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A topological application of flat morasses

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Abstract. We define combinatorial structures which we refer to as *flat morasses*, and use them to construct a Lindelöf space with points G_δ of cardinality $\aleph_\omega$, consistent with GCH. The construction reveals, it is hoped, that flat morasses are a tool worth adding to the kit of any user of set theory.

1. Introduction

1.1. *Flat morasses.* The *flat morasses* used in this paper resemble gap-1 simplified morasses, in that they provide a method of building a large structure out of small pieces along a short directed set.

Morasses—first the concrete morasses of Jensen (see a description, for the gap-1 case, in [1]) and later simplified morasses, whose existence was proved equivalent to that of concrete morasses of the same gap-number by Velleman, Jensen, Morgan and others—provide a technology for, for example, using countable pieces to build a structure of size ω_2, ω_3 or more along a direct system of length ω_1. The recursion along the direct system has two stages, the successor stage and the limit stage. The limit stage is trivially easy, simply involving taking a direct limit; at the successor stage care is typically required, as one has to amalgamate several simple structures together to make a larger structure.

In the gap-1 case (at least in the simplified setting; from here on we drop the word "simplified"), in which one is using pieces of size κ (for some κ) and a recursion of length κ^+ to build a structure of size κ^{++}, this amalgamation is still of a fairly simple kind; and there are many applications of gap-1 morasses in the literature. But the amalgamation step in the application of a higher-gap morass presents a forbidding appearance (here one is skipping more cardinals, to build a structure of size κ^{+++} or higher). Though

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the structure of such a morass is extraordinarily beautiful, it is still rather complex.

Flat morasses will therefore, we hope, be of some use. While much less elegant than the higher-gap morasses, they are—at least in the opinion of the author—not much more complex than morasses of gap 1, and allow one to build structures of size κ^{n+}, for any value of n, using pieces of size κ, consistent with GCH. The author originally extracted them from an attempt to apply higher-gap morasses to the problem described in the next section, but in this paper a gap-n flat morass is derived rather more simply. While this certainly shows them to be consistent with ZFC plus GCH, it leaves open the precise circumstances under which they exist.

For those who are familiar with the use of gap-1 morasses, it may be worth describing in a little more detail how flat morasses differ from them. What does one sacrifice, in order to be able to build larger structures?

The most obvious thing is that the amalgamation step, in the use of a flat morass, is less neat than it is in a gap-1 morass. At the amalgamation stage in a gap-1 morass, one has a structure $\mathfrak{A}_\alpha$ on an ordinal θ_α, one has two maps from θ_α to another ordinal $\theta_{\alpha+1}$, one of which is the identity on θ_α, and the other is a shift map f such that

$$f : \gamma \mapsto \begin{cases} \gamma & \text{if } \gamma < \xi, \\ \theta_\alpha + \delta & \text{if } \gamma = \xi + \delta \end{cases}$$

for some ordinal ξ, and one must build $\mathfrak{A}_{\alpha+1}$ in such a way that these two maps both embed $\mathfrak{A}_\alpha$ in it. In a flat morass, though, there are more than two maps, they may not all have the same domain, and they certainly do not overlap this neatly.

The second, perhaps less important, thing is there is less control over the limit stage, in that it is harder to say what a limit stage is; this difficulty is visible in the preamble to condition 2(d)(ii) in the definition of a flat morass in Section 2.2.

1.2. *Lindelöf spaces of countable pseudocharacter.* Recall that a topological space X is *Lindelöf* iff every open cover of X has a countable subcover.

Arkhangel'skiĭ's striking theorem that every Lindelöf first countable space has cardinality $\leq 2^{\aleph_0}$ raises the question: what happens if we weaken the hypothesis of first countability to countable pseudocharacter (that is, the hypothesis that each point is a G_δ)?

The situation is entirely different. Shelah proved in [8] that consistently, one can have a Lindelöf space of countable pseudocharacter with cardinality $\aleph_2$ under GCH. Hajnal and Juhász give a proof of this theorem in [4]. This proof can be cast as an application of a gap-1 morass. Gorelić in [3] generalised this to cardinality $2^{\aleph_1}$, with $2^{\aleph_1}$ being arbitrarily large; this result can be proved using a generalisation of a gap-1 morass.

In this paper we consistently obtain an example of size $\aleph_\omega$ with GCH, using the already-mentioned flat morasses.

There remain the following questions.

QUESTION 1.2.1. *Can it be proved from ZFC together with GCH that flat morasses exist?*

QUESTION 1.2.2. *Is it consistent with ZFC together with GCH that there does not exist a Lindelöf space of cardinality $\aleph_\omega$ having countable pseudocharacter?*

Is it consistent that there exists one of cardinality $\beth_\alpha$, for all countably infinite α, or of cardinality $\beth_{\omega_1}$?

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2. Flat morasses

2.1. *Gap-1 morasses.* For the purposes of comparison, we first recall the definition of a gap-1 morass, and some of the major properties.

DEFINITION 2.1.1. Suppose λ is a regular uncountable cardinal. A *gap*-1 *morass of height* λ is a pair $\mathcal{M} = \langle \phi, \mathcal{G} \rangle$ such that

1. (a) ϕ is an ordinal-valued function on $\lambda + 1$.
   (b) If $\alpha < \lambda$, then $\phi(\alpha)$, which we write as ϕ_α, is $< \lambda$.
   (c) $\phi_\lambda = \lambda^+$.
   (d) If $\alpha \le \beta$, then $\phi_\alpha \le \phi_\beta$.
2. $\mathcal{G}$ is a function with domain $(\lambda + 1) \times (\lambda + 1)$ such that
   (a) $\mathcal{G}(\alpha, \beta)$ (which we write $\mathcal{G}_{\alpha,\beta}$) is non-empty iff $\alpha \le \beta$.
   (b) If $f \in \mathcal{G}_{\alpha,\beta}$, then f is a one-to-one order-preserving function from ϕ_α to ϕ_β, and $\mathcal{G}_{\alpha,\alpha}$ has exactly one element, the identity on ϕ_α.
   (c) (*Compositionality*)
      (i) If $f \in \mathcal{G}_{\alpha,\beta}$ and $g \in \mathcal{G}_{\beta,\gamma}$, then $g \circ f \in \mathcal{G}_{\alpha,\gamma}$.
      (ii) If $\alpha \le \beta \le \gamma$ and $h \in \mathcal{G}_{\alpha,\gamma}$, then there exist $f \in \mathcal{G}_{\alpha,\beta}$ and $g \in \mathcal{G}_{\beta,\gamma}$ such that $h = g \circ f$.
   (d) (*Covering at Limits*)
      (i) If F is a finite subset of ϕ_γ, where γ is a limit ordinal, then there exist $\alpha < \gamma$ and $f \in \mathcal{G}_{\alpha,\gamma}$ such that $F \subseteq \operatorname{ran} f$.

(ii) If γ is a limit, $\alpha, \beta < \gamma$, $f \in \mathcal{G}_{\alpha,\gamma}$ and $g \in \mathcal{G}_{\beta,\gamma}$, then there exist $\delta < \gamma$, $h \in \mathcal{G}_{\delta,\gamma}$, $f' \in \mathcal{G}_{\alpha,\delta}$, and $g' \in \mathcal{G}_{\beta,\delta}$ such that $f = h \circ f'$ and $g = h \circ g'$.

(e) (*Local Smallness*) If $\alpha, \beta < \lambda$, then $|\mathcal{G}_{\alpha,\beta}| < \lambda$.

(f) For each $\alpha < \lambda$, $\mathcal{G}_{\alpha,\alpha+1}$ has exactly two elements:

(i) the identity on ϕ_α,

(ii) a *shift map* σ having the property that for some $\eta < \phi_\alpha$, $\sigma(\xi) = \xi$ if $\xi < \eta$, and $\sigma(\eta + \zeta) = \phi_\alpha + \zeta$ for all ζ.

This structure has many beautiful properties, of which we mention two:

1. (*Coherence*) If $\alpha \leq \beta$, and f, $g \in \mathcal{G}_{\alpha,\beta}$, then there exists $\eta \leq \phi_\alpha$ such that

(a) for all $\xi < \eta$, $f(\xi) = g(\xi)$,

(b) for all γ, $f(\eta + \gamma) \notin \operatorname{ran} g$, and for all γ, $g(\eta + \gamma) \notin \operatorname{ran} f$.

Therefore, on $\operatorname{ran} f \cap \operatorname{ran} g$, f^{-1} and g^{-1} are equal; and $\operatorname{ran} f \cap \operatorname{ran} g$ is an initial segment of both $\operatorname{ran} f$ and $\operatorname{ran} g$.

2. (*Offset Condition*) If $\alpha \leq \beta \leq \gamma$, $f \in \mathcal{G}_{\alpha,\gamma}$, $g \in \mathcal{G}_{\beta,\gamma}$, then $\operatorname{ran} f \cap \operatorname{ran} g$ is an initial segment of $\operatorname{ran} f$; and there exists $f' \in \mathcal{G}_{\alpha,\beta}$ such that $\operatorname{ran} f \cap \operatorname{ran} g$ is an initial segment of $\operatorname{ran}(g \circ f')$.

A comment is in order about the purpose of some of the clauses in the definition.

Remember that the function of a morass is to perform constructions by recursion. The idea is that we recursively define some first-order structure on ϕ_α, such that the maps in the various $\mathcal{G}_{\alpha,\beta}$ are embeddings.

The condition of *Covering at Limits* ensures that if γ is a limit, then the structure we build on ϕ_γ is the direct limit along the morass of the previously-built structures. In particular, the structure on $\phi_\lambda = \lambda^+$ is a direct limit. In other words, we do not need to exercise any care at limit stages.

The special form of $\mathcal{G}_{\alpha,\alpha+1}$, which, unfortunately, we will not be able to imitate closely in the definition of a flat morass, makes successor stages particularly easy.

The condition of *Coherence* reassures us that we can arrange that all the maps in the $\mathcal{G}_{\alpha,\beta}$ are embeddings of our structures; that two maps f and g will not impose incompatible requirements.

The *Offset Condition* tells us that the ranges of two morass functions f and g can overlap in only a few ways. A generalised offset condition will play an important role in the construction we will perform.

2.2. *The definition of a flat morass.* We now define the structure we will be using.

Suppose κ is an infinite cardinal. Then an *gap-$(n-1)$ flat morass of height κ^+* is a pair $\langle \theta, \mathcal{F} \rangle$ such that

1. There is a partial order $\mathfrak{C}$, with order relation $\leq_{\mathfrak{C}}$, and with greatest element the ordinal κ^+, such that:
   (a) $\mathfrak{C} \setminus \{\kappa^+\}$ is directed and well-founded.
   (b) θ is an ordinal-valued function on $\mathfrak{C}$.
   (c) If $\alpha \in \mathfrak{C} \setminus \{\kappa^+\}$, then $\theta(\alpha)$, which we write as θ_α, is $< \kappa^+$.
   (d) $\theta_{\kappa^+} = \kappa^{n+}$.
   (e) If $\alpha \leq_{\mathfrak{C}} \beta$, then $\theta_\alpha \leq \theta_\beta$. [This will follow from the other conditions.]
   (f) For all $\alpha \in \mathfrak{C} \setminus \{\kappa^+\}$, $\{\beta \in \mathfrak{C} : \beta \leq \alpha\}$ has cardinality $\leq \kappa$.
2. $\mathcal{F}$ is a function with domain $\mathfrak{C} \times \mathfrak{C}$ such that
   (a) $\mathcal{F}(\alpha, \beta)$ (which we write $\mathcal{F}_{\alpha,\beta}$) is non-empty iff $\alpha \leq_{\mathfrak{C}} \beta$.
   (b) If $f \in \mathcal{F}_{\alpha,\beta}$, then f is a one-to-one order-preserving function from θ_α to θ_β, which is onto only if $\alpha = \beta$.
   $\mathcal{F}_{\alpha,\alpha}$ has a unique element, namely the identity on θ_α.
   (c) (*Compositionality*)
      (i) If $f \in \mathcal{F}_{\alpha,\beta}$ and $g \in \mathcal{F}_{\beta,\gamma}$, then $g \circ f \in \mathcal{F}_{\alpha,\gamma}$.
      (ii) If $\alpha \leq \beta \leq \gamma$, and $h \in \mathcal{F}_{\alpha,\gamma}$, then there exist $f \in \mathcal{F}_{\alpha,\beta}$ and $g \in \mathcal{F}_{\beta,\gamma}$ such that $h = g \circ f$.
      (iii) If $f \in \mathcal{F}_{\alpha,\gamma}$, $g \in \mathcal{F}_{\beta,\gamma}$, and $\operatorname{ran} f \subseteq \operatorname{ran} g$, then there exists $f' \in \mathcal{F}_{\alpha,\beta}$ such that $f = g \circ f'$. [This follows from the Offset Condition.] We write f' as (f/g).
   (d) (*Covering at Limits*)
      (i) If F is a finite subset of κ^{n+}, then there exist $\alpha < \kappa^+$ and $f \in \mathcal{F}_{\alpha,\kappa^+}$ such that $F \subseteq \operatorname{ran} f$.
      (ii) Suppose $\alpha \in \mathfrak{C}$. Then we say α is a *limit in the morass structure* iff it satisfies the following conditions:
         A. For every pair $x, y \in \theta_\alpha$, there exist $\beta <_{\mathfrak{C}} \alpha$ and $f \in \mathcal{F}_{\beta,\alpha}$ such that x, $y \in \operatorname{ran} f$.
         B. For every pair $f, g \in \bigcup_{\beta<\alpha} \mathcal{F}_{\beta,\alpha}$, there exists $h \in \bigcup_{\beta<\alpha} \mathcal{F}_{\beta,\alpha}$ such that $\operatorname{ran} h \supseteq \operatorname{ran} f$, $\operatorname{ran} g$.

         Then the collection of all sets $\operatorname{ran} f$ such that there exists $\alpha < \kappa^+$ such that α is a limit in the morass structure and $f \in \mathcal{F}_{\alpha,\kappa^+}$, is stationary in $[\kappa^{n+}]^{\leq\kappa}$.
   (e) (*Local Smallness*) If $\alpha, \beta < \kappa^+$, then $\mathcal{F}_{\alpha,\beta}$ has cardinality $\leq \kappa$.

(f) (*Coherence*) Suppose $f, g \in \mathcal{F}_{\alpha,\beta}$, and F is a finite subset of $\operatorname{ran} f \cap \operatorname{ran} g$. Then there exist $\gamma \leq \alpha$, $f', g' \in \mathcal{F}_{\gamma,\alpha}$ and a finite subset G of θ_γ such that

$$F = f \circ f'(G) = g \circ g'(G).$$

[This follows from the next condition.]

(g) (*Offset Condition*) Suppose $f \in \mathcal{F}_{\alpha,\delta}$, $g \in \mathcal{F}_{\beta,\delta}$. Then there exist $\gamma \leq \alpha, \beta$, $f' \in \mathcal{F}_{\gamma,\alpha}$, $g' \in \mathcal{F}_{\gamma,\beta}$ and $h \in \mathcal{F}_{\gamma,\delta}$ such that

(i) $\operatorname{ran} f \cap \operatorname{ran} g = \operatorname{ran} h$.

(ii) $h = f \circ f' = g \circ g'$.

We write h as $f \sqcap g$. We write (f/g) for $f' = (f \sqcap g/g)$, and (g/f) for $g' = (f \sqcap g/f)$.

Notice that a gap-1 flat morass is not necessarily a gap-1 morass, because of condition 2(f) in the definition of a gap-1 morass, and a gap-1 morass is not necessarily a gap-1 flat morass, because of condition 2(d)(ii) in the definition of a flat morass.

2.3. *Conditions for the existence of a flat morass.* We show that a flat gap-$(n-1)$ morass of height κ^+ exists given the assumption that for each $j \leq n-2$, $\square_{\kappa^{(j+1)+}}$ holds, and so we assume the existence of sequences as follows:

Definition 2.3.1. For each $j \leq n-2$, let $\langle C^{j+1}_\alpha : \alpha \in \lim \cap \kappa^{(j+2)+} \rangle$ be a sequence such that

- If α is a limit in $\kappa^{(j+2)+}$, then C^{j+1}_α is closed unbounded in α.
- If α is a limit in $\kappa^{(j+2)+}$ and the cofinality of α is less than $\kappa^{(j+1)+}$, then the order-type of C^{j+1}_α is less than $\kappa^{(j+1)+}$.
- If α is a limit in $\kappa^{(j+2)+}$ and β is a limit point of C^{j+1}_α, then $C^{j+1}_\beta = C^{j+1}_\alpha \cap \beta$.

We note that this assumption is provable under $V = L$.

Also, for each $j \leq n-1$ and $\alpha \in [\kappa^{j+}, \kappa^{(j+1)+})$, let ψ^j_α be a bijection between κ^{j+} and α.

2.4. *The construction of the flat morass.* Let us say that a subset A of κ^{n+} of cardinality κ is *rounded* iff

(i) $A \cap \kappa^+$ is an ordinal.

(ii) A is closed under both ψ^j_α and $(\psi^j_\alpha)^{-1}$ for all $\alpha \in A \cap [\kappa^{j+}, \kappa^{(j+1)+})$, for all $j \leq n-1$.

(iii) For all $j \in [1, n-1]$, for all limits $\mu \in [\kappa^{j+}, \kappa^{(j+1)+})$, if $A \cap \mu$ is cofinal in μ then A contains the subset of C^j_μ enumerated by $A \cap \kappa^{j+}$ (in the sense that if C^j_μ is enumerated in order as $\{\eta_\alpha : \alpha < \lambda\}$, then A contains $\{\eta_\alpha : \alpha \in \lambda \cap A\}$).

Let us say that A is *weakly rounded* iff it satisfies conditions (ii) and (iii) in the above list, but (iii) holds only for ordinals $\mu \in [\kappa^{j+}, \kappa^{(j+1)+})$ (for $j \leq n-1$) such that μ is less than the supremum of $A \cap \kappa^{(j+1)+}$.

LEMMA 2.4.1. *The collection of weakly rounded sets contains a closed unbounded subset of* $[\kappa^{n+}]^{\kappa}$.

Proof. Suppose A is the intersection of an elementary submodel $\mathfrak{M}$ of size κ of some large enough $H(\theta)$ with κ^{n+}, where $\langle\langle C^j_\alpha : \alpha \in \lim \cap \kappa^{(j+1)+}\rangle : j \in [1, n-1]\rangle$ and $\langle \psi^j_\alpha : j \leq n-1,\ \alpha \in [\kappa^{j+}, \kappa^{(j+1)+})\rangle$ both belong to $\mathfrak{M}$ and $\kappa \subseteq \mathfrak{M}$.

Then it is clear that $A \cap \kappa^+$ is an ordinal, and that, by elementarity, A is closed under ψ^j_α and $(\psi^j_\alpha)^{-1}$, for all $j \leq n-1$ and for all $\alpha \in A \cap [\kappa^{j+}, \kappa^{(j+1)+})$.

Now we verify the modified condition (iii).

Suppose μ is a limit in $[\kappa^{j+}, \kappa^{(j+1)+})$ for some $j \leq n-1$, $A \cap \mu$ is cofinal in μ, and μ is less than the supremum of $A \cap \kappa^{(j+1)+}$.

Then either $\mu \in A$, in which case the subset of C^j_μ enumerated by $A \cap \kappa^{j+}$ is included in A by elementarity, or C_μ is $C_{\mu^*} \cap \mu$, where μ^* is the least element of A greater than μ. Hence $A \cap C_\mu = A \cap C_{\mu^*}$, so again the subset of C^j_μ enumerated by $A \cap \kappa^{j+}$ is included in A. ∎

Define $\mathfrak{C}$ so that

$$\mathfrak{C} = \{\kappa^+\} \cup \{A \subseteq \kappa^{n+} : A \text{ is rounded}\},$$

with $\alpha \leq_{\mathfrak{C}} \beta$ iff $\beta = \kappa^+$ or $\alpha \subseteq \beta$.

Define θ_{κ^+} to be κ^{n+}.

If α is a rounded set, define θ_α to be the order-type of α. (It will follow from the Claim in the proof of condition 1(f), in the next section, that $|\mathfrak{C}| = \kappa^{n^+}$.)

If $\alpha \in \mathfrak{C} \setminus \{\kappa^+\}$, define $\mathcal{F}_{\alpha,\kappa^+}$ to be the set whose only element is the function enumerating the elements of α in order.

If $\alpha, \beta \in \mathfrak{C} \setminus \{\kappa^+\}$ and $\alpha \leq \beta$, define $\mathcal{F}_{\alpha,\beta}$ to be the set of all f such that there exist $g \in \mathcal{F}_{\beta,\kappa^+}$ and $h \in \mathcal{F}_{\alpha,\kappa^+}$ such that $h = g \circ f$. (Thus $\mathcal{F}_{\alpha,\beta}$ has in fact just one element.)

2.5. *The flat morass conditions.* We check that $\langle \theta, \mathcal{F}\rangle$ is a flat morass of height κ^+, by checking the clauses in the definition.

1. (a) That $\mathfrak{C} \setminus \{\kappa^+\}$ is directed will follow once we have established condition 2(d)(ii). By the claim made in the proof of clause (f), if $\alpha <_{\mathfrak{C}} \beta$, then for all j, $\sup \alpha \cap \kappa^{j+} \leq \sup \beta \cap \kappa^{j+}$, and for some j, $\sup \alpha \cap \kappa^{j+} < \beta \cap \kappa^{j+}$—for otherwise we would have $\alpha = \beta$, contradicting $\alpha <_{\mathfrak{C}} \beta$. So $\mathfrak{C}$ is well-founded.

(b) Obvious.

(c) Obvious.

(d) By construction.

(e) It is unnecessary to check this condition.

(f) Claim. *Suppose $\alpha \in \mathfrak{C} \setminus \{\kappa^+\}$. Then α is determined by the sequence*

$$\langle \sup\{\gamma + 1 : \gamma \in \alpha \cap \kappa^{j+}\} : j \leq n \rangle.$$

Proof of Claim. Assume that

$$\langle \sup\{\gamma + 1 : \gamma \in \alpha \cap \kappa^{j+}\} : j \leq n \rangle = \langle \sup\{\gamma + 1 : \gamma \in \beta \cap \kappa^{j+}\} : j \leq n \rangle.$$

We prove by induction on j that

$$\alpha \cap \kappa^{j+} = \beta \cap \kappa^{j+} \quad \text{for each } j.$$

Now, $\kappa \subseteq \alpha, \beta$, because α and β are rounded. So the case for $j = 0$ is done. If $j = 1$, then $\alpha \cap \kappa^+$ and $\beta \cap \kappa^+$ are both ordinals, and so are equal. So suppose the case for $j \geq 1$ to be done; we do the case for $j + 1$. Let $\lambda = \sup\{\gamma + 1 : \gamma \in \alpha \cap \kappa^{(j+1)+}\}$.

First suppose that λ is a limit. Then because α and β are both rounded, α and β both contain the subset of C^j_λ enumerated by $\alpha \cap \kappa^{j+} = \beta \cap \kappa^{j+}$; hence $\alpha \cap \beta$ is cofinal in both $\alpha \cap \kappa^{(j+1)+}$ and $\beta \cap \kappa^{(j+1)+}$. Now suppose $\xi \in (\alpha \cap \beta) \cap [\kappa^{j+}, \kappa^{(j+1)+})$. Then $\alpha \cap \xi = \beta \cap \xi$, because, by roundedness, α and β are closed under the functions ψ^j_ξ and $(\psi^j_\xi)^{-1}$. Because $\alpha \cap \beta$ is cofinal in both $\alpha \cap \kappa^{(j+1)+}$ and $\beta \cap \kappa^{(j+1)+}$, it now follows that $\alpha \cap \kappa^{(j+1)+} = \beta \cap \kappa^{(j+1)+}$.

Now suppose that λ is a successor $\xi + 1$. Then again $\xi \in \alpha \cap \beta$, and because α and β are both closed under ψ^j_ξ and $(\psi^j_\xi)^{-1}$, $\alpha \cap \kappa^{(j+1)} = \beta \cap \kappa^{(j+1)}$. ■

Now we apply the claim. Suppose $f \in \mathcal{F}_{\alpha,\kappa^+}$ for some $\alpha <_{\mathfrak{C}} \kappa^+$. Then f is the function from θ_α into κ^{n+} enumerating α in order. Hence f is determined by the sequence

$$\langle \sup\{\gamma + 1 : \gamma \in \operatorname{ran} f \cap \kappa^{j+}\} : j \leq n \rangle.$$

Now, given $\beta \leq_{\mathfrak{C}} \alpha$, $f \in \mathcal{F}_{\beta,\alpha}$ and $h \in \mathcal{F}_{\alpha,\kappa^+}$, we have $\operatorname{ran} f \subseteq \theta_\alpha = \operatorname{dom} h$, and $\operatorname{ran}(h \circ f)$ is determined by

$$\langle \sup\{\gamma + 1 : \gamma \in \operatorname{ran}(h \circ f) \cap \kappa^{j+}\} : j \leq n \rangle,$$

so $\operatorname{ran} f$ is determined by the sequence

$$\langle \sup\{\gamma + 1 : \gamma \in \operatorname{ran} f \cap \theta^j_\alpha\} : j \leq n \rangle.$$

Since $|\theta_\alpha| \leq \kappa$, there are only $\leq\kappa$-many possible such sequences; hence only $\leq\kappa$-many possible such f, hence only $\leq\kappa$-many possible β.

2. (a) Obvious. Note that for all α, $\mathcal{F}_{\alpha,\alpha}$ is non-empty.

(b) By definition.

(c) (Compositionality) (i) Suppose $f \in \mathcal{F}_{\alpha,\beta}$ and $g \in \mathcal{F}_{\beta,\gamma}$. Then each of $\mathcal{F}_{\alpha,\kappa^+}$, $\mathcal{F}_{\beta,\kappa^+}$ and $\mathcal{F}_{\gamma,\kappa^+}$ has respectively a unique element f', g' and h'. So $f' = g' \circ f$ and $g' = h' \circ g$, so $g' = (h' \circ g) \circ f = h' \circ (g \circ f)$, so $g \circ f \in \mathcal{F}_{\alpha,\gamma}$.

(ii) If $\alpha \leq \beta \leq \gamma$ and $h \in \mathcal{F}_{\alpha,\gamma}$, let f', g' and k' be respectively the unique elements of $\mathcal{F}_{\alpha,\kappa^+}$, $\mathcal{F}_{\beta,\kappa^+}$ and $\mathcal{F}_{\gamma,\kappa^+}$. Then $f' = k' \circ h$. Let f and g be such that $f' = g' \circ f$ and $g' = k' \circ g$. Thus $f \in \mathcal{F}_{\alpha,\beta}$ and $g \in \mathcal{F}_{\beta,\gamma}$. Then $f' = h' \circ g \circ f$, so $g \circ f = h$, as required.

(iii) It is unnecessary to check this condition.

(d) (Covering) (i) This follows from condition (ii).

(ii) Suppose $\mathcal{C}$ is a closed unbounded set in $[\kappa^{n+}]^\kappa$, every member of which is weakly rounded. We find an element of $\mathfrak{C}$ belonging to $\mathcal{C}$.

We build an increasing sequence $\langle C_\alpha : \alpha \leq \kappa^+\rangle$ of elements of $\mathcal{C}$ such that

- for all α, for $j \leq n$, $\overline{C}_\alpha \subseteq C_{\alpha+1}$,
- for each limit λ, $C_\lambda = \bigcup_{\alpha<\lambda} C_\alpha$.

Now let $\mathfrak{M}$ be an elementary substructure of cardinality κ of a sufficiently large H_ν, including κ as a subset, and having the following sets as elements:

- $\langle \theta, \mathcal{F}\rangle$,
- $\langle\langle C^j_\alpha : \alpha \in \lim \cap\, \kappa^{(j+1)+}\rangle : j \in [1, n]\rangle$ and
- $\langle C_\alpha : \alpha \leq \kappa^+\rangle$.

Let $\mu = \mathfrak{M} \cap \kappa^+$; then μ is a limit, and $C_{\kappa^+} \cap \mathfrak{M} = C_\mu$.

We show that C_μ is rounded. We check the last condition of roundedness. Suppose $\lambda = \sup(C_\mu \cap \kappa^{(j+1)+})$ for some $j \leq n-1$. Obviously all the cardinals κ^{j+} for $j \leq n$ belong to $\mathfrak{M}$. Hence $\mathfrak{M} \cap (\kappa^{n+} + 1) \setminus \lambda$ is non-empty, and so has a least element, which we refer to as λ^*.

Since C_μ is cofinal in λ, by elementarity C_{κ^+} is cofinal in λ^*, and indeed $\lambda^* \in \overline{C}_{\kappa^+} \setminus C_{\kappa^+}$, and so the cofinality of λ^* is κ^+. Let $\lambda_\alpha = \sup(\lambda^* \cap C_\alpha)$ for all α (so that $\lambda = \lambda_\mu$ and $\lambda^* = \lambda_{\kappa^+}$). For all α, $C_{\alpha+1}$ contains the subset of $C^j_{\lambda_\alpha}$ enumerated by $C_\alpha \cap \kappa^{j+}$, because $C_{\alpha+1}$ is weakly rounded.

Now the set $\langle \lambda_\alpha : \alpha < \kappa^+\rangle$ is closed unbounded in λ_{κ^+}. Hence it meets $C^j_{\lambda_{\kappa^+}}$ in a closed unbounded set which is an element of $\mathfrak{M}$. It follows that λ_μ belongs to this closed unbounded set, and so since $C^j_{\lambda_\mu} = C^j_{\lambda_{\kappa^+}} \cap \lambda_\mu$, C_μ includes a closed unbounded subset of $C^j_{\lambda_\mu}$, and this closed unbounded set includes a cofinal set of limit ordinals. It now follows, since $C_\mu \cap \kappa^{j+} = \bigcup_{\alpha<\mu} C_\alpha \cap \kappa^{j+}$, and since the subset of $C^j_{\lambda_\mu}$ enumerated by $C_\mu \cap \kappa^{j+}$ is the union of the corresponding sets for the $C^j_{\lambda_\alpha}$ for $\alpha \in \mathfrak{M} \cap \kappa^+$, that the subset of $C^j_{\lambda_\mu}$ enumerated by $C_\mu \cap \kappa^{j+}$ is included in C_μ as required. Thus C_μ is rounded, and is therefore an element of $\mathfrak{C}$.

(e) (Local Smallness) In fact, $\mathcal{F}_{\alpha,\beta}$ has only one element.

(f) It is not necessary to prove this.

(g) (Offset Condition) Suppose $f \in \mathcal{F}_{\alpha,\delta}$, $g \in \mathcal{F}_{\beta,\delta}$, and $k \in \mathcal{F}_{\delta,\kappa^+}$. Then $k \circ f$ enumerates α and $k \circ g$ enumerates β. Let $\gamma = \alpha \cap \beta$. We check that γ is rounded. Clearly $\gamma \cap \kappa^+$ is an ordinal, and γ is closed under the actions of ψ^j_ξ and $(\psi^j_\xi)^{-1}$ for all $\xi \in \gamma$.

Now suppose that $\gamma \cap \mu$ is cofinal in μ, where μ is a limit in $[\kappa^{j+}, \kappa^{(j+1)+})$ for $j \geq 1$. Then $\alpha \cap \mu$ contains the subset of C^j_μ enumerated by $\alpha \cap \kappa^{j+}$, and $\beta \cap \mu$ contains the subset of C^j_μ enumerated by $\beta \cap \kappa^{j+}$. Hence $\gamma = \alpha \cap \beta$ contains the subset of C^j_μ enumerated by $\alpha \cap \beta \cap \kappa^{j+} = \gamma \cap \kappa^{j+}$.

As a final comment on this construction, the flat morass constructed has two strange properties: firstly, $\mathcal{F}_{\alpha,\beta}$ has at most one element for any α and β, and secondly, $\mathfrak{C}$ is not well-ordered. This motivates the following question:

Question 2.5.1. *Assume that there exists a gap-n morass of height κ^+.* [*Such have not been defined in this paper*; *we refer the reader to the literature*, *in particular*, *for $n > 2$, to C. Morgan's 1989 Oxford D. Phil. thesis*, *or to* [7].]

Is there a gap-n flat morass of height κ^+ for which the associated directed system $\mathfrak{C}$ is in fact the ordinal $\kappa^+ + 1$?

Can one additionally obtain the property of finite covering, *that for all $\alpha \in \kappa^+ + 1 = \mathfrak{C}$, one of the following conditions must hold*:

(i) *for every finite subset F of θ_α, there exist $\beta <_{\mathfrak{C}} \alpha$ and $f \in \mathcal{F}_{\beta,\alpha}$ such that $F \subseteq \operatorname{ran} f$, or*

(ii) *there exist finitely many ordinals $\gamma_1, \ldots, \gamma_k < \alpha$ and elements $f_i \in \mathcal{F}_{\gamma_i,\alpha}$ such that for all $\beta < \alpha$, for all $f \in \mathcal{F}_{\beta,\alpha}$, there exists i such that* $\operatorname{ran} f \subseteq \operatorname{ran} f_i$*?*

Can we achieve that the finite number k, in condition (ii) *above*, *is $n+1$?*

The flat morass constructed in this paper has, in the special case $\kappa = \omega$ and $n = 2$, the property that for all $\alpha < \omega_2$, the set $\{\operatorname{ran} f \cap \alpha : f \in \bigcup_{\beta \in \mathfrak{C} \setminus \{\kappa^+\}} \mathcal{F}_{\beta,\kappa^+}\}$ has cardinality ω_1. This is, in the terminology of [2], a thin stationary subset of $[\omega_2]^\omega$. It is proved in that paper that such sets may not exist if CH fails; thus the kind of flat morass constructed here is independent of ZFC.

3. The topological application

3.1. *Lindelöf graphs.* We will now begin the construction of an oriented graph on a set X_{ω_1} of size ω_n, that is to say, a function

$$\mathfrak{G} : (X_{\omega_1})^2 \to \{0, 1\}.$$

We will construct $\mathfrak{G}$ using a flat gap-$(n-1)$ morass $\langle \theta, \mathcal{F} \rangle$ of height ω_1; let $\mathfrak{C} = \operatorname{dom} \theta$.

We will impose conditions on this graph which will enable us to construct from it a regular Lindelöf space with points G_δ, by the method described in [5], which we now summarize.

We define the notion of an oriented graph as usual:

DEFINITION 3.1.1. If B is a set, a function $\mathfrak{G} : B^2 \to \{0,1\}$ is said to be an *oriented graph* on B (or, in this paper, simply a *graph*).

If there exists a set $\breve{B}$ such that $B = \breve{B} \times \{0,1\}$, then $\mathfrak{G}$ is *mirrored* iff for all $x \in B$ and $y \in \breve{B}$, $\mathfrak{G}(x, \langle y, 0\rangle) = \mathfrak{G}(x, \langle y, 1\rangle)$, and for all $y \in \breve{B}$, $\mathfrak{G}(\langle y, \epsilon\rangle, \langle y, \zeta\rangle) = \epsilon$.

If there exists a set $\breve{B}$ such that $B = \breve{B} \times \{0,1\}$, then if $x = \langle y, \epsilon\rangle$, we write x^0 for $\langle y, 0\rangle$ and x^1 for $\langle y, 1\rangle$, and we say $x \sim z$ iff $x^0 = z^0$.

Given a mirrored graph $\mathfrak{G}$ on a set B, we can define two topologies on B as follows:

DEFINITION 3.1.2. For $i \in \{0,1\}$, we define a topology τ^i by declaring the following subsets of B to be subbasic open sets: for each x, the set $U_{x,i,i} = \{x^i\} \cup \{y \not\sim x : \mathfrak{G}(x,y) = i\}$, the set $U_{x,i,1-i} = \{y : \mathfrak{G}(x,y) = 1-i\} \setminus \{x^0, x^1\}$, and the singleton set $\{x^{1-i}\}$.

Note that, in the coding used, the *first* argument of the function $\mathfrak{G}$ codes clopen sets, while the *second* argument codes points; and the function $\mathfrak{G}$ describes the incidence relation between points and open sets.

Note that if $x \not\sim y$, then $y^0 \in U_{x,i,j}$ iff $y^1 \in U_{x,i,j}$.

Of course topological properties of these two topological spaces will depend on graph-theoretic properties of $\mathfrak{G}$. We define a couple of possible properties of $\mathfrak{G}$:

DEFINITION 3.1.3. Let B be a set. Then a mirrored graph $\mathfrak{G} : B^2 \to \{0,1\}$ is *Hausdorff* iff for all $x, y \in B$ such that $x \not\sim y$, there exists $z \in B$ such that $z \not\sim x, y$, $\mathfrak{G}(z,x) = 1$ and $\mathfrak{G}(z,y) = 0$.

The graph $\mathfrak{G}$ is *Lindelöf* iff the topologies τ^0 and τ^1 are Lindelöf.

The theorem suggested by the terminology is true; and even more, given that τ^i is Lindelöf, we can deduce that τ^{1-i} has points G_δ, by an adaptation of the arguments in [5, Lemmas 1.4 and 1.5]:

PROPOSITION 3.1.4. *If a graph $\mathfrak{G}$ on B^2 is Hausdorff and Lindelöf, then both topologies τ^0 and τ^1 turn B into a regular Lindelöf space with points G_δ.*

Proof. We show that if τ^0 is Lindelöf, then τ^1 has points G_δ.

Certainly, for any point x, $\{x^0\}$ is G_δ in τ^1, because it is open. So we show that $\{x^1\}$ is G_δ.

Consider the family

$$\mathcal{U} = \{x^1\} \cup \{U_{x^0,0,0}\} \cup \{U_{z,0,0} : z \not\sim x,\ x \notin U_{z,0,0}\}.$$

Since $\mathfrak{G}$ is Hausdorff, this is a τ^0-open cover of X. Let $\mathcal{V}$ be a countable subcover. Let

$$\mathcal{V}' = \{U_{y,1,0} : U_{y,0,0} \in \mathcal{V}\}.$$

Then $x^1 \notin \bigcup \mathcal{V}'$, and $\mathcal{V}'$ covers all but countably many points of B. Since $\mathcal{V}'$ is a family of τ^1-clopen sets, and τ^1 is Hausdorff, we can now see that $\{x^1\}$ is a G_δ, as required. ∎

So, we set out to construct a Hausdorff, Lindelöf graph on a set of size ω_n.

3.2. *The construction: motivation and preliminary definitions.* We will perform this construction by recursion along the flat morass $\langle \theta, \mathcal{F} \rangle$ assumed at the beginning of Section 3.1, as follows. First, we define, for each $\alpha \in \mathfrak{C}$, the domain of the graph associated with α.

Definition 3.2.1. Suppose $\alpha \in \mathfrak{C}$. We define

$$X_\alpha = \bigcup_{\beta<\alpha} \mathcal{F}_{\beta,\alpha} \times \theta_\alpha \times \{0,1\}.$$

For $\alpha \neq \omega_1$, we define

$$Y_\alpha = \bigcup_{\beta\leq\alpha} \mathcal{F}_{\beta,\alpha} \times \theta_\alpha \times \{0,1\}.$$

If $\alpha <_{\mathfrak{C}} \beta$ and $f \in \mathcal{F}_{\alpha,\beta}$, we define $f^* : Y_\alpha \to X_\beta$ as follows:

$$f^*(g,\xi,q) = \langle f \circ g, f(\xi), q \rangle.$$

If $x = \langle g,\xi,q \rangle$ and $y = \langle g',\xi',q' \rangle$ are elements of X_α or Y_α, say $x \sim y$ iff $g = g'$ and $\xi = \xi'$.

So, we must construct mirrored graphs $\mathfrak{G}_\alpha$ on X_α and $\mathfrak{H}_\alpha$ on Y_α extending $\mathfrak{G}_\alpha$ such that for all $f \in \mathcal{F}_{\alpha,\beta}$, f^* embeds $\mathfrak{H}_\alpha$ in $\mathfrak{G}_\beta$, in the obvious sense:

Definition 3.2.2. Suppose $\mathfrak{G}_\alpha$ is a graph on X_α, $\mathfrak{H}_\alpha$ is a graph on Y_α, and $\mathfrak{G}_\beta$ is a graph on X_β, where $\alpha <_{\mathfrak{C}} \beta$. Then $\mathfrak{G}_\alpha$ *embeds in* $\mathfrak{H}_\alpha$ iff $\mathfrak{H}_\alpha \restriction X_\alpha \times X_\alpha = \mathfrak{G}_\alpha$, and f^* *embeds* $\mathfrak{H}_\alpha$ in $\mathfrak{G}_\beta$ iff for all $x, y \in Y_\alpha$, $\mathfrak{G}_\beta(f^*(x), f^*(y)) = \mathfrak{H}_\alpha(x,y)$.

The ultimate aim, of course, is the construction of the graph $\mathfrak{G}_{\omega_1}$ on X_{ω_1}, which is a set of size ω_n.

However, if we are attempting to construct $\mathfrak{G}_\alpha$, having already constructed $\mathfrak{H}_\beta$ for $\beta < \alpha$, we had better be confident that the proposed embeddings of the $\mathfrak{H}_\beta$ in $\mathfrak{G}_\alpha$ are compatible, in the following sense:

Definition 3.2.3. A sequence of graphs $\mathfrak{H}_\beta$ on Y_β, for $\beta <_{\mathfrak{C}} \alpha$, is *compatible with the morass structure* iff for all $\beta, \gamma <_{\mathfrak{C}} \alpha$, for all $f \in \mathcal{F}_{\gamma,\alpha}$ and $g \in \mathcal{F}_{\beta,\alpha}$, for all $w, x \in Y_\gamma$ and $y, z \in Y_\beta$, if $f^*(w) = g^*(y)$ and $f^*(x) = g^*(z)$, then

$$\mathfrak{H}_\gamma(w,x) = \mathfrak{H}_\beta(y,z).$$

How can we be sure that a sequence of graphs will be compatible with the morass structure? Recall that, by the Offset Condition, if f and g are as in the last definition, there exist $\delta \leq \beta, \gamma$, $f' \in \mathcal{F}_{\delta,\gamma}$, $g' \in \mathcal{F}_{\delta,\beta}$ and $h \in \mathcal{F}_{\delta,\alpha}$ such that

- $\operatorname{ran} h = \operatorname{ran} f \cap \operatorname{ran} g$,
- $h = f \circ f' = g \circ g'$.

It easily follows that

- $\operatorname{ran} h^* = \operatorname{ran} f^* \cap \operatorname{ran} g^*$,
- $h^* = f^* \circ f'^* = g^* \circ g'^*$.

Thus we can see the following:

LEMMA 3.2.4. *If, for each $\beta <_{\mathfrak{C}} \alpha$,*

- *$\mathfrak{G}_\beta$ is a graph on X_β,*
- *$\mathfrak{H}_\beta$ is a graph on Y_β,*
- *$\mathfrak{G}_\beta$ is embedded in $\mathfrak{H}_\beta$,*

and if, for each $\gamma <_{\mathfrak{C}} \beta <_{\mathfrak{C}} \alpha$ and $f \in \mathcal{F}_{\gamma,\beta}$, f^* embeds $\mathfrak{H}_\gamma$ in $\mathfrak{G}_\beta$, then the sequence of graphs $\langle \mathfrak{H}_\beta : \beta <_{\mathfrak{C}} \alpha \rangle$ is compatible with the morass structure.

It is now easy enough to see how to construct $\mathfrak{G}_\alpha$, given $\mathfrak{H}_\beta$ for $\beta <_{\mathfrak{C}} \alpha$. We must simply ensure that if $f \in \mathcal{F}_{\beta,\alpha}$, then f^* embeds $\mathfrak{H}_\beta$ in $\mathfrak{G}_\alpha$. For some values of α—in particular, by the Covering Condition, for the crucial value $\alpha = \omega_1$—this will completely define $\mathfrak{G}_\alpha$. For other values of α, there may be pairs $\langle x, y \rangle \in {X_\alpha}^2$ such that there does not exist any $f \in \mathcal{F}_{\beta,\alpha}$, for $\beta < \alpha$, such that $x, y \in \operatorname{ran} f^*$, and so there may be a bit of additional work to do.

Once $\mathfrak{G}_\alpha$ has been constructed, we build $\mathfrak{H}_\alpha$, which differs from $\mathfrak{G}_\alpha$ because $Y_\alpha \setminus X_\alpha$ is (if $\alpha <_{\mathfrak{C}} \omega_1$) countably infinite.

We use these countably many new points for two purposes.

The first is to index new open sets whose purpose is to ensure that $\mathfrak{H}_\alpha$ is Hausdorff. If we do this for every $\alpha <_{\mathfrak{C}} \omega_1$, then this will (by the Covering Condition) be enough to guarantee that $\mathfrak{G}_{\omega_1}$ is Hausdorff.

The second purpose is more subtle and harder to describe, and is to do with ensuring that $\mathfrak{G}_{\omega_1}$ is Lindelöf.

So, how do we make sure that $\mathfrak{G}_{\omega_1}$ is Lindelöf?

Suppose that, at the end of the construction, $\mathcal{U}$ is a cover of X_{ω_1} in the topology τ^i. Without loss of generality, $\mathcal{U}$ is a cover by basic open sets; we can therefore code it by a subset $\ulcorner\mathcal{U}\urcorner$ of $[X_{\omega_1} \times \{0,1\}]^{<\omega}$, such that $\{\langle x_j, \epsilon_j \rangle : j < r\} \in \ulcorner\mathcal{U}\urcorner$ iff $\bigcap_{j<r} U_{x_j,i,\epsilon_j} \in \mathcal{U}$. Then by the condition of Covering at Limits, there will exist $\alpha <_{\mathfrak{C}} \omega_1$ and $f \in \mathcal{F}_{\alpha,\omega_1}$ such that $\ulcorner\mathcal{V}\urcorner = \ulcorner\mathcal{U}\urcorner \cap [\operatorname{ran} f^* \times \{0,1\}]^{<\omega}$ codes a cover of $\operatorname{ran} f^*$. Let us define $\mathcal{U}_\alpha$ so

that $\mathcal{V} = f^*(\mathcal{U}_\alpha)$, where
$$\ulcorner f^*(\mathcal{U}_\alpha)\urcorner = \{\{\langle f^*(x), \epsilon\rangle : \langle x, \epsilon\rangle \in A\} : A \in \ulcorner \mathcal{U}_\alpha \urcorner\}.$$
Then, to ensure that $\mathcal{V}$ is a countable subcover, we will want to have ensured that if $\mathcal{U}_\alpha$ is a cover of X_α, then, for all $\beta >_{\mathfrak{C}} \alpha$ and for all $f \in \mathcal{F}_{\alpha,\beta}$, $f^*(\mathcal{U}_\alpha)$ is a cover of X_β, and, indeed, of Y_β.

We will only be interested in doing this when $\operatorname{ran} f^*$ belongs to some large closed unbounded set, and then we will be able to assume that α is a limit in the morass structure; and we will arrange for $f^*(\mathcal{U}_\alpha)$ to be a cover of X_{ω_1} by imposing the following condition:

Definition 3.2.5. Suppose X is one of X_α or Y_α, and that $\mathfrak{G}$ is a mirrored graph on X. Let $Z \subseteq X$. Then $\mathfrak{G}$ *reflects to* Z iff, for all $x \in X$, there exists $z \in Z$ such that for all $y \in Z$, $\mathfrak{G}(y, x) = \mathfrak{G}(y, z)$. We say that z is an *image of* x *in* Z.

It is then enough for us to guarantee that all $\mathfrak{G}_\alpha$ are mirrored, and that $\mathfrak{G}_{\omega_1}$ reflects to $\operatorname{ran} f^*$, to ensure that $f^*(\mathcal{U}_\alpha)$ is a cover of X_{ω_1}, and that therefore $\mathcal{U}$ has a countable open subcover (namely $f^*(\mathcal{U}_\alpha)$).

This will depend crucially on the Offset Property.

So, we wish the sequence of graphs $\mathfrak{G}_\alpha$, $\mathfrak{H}_\alpha$ to have the following property:

Definition 3.2.6. Suppose $\mathfrak{D}$ is a downward-closed subset of $\mathfrak{C}$ (that is, for each $p \in \mathfrak{D}$ and $q <_{\mathfrak{C}} p$, $q \in \mathfrak{D}$). Suppose graphs $\mathfrak{G}_\alpha$ on X_α and $\mathfrak{H}_\alpha$ on Y_α have been constructed for $\alpha \in \mathfrak{D}$. Then we say they are *reflective* iff they satisfy the following conditions:

- For each $\alpha \in \mathfrak{D}$, provided α is a limit in the morass structure, $\mathfrak{H}_\alpha$ reflects to X_α.
- If $\beta <_{\mathfrak{C}} \alpha \in \mathfrak{D}$ and $f \in \mathcal{F}_{\beta,\alpha}$, then $\mathfrak{G}_\alpha$ and $\mathfrak{H}_\alpha$ reflect to $\operatorname{ran} f^*$.

We must address the question of how we are to construct our graphs to be reflective. It would appear to be easy enough to construct $\mathfrak{H}_\alpha$ from $\mathfrak{G}_\alpha$ so that $\mathfrak{H}_\alpha$ reflects in X_α.

The question of making sure that, if $\beta <_{\mathfrak{C}} \alpha$ and $f \in \mathcal{F}_{\beta,\alpha}$, $\mathfrak{G}_\alpha$ reflects to $\operatorname{ran} f^*$, is rather more difficult, for the following reason.

Suppose we are given $\beta_1, \beta_2, \beta_3 <_{\mathfrak{C}} \alpha$ and $f_i \in \mathcal{F}_{\beta_i,\alpha}$ for each i. Suppose we have $x \in X_\alpha$, and we are trying to choose $z_i \in \operatorname{ran} f_i^*$, for each i, so that z_i is an image of x in $\operatorname{ran} f_i^*$. We might begin as follows.

First, choose some $z_1 \in \operatorname{ran} f_1^*$, and promise to ensure that z_1 shall be an image of x in $\operatorname{ran} f_1^*$. Next, observe that, assuming the family of graphs constructed so far is reflective, there is $z_{1,2} \in \operatorname{ran}(f_1 \sqcap f_2)^*$ such that $z_{1,2}$ is an image of z_1 in $\operatorname{ran}(f_1 \sqcap f_2)^*$ ([1]). Define z_2 to be $z_{1,2}$. Next, try to

([1]) For, $z_1 \in X_\alpha$, because $\operatorname{ran} f_1^* \subseteq X_\alpha$. Suppose γ is such that $(f_1 \sqcap f_2) \in \mathcal{F}_{\gamma,\alpha}$. Then $\gamma <_{\mathfrak{C}} \alpha$. By reflectivity, the graph $\mathfrak{G}_\alpha$ reflects to $\operatorname{ran}(f_1 \sqcap f_2)^*$. Thus there exists an image $z_{1,2}$ of z_1 in $\operatorname{ran}(f_1 \sqcap f_2)^*$.

define z_3. Observe that there exist $z_{1,3} \in \operatorname{ran}(f_1 \sqcap f_3)^*$ such that $z_{1,3}$ is an image of z_1 in $\operatorname{ran}(f_1 \sqcap f_3)^*$ and $z_{2,3} \in \operatorname{ran}(f_2 \sqcap f_3)^*$ such that $z_{2,3}$ is an image of z_2 in $\operatorname{ran}(f_2 \sqcap f_3)^*$.

But we now cannot proceed further; we cannot simply identify z_3 as one of $z_{1,3}$ or $z_{2,3}$; we need some kind of mixture of the two. What we need, in fact, is the following condition:

Definition 3.2.7. Suppose $\mathfrak{G}$ is a graph on X, where X is one of X_α or Y_α. Then we say it *admits hybridization* under the following circumstances.

Suppose $\beta_i <_{\mathfrak{C}} \alpha \in \mathfrak{D}$, for $i = 1, \ldots, n$, $f_i \in \mathcal{F}_{\beta_i,\alpha}$, and $z_i \in \operatorname{ran} f_i^*$ such that for all i and j and $y \in \operatorname{ran} f_i^* \cap \operatorname{ran} f_j^*$, $\mathfrak{G}(y, z_i) = \mathfrak{G}(y, z_j)$. Then there exists $z \in X$ such that for all i, z_i is an image of z in $\operatorname{ran} f_i^*$.

This concept is related to the notion of a *twin* in [8].

If we know that all graphs $\mathfrak{H}_\beta$, for $\beta < \alpha$, admit hybridization, then we may hope to be able to construct a graph $\mathfrak{G}_\alpha$ preserving the condition of reflectivity.

We use the new points in $Y_\alpha \setminus X_\alpha$ to ensure that $\mathfrak{H}_\alpha$ admits hybridization.

We now perform the construction.

3.3. *Performing the construction.* We recursively construct graphs $\mathfrak{G}_\alpha$ on X_α and $\mathfrak{H}_\alpha$ on Y_α, for $\alpha \in \mathfrak{C}$, satisfying the following inductive hypotheses:

Assumption 3.3.1. *Suppose $\mathfrak{D}$ is a downward-closed subset of $\mathfrak{C}$ such that for all $\alpha \in \mathfrak{D}$, $\mathfrak{G}_\alpha$ and $\mathfrak{H}_\alpha$ have been defined. Then*:

- *Each $\mathfrak{G}_\alpha$ embeds in the corresponding $\mathfrak{H}_\alpha$.*
- *If $\beta <_{\mathfrak{C}} \alpha \in \mathfrak{D}$ and $f \in \mathcal{F}_{\beta,\alpha}$, then f^* embeds $\mathfrak{H}_\beta$ in $\mathfrak{G}_\alpha$.*
- *The family of graphs $\mathfrak{G}_\alpha$ and $\mathfrak{H}_\alpha$, for $\alpha \in \mathfrak{D}$, is reflective.*
- *Each $\mathfrak{H}_\alpha$ is Hausdorff.*
- *Each $\mathfrak{H}_\alpha$ admits hybridization.*

In this section, we assume that these hypotheses hold for $\mathfrak{D} = \{\beta : \beta <_{\mathfrak{C}} \alpha\}$ and give the construction of $\mathfrak{G}_\alpha$ and $\mathfrak{H}_\alpha$; in the next section we will show that the inductive hypotheses are preserved.

We first dismiss an easy special case.

$\mathfrak{G}_\alpha$: *the base case.* If α is a minimal element of $\mathfrak{C}$, then X_α is empty; so $\mathfrak{G}_\alpha$ will be the empty graph.

$\mathfrak{G}_\alpha$: *the limit case.* The next case we consider is that in which α is a limit in the morass structure, which implies that

$$X_\alpha \times X_\alpha = \bigcup_{\beta <_{\mathfrak{C}} \alpha} \bigcup_{f \in \mathcal{F}_{\beta,\alpha}} \operatorname{ran} f^* \times \operatorname{ran} f^*.$$

Then we simply define $\mathfrak{G}_\alpha$ so that for all $f \in \mathcal{F}_{\beta,\alpha}$, for all $x, y \in Y_\beta$, $\mathfrak{G}_\alpha(f(x), f(y)) = \mathfrak{H}_\beta(x, y)$. We postpone till the next section the argument that this function is well-defined; but we observe that if so, $\mathfrak{G}_\alpha$ is now a total function on $X_\alpha \times X_\alpha$.

We note that, by the Covering Condition, this defines $\mathfrak{G}_{\omega_1}$ for us, given all $\mathfrak{G}_\alpha$, $\mathfrak{H}_\alpha$ for $\alpha <_{\mathfrak{C}} \omega_1$.

$\mathfrak{G}_\alpha$: *the general case.* Now we show how to define $\mathfrak{G}_\alpha$ in the general case. We will be taking care to preserve the hypothesis of reflectivity. We will assume that $\alpha <_{\mathfrak{C}} \omega_1$ (since the case $\alpha = \omega_1$ has already been dealt with).

Given $x \in X_\alpha$, we define the function $y \mapsto \mathfrak{G}_\alpha(y, x)$. And, of course, we ensure (without difficulty) that if $x \sim x'$, then $\mathfrak{G}_\alpha(y, x) = \mathfrak{G}_\alpha(y, x')$.

We enumerate the set $\bigcup_{\beta <_{\mathfrak{C}} \alpha} \mathcal{F}_{\beta,\alpha}$ in order-type ω as $\{f_n : n \in \omega\}$, in such a way that if there exist $\beta <_{\mathfrak{C}} \alpha$ and $f \in \mathcal{F}_{\beta,\alpha}$ such that $x \in \operatorname{ran} f^*$, then $x \in \operatorname{ran} f_0^*$. For each n, define β_n so that $f_n \in \mathcal{F}_{\beta_n,\alpha}$.

We define the function $y \mapsto \mathfrak{G}_\alpha(y, x)$, for $y \in \operatorname{ran} f_n^*$, by recursion on n.

For the base case, if $x \in \operatorname{ran} f_0^*$ (let us say $x = f_0^*(z_0)$), and $y \in \operatorname{dom} f_0^*$, then we simply define $\mathfrak{G}_\alpha(f_0^*(y), x)$ to be $\mathfrak{H}_{\beta_0}(y, z_0)$. If $x \notin \operatorname{ran} f_0^*$, then choose some $z_0 \in \operatorname{dom} f_0^*$, and let $\mathfrak{G}_\alpha(f_0^*(y), x) = \mathfrak{H}_{\beta_n}(y, z_0)$.

Now suppose $n > 0$. Again, if $x \in \operatorname{ran} f_n^*$, say $x = f_n^*(z_n)$, and $y \in \operatorname{dom} f_n^*$, then we define $\mathfrak{G}_\alpha(f_n^*(y), x)$ to be $\mathfrak{H}_{\beta_n}(y, z_n)$. If $x \notin \operatorname{ran} f_n^*$, we use the condition of admitting hybridization as follows. For each $i < n$, suppose $f_i \sqcap f_n \in \mathcal{F}_{\beta_{i,n},\alpha}$. Let $f_i \sqcap f_n = f_i \circ (f_n/f_i)$. (Recall, from condition 2(g) in the definition of a flat morass, that $f_i \sqcap f_n$ is a function, indeed an element of the flat morass, whose range is $\operatorname{ran} f_i \cap \operatorname{ran} f_n$; and f_n/f_i is defined to be the function satisfying $f_i \circ (f_n/f_i) = f_i \sqcap f_n$.) Then, by the inductive hypothesis of reflectivity, there exists $z_{i,n} \in Y_{\beta_{i,n}}$ such that $(f_n/f_i)^*(z_{i,n})$ is an image of z_i in $\operatorname{ran} (f_n/f_i)^*$.

Now consider the functions (f_i/f_n), the points z_i for $i < n$, and the points $(f_i/f_n)^*(z_{i,n})$. By the condition of admitting hybridization, as applied to the graph $\mathfrak{H}_{\beta_n}$ on $\operatorname{dom} f_n^*$, we find an element z_n of Y_{β_n} (depending, of course, on the z_i for $i < n$) such that for each $i < n$, $(f_i/f_n)^*(z_{i,n})$ is an image of z_n in $\operatorname{ran} (f_i/f_n)^*$.

Now, for each $y \in Y_{\beta_n}$, define $\mathfrak{G}_\alpha(f_n^*(y), x)$ to be $\mathfrak{G}_{\beta_n}(y, z_n)$.

Finally, suppose $y \notin \operatorname{ran} f_n^*$ for any n. Then let $\mathfrak{G}_\alpha(y, x) = 0$.

Having defined $\mathfrak{G}_\alpha$, we now define $\mathfrak{H}_\alpha$.

$\mathfrak{H}_\alpha$: *preliminaries.* Recall that we wish to ensure that $\mathfrak{H}_\alpha$ is Hausdorff and admits hybridization.

If α is a limit in the morass structure, we do not explicitly construct $\mathfrak{H}_\alpha$ to admit hybridization; instead we build it to reflect to X_α.

Accordingly, at stage α we define a *Hausdorffness problem* to be a pair $\langle x, y\rangle$ of distinct elements of Y_α such that $x \not\sim y$.

We define a *reflection problem* to be

- $\emptyset$, if α is a limit in the morass structure;
- otherwise, a finite set $\{S_1, \dots, S_n\}$, where $S_i = \langle f_i, z_i\rangle$, and there is $\beta_i < \alpha$ such that $f_i \in \mathcal{F}_{\beta_i,\alpha}$, $z_i \in Y_{\beta_i}$, and for all i and j, if $f_i^*(y) = f_j^*(z)$, then $\mathfrak{G}_{\beta_i}(y, z_i) = \mathfrak{G}_{\beta_j}(z, z_j)$.

We list all Hausdorffness problems as $\langle P_n : n \in \omega\rangle$, and all reflection problems as $\langle Q_n : n \in \omega\rangle$. We allow the possibility that these lists may include repetitions.

List $Y_\alpha \setminus X_\alpha$ as $\langle x_n : n \in \omega\rangle$, so that if $P_n = \langle p_n, q_n\rangle$, then $x_n \not\sim p_n, q_n$.

$\mathfrak{H}_\alpha$: *new open sets.* Suppose P_n is the pair $\langle p_n, q_n\rangle$. We define $\mathfrak{H}_\alpha(x_n, y) = 1$ for all $y \sim p_n$, and $\mathfrak{H}_\alpha(x_n, y) = 0$ for all $y \not\sim p_n$.

$\mathfrak{H}_\alpha$: *new points.* If there exists $m < n$ such that $x_m \sim x_n$, then we will already have defined $\mathfrak{H}_\alpha(y, x_n)$ for all y, so no action is necessary.

So suppose that for all $m < n$, $x_m \not\sim x_n$.

Suppose α is a limit in the morass structure. Then we choose some $x \in X_\alpha$ and, for all y, and for all $x' \sim x_n$, define $\mathfrak{H}_\alpha(y, x')$ to be $\mathfrak{H}_\alpha(y, x)$ (which has already been defined).

Now suppose α is not a limit in the morass structure, and suppose Q_n is the reflection problem $\langle S_1, \dots, S_{k_n}\rangle$. We write S_i as $\langle f_{-i}, z_{-i}\rangle$. (Notice the change in notation from the definition above of a reflection problem; this makes the induction below easier to describe.)

Now we define the function $y \mapsto \mathfrak{H}_\alpha(y, x_n)$ by imitating the procedure for defining the function $y \mapsto \mathfrak{G}(y, x)$, for $x \in X_\alpha$, in the following way.

We enumerate the set $\bigcup_{\beta<\alpha} \mathcal{F}_{\beta,\alpha}$ in order-type ω as $\{f_m : m \in \omega\}$. For each $m \geq -k_n$, define β_m so that $f_m \in \mathcal{F}_{\beta_m,\alpha}$.

We define the function $y \mapsto \mathfrak{H}_\alpha(y, x_n)$, for $y \in \operatorname{ran} f_m^*$, by recursion on m, simultaneously choosing $z_m \in \operatorname{ran} f_m^*$ (for $m \geq 0$) to be an image of x_n in $\operatorname{ran} f_m^*$.

For the base case, which is $m < 0$, if $y \in \operatorname{ran} f_m^*$, let $\mathfrak{H}_\alpha(y, x_n) = \mathfrak{G}_\alpha(y, z_m)$.

Now suppose $m \geq 0$. We use the condition of admitting hybridization as follows. For each $i < m$, suppose $f_i \sqcap f_m \in \mathcal{F}_{\beta_{i,m},\alpha}$. Let $f_i \sqcap f_m = f_i \circ (f_m/f_i)$. Then, by the inductive hypothesis of reflectivity, there exists $z_{i,m} \in Y_{\beta_{i,m}}$ such that $(f_m/f_i)^*(z_{i,m})$ is an image of $(f_i^*)^{-1}(z_i)$ in $\operatorname{ran}(f_m/f_i)^*$.

Now consider the functions (f_i/f_m), and the points $(f_i/f_m)^*(z_{i,n})$. By the condition of admitting hybridization, as applied to the graph $\mathfrak{H}_{\beta_m}$ on $\operatorname{dom} f_m^*$, we find an element $\check{z}_m$ of Y_{β_n} such that for each $i < m$, $(f_i/f_m)^*(z_{i,m})$ is an image of $\check{z}_m$ in $\operatorname{ran}(f_i/f_m)^*$. Let $z_m = f_m^*(\check{z}_m)$.

Now, for each $y \in \operatorname{ran} f_m^*$, define $\mathfrak{H}_\alpha(y, x_n)$ to be $\mathfrak{G}_\alpha(y, z_m)$. Let $\mathfrak{H}_\alpha({x_n}^0, x_n) = 0$, and let $\mathfrak{H}_\alpha({x_n}^1, x_n) = 1$. Finally, if $y \not\sim x_n$ and $y \notin \operatorname{ran} f_n^*$ for any m then let $\mathfrak{H}_\alpha(y, x_n) = 0$.

Now for all $x' \sim x_n$, let $\mathfrak{H}_\alpha(y, x') = \mathfrak{H}_\alpha(y, x_n)$.

This completes the definitions of $\mathfrak{G}_\alpha$ and $\mathfrak{H}_\alpha$. We now show that they have the properties required.

3.4. *Checking the inductive hypotheses.* We prove that if $\alpha \in \mathfrak{C}$, and if $\mathfrak{G}_\beta$ and $\mathfrak{H}_\beta$ all satisfy the inductive hypotheses for $\beta <_{\mathfrak{C}} \alpha$, then the inductive hypotheses are preserved at stage α.

LEMMA 3.4.1. *If*

$$X_\alpha \times X_\alpha = \bigcup_{\beta <_{\mathfrak{C}} \alpha} \bigcup_{f \in \mathcal{F}_{\beta,\alpha}} \operatorname{ran} f^* \times \operatorname{ran} f^*,$$

then $\mathfrak{G}_\alpha$ is well-defined and mirrored.

Proof. The required result is that if $f^*(w) = g^*(y)$ and $f^*(x) = g^*(z)$, where $f \in \mathcal{F}_{\beta,\alpha}$ and $g \in \mathcal{F}_{\gamma,\alpha}$, then $\mathfrak{H}_\beta(w, x) = \mathfrak{H}_\gamma(y, z)$.

But this follows from the fact (see condition 2(g) in the definition of a flat morass) that, if $f \sqcap g \in \mathcal{F}_{\delta,\alpha}$ and $f \sqcap g = f \circ (g/f) = g \circ (f/g)$, then $(g/f)^*$ embeds $\mathfrak{H}_\delta$ in $\mathfrak{H}_\beta$ and $(f/g)^*$ embeds $\mathfrak{H}_\delta$ in $\mathfrak{H}_\gamma$; and moreover there exist u and v such that $(g/f)^*(u) = w$, $(g/f)^*(v) = x$, $(f/g)^*(u) = y$ and $(f/g)^*(v) = z$. ∎

By a similar argument we can confirm the general case:

LEMMA 3.4.2. *$\mathfrak{G}_\alpha$ and $\mathfrak{H}_\alpha$ are well-defined and mirrored.*

LEMMA 3.4.3. *If $\beta < \alpha$ and $f \in \mathcal{F}_{\beta,\alpha}$, then f^* embeds $\mathfrak{H}_\beta$ in $\mathfrak{G}_\alpha$.*

Proof. This is explicit in the construction. ∎

LEMMA 3.4.4. *Suppose $\epsilon <_{\mathfrak{C}} \omega_1$. Then the family of graphs $\mathfrak{G}_\beta$ and $\mathfrak{H}_\beta$, for $\beta \leq_{\mathfrak{C}} \epsilon$, is reflective.*

Proof. If $\alpha \leq_{\mathfrak{C}} \epsilon$ is a limit in the morass structure, then we explicitly arranged that $\mathfrak{H}_\alpha$ should reflect to X_α.

For, suppose $z \in Y_\alpha \setminus X_\alpha$. Then, if we enumerate $Y_\alpha \setminus X_\alpha$ as $\langle x_n : n \in \omega \rangle$, as described in the section "$\mathfrak{H}_\alpha$: preliminaries", then $z = x_n$ for some n.

If there does not exist $m < n$ such that $x_m \sim x_n$, then, in the section headed "$\mathfrak{H}_\alpha$: new points", in the case where α is a limit in the morass structure, we chose a point $x \in X_\alpha$, and, for all y and $x' \sim x_n$, defined $\mathfrak{H}_\alpha(y, x')$ to be $\mathfrak{H}_\alpha(y, x)$. But $\mathfrak{H}_\alpha(y, x)$ is equal to $\mathfrak{G}_\alpha(y, x)$, since $\mathfrak{G}_\alpha \subseteq \mathfrak{H}_\alpha$ and $y, x \in X_\alpha = \operatorname{dom} \mathfrak{G}_\alpha$.

Consulting Definition 3.2.5, we see that we have determined that x should be an image of y in X_α. Moreover, x is also an image of any $x' \sim y$.

Thus, for every $y \in Y_\alpha \setminus X_\alpha$, y possesses an image in X_α. Hence, $\mathfrak{H}_\alpha$ reflects to X_α.

Now suppose that β and α are any two elements of $\mathfrak{C}$ below ϵ, and that $\beta <_{\mathfrak{C}} \alpha$. We show by induction on α that $\mathfrak{H}_\alpha$ reflects to $\operatorname{ran} f^*$.

Suppose that $f \in \mathcal{F}_{\beta,\alpha}$, and that $z \in Y_\alpha$. We show that z has an image in $\operatorname{ran} f^*$. First suppose that $z \in X_\alpha$.

If α is a limit in the morass structure, then by condition 2(d)(ii) in the definition of a flat morass in Section 2.2, there exist $\gamma <_{\mathfrak{C}} \alpha$ and $g \in \mathcal{F}_{\gamma,\alpha}$ such that $z \in \operatorname{ran} g^*$, and there exist δ and h such that $\gamma, \beta <_{\mathfrak{C}} \delta <_{\mathfrak{C}} \alpha$, $h \in \mathcal{F}_{\delta,\alpha}$, and $\operatorname{ran} f, \operatorname{ran} g \subseteq \operatorname{ran} h$.

Hence $z \in \operatorname{ran} h^*$. Now an appeal to the inductive hypothesis for δ tells us that $(h^*)^{-1} z$ has an image in $\operatorname{ran}(f/h)^*$. Hence z has an image in $\operatorname{ran} f^*$.

If α is not a limit in the morass structure, we examine the section of the construction of $\mathfrak{G}_\alpha$ headed "$\mathfrak{G}_\alpha$: general case". In that section, we enumerated $\bigcup_{\gamma <_{\mathfrak{C}} \alpha} \mathcal{F}_{\gamma,\alpha}$ in order-type ω as $\langle f_n : n \in \omega \rangle$. Suppose $f = f_n$. Then we chose a point $z_n \in Y_\beta$, and defined $\mathfrak{G}_\alpha(f^*(y), z)$ to be $\mathfrak{H}_\beta(y, z_n)$, for all $y \in Y_\beta$; that is, for all $y \in \operatorname{ran} f^*$, $\mathfrak{G}_\alpha(y, z) = \mathfrak{G}_\alpha(y, f^*(z_n))$. Thus in this case also, z has an image in $\operatorname{ran} f^*$.

We have now dealt with the case where $z \in X_\alpha$. Now suppose $z \in Y_\alpha \setminus X_\alpha$. Turning to the part of the section "$\mathfrak{H}_\alpha$: new points", we find, again, that we have listed $\bigcup_{\gamma <_{\mathfrak{C}} \alpha} \mathcal{F}_{\gamma,\alpha}$ in order-type ω as $\langle f_n : n \geq -m \rangle$ for some m, and, if $f = f_n$, then we have chosen a point $\check{z}_n \in Y_\beta$ and defined $\mathfrak{H}_\alpha(y, z)$ to be $\mathfrak{G}_\alpha(y, f^*(\check{z}_n))$, which, as we have already seen, is equal to $\mathfrak{H}_\beta((f^*)^{-1}(y), \check{z}_n)$, for all $y \in \operatorname{ran} f^*$. That is, we have provided z with an image in $\operatorname{ran} f^*$.

We now see that the family of graphs $\mathfrak{G}_\beta$ and $\mathfrak{H}_\beta$, for $\beta \leq_{\mathfrak{C}} \alpha$, is indeed reflective. ∎

LEMMA 3.4.5. *$\mathfrak{H}_\alpha$ is Hausdorff.*

Proof. Let p, q be distinct elements of Y_α, and suppose $p \not\sim q$. Then the pair $\langle p, q \rangle$ is a Hausdorffness problem. So, by the construction, there is a point $x \in Y_\alpha \setminus X_\alpha$ such that $\mathfrak{H}_\alpha(x, p) = 1$ and $\mathfrak{H}_\alpha(x, q) = 0$. ∎

LEMMA 3.4.6. *$\mathfrak{H}_\alpha$ admits hybridization.*

Proof. If α is not a limit in the morass structure, then whenever $S_1, \ldots, S_k$ is a finite sequence of pairs, with S_i having the form $\langle f_i, z_i \rangle$, where $f_i \in \bigcup_{\beta < \alpha} \mathcal{F}_{\beta,\alpha}$ and $z_i \in \operatorname{ran} f_i^*$, and where for all $y \in \operatorname{ran} f_i^* \cap \operatorname{ran} f_j^*$, $\mathfrak{G}_\alpha(y, z_i) = \mathfrak{G}_\alpha(y, z_j)$, then $\langle S_1, \ldots, S_n \rangle$ is a reflectivity problem.

Hence, there exists $x \in Y_\alpha \setminus X_\alpha$ such that for each i, for all $y \in \operatorname{ran} f_i^*$, $\mathfrak{H}_\alpha(y, x) = \mathfrak{H}_\alpha(y, z_i)$.

If α is a limit in the morass structure, suppose $S_1, \ldots, S_n$ is a sequence of pairs as above. Then there exists $f \in \bigcup_{\beta <_{\mathfrak{C}} \alpha} \mathcal{F}_{\beta,\alpha}$ such that for all i,

$\operatorname{ran} f_i \subseteq \operatorname{ran} f$. Suppose $f \in \mathcal{F}_{\gamma,\alpha}$, and for each i, $f_i = f \circ g_i$ and $z_i = f(w_i)$.

Then, recalling that f^* embeds $\mathfrak{H}_\gamma$ in $\mathfrak{H}_\alpha$, we can apply the inductive hypothesis, that $\mathfrak{H}_\gamma$ admits hybridization, to the sequence of pairs $\langle g_1, w_1\rangle, \ldots, \langle g_n, w_n\rangle$. ∎

We have now established that the inductive hypotheses are preserved, and so the graphs $\mathfrak{G}_\alpha$ and $\mathfrak{H}_\alpha$ can be constructed for all α.

3.5. *Properties of the graphs.* Finally, we require to show that $\mathfrak{G} = \mathfrak{G}_{\omega_1}$ is Hausdorff and Lindelöf.

Proposition 3.5.1. *$\mathfrak{G}$ is Hausdorff.*

Proof. Suppose $x, y \in X_{\omega_1}$ and $x \not\sim y$. Then by the Covering Property we can find $\alpha < \omega_1$ and $f \in \mathcal{F}_{\alpha,\omega_1}$ such that $x, y \in \operatorname{ran} f^*$.

Let $x = f^*(x')$ and $y = f^*(y')$. Then $x' \not\sim y'$. Thus, because $\mathfrak{H}_\alpha$ is Hausdorff, there exists $z' \not\sim x', y'$ such that $\mathfrak{H}_\alpha(z', x') = 0$ and $\mathfrak{H}_\alpha(z', y') = 1$. Recall that f^* embeds $\mathfrak{H}_\alpha$ in $\mathfrak{G}_{\omega_1} = \mathfrak{G}$. Let $z = f^*(z')$. Then $\mathfrak{G}(z, x) = 0$ and $\mathfrak{G}(z, y) = 1$, as required. ∎

Proposition 3.5.2. *$\mathfrak{G}$ is Lindelöf.*

Proof. Let $\mathcal{U}$ be a cover of $\mathfrak{G}$ by τ^0-basic open sets. (The argument for τ^1 will be similar.) Note that for all $x, y \in X_{\omega_1}$, there exist $\beta < \omega_1$ and $f \in \mathcal{F}_{\beta,\omega_1}$ such that $x, y \in \operatorname{ran} f^*$, and moreover if $f, g \in \bigcup_{\beta<\omega_1} \mathcal{F}_{\beta,\omega_1}$, then there exist $\gamma < \omega_1$ and $h \in \mathcal{F}_{\gamma,\omega_1}$ such that $\operatorname{ran} f, \operatorname{ran} g \subseteq \operatorname{ran} h$.

Let μ be a large enough cardinal. Then the set of all sets $\mathfrak{M} \cap \omega_n$ such that $\mathfrak{M}$ is a countable elementary substructure of H_μ containing any given countable selection of sets, is closed unbounded in $[\omega_n]^\omega$.

Accordingly, since our flat morass $\mathcal{M}$ has Covering at Limits, there exist $\alpha < \omega_1$ and $f \in \mathcal{F}_{\alpha,\omega_1}$ such that

- α is a limit in the morass structure (by consideration of elementarity and the first paragraph of this proof),
- $\ulcorner\mathcal{U}\urcorner \cap [f^*(X_\alpha) \times \{0,1\}]^{<\omega}$ codes a cover $f^*(X_\alpha)$.
  Equivalently, $\mathcal{U}_\alpha$ is a cover of X_α, where $\ulcorner\mathcal{U}_\alpha\urcorner$ is defined as above so that
  $$\ulcorner\mathcal{U}\urcorner \cap [\operatorname{ran} f^* \times \{0,1\}]^{<\omega} = \{\{\langle f^*(x), \epsilon\rangle : \langle x, \epsilon\rangle \in A\} : A \in \ulcorner\mathcal{U}_\alpha\urcorner\}.$$

Now, $\mathfrak{G} = \mathfrak{G}_{\omega_1}$ reflects to $\operatorname{ran} f^*$. Since $\mathfrak{H}_\alpha$ reflects to $\mathfrak{G}_\alpha$, it follows that the family $\mathcal{V}$ coded by $\ulcorner\mathcal{U}\urcorner \cap [\operatorname{ran} f^* \times \{0,1\}]^{<\omega}$ is a cover of X_{ω_1}.

For, suppose $x \in X_{\omega_1}$. Let z be an image of x in X_α. Then for all $y \in X_\alpha$, $\mathfrak{G}(f^*(y), x) = \mathfrak{G}_\alpha(y, z)$.

Without loss of generality, $z = z^0$, because if $w \not\sim z$, then $\mathfrak{G}_\alpha(w, z^0) = \mathfrak{G}_\alpha(w, z^1)$; while $\mathfrak{G}_\alpha(z^i, z^j) = i$; so z^0 and z^1 are both images of x.

Now $\mathcal{U}_\alpha$ covers X_α; say

$$z \in B = \bigcap_{i=1}^{n} U_{w_i,0,j_i} \in \mathcal{U}_\alpha.$$

If $w_i \not\sim z$, then since $z \in U_{w_i,0,j_i}$, we have $\mathfrak{G}_\alpha(w_i, z) = j_i$, so $\mathfrak{G}(f^*(w_i), x) = j_i$, and thus $x \in U_{f^*(w_i),0,j_i}$.

If $w_i \sim z$, then in fact $j_i = 0$, and without loss of generality $w_i = z$, since $U_{z^0,0,j_i} = U_{z^1,0,j_i}$. Now $\mathfrak{G}_\alpha(z, z) = 0$ since $z = z^0$, so $\mathfrak{G}(f^*(z), x) = 0$ since z is an image of z, and hence $x \in U_{f^*(z),0,0} = U_{f^*(w_i),0,j_i}$.

It follows that

$$x \in \bigcap_{i=1}^{n} U_{f^*(w_i),0,j_i},$$

and this set is an element of $\mathcal{V}$.

Now $\mathcal{U}_\alpha$ is obviously countable, since X_α is countable; hence $\mathcal{V}$ is countable, and $\mathcal{U}$ has a countable subcover. ∎

3.6. *Conclusion.* So, to summarise:

THEOREM 3.6.1. *Suppose that there exists an gap-$(n-1)$ flat morass of height ω_1. Then there is a Lindelöf Tikhonov space with points G_δ of size $\aleph_n$.*

The following now follows trivially:

THEOREM 3.6.2. *Suppose that for all n, there exists a gap-$(n-1)$ flat morass of height ω_1. Then there is a Lindelöf Tikhonov space with points G_δ of size $\aleph_\omega$.*

Proof. Take the disjoint union of spaces X_n, where X_n is a Lindelöf Tikhonov space with points G_δ. ∎

Since the condition of the above theorem holds in L, it is easy to see the following:

THEOREM 3.6.3. *It is consistent that there is a Lindelöf Tikhonov space with points G_δ of size $\beth_\omega$.*

The obvious next question is whether this can be improved: for instance, whether, for all countable α, it is consistent that there is a Lindelöf Tikhonov space with points G_δ of size $\beth_\alpha$. The author does not know whether this could be done for $\alpha < \omega_1$ using the methods of this paper; it seems reasonably clear that it would fail for $\alpha \geq \omega_1$.

We repeat here a question we stated in the introduction:

QUESTION 3.6.4. *Can it be proved from ZFC together with GCH that there is a Lindelöf space with countable pseudocharacter having cardinality $\aleph_\omega$?*

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