Proof. If X is in $\mathcal{C}_1$, then its rational homology is trivial and hence its rational Euler characteristic $\chi_0(X)$ is zero. Therefore X belongs to every dense triangulated subcategory of $\mathcal{C}_0$. It is easy to see that the spectrum $S \vee \Sigma S$ belongs to $\mathcal{C}_0^k - \mathcal{C}_1$ for all $k \geq 0$, therefore the containment $\mathcal{C}_1 \subseteq \mathcal{C}_0^k$ is proper. ∎

Motivated by this proposition, we wondered if it is true that for all non-negative integers n and k,

$$\mathcal{C}_{n+1} \subsetneq \mathcal{C}_n^k \subseteq \mathcal{C}_n.$$

We now proceed to show that this is indeed the case. For better clarity we separate the cases $n = 1$ and $n > 1$.

3.5.1. *An Atiyah–Hirzebruch spectral sequence.* We prove that $\mathcal{C}_2 \subsetneq \mathcal{C}_1^k$ for all $k \geq 0$. Our main tool will be an Atiyah–Hirzebruch spectral sequence. We begin with two lemmas: the first one is an elementary algebraic fact and the second one is a standard topological fact.

Lemma 3.23. *If* $A := \cdots \to 0 \to A_1 \to \cdots \to A_k \to 0 \to \cdots$ *is a bounded chain complex of finite* p*-groups*, *then*

$$\sum_i (-1)^i \log_p |A_i| = \sum_i (-1)^i \log_p |H_i(A)|.$$

Proof. This is left as an easy exercise to the reader.

Lemma 3.24. *The thick subcategory* $\mathcal{C}_2$ *consists of all finite* p*-torsion spectra whose complex* K*-theory is trivial.*

Proof. We use a result of Adams [Ada69] which states that the complex K-theory localised at p splits as a wedge of suspensions of $E(1)$. More precisely, $K_p = \bigvee_{i=0}^{p-2} \Sigma^{2i} E(1)$. In particular, $\langle K_p \rangle = \langle E(1) \rangle$. With this at hand, we get the following equalities of Bousfield classes:

$$\langle K \rangle = \bigvee_p \langle K_p \rangle = \bigvee_p \langle E(1) \rangle = \bigvee_p (\langle K(0) \rangle \vee \langle K(1) \rangle).$$

The last equality follows from [Rav84, Theorem 2.1(d)]. Now it is clear from these equations that for X finite and p-torsion, $K_* X = 0$ if and only if $K(1)_* X = 0$. ∎

Proposition 3.25. $\mathcal{C}_2$ *is properly contained in every dense triangulated subcategory of* $\mathcal{C}_1$, *i.e.*, $\mathcal{C}_2 \subsetneq \mathcal{C}_1^k$ *for all* $k \geq 0$.

Proof. Recall that $\mathcal{C}_1^k$ is the collection of p-torsion spectra X for which $\chi_1(X)$ is divisible by k. So it is clear that $X \in \mathcal{C}_1^k$ for all k if and only if $\chi_1(X) = 0$. Therefore by Lemma 3.24 we have to show that if X is a finite p-torsion spectrum for which $K_*(X) = 0$, then $\chi_1(X) := \sum (-1)^i \log_p |H\mathbb{Z}_i X| = 0$.

Recall that the spectrum K (also denoted by BU) is a ring spectrum whose coefficient group is given by the complex Bott periodicity theorem; $K_* = \mathbb{Z}[u, u^{-1}]$ where $|u| = 2$. We make use of the Atiyah–Hirzebruch spectral sequence

$$E^2_{s,t} = H_s(X; K_t) \;\Rightarrow\; K_{s+t} X$$

converging strongly to the K-theory of X. The differentials (d_r) in this spectral sequence have bidegrees $|d_r| = (-r, r-1)$. Note that X being a finite spectrum, the E^2 page is concentrated in a vertical strip of finite width (see Fig. 1) and therefore the spectral sequence collapses after a finite stage. Since it converges to $K_*(X)$, which is zero by hypothesis, we conclude that for all sufficiently large n, $E^n = E^\infty = 0$.

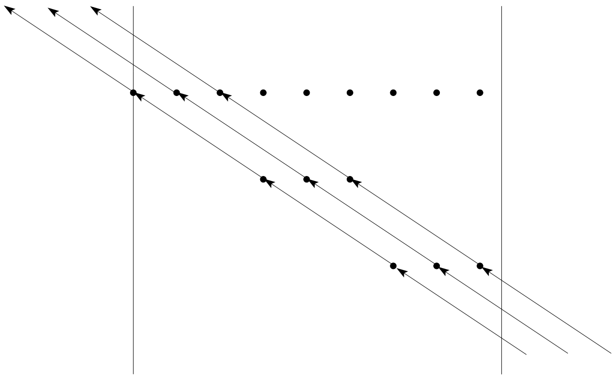

Fig. 1. Atiyah–Hirzebruch spectral sequence: E^3-term

Next we claim that the function

$$n \mapsto \sum_i (-1)^i \log_p |E^n_{i,0}|$$

is constant.

Assuming this claim, we will finish the proof of the proposition. When $n = 2$, this function takes the value $\sum_i (-1)^i \log_p |H\mathbb{Z}_i\, X| = \chi_1(X)$ and for large enough n, it takes the value 0 (since $E^n = E^\infty = 0$). Since the function is constant (by the above claim), we get $\chi_1(X) = 0$.

Now we prove our claim. First note that this spectral sequence is a module over the coefficient ring $K_* \cong \mathbb{Z}[u, u^{-1}]$. Therefore the differentials commute with this ring action. Also since u is a unit, it acts isomorphically on the spectral sequence, and therefore induces periodicity on E^4: $E^4_{i,*} \cong E^4_{i-2,*}$. Also note that just for degree reasons, all the even differentials are zero. So at E_3, where the first potential non-zero differentials occur, the alternating sum $\sum_i (-1)^i \log_p |E^3_{i,0}|$ can be broken into three parts as shown in the equation below.

$$\sum_i (-1)^i \log_p |E^3_{i,0}| = \sum_{i \equiv 0\,(3)} + \sum_{i \equiv 1\,(3)} + \sum_{i \equiv 2\,(3)} .$$

Now because of the periodicity of the E^3 page, we can assemble the terms in the above equation along three parallel lines in the E^3 page, where the term with congruence class l modulo three corresponds to the line with t-intercept l; see Fig. 1. Now we can apply Lemma 3.23 along each of these lines which are bounded complexes of finite abelian p-groups to pass to the homology groups. Invoking the periodicity of the differentials again, we conclude that the new alternating sum thus obtained is equal to $\sum_i (-1)^i \log_p |E^4_{i,0}|$. Now an easy induction will complete the proof of the claim: At (E^r, d_r), for r odd, we decompose the alternating sum $\sum_i (-1)^i \log_p |E^r_{i,0}|$ into r parts, one for each congruence class modulo r, and assemble these terms along r parallel lines on the E^r page such that the term whose congruence class is l modulo r corresponds to the line with t-intercept l. Periodicity of the differentials can be used (as before) to complete the induction step.

So we have shown that $\mathcal{C}_2 \subseteq \mathcal{C}_1^k$ for all $k \geq 0$. To see that this inclusion is strict, observe that the p-torsion spectrum $M(p) \vee \Sigma M(p)$ belongs to $\mathcal{C}_1^k - \mathcal{C}_2$ for all k. ∎

Corollary 3.26. *$K_0(\mathcal{C}_1/C_2) \cong \mathbb{Z}$ generated by the Grothendieck class of the image of the Moore spectrum under the quotient functor $\mathcal{C}_1 \to \mathcal{C}_1/\mathcal{C}_2$.*

Proof. On applying $K_0(-)$ to the sequence $\mathcal{C}_2 \to \mathcal{C}_1 \to \mathcal{C}_1/\mathcal{C}_2 \to 0$, we get an exact sequence of abelian groups: $K_0(\mathcal{C}_2) \to K_0(\mathcal{C}_1) \to K_0(\mathcal{C}_1/\mathcal{C}_2) \to 0$; see (3). The first map in this sequence is the zero map by Proposition 3.25, and $K_0(\mathcal{C}_1)$ was shown to be infinite cyclic on $[M(p)]$. So the corollary follows by combining these two facts. ∎

The inclusion $\mathcal{C}_2 \subsetneq \mathcal{C}_1^k$ for all k, which we have just established, gives some new evidence for Conjecture 3.18 of Adams. To see this, first note that since $M(p, v_1)$ is a cofibre of an even degree self map of $M(p)$, $\chi_1(M(p, v_1)) = 0$. Now if $M(p, v_1)$ generates $\mathcal{C}_2$ by cofibre sequences (Adam's conjecture), then $\chi_1(X) = 0$ for all $X \in \mathcal{C}_2$. This now clearly implies that $\mathcal{C}_2 \subsetneq \mathcal{C}_1^k$ for all k.

One can try to test this conjecture by asking whether we can generate $M(p^i, v_1^j)$ (whenever it exists) from $M(p, v_1)$. Results on nilpotence and periodicity [HS98] give the following partial answer to this question.

Proposition 3.27. *For every fixed positive integer $k > 0$, there exist infinitely many positive integers j for which $M(p^k, v_1^j)$ can be generated from $M(p, v_1)$ using cofibre sequences.*

We leave the proof of this well known proposition to the reader.

All these results give only some evidence for Adams's conjecture. The conjecture, however, still remains open.

3.5.2. *A Bockstein spectral sequence.* Our goal now is to prove $\mathcal{C}_{n+1} \subsetneq \mathcal{C}_n^k$ for all n and k. We mimic our strategy for the case $n = 1$ by replacing the Atiyah–Hirzebruch spectral sequence with a Bockstein spectral sequence. The above inclusion is an easy corollary of the following theorem.

Theorem 3.28. *If X is a spectrum in $\mathcal{C}_{n+1}$, then*

$$\chi_n(X) := \sum (-1)^i \log_p |\mathrm{BP}\langle n-1\rangle_i X| = 0.$$

Corollary 3.29. *$\mathcal{C}_{n+1} \subsetneq \mathcal{C}_n^k$ for all $k \geq 0$ and all $n \geq 1$.*

Proof. By the above theorem, for a spectrum X in $\mathcal{C}_{n+1}$, $\chi_n(X) = 0$. Therefore X belongs to $\mathcal{C}_n^k$ for all k. The spectrum $F \vee \Sigma F$, where F is a type-n spectrum, belongs to $\mathcal{C}_n^k - \mathcal{C}_{n+1}$. So $\mathcal{C}_{n+1} \subsetneq \mathcal{C}_n^k$ for all $k \geq 0$. ∎

We now outline the strategy for proving the above theorem. This is very similar to the proof of Proposition 3.25. First recall that $\mathcal{C}_{n+1}$ can also be characterised as the collection of finite p-local spectra that are acyclic with respect to $E(n)$. So we seek a strongly convergent spectral sequence

$$E_r^{**} \;\Rightarrow\; E(n)_*X,$$

whose E_1 term is built out of $\mathrm{BP}\langle n-1\rangle_*X$. We show that a certain Bockstein spectral sequence has this property and that it collapses after a finite stage. Finally, we work backward to conclude that $\chi_n X = 0$.

Now we proceed to construct such a spectral sequence. Fix an integer $n \geq 1$ and recall that

$$E(n) = v_n^{-1}\mathrm{BP}\langle n\rangle = \operatorname*{hocolim}_{v_n} \mathrm{BP}\langle n\rangle.$$

This gives a sequence of cofibre sequences that fit into a diagram extending to infinity in both directions as shown below:

$$\begin{tikzcd}
\cdots \arrow[r, "v_n"] & \Sigma^{|v_n|}\mathrm{BP}\langle n\rangle \arrow[r, "v_n"] \arrow[d] & \mathrm{BP}\langle n\rangle \arrow[r, "v_n"] \arrow[d] & \Sigma^{-|v_n|}\mathrm{BP}\langle n\rangle \arrow[r, "v_n"] \arrow[d] & \cdots\cdots \arrow[r] & E(n) \\
\cdots & \Sigma^{|v_n|}\mathrm{BP}\langle n-1\rangle \arrow[ul, "\circ"] & \mathrm{BP}\langle n-1\rangle \arrow[ul, "\circ"] & \Sigma^{-|v_n|}\mathrm{BP}\langle n-1\rangle \arrow[ul, "\circ"] & \cdots &
\end{tikzcd}$$

Since both the functors $(-) \wedge X$ and $\pi_*(-)$ commute with the functor $\mathrm{hocolim}_{v_n}(-)$, smashing the above diagram with X and taking π_* gives an exact couple of graded abelian groups:

$$\begin{tikzcd}
\cdots \arrow[r, "v_n"] & \Sigma^{|v_n|}\mathrm{BP}\langle n\rangle_*X \arrow[r, "v_n"] \arrow[d] & \mathrm{BP}\langle n\rangle_*X \arrow[r, "v_n"] \arrow[d] & \Sigma^{-|v_n|}\mathrm{BP}\langle n\rangle_*X \cdots \arrow[r] \arrow[d] & E(n)_*X \\
\cdots & \Sigma^{|v_n|}\mathrm{BP}\langle n-1\rangle_*X \arrow[ul, "\circ"] & \mathrm{BP}\langle n-1\rangle_*X \arrow[ul, "\circ"] & \Sigma^{-|v_n|}\mathrm{BP}\langle n-1\rangle_*X \arrow[ul, "\circ"] &
\end{tikzcd}$$

This exact couple gives rise to a (Bockstein) spectral sequence $E_r^{*,*}$ in the usual way. We choose a convenient grading so that the E_1 term is concentrated in a horizontal strip of finite width, i.e., $E_1^{*,q} = 0$ for $|q| \gg 0$ (see Fig. 2). This can be ensured by setting

$$D_1^{p,q} = \Sigma^{-p|v_n|}\mathrm{BP}\langle n\rangle_{-p|v_n|+q}X, \qquad E_1^{p,q} = \Sigma^{-p|v_n|}\mathrm{BP}\langle n-1\rangle_{-p|v_n|+q}X.$$

With this grading one can easily verify that the differentials have bidegrees given by $|d_r| = (-r, -r|v_n| - 1)$. It is clear from Figure 2 that after a finite stage all the differentials exit the horizontal strip and therefore the spectral sequence collapses. Another important fact that we need about this Bockstein spectral sequence is the periodicity of all the differentials. More precisely, for all integers p and $r \geq 0$, $E_r^{p,*} = E_r^{p+1,*}$, and further the

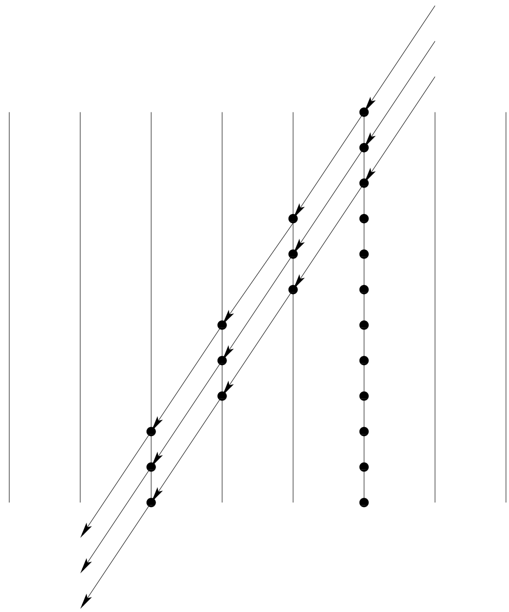

Fig. 2. Bockstein spectral sequence: E_1-term

following diagram commutes:

$$\begin{tikzcd}
E_r^{p,*} \arrow[r, "d_r"] \arrow[d, "="] & E_r^{p-r,*-1-r|v_n|} \arrow[d, "="] \\
E_r^{p+1,*} \arrow[r, "d_r"] & E_r^{p+1-r,*-1-r|v_n|}
\end{tikzcd}$$

Now we show that this spectral sequence converges strongly to $E(n)_*(X)$. For this, we make use of a theorem of Boardman [Boa99]. Before we can state his result we have to recall some of his terminology. Consider an exact couple of graded abelian groups:

$$\begin{tikzcd}
\cdots \arrow[r, "i"] & A^{s+1} \arrow[r, "i"] \arrow[d, "j"] & A^s \arrow[r, "i"] \arrow[d, "j"] & A^{s-1} \arrow[r, "i"] \arrow[d, "j"] & A^{s-2} \arrow[r, "i"] \arrow[d, "j"] & \cdots \arrow[r] & A^{-\infty} \\
\cdots & E^{s+1} \arrow[ul, "k"] & E^s \arrow[ul, "k"] & E^{s-1} \arrow[ul, "k"] & E^{s-2} \arrow[ul, "k"] & \cdots\cdots &
\end{tikzcd}$$

This gives the following filtration of the groups E^s by cycles and boundaries of the differentials in the spectral sequence that arises from this exact couple:

$$0 = B_1^s \subset B_2^s \subset \cdots \subset Z_2^s \subset Z_1^s = E^s,$$

where $Z_r^s := k^{-1}(\mathrm{Im}\,[i^{(r-1)} : A^{s+r} \to A^{s+1}])$ and $B_r^s := j\ker[i^{(r-1)} : A^s \to A^{s-r+1}]$.

Associated to this exact couple and the resulting spectral sequence, [Boa99] defines the following groups (these definitions also hold when $n = \infty$):

- $A^{-\infty} := \operatorname{colim}_s A^s, \quad A^{\infty} := \lim_s A^s, \quad RA^{\infty} := \operatorname{Rlim}_s A^s,$
- $K_n A^s := \ker[i^{(n)} : A^s \to A^{s-n}], \quad \mathrm{Im}^r A^s := \mathrm{Im}[i^{(r)} : A^{s+r} \to A^s],$
- $K_n \mathrm{Im}^r A^s := K_n A^s \cap \mathrm{Im}^r A^s,$
- $W = \operatorname{colim}_s \operatorname{Rlim}_r K_\infty \mathrm{Im}^r A^s,$
- $RE_\infty^s = \operatorname{Rlim}_r Z_r^s.$

The main theorem of [Boa99] then states:

Theorem 3.30 ([Boa99, Theorem 8.10]). *The spectral sequence arising from the above exact couple converges strongly to $A^{-\infty}$ if the obstruction groups A^∞, RA^∞, W, and RE_∞ are zero.*

We recall a few elementary facts about inverse limits before we can apply this theorem to our Bockstein spectral sequence. The proofs follow quite easily from the universal properties of these limit functors. Parts of this lemma can also be derived from the more general Mittag-Leffler condition.

Lemma 3.31. *Consider a sequence of (graded) groups and homomorphisms:*

$$\cdots \to A^{s+1} \to A^s \to A^{s-1} \to \cdots .$$

Then the following statements hold.

- *If, for some integer M, the composite of M consecutive maps in this sequence is always null, then $\lim_s A^s = \operatorname{Rlim}_s A^s = 0$.*
- *If there is an integer L such that for all $s \geq L$ the map $A^{s+1} \to A^s$ is the identity map, then $\lim_s A^s = A^L$ and $\operatorname{Rlim}_s A^s = 0$.*

We now show that our Bockstein spectral sequence converges strongly to $E(n)_* X$ by verifying the hypothesis of Boardman's theorem. So we apply his theorem to the sequence

$$(5) \qquad \cdots \xrightarrow{v_n} \Sigma^{|v_n|}\mathrm{BP}\langle n\rangle \xrightarrow{v_n} \mathrm{BP}\langle n\rangle \xrightarrow{v_n} \Sigma^{-|v_n|}\mathrm{BP}\langle n\rangle \xrightarrow{v_n} \cdots .$$

For brevity, we shall denote $\Sigma^{-s|v_n|}\mathrm{BP}\langle n\rangle_* X$ by A^s.

(a) $A^\infty = 0$: By Lemma 3.31, all we need to show is that there is some integer M such that the composite of any M consecutive maps in the sequence (5) is zero. X being a spectrum in $\mathcal{C}_{n+1}$, $\mathrm{BP}\langle n\rangle_* X$ is concentrated only in a finite range. Now since v_n is a graded map of degree $2(p^n - 1)$, a sufficiently large iterate of v_n vanishes, hence $A^\infty = \lim_s \Sigma^{-p|v_n|}\mathrm{BP}\langle n\rangle_* X = 0$.

(b) $RA^\infty = 0$: This is also immediate from the first part of Lemma 3.31 because we have already seen that a sufficiently large iterate of v_n is zero in part (a).

(c) $W = 0$: The colimit of the sequence (5) is $E(n)_*X$ and this latter group is zero by hypothesis. It now follows that $K_\infty A^s = A^s$. We make use of this fact in the third equality below.

$$\begin{aligned} W &= \operatorname*{colim}_s \operatorname*{Rlim}_r (K_\infty \mathrm{Im}^r A^s) = \operatorname*{colim}_s \operatorname*{Rlim}_r (K_\infty A^s \cap \mathrm{Im}^r A^s) \\ &= \operatorname*{colim}_s \operatorname*{Rlim}_r (A^s \cap \mathrm{Im}^r A^s) = \operatorname*{colim}_s \operatorname*{Rlim}_r \mathrm{Im}^r A^s \\ &= \operatorname*{colim}_s \operatorname*{Rlim}_r (\cdots 0 \to 0 \to \cdots \subseteq \mathrm{Im}^2 A^s \subseteq \mathrm{Im}^1 A^s) = \operatorname*{colim}_s 0 = 0. \end{aligned}$$

(d) $RE_\infty = 0$: This follows from the fact that the spectral sequence collapses after a finite stage. For each fixed s, all inclusions in the sequence

$$\cdots \xrightarrow{=} Z^s_{r+1} \xrightarrow{=} Z^s_r \subseteq \cdots \subseteq Z^s_2 \subseteq Z^s_1 = E^s_1$$

become equalities eventually. So we now invoke the second part of Lemma 3.31 to conclude that $RE^s_\infty = \mathrm{Rlim}_r Z^s_r = 0$.

So we have shown that all the obstruction groups vanish and therefore Boardman's theorem tells us that our spectral sequence converges strongly to $E(n)_*X$.

We now move to the final part of the theorem: $\chi_n(X) = 0$. We mimic the argument given for the fact "$\mathcal{C}_2 \subseteq \mathcal{C}_1^k$". The crux of the proof lies in the key observation that the q component of the differential d_r is always an odd number $(-r|v_n| - 1)$. We claim that the function

$$k \overset{\phi}{\mapsto} \sum (-1)^i \log_p |E_k^{0,i}|$$

is constant. Toward this, we decompose the sum $\sum(-1)^i \log_p |E_1^{0,i}|$ into $t := |v_n| + 1$ parts as follows:

$$\sum_i (-1)^i \log_p |E_1^{i,0}| = \sum_{i \equiv 0\,(t)} + \sum_{i \equiv 1\,(t)} + \cdots + \sum_{i \equiv t-1\,(t)} .$$

Now the periodicity of all the differentials coupled with the fact that $|v_n|+1$ is an odd number will enable us to assemble all these terms on the right hand side of this equation along t parallel complexes of differentials on the E_1 term. (The term corresponding to the congruence class l modulo t will correspond to the parallel complex with q-intercept l; see Fig. 2.) We can now apply Lemma 3.23 to each of these complexes and pass on to the homology groups without changing the underlying alternating sum. Again using the periodicity of the differentials, we can reassemble all the terms after taking homology to obtain $\phi(2)$. This shows that $\phi(1) = \phi(2)$. Now a straightforward induction will complete the proof of the claim.

Finally, to see that $\chi_n(X) = 0$, observe that when $k = 1$, ϕ takes the value $\chi_n(X)$, and for all sufficiently large values of k, ϕ is zero because our spectral sequence collapses at a finite stage and converges strongly to $E(n)_*X$, which is zero by hypothesis. Since we know that ϕ is constant, this completes the proof of the theorem.

3.6. Questions

3.6.1. *Grothendieck groups and Adams's conjecture.* The biggest problem that needs to be settled is the classification of all triangulated subcategories of $\mathcal{F}_p$. We believe that such a classification will reveal some hidden ideas behind the chromatic tower which might shed some new light on the stable homotopy category. We have seen that $K_0(\mathcal{C}_0)$ is infinite cyclic with the sphere as the generator, and $K_0(\mathcal{C}_1)$ is infinite cyclic with the mod-p Moore spectrum as the generator. Computing the Grothendieck groups of higher thick subcategories ($\mathcal{C}_n$, $n \geq 2$), as we have seen, is much more complicated. In particular Adams's conjecture remains open.

Another interesting and related question at this point is the following. In $\mathcal{C}_n$, what is the triangulated subcategory generated by atomic spectra X of type n for which $e(X) = k$ (k some fixed positive integer)? By [Xu95], this is the whole of $\mathcal{C}_n$ if $k = 1$; otherwise, this will be some dense triangulated subcategory of $\mathcal{C}_n$. The immediate question that springs out now is whether these triangulated subcategories, when $n = 2$, are precisely the subcategories $\mathcal{C}_2^k$. A non-affirmative answer to this question will settle Adams's conjecture in the negative. Similarly when $n = 1$, it will be interesting to match these dense triangulated subcategories with the subcategories $\mathcal{C}_1^k$.

One can also try to investigate some properties of these Grothendieck groups. For instance, are they finitely generated? are they torsion-free?

3.6.2. *Euler characteristics and the lattice of triangulated subcategories.* It would be interesting to find an Euler characteristic defined on $\mathcal{C}_n$ that is not a multiple of χ_n (see equation (4)). Such an Euler characteristic function might tell us something new about $K_0(\mathcal{C}_n)$.

Recall that l_n was defined to be the generator of the image of $\phi_n : K_0(\mathcal{C}_n) \to \mathbb{Z}$. It is clear that $l_n = 1$ if $V(n)$ exists. In general, l_n can be a very large integer and not much is known about it. The following is a conjecture of Ravenel (personal communication): If $(p, f) \neq (2, 1)$, then p^f divides $l_{(p-1)f+1}$.

We have seen that the triangulated subcategories $\mathcal{C}_n^k$ are sandwiched between $\mathcal{C}_{n+1}$ and $\mathcal{C}_n$. Is the same true for all triangulated subcategories of $\mathcal{F}_p$?

These are some questions which we think merit further study in this direction.

4. TRIANGULATED SUBCATEGORIES OF PERFECT COMPLEXES

For the rest of this paper we work in the derived categories of rings. The subsections that follow are organised as follows. We begin with a quick recap of the derived category in the next subsection. In Subsection 4.2 we review some basic algebraic K-theory and connect it to the problem of classifying triangulated subcategories. We then start applying Thomason's Theorem 2.5 to classify triangulated subcategories of perfect complexes over PIDs, Artin rings, and some non-noetherian rings. We end with some questions.

Unless stated otherwise, all rings will be assumed to be commutative with a unit.

4.1. The derived category. There are many beautiful constructions of the derived category; see [Wei94] for the classical approach, or [Hov99] for a model category theoretic approach. We briefly review some preliminaries on the derived category $D(R)$ of a commutative ring R. It is obtained from the category of unbounded chain complexes of R-modules and chain maps by inverting the *quasi-isomorphisms* (maps that induce an isomorphism in homology). $D(R)$ is a tensor triangulated category with the derived tensor product as the smash product and the ring R (in degree 0) as the unit object. It is a standard fact that the small objects of $D(R)$ are precisely those complexes that are quasi-isomorphic to *perfect complexes* (bounded chain complexes of finitely generated projective R-modules); see [Chr98, Prop. 9.6] for a nice proof. It follows from [Wei94, Corollary 10.4.7] that the full subcategory of small objects in $D(R)$ is equivalent (as a triangulated category) to the chain homotopy category of perfect complexes. The latter will be denoted by $D^b(\operatorname{proj} R)$ and it provides a nice framework for studying small objects. The full subcategory of small objects in $D(R)$ can also be characterised as the thick subcategory generated by R; see [Chr98, Prop. 9.6].

4.2. Algebraic K-theory of rings. Now we recall some classical algebraic K-theory of rings and connect it to the problem of classifying (dense) triangulated subcategories of perfect complexes. If R is any commutative ring, the *K-group of the ring R*, which is denoted by $K_0(R)$, is defined to be the free abelian group on the isomorphism classes of finitely generated projective modules modulo the subgroup generated by the relations $[P] - [P \oplus Q] - [Q] = 0$, where P and Q are finitely generated projective R-modules. A folklore result says that these K-groups are isomorphic to the Grothendieck groups of $D^b(\operatorname{proj} R)$.

PROPOSITION 4.1 (well known). *If R is any ring* (*not necessarily commutative*), *then there is a natural isomorphism of abelian groups*

$$K_0(R) \cong K_0(D^b(\operatorname{proj} R)).$$

Proof (sketch). Let A denote the free abelian group on the isomorphism classes of perfect complexes in the derived category of R and let B denote the free abelian group on the isomorphism classes of finitely generated projective R-modules. Since the Grothendieck groups under consideration are quotients of these free groups, we define maps on A and B that descend to give the desired bijections. Define $f : A \to B$ by $f(\langle X \rangle) = \sum_{i\in\mathbb{Z}}(-1)^i\langle X_i\rangle$ and $g : B \to A$ by $g(\langle M \rangle) = \langle M[0]\rangle$. Now one can easily verify that these map descend to the Grothendieck groups and that the descended maps are inverses of each other. ∎

The importance of this folklore result, for our purpose, can be seen from the following observation.

Remark 4.2. This folklore result, along with Theorem 2.5 of Thomason, connects the problem of classifying dense triangulated subcategories in $D^b(\operatorname{proj} R)$ with the K-theory of R. More precisely, there is a 1-1 correspondence between the subgroups of $K_0(R)$ and the dense triangulated subcategories of $D^b(\operatorname{proj} R)$. So this leads us naturally to algebraic K-theory—a subject that has been extensively studied.

For the remainder of this section, we review some well known computations of K-groups of rings that will be relevant to us.

Example 4.3. If R is any commutative local ring, then every finitely generated projective R-module is free. Thus the monoid $\operatorname{proj} R$ (the category of finitely generated projective R-modules) is equivalent to the monoid consisting of the whole numbers. The Grothendieck group of the latter is clearly isomorphic to $\mathbb{Z}$. Thus for local rings,

$$K_0(D^b(\operatorname{proj} R) \cong K_0(R) \cong \mathbb{Z}.$$

Similarly, it is easy to see that if R is any principal ideal domain, then $K_0(R) \cong \mathbb{Z}$.

Now we state some results on the K-groups of some low-dimensional commutative rings. These results are of interest because they connect K-groups and classical Picard groups of rings. Before we state the result, we need to recall the definition of the Picard groups of rings.

Definition 4.4. The *Picard group* $\operatorname{Pic}(R)$ of a ring R is defined to be the group of isomorphism classes of R-modules that are invertible under the tensor product. Similarly the *Picard group* of $D(R)$, denoted by $\operatorname{Pic}(D(R))$, is the group of isomorphism classes of objects in $D(R)$ that are invertible under the derived tensor product. In both these cases, note that the ring R acts as the identity element.

Theorem 4.5 ([Wei03]). *Let* $[\operatorname{Spec}(R), \mathbb{Z}]$ *denote the additive group of continuous functions from* $\operatorname{Spec}(R)$ *to the ring of integers with the discrete* *topology. Then the following holds*:

(1) *For every* 0-*dimensional ring* R, $K_0(D^b(\operatorname{proj} R)) \cong [\operatorname{Spec}(R), \mathbb{Z}]$.
(2) *For every* 1-*dimensional noetherian ring*,

$$K_0(D^b(\operatorname{proj} R)) \cong \operatorname{Pic}(D(R)).$$

Proof. The first statement is Pierce's theorem; see [Wei03, Theorem 2.2.2]. The second statement can be seen as a corollary of a theorem due to Fausk [Fau03]: There is a natural split short exact sequence (for any commutative ring)

$$0 \to \operatorname{Pic}(R) \to \operatorname{Pic}(D(R)) \to [\operatorname{Spec}(R), \mathbb{Z}] \to 0.$$

Therefore $\operatorname{Pic}(D(R)) \cong \operatorname{Pic}(R) \oplus [\operatorname{Spec}(R), \mathbb{Z}]$. It is shown in [Wei03, Corollary 2.6.3] that

$$K_0(D^b(\operatorname{proj} R)) \cong \operatorname{Pic}(R) \oplus [\operatorname{Spec}(R), \mathbb{Z}].$$

So the second statement follows by combining these two results. ∎

With these and related results from algebraic K-theory, one can compute the K-groups of various families of rings and that will help understand the dense triangulated subcategories of perfect complexes over all those rings. However, in order to classify all triangulated subcategories of perfect complexes, one has to compute the Grothendieck groups of all thick subcategories of these complexes.

We now recall some definitions and a theorem due to Hopkins and Neeman [Nee92] which classifies the thick subcategories of perfect complexes over a noetherian ring.

Definition 4.6 ([Nee92]). Given a perfect complex X in $D(R)$, define the *support* of X, denoted by $\operatorname{Supp}(X)$, to be the set $\{p \in \operatorname{Spec}(R) : X \otimes R_p \neq 0\}$, where R_p is the localisation of R.

Definition 4.7. A subset of $\operatorname{Spec}(R)$ is said to be *closed under specialisation* if it is a union of closed sets under the Zariski topology. Equivalently, and more explicitly, a subset S of $\operatorname{Spec}(R)$ is specialisation-closed if whenever a prime ideal p is in S, then so is every prime ideal q that contains p.

Now we are ready to state the celebrated thick subcategory theorem of Hopkins and Neeman.

Theorem 4.8 ([Nee92]). *If R is any noetherian ring, then there is a natural order preserving bijection between the sets*

$$\{\textit{thick subcategories } \mathcal{A} \textit{ of } D^b(\operatorname{proj} R)\}$$
$$f \downarrow \uparrow g$$
$$\{\textit{subsets } S \textit{ of } \operatorname{Spec}(R) \textit{ that are closed under specialisation}\}.$$

The map f sends a thick subcategory $\mathcal{A}$ to $\bigcup_{X\in\mathcal{A}} \operatorname{Supp}(X)$, and the map g sends a specialisation-closed subset S to the thick subcategory $\mathcal{T}_S := \{X \in D^b(\operatorname{proj} R) : \operatorname{Supp}(X) \in S\}$.

The following corollary is immediate from the above theorem.

Corollary 4.9. *Every thick subcategory of $D^b(\operatorname{proj} R)$ is a thick ideal.*

Corollary 4.10. *Let R be any commutative ring such that $K_0(R) \cong \mathbb{Z}$. Then every triangulated subcategory of $D^b(\operatorname{proj} R)$ is a triangulated ideal.*

Proof. Since we know that thick subcategories of $D^b(\operatorname{proj} R)$ are thick ideals, they can be viewed as triangulated modules over $D^b(\operatorname{proj} R)$. Now we can apply Theorem 2.12 to conclude that every triangulated subcategory is also a triangulated submodule because $K_0(D^b(\operatorname{proj} R)) \cong \mathbb{Z}$. ∎

It is clear that the proofs of the above corollaries generalise to prove the following proposition.

Proposition 4.11. *Let $\mathcal{T}$ be a tensor triangulated category. If the unit object S is small and generates $\mathcal{T}$, then every thick subcategory of compact objects is a thick ideal. Moreover, if the Grothendieck ring of the compact objects in $\mathcal{T}$ is isomorphic to $\mathbb{Z}$, then every triangulated subcategory of compact objects is a triangulated ideal.*

We will now apply Thomason's recipe to classify the triangulated subcategories of perfect complexes over some commutative rings: principal ideal domains, Artin rings, and non-noetherian rings with a unique prime ideal.

4.3. Principal ideal domains. We first set up a few notations and recall some definitions and basic facts about PIDs.

For any element x in R, we define the mod-x Moore complex $M(x)$, in analogy with the stable homotopy category of spectra, to be the cofibre of the self map (of degree 0)

$$R \xrightarrow{x} R$$

in $D(R)$.

Now we recall the notion of the length of a module. For an R-module M, a chain

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_r = 0$$

is called a *composition series* if each M_i/M_{i+1} is a simple module (one that does not have any non-trivial submodules). The *length* of a module, denoted by $l(M)$, is defined to be the length of any composition series of M. The fact that this is well defined is part of the Jordan–Hölder theorem.

The function $l(-)$ is an additive function on the subcategory of R-modules which have finite length, i.e., if

$$0 \to M_1 \to \cdots \to M_s \to 0$$

is an exact sequence of R-modules of finite length, then

$$\sum_i (-1)^i l(M_i) = 0.$$

Also note that when R is a PID, every finitely generated torsion module has finite length. (This can be seen as an immediate consequence of the structure theorem for finitely generated modules over a PID.)

Finally, we need the following easy exercise. If p and q are two distinct (non-zero) prime elements in a PID R, then for any $i \geq 1$,

$$R/(p^i) \otimes_R R_{(q)} = \begin{cases} R/(p^i) & \text{when } p = q, \\ 0 & \text{when } p \neq q. \end{cases}$$

Now we are ready to compute the Grothendieck groups for thick subcategories of perfect complexes over a PID. Given a subset S of $\operatorname{Spec}(R)$ that is closed under specialisation, the thick subcategory that corresponds to the subset S (under the Hopkins–Neeman bijection) will be denoted by $\mathcal{T}_S$.

Theorem 4.12. *Let R be a PID and let S be a specialisation-closed subset of* $\operatorname{Spec}(R)$*. Then we have the following.*

(1) *If $S = \operatorname{Spec}(R)$, then $K_0(\mathcal{T}_S)$ is an infinite cyclic group generated by R.*
(2) *If $S \neq \operatorname{Spec}(R)$, then $K_0(\mathcal{T}_S)$ is a free abelian group on the Moore complexes $M(p)$, for $p \in S$.*

Proof. The first part follows from the fact that every finitely generated projective module over a PID is free; consequently, its Grothendieck group is an infinite cyclic group (see Proposition 4.1 and Example 4.3). For the second part, first note that a specialisation-closed subset $S \neq \operatorname{Spec}(R)$ is a subset of maximal ideals in R (because non-zero prime ideals in a PID are also maximal). For each prime element p in S, define an Euler characteristic function $\lambda_p : \mathcal{T}_S \to \mathbb{Z}$ by

$$\lambda_p(X) := \sum_i (-1)^i l[H_i(X) \otimes_R R_{(p)}].$$

(Since S does not contain (0), it follows that $H_*(X)$ is a torsion R- module. Therefore $\lambda_p(-)$ is a well defined Euler characteristic function.)

Also, since $\lambda_p(M(q)) = \delta_p^q$, the Euler characteristic map

$$\bigoplus_{p \in S} \lambda_p : K_0(\mathcal{T}_S) \to \bigoplus_{p \in S} \mathbb{Z}$$

is clearly surjective.

To see that this map is injective, it suffices to show that every complex in $\mathcal{T}_S$ can be generated by the set $\{M(p) : p \in S\}$ using cofibre sequences. We do this by induction on $\sum_i l[H_i(-)]$. If $X \in \mathcal{T}_S$ is such that $\sum_i l[H_i(X)] = 1$,

$$\begin{tikzcd} \vdots \arrow[r] \arrow[d] & \vdots \arrow[d] \\ 0 \arrow[r] \arrow[d] & P_2 \arrow[d] \\ R \arrow[r, "a"] \arrow[d, "p"'] & P_1 \arrow[d, "c"] \\ R \arrow[r, "b"] \arrow[d] & P_0 \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

Fig. 3. Killing a homology class in the Hurewicz dimension

then there exists an integer j such that $H_i(X) = 0$ for all $i \neq j$, and $H_j(X) = R/(p)$ for some prime p in R (because every simple module over a PID is of the form $R/(p)$ for some prime p). Such an X is clearly quasi-isomorphic to $M(p)$. Now consider an $X \in \mathcal{T}_S$ for which $\sum_i l[H_i(X] > 1$. Without loss of generality we can assume that up to suspension X is of the form

$$\cdots \to 0 \to P_k \to \cdots \to P_1 \to P_0 \to 0 \to \cdots$$

with $H_0(X) \neq 0$ (otherwise, we can replace X with a quasi-isomorphic complex in $\mathcal{T}_S$ which has this property). Now pick a non-zero element in $H_0(X)$ and represent it with a cycle t. Since $H_0(X)$ is a torsion module, there exists a prime p and a positive integer k such that $p^k t = 0$ in homology. Replacing t with $p^{k-1}t$, we can assume that $pt = 0$ in homology, which means pt is a boundary. So there is an element $y \in P_1$ which maps under the differential to pt. Consider Fig. 3 where $a(1) = y$, $c(y) = pt$, and $b(1) = t$. This diagram shows a chain map between the two complexes in $\mathcal{T}_S$ such that the induced map in homology in dimension 0 sends 1 to t (by construction). So the class t is killed. Now if we extend this morphism to a triangle ($M(p) \to X \xrightarrow{d} Y \to \Sigma M(p)$) and look at the long exact sequence in homology, it is clear that X and Y have the same homology in all dimensions except dimension 0. In dimension 0, part of the long exact sequence gives a short exact sequence $0 \to R/(p) \to H_0(X) \to H_0(Y) \to 0$. Therefore $l[H_0(Y)] = l[H_0(X)] - 1$. By the induction hypothesis we know that Y can be generated by the set $\{M(p) : p \in S\}$ using cofibre sequences. The above exact triangle then tells us that X can also be generated using cofibre sequences in this way. So we are done. ∎

COROLLARY 4.13. *If R is any PID, then* $\operatorname{Pic}(D(R)) \cong \mathbb{Z}$.

Proof. Since $K_0(D^b(\operatorname{proj} R)) \cong \mathbb{Z}$, the corollary follows by invoking Theorem 4.5. ∎

Classification of the triangulated subcategories of $D^b(\operatorname{proj} R)$ when R is a PID. There are two families of triangulated subcategories: triangulated subcategories that correspond to $S = \operatorname{Spec}(R)$ and the ones that correspond to $S \neq \operatorname{Spec}(R)$ (subsets of maximal ideals).

1. $S = \operatorname{Spec}(R)$: Consider the Euler characteristic function

$$\chi(X) = \sum_{-\infty}^{\infty} (-1)^i \dim_F \{H_i(X) \otimes_R F\},$$

where F is the field of fractions of our domain R. For every integer k, we define

$$D_k = \{X : \chi(X) \equiv 0 \bmod k\}.$$

These are all the triangulated subcategories that are dense in $D^b(\operatorname{proj} R)$.

2. $S \neq \operatorname{Spec}(R)$: Given such a subset S and a subgroup H of $\bigoplus_{p\in S} \mathbb{Z}$, we define

$$\mathcal{T}(S,H) = \Big\{ X \in \mathcal{T}_S : \Big(\bigoplus_{p\in S} \lambda_p\Big)(X) \in H \Big\}.$$

These are all the triangulated subcategories that are dense in $\mathcal{T}_S$.

It is clear from Theorem 4.12 and Theorem 2.5 that every triangulated subcategory of $D^b(\operatorname{proj} R)$ is of one of these two types.

Here is an interesting consequence of the above theorem.

COROLLARY 4.14. *Let X and Y be perfect complexes over a PID. Then Y can be generated from X using cofibrations if and only if*

- $\operatorname{Supp}(Y) \subseteq \operatorname{Supp}(X)$,
- *If* $(0) \in \operatorname{Supp}(X)$, *then* $\lambda_0(X)$ *divides* $\lambda_0(Y)$; *otherwise*, $\lambda_p(X)$ *divides* $\lambda_p(Y)$ *for all* $p \in \operatorname{Supp}(X)$.

4.4. Product of rings: Artin rings. We now address the following question.

QUESTION. Suppose a commutative ring R is a direct product of rings, say

$$R \cong R_1 \times \cdots \times R_k,$$

and suppose that we have a classification of all the triangulated subcategories of $D^b(\operatorname{proj} R_i)$ for all i. Using this information, how can we get a classification of all triangulated subcategories of $D^b(\operatorname{proj} R)$? More generally, one can ask how the spectral theory of R and the spectral theories of the rings R_i are related.

Before we go further, we remark that by the obvious induction, it suffices to consider only two components ($R \cong R_1 \times R_2$). We collect some standard facts about products of triangulated categories that will be needed.

LEMMA 4.15. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be triangulated categories. Then we have the following.*

- *The product category $\mathcal{T}_1 \times \mathcal{T}_2$ admits a triangulated structure that has the following universal property: Given any triangulated category $\mathcal{D}$ and some triangulated functors $\mathcal{T}_1 \leftarrow \mathcal{D} \rightarrow \mathcal{T}_2$, there exists a unique triangulated functor $\mathcal{F} : \mathcal{D} \rightarrow \mathcal{T}_1 \times \mathcal{T}_2$ making the following diagram of triangulated functors commutative:*

$$\begin{tikzcd} & \mathcal{D} \arrow[dl] \arrow[d, dotted, "F"] \arrow[dr] & \\ \mathcal{T}_1 & \mathcal{T}_1 \times \mathcal{T}_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"'] & \mathcal{T}_2 \end{tikzcd}$$

- *Every thick (resp. localising) subcategory of $\mathcal{T}_1 \times \mathcal{T}_2$ is of the form $\mathcal{B}_1 \times \mathcal{B}_2$, where $\mathcal{B}_i$ is a thick (resp. localising) subcategory in $\mathcal{T}_i$.*
- *If both $\mathcal{T}_1$ and $\mathcal{T}_2$ are essentially small triangulated (resp. tensor triangulated) categories, then $K_0(\mathcal{T}_1 \times \mathcal{T}_2) \cong K_0(\mathcal{T}_1) \times K_0(\mathcal{T}_2)$ as groups (resp. rings).*

REMARK 4.16. In contrast with the thick subcategories, not every triangulated subcategory of $\mathcal{T}_1 \times \mathcal{T}_2$ is of the form $\mathcal{A}_1 \times \mathcal{A}_2$, where $\mathcal{A}_i$ is a triangulated subcategory of $\mathcal{T}_i$. This is clear from the third part of the above lemma because not every subgroup of the product group $K_0(\mathcal{T}_1) \times K_0(\mathcal{T}_2)$ is a product of subgroups.

Now we relate the category of perfect complexes over R and those over the rings R_i.

PROPOSITION 4.17. *There is a natural equivalence of triangulated categories*

$$D^b(\mathrm{proj}\, R) \simeq D^b(\mathrm{proj}\, R_1) \times D^b(\mathrm{proj}\, R_2).$$

In particular, $K_0[D^b(\mathrm{proj}\, R)] \cong K_0[D^b(\mathrm{proj}\, R_1)] \times K_0[D^b(\mathrm{proj}\, R_2)]$.

Proof. Note that every module M over $R_1 \times R_2$ is a direct sum (in the category of $R_1 \times R_2$-modules)

$$M \cong P_1 \oplus P_2,$$

where P_i is an R_i-module: Take $P_1 = \langle (1,0) \rangle M$ and $P_2 = \langle (0,1) \rangle M$, where P_i is also regarded as an $R_1 \times R_2$-module via the projection maps $R_1 \times R_2 \rightarrow R_i$. Now one can verify easily that this decomposition is functorial and sends finitely generated (resp. projective) modules to finitely generated (resp. projective) modules. Therefore every complex in $D^b(\mathrm{proj}\, R_1 \times R_2)$ splits as a direct sum of two complexes, each in $D^b(\operatorname{proj} R_i)$. Conversely, given a pair of complexes (X_1, X_2) with X_i in $D^b(\operatorname{proj} R_i)$, their direct sum $X_1 \oplus X_2$ is clearly a complex in $D^b(\operatorname{proj} R_1 \times R_2)$. Now it can be verified that these two functors establish the desired equivalence of triangulated categories. The second statement follows immediately from Lemma 4.15. ∎

In view of this proposition, the problem of classifying triangulated subcategories in $D^b(\operatorname{proj} R_1 \times R_2)$ boils down to classifying triangulated subcategories of $D^b(\operatorname{proj} R_1) \times D^b(\operatorname{proj} R_2)$. So, following Thomason's recipe, we need to classify the thick subcategories of $D^b(\operatorname{proj} R_1) \times D^b(\operatorname{proj} R_2)$ and compute their Grothendieck groups. This is given by Lemma 4.15: Every thick subcategory $\mathcal{T}$ of $D^b(\operatorname{proj} R_1) \times D^b(\operatorname{proj} R_2)$ is of the form $\mathcal{T}_1 \times \mathcal{T}_2$ where $\mathcal{T}_i$ is thick in $D^b(\operatorname{proj} R_i)$, and $K_0(\mathcal{T}) \cong K_0(\mathcal{T}_1 \times \mathcal{T}_2) \cong K_0(\mathcal{T}_1) \times K_0(\mathcal{T}_2)$.

4.4.1. *Artin rings.* We will now apply these ideas to Artin rings. Recall that a ring is said to be *Artinian* if every descending chain of ideals terminates. Artin rings can be characterised as zero-dimensional noetherian rings. They have the following structure theorem.

THEOREM 4.18 ([AM69]). *Every Artin ring R is isomorphic to a finite direct product of Artin local rings. Moreover, the number of local rings that appear in this isomorphism is equal to the cardinality of* $\operatorname{Spec}(R)$.

Thus $R \cong \prod_{i=1}^{n} R_i$, where each R_i is an Artin local ring (n is the cardinality of $\operatorname{Spec}(R)$). We have seen above that there is an equivalence of triangulated categories

$$D^b(\operatorname{proj} R) \cong \prod_{i=1}^{n} D^b(\operatorname{proj} R_i).$$

So by the above discussion, we just have to compute the Grothendieck groups of the thick subcategories of $D^b(\operatorname{proj} R_i)$. But since each R_i is an Artinian local ring, $\operatorname{Spec}(R_i)$ is a one-point space. This implies (by the Hopkins–Neeman theorem) that the only non-zero thick subcategory is $D^b(\operatorname{proj} R_i)$ itself, whose Grothendieck group is well known to be infinite cyclic. Therefore we have

$$K_0(\mathcal{T}_S) \cong \bigoplus_{p_i \in S} K_0(D^b(\operatorname{proj} R_i)) \cong \bigoplus_{p_i \in S} \mathbb{Z}.$$

The universal Euler characteristic function that gives this isomorphism is $\bigoplus_{p_i \in S} \Lambda_{p_i}$ where $\Lambda_{p_i}(X) = \sum_{t=-\infty}^{\infty} (-1)^t \dim_{R_i/p_i} H_t(X \otimes R_i/p_i)$.

We now record the classification of triangulated subcategories of perfect complexes over Artin rings.

THEOREM 4.19. *Let R be any Artin ring and let $R = \prod_i R_i$ be its unique decomposition into Artin local rings. For every subset S of* $\operatorname{Spec}(R)$ *and* *every subgroup H of $\bigoplus_{p_i \in S} \mathbb{Z}$, define*

$$\mathcal{T}(S,H) := \Big\{ X \in \mathcal{T}_S : \Big(\bigoplus_{p_i \in S} \Lambda_{p_i} \Big)(X) \in H \Big\}.$$

This is a complete list of triangulated subcategories of $D^b(\operatorname{proj} R)$. Further, every dense triangulated subcategory of $\mathcal{T}_S$ is a triangulated ideal if and only if S is a one-point space.

Remark 4.20. It is clear from the proof that this theorem also holds whenever $R \cong \prod_{i=1}^n R_i$, where each of the rings R_i has exactly one prime ideal.

We now derive some easy consequences of the above theorem.

Corollary 4.21. *Let X and Y be perfect complexes over an Artin ring. Then Y can be generated from X using cofibrations if and only if*

- $\operatorname{Supp}(Y) \subseteq \operatorname{Supp}(X)$,
- *$\Lambda_{p_i}(X)$ divides $\Lambda_{p_i}(Y)$ for all $p_i \in \operatorname{Supp}(X)$, where $\Lambda_{p_i}(X) = \sum (-1)^i \dim_{R_i/p_i} H_i(X \otimes R_i/p_i)$.*

Corollary 4.22. *An Artin ring R is local if and only if every dense triangulated subcategory of $D^b(\operatorname{proj} R)$ is a triangulated ideal.*

Proof. By Theorem 2.11, it is clear that every dense triangulated subcategory is a triangulated ideal if and only if every subgroup of $K_0(D^b(\operatorname{proj} R)) \cong \prod_{p \in \operatorname{Spec}(R)} \mathbb{Z}$ is also an ideal. Clearly the latter happens if and only if $|\operatorname{Spec}(R)| = 1$, or equivalently if R is local. ∎

4.5. Non-noetherian rings. In order to study the problem of classifying triangulated subcategories in the non-noetherian case, we need a thick subcategory theorem for $D^b(\operatorname{proj} R)$, when R is a non-noetherian ring. This is given by a result of Thomason, which is a far-reaching generalisation of the Hopkins–Neeman theorem to schemes. We now state this theorem for commutative rings.

Theorem 4.23 ([Tho97]). *Let R be any commutative ring. Then there is a natural order preserving bijection between the sets*

$$\{\textit{thick subcategories } \mathcal{A} \textit{ of } D^b(\operatorname{proj} R)\}$$

$$f \downarrow \quad \uparrow g$$

$$\{\textit{subsets } S \textit{ of } \operatorname{Spec}(R) \textit{ such that } S = \textstyle\bigcup_\alpha V(I_\alpha), \ \textit{where } I_\alpha \textit{ is finitely generated}\}.$$

The map f sends a thick subcategory $\mathcal{A}$ to the set $\bigcup_{X \in \mathcal{A}} \operatorname{Supp}(X)$ and the map g sends S to the thick subcategory $\{X \in D^b(\operatorname{proj} R) : \operatorname{Supp}(X) \in S\}$.

REMARK 4.24. The subsets of $\operatorname{Spec}(R)$ in this corollary which determine the thick subcategories of perfect complexes will be called *thick supports.* If R is noetherian, every thick support is a specialisation-closed subset (since ideals in a noetherian ring are finitely generated), therefore the above corollary recovers the Hopkins–Neeman thick subcategory theorem.

We do not know much about the K-theory of thick subcategories over non-noetherian rings, except in the simplest case where the rings have only one prime ideal, e.g., $R = \mathbb{F}_2[X_2, X_3, \ldots]/(X_2^2, X_3^3, \ldots)$.

REMARK 4.25. The geometry of these rings is very simple—just a one point space, however, their derived categories can be very mysterious and extremely complicated. Amnon Neeman [Nee00] showed that the derived category of the above ring has uncountably many Bousfield classes—a striking contrast with the noetherian result where the Bousfield classes are known to be in bijection with the subsets of $\operatorname{Spec}(R)$ [Nee92]. Despite this incredible complexity in the derived categories of such rings, K-theory does enable us to classify all the triangulated subcategories of perfect complexes.

PROPOSITION 4.26. *Let R be any commutative ring with a unique prime ideal p. Then every triangulated subcategory of $D^b(\operatorname{proj} R)$ is a triangulated ideal and is of the form*

$$\mathcal{D}_m = \{X \in D^b(\operatorname{proj} R) : \Lambda(X) \equiv 0 \bmod m\}$$

for some non-negative integer m, where

$$\Lambda(X) = \sum_{-\infty}^{\infty} (-1)^i \dim_{R/p} H_i(X \otimes R/p).$$

Proof. Note that any such R is, in particular, a local ring. Therefore,

$$K_0(D^b(\operatorname{proj} R)) \ (\cong K_0(R)) \cong \mathbb{Z}.$$

It is easily verified that the given Euler characteristic function gives this isomorphism. Moreover, since R has a unique prime ideal, it is clear from Corollary 4.23 that there are no non-trivial thick subcategories in $D^b(\operatorname{proj} R)$. So the dense triangulated subcategories in $D^b(\operatorname{proj} R)$ are all the triangulated subcategories in $D^b(\operatorname{proj} R)$. It is clear that these are all triangulated ideals. This completes the proof of the proposition. ∎

4.6. Questions

4.6.1. *Algebraic K-theory for thick subcategories.* The key to the problem of classifying the triangulated subcategories of perfect complexes lies in the algebraic K-theory of thick subcategories. In Section 4.2 we have summarised a few results from classical algebraic K-theory. They concerned the Grothendieck groups of $D^b(\operatorname{proj} R)$. So we now ask if such results also hold for thick subcategories of $D^b(\operatorname{proj} R)$. We ask a very specific question to make this point clear. It is well known that if J is the nilradical of R, then

$$K_0(D^b(\operatorname{proj} R)) \cong K_0(D^b(\operatorname{proj} R/J)).$$

Since every prime ideal contains the nilradical, the quotient map $R \to R/J$ induces a homeomorphism on prime spectra: $\operatorname{Spec}(R) \cong \operatorname{Spec}(R/J)$. This homeomorphism implies that the lattice of specialisation-closed subsets of $\operatorname{Spec}(R)$ is isomorphic to that of $\operatorname{Spec}(R/J)$. Now if R is noetherian, we can invoke the Hopkins–Neeman thick subcategory theorem to conclude that the same is true for the lattices of thick subcategories of perfect complexes over R and R/J. Now the question arises whether the thick subcategories that correspond to each other under this isomorphism have isomorphic Grothendieck groups.

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Received 4 August 2005;
in revised form 5 October 2005