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More on the Ehrenfeucht–Fraïssé game of length ω_1

by

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Abstract. By results of [9] there are models $\mathfrak{A}$ and $\mathfrak{B}$ for which the Ehrenfeucht–Fraïssé game of length ω_1, $\mathrm{EFG}_{\omega_1}(\mathfrak{A},\mathfrak{B})$, is non-determined, but it is consistent relative to the consistency of a measurable cardinal that no such models have cardinality $\leq \aleph_2$. We now improve the work of [9] in two ways. Firstly, we prove that the consistency strength of the statement "CH and $\mathrm{EFG}_{\omega_1}(\mathfrak{A},\mathfrak{B})$ is determined for all models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\aleph_2$" is that of a weakly compact cardinal. On the other hand, we show that if $2^{\aleph_0} < 2^{\aleph_3}$, T is a countable complete first order theory, and one of

(i) T is unstable,
(ii) T is superstable with DOP or OTOP,
(iii) T is stable and unsuperstable and $2^{\aleph_0} \leq \aleph_3$,

holds, then there are $\mathcal{A},\mathcal{B} \models T$ of power $\aleph_3$ such that $\mathrm{EFG}_{\omega_1}(\mathcal{A},\mathcal{B})$ is non-determined.

This paper is a continuation of [9]. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two first order structures of the same vocabulary L. We denote the domains of $\mathfrak{A}$ and $\mathfrak{B}$ by A and B respectively. All vocabularies are assumed to be relational. The *Ehrenfeucht–Fraïssé game* of length γ of $\mathfrak{A}$ and $\mathfrak{B}$ denoted by $\mathrm{EFG}_\gamma(\mathfrak{A},\mathfrak{B})$ is defined as follows: There are two players called $\forall$ and $\exists$. First $\forall$ plays x_0 and then $\exists$ plays y_0. After this $\forall$ plays x_1, and $\exists$ plays y_1, and so on. If $\langle (x_\beta, y_\beta) : \beta < \alpha\rangle$ has been played and $\alpha < \gamma$, then $\forall$ plays x_α after which $\exists$ plays y_α. Eventually a sequence $\langle (x_\beta, y_\beta) : \beta < \gamma\rangle$ has been played. The rules of the game say that both players have to play elements of $A \cup B$. Moreover, if $\forall$ plays his x_β in A (resp. B), then $\exists$ has to play his y_β in B (resp. A). Thus the sequence $\langle (x_\beta, y_\beta) : \beta < \gamma\rangle$ determines a relation $\pi \subseteq A \times B$. Player $\exists$ wins this round of the game if π is a partial isomorphism. Otherwise

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$\forall$ wins. The notion of winning strategy is defined in the usual manner. The game $\mathrm{EFG}_\gamma^\delta(\mathfrak{A},\mathfrak{B})$ is defined like $\mathrm{EFG}_\gamma(\mathfrak{A},\mathfrak{B})$ except that the players play sequences of length $< \delta$ at a time. Thus $\mathrm{EFG}_\gamma(\mathfrak{A},\mathfrak{B})$ is the same game as $\mathrm{EFG}_\gamma^2(\mathfrak{A},\mathfrak{B})$.

It was proved in [9] that, assuming $\square_{\omega_1}$, there are models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\aleph_2$ such that the game $\mathcal{G}_{\omega_1}(\mathfrak{A},\mathfrak{B})$ is non-determined. In this paper we weaken the assumption $\square_{\omega_1}$, to "ω_2 is not weakly compact in L" (Corollary 8), but we can do this only if we assume CH. We do not know if this is possible without CH. In the other direction, it was proved in [9] that if the non-ω_1-stationary ideal on ω_2 has a σ-closed dense subset, then the game $\mathrm{EFG}_{\omega_1}(\mathfrak{A},\mathfrak{B})$ is determined for all $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\le \aleph_2$. The assumption is equiconsistent with the existence of a measurable cardinal. In this paper we weaken the assumption to a condition which is consistent relative to the existence of a weakly compact cardinal (Corollary 13). Thus we establish:

THEOREM 1. *The following statements are equiconsistent relative to ZFC*:

1. *There is a weakly compact cardinal.*
2. *CH and* $\mathrm{EFG}_{\omega_1}(\mathfrak{A},\mathfrak{B})$ *is determined for all models* $\mathfrak{A}$ *and* $\mathfrak{B}$ *of cardinality* $\aleph_2$.

In [9] we proved in ZFC that there are structures $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\aleph_3$ with one binary predicate such that the game $\mathrm{EFG}_{\omega_1}(\mathfrak{A},\mathfrak{B})$ is non-determined. We now improve this result under some cardinal arithmetic assumptions. We prove:

THEOREM 2. *Assume that* $2^\omega < 2^{\omega_3}$ *and* T *is a countable complete first order theory. Suppose that one of* (i)–(iii) *below holds. Then there are* $\mathcal{A},\mathcal{B} \models T$ *of power* ω_3 *such that for all cardinals* $1 < \theta \le \omega_3$, $\mathrm{EFG}_{\omega_1}^\theta(\mathcal{A},\mathcal{B})$ *is non-determined.*

(i) T *is unstable.*
(ii) T *is superstable with DOP or OTOP.*
(iii) T *is stable and unsuperstable and* $2^\omega \le \omega_3$.

This result complements the result in [9] that if T is an ω-stable first order theory with NDOP, then $\mathrm{EFG}_{\omega_1}(\mathfrak{A},\mathfrak{B})$ is determined for all models $\mathfrak{A}$ of T and all models $\mathfrak{B}$. This is actually true under the weaker assumption that T is superstable with NDOP and NOTOP.

NOTATION. We follow Jech [6] in set-theoretic notation. We use S_n^m to denote the set $\{\alpha < \omega_m : \mathrm{cof}(\alpha) = \omega_n\}$. Closed unbounded sets are called *cub sets*. A set of ordinals is λ-*closed* if it is closed under supremums of ascending λ-sequences $\langle \alpha_i : i < \lambda\rangle$ of its elements. A subset of a cardinal is λ-*stationary* if it meets every λ-closed unbounded subset of the cardinal.

1. Getting a weakly compact cardinal. In this section we show that if CH holds and $\mathrm{EFG}_{\omega_1}(\mathfrak{A},\mathfrak{B})$ is determined for all models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\aleph_2$, then ω_2 is weakly compact in L (Corollary 8). We use the results from [8] that if ω_2 is not weakly compact in L, then there is a bistationary $S \subseteq S^2_0$ such that for all $\alpha < \omega_2$ either $\alpha \cap S$ or $\alpha \setminus S$ is non-stationary.

If I is a linear order, we use $(I)^*$ to denote the reverse order of I. We call a sequence $s = (s_\xi)_{\xi<\zeta}$ a *coinitial sequence of length* ζ in I if it is decreasing in I and has no lower bound in I. The *coinitiality* $\mathrm{coinit}(I)$ of a linear order I is the smallest length of a coinitial sequence in I.

Let $\theta = \omega + ((\omega_1)^* + \omega)\cdot\omega_1$.

LEMMA 3. *There is a dense linear order I such that*:

(i) $|I| = \aleph_1$.
(ii) $\mathrm{coinit}(I) = \aleph_0$,
(iii) $I\cdot(\alpha+1) \cong I$ *for all* $\alpha \le \omega_1$.
(iv) $I \cong I\cdot\omega + I\cdot(\omega_1)^*$.
(v) $I\cdot\theta + I \cong I$.

Proof. This is like Lemma 7.17 in [10]. If J_1 and J_2 are linear orders, let $H(J_1, J_2)$ be the set of $f : n_f \to J_1 \cup J_2$, where $n_f < \omega$ is even, $f(2i) \in J_1$ and $f(2i+1) \in J_2$ for all $i < n_f$. We can make $H(J_1,J_2)$ a linear order by ordering the functions lexicographically, i.e.

$$f \le g \ \Leftrightarrow\ (\exists m \le n_f)((\forall i < m)(f(i) = g(i))\ \&\ (m < n_f \to f(m) < g(m))).$$

Let $I_0 = H(\mathbb{Q}, \omega+(\omega_1)^*)$ and $I_1 = H(I_0, \omega_1)$. Thus $I_0 \cong (1+I_0)\cdot(\omega+(\omega_1)^*)\cdot\mathbb{Q}$ and $I_1 \cong (1+I_1)\cdot\omega_1\cdot I_0$. By using $\mathbb{Q} \cong \mathbb{Q}+1+\mathbb{Q}$, $\omega = 1+\omega$ and $\omega_1 = 1+\omega_1$, one easily gets the following, first for I_0, and then for I_1:

$$I_0 \cong I_0 + 1 + I_0, \qquad I_1 \cong I_1 + 1 + I_1. \tag{1}$$

Let I be the set of $f : \omega \to I_1 \cup \theta$, where $f(2i) \in I_1$ and $f(2i+1) \in \theta$ for all $i < \omega$ ordered lexicographically. Thus $I \cong I\cdot\theta\cdot I_1$. In fact, I is of the form $J\cdot\mathbb{Q}$, so (ii) is true. By (1) and $\theta \cong 1+\theta$ one gets (v) immediately. As $I \cong I\cdot\theta\cdot(1+I_1)\cdot\omega_1\cdot I_0$, from (v) we easily get (iii) for $\alpha = \omega_1$. From this and $\alpha+\omega_1 = \omega_1$ we immediately get (iii) for $\alpha < \omega_1$. Note that $\theta \cong \omega+(\omega_1)^*+\theta$. If we combine this with $I \cong I\cdot\theta\cdot I_1$ and $(\omega_1)^* \cong (\omega_1)^* + 1$, we get (iv).

As to (i), we only have $|I| = 2^\omega$. We use this lemma in a context where CH is assumed, so we could simply assume it here. But actually the lemma is true without CH, as we can construct I in L. Then $|I| = \aleph_1$. Note that our I_0 and I_1 are in L, and the only property of ω_1 that we used was that it is a limit ordinal. ∎

In the following, I denotes the dense linear ordering of Lemma 3.

DEFINITION 4. Suppose $S \subseteq S^2_0$. We define
$$\Phi(S) = \sum_{i<\omega_2} \eta_i,$$
where
$$\eta_i = \begin{cases} I \cdot (\omega_1)^* & \text{if } i \in S, \\ I & \text{if } i \notin S. \end{cases}$$
Let $\Phi_{\alpha,\beta}(S)$ be the suborder $\sum_{\alpha \le i < \beta} \eta_i$ of $\Phi(S)$. The *rank* of $x \in \Phi(S)$ is the least α such that $x \in \Phi_{\alpha,\alpha+1}(S)$. We denote this α by $\mathrm{rnk}(\Phi(S), x)$.

LEMMA 5. *Assume $S \subseteq S^2_0$ is such that there is no $\alpha \in S^2_1$ with both $S \cap \alpha$ and $(\alpha \cap S^2_0) \setminus S$ stationary. Then*
$$\Phi_{\alpha,\beta+1}(S) \cong I$$
whenever $\alpha < \beta < \omega_2$ and $\alpha \notin S$.

Proof. This is like Lemma 7.20 in [10]. We use Lemma 3 and induction on β.

Let us first assume $\beta \notin S$. If β is a successor ordinal, then $\Phi_{\alpha,\beta+1}(S) \cong I + I = I$ by (iii). If β has cofinality ω, then $\Phi_{\alpha,\beta+1}(S) \cong I \cdot \omega + I \cong I$. If β has cofinality ω_1 and $\beta \cap S$ is non-stationary, then $I \cong I \cdot \omega_1 + I \cong I$. Finally, if β has cofinality ω_1 and $\beta \setminus S$ is non-stationary, then $I \cong I \cdot \theta + I \cong I$, by (v).

Let us then assume $\beta \in S$. Thus β has cofinality ω. Therefore $\Phi_{\alpha,\beta+1}(S) \cong I \cdot \omega + I \cdot (\omega_1)^* \cong I$, by (iv). ∎

LEMMA 6. *Assume $S \subseteq S^2_0$ is such that there is no $\alpha \in S^2_1$ with both $S \cap \alpha$ and $(\alpha \cap S^2_0) \setminus S$ stationary. Then $\Phi_{0,\alpha}(S) \cong \Phi_{0,\alpha}(\emptyset)$ whenever $\alpha \in S^2_1$ and $S \cap \alpha$ is not stationary.*

Proof. Let $(\alpha_\xi)_{\xi<\omega_1}$ by a continuously increasing cofinal sequence in α such that $\alpha_\xi \notin S$ for all $\xi < \omega_1$. By Lemma 5 there is an isomorphism
$$f_\xi : \Phi_{\alpha_\xi,\alpha_{\xi+1}+1}(S) \to \Phi_{\alpha_\xi,\alpha_{\xi+1}+1}(\emptyset).$$
Let $f = \bigcup_{\xi<\omega_1} f_\xi$. This is the required isomorphism. ∎

PROPOSITION 7. *Assume CH and that there is $S \subseteq S^2_0$ such that both S and $S^2_0 \setminus S$ are stationary but there is no $\alpha \in S^2_1$ with both $S \cap \alpha$ and $(\alpha \cap S^2_0) \setminus S$ stationary. Then there are models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\aleph_2$ such that* $\mathrm{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ *is non-determined.*

Proof. We may assume that $\{\alpha \in S^2_1 : \alpha \cap S \text{ is non-stationary}\}$ is stationary, for otherwise we work with $S' = S^2_0 \setminus S$. Let $\mathfrak{A} = \Phi(S)$ and $\mathfrak{B} = \Phi(\emptyset)$. We first show that $\exists$ cannot have a winning strategy in $\mathrm{EFG}_{\omega+\omega+1}(\mathfrak{A}, \mathfrak{B})$. Suppose τ is a strategy of $\exists$. Let C be the cub of ordinals $\alpha < \omega_2$ such that if during the first ω rounds of the game, $\forall$ plays elements of the models of rank $< \alpha$, then so does $\exists$ following τ. Let $\delta \in C \cap S$. Let $(\delta_n)_{n<\omega}$ be an increasing cofinal sequence in δ. Now we let $\forall$ play against τ as follows: On round $n < \omega$ we let $\forall$ play some element of $\mathfrak{A}$ if n is even, and of $\mathfrak{B}$ if n is odd, of rank δ_n. During rounds $\omega + n$, $n < \omega$, we let $\forall$ play a coinitial sequence of length ω in $\Phi_{\delta,\delta+1}(\emptyset) \subseteq \mathfrak{B}$. As $\mathrm{coinit}(\Phi_{\delta,\delta+1}(S)) = \omega_1$, the game is lost for $\exists$. So τ could not be a winning strategy.

Suppose then ϱ is a strategy of $\forall$. We show that it cannot be a winning strategy. By CH we have an ω_1-cub set D of ordinals $\delta < \omega_2$ such that if $\exists$ plays only elements of rank $< \delta$, then ϱ directs $\forall$ to play also elements of rank $< \delta$ only. Let $\delta \in D \cap S^2_1$ be such that $\delta \cap S$ is non-stationary. By Lemma 6 there is an isomorphism $f : \Phi_{0,\alpha}(S) \to \Phi_{0,\alpha}(\emptyset)$. Now $\exists$ can beat ϱ by using f. ■

Using the result from [8] referred to above, we now get:

Corollary 8. *If CH holds and* $\mathrm{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ *is determined for all models* $\mathfrak{A}$ *and* $\mathfrak{B}$ *of cardinality* $\aleph_2$, *then* ω_2 *is weakly compact in* L.

2. Getting determinacy from a weakly compact cardinal. In this section we show that if κ is weakly compact, then there is a forcing extension in which the game $\mathrm{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is determined for all $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\le \aleph_2$.

We shall consider models $\mathfrak{A}, \mathfrak{B}$ of cardinality $\aleph_2$, so we may as well assume they have ω_2 as universe. For such a model $\mathfrak{A}$ and any ordinal $\alpha < \omega_2$ we let $\mathfrak{A}_\alpha$ denote the structure $\mathfrak{A} \cap \alpha$, and similarly for $\mathfrak{B}_\alpha$. Let us first recall the following basic fact from [9]:

Lemma 9 ([9]). *Suppose* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are structures of cardinality* $\aleph_2$. *If* $\forall$ *does not have a winning strategy in* $\mathrm{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$, *then*

$$S = \{\alpha : \mathfrak{A}_\alpha \cong \mathfrak{B}_\alpha\}$$

is ω_1*-stationary.*

This shows that to get determinacy of $\mathrm{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ it suffices to give a winning strategy of $\exists$ under the assumption that the above set S is ω_1-stationary. In [9] an assumption $I^*(\omega)$ was used. It says that the non-ω_1-stationary ideal on ω_2 has a σ-closed dense set. The rough idea was that $\exists$ uses the Pressing Down Lemma on S to "normalize" his moves so that he always has an ω_1-stationary set of possible continuations of the game. We now use the same idea. The hypothesis $I^*(\omega)$ is equiconsistent with a measurable cardinal. Since we assume only the consistency of a weakly compact cardinal, we have to work more.

Suppose κ is a weakly compact cardinal. Let $\mathcal{I}$ denote the Π^1_1-ideal on κ, i.e., the ideal of subsets of κ generated by the sets $\{\alpha : (H(\alpha), \varepsilon, A \cap H(\alpha)) \models \neg\phi\}$, where $A \subseteq H(\kappa)$ and ϕ is a Π^1_1-sentence such that

$(H(\kappa), \varepsilon, A) \models \phi$. We collapse κ to ω_2 and then force a cub to the complement of every set $S \subseteq S^2_1$ in $\mathcal{I}$. In the resulting model the above "normalization" strategy of $\exists$ works even though the non-ω_1-stationary ideal on ω_2 may not have a σ-closed dense set.

Definition 10. Let $\mathcal{F}$ be a set of cardinality κ of regressive functions $\kappa \to \kappa$ and $S \subseteq \kappa$. The game $\mathrm{PDG}_{\omega_1}(S, \mathcal{F})$ has two players called $\forall$ and $\exists$. They alternately play ω_1 rounds. During each round $\forall$ first chooses $f_i \in \mathcal{F}$. Then $\exists$ chooses a subset S_i of $\bigcap_{j<i} S_j$ (of S if $i = 0$) such that S_i is unbounded in κ and f_i is constant on S_i. Player $\exists$ wins if he can play all ω_1 moves following the rules.

Lemma 11. *Suppose $S = \{\alpha < \omega_2 : \alpha \neq 0, \mathfrak{A}_\alpha \cong \mathfrak{B}_\alpha\}$ and $h_\alpha : \mathfrak{A}_\alpha \cong \mathfrak{B}_\alpha$ for $\alpha \in S$. Let*

$$\mathcal{F} = \{f_\alpha : \alpha \in S\} \cup \{g_\alpha : \alpha \in S\},$$

where $f_\alpha : \omega_2 \to \omega_2$ is the regressive function mapping ξ ($\neq 0$) to $h_\xi(\alpha)$ if $\xi > \alpha$, and to 0 otherwise, and g_α is the regressive function mapping ξ ($\neq 0$) to $(h_\xi)^{-1}(\alpha)$ if $\xi > \alpha$, and to 0 otherwise. Suppose $\exists$ has a winning strategy in $\mathrm{PDG}_{\omega_1}(S, \mathcal{F})$. Then $\exists$ has a winning strategy in the game $\mathrm{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$.

Proof. Let $\mathcal{H} = \{h_\alpha : \alpha \in S\}$, where $h_\alpha : \mathfrak{A}_\alpha \cong \mathfrak{B}_\alpha$ for $\alpha \in S$. Let τ be a winning strategy of $\exists$ in the game $\mathrm{PDG}_{\omega_1}(S, \mathcal{F})$. Suppose the sequence $\langle (x_i, y_i) : i < \alpha \rangle$ has been played, where $\alpha < \omega_1$, x_i denotes a move of $\forall$ and y_i a move of $\exists$. Suppose $\forall$ plays next x_α. During the game $\exists$ also plays $\mathrm{PDG}_{\omega_1}(S, \mathcal{F})$. Let us denote his moves in $\mathrm{PDG}_{\omega_1}(S, \mathcal{F})$ by S_i. Thus $S_j \subseteq S_i$ for $i < j < \alpha$. The point of the sets S_i is that $\exists$ has taken care that for all $i < \alpha$ and $j \in S_i$ we have $y_i = h_j(x_i)$ or $x_i = h_j(y_i)$ depending on whether $x_i \in \mathfrak{A}$ or $x_i \in \mathfrak{B}$. Let $S'_\alpha = \bigcap_{i<\alpha} S_i \setminus \alpha$. The winning strategy τ gives an $S_\alpha \subseteq S'_\alpha$ and a y_α such that $f_i(x_\alpha) = y_\alpha$ for all $i \in S_\alpha$ if $x_\alpha \in \mathfrak{A}$, and $g_i(x_\alpha) = y_\alpha$ for all $i \in S_\alpha$ if $x_\alpha \in \mathfrak{B}$. This element y_α is the next move of $\exists$. Using this strategy $\exists$ cannot lose and hence wins. ∎

Theorem 12. *It is consistent relative to the consistency of a weakly compact cardinal that for every ω_1-stationary $S \subseteq \omega_2$ and every set $\mathcal{F}$ of cardinality $\aleph_2$ of regressive functions $\omega_2 \to \omega_2$, $\exists$ has a winning strategy in the game $\mathrm{PDG}_{\omega_1}(S, \mathcal{F})$.*

Proof. We may assume GCH. Suppose κ is weakly compact. Let $\mathbb{Q}$ be the Levy collapse of κ to $\aleph_2$. In $V^{\mathbb{Q}}$ we define by induction a sequence $\mathbb{P}_\alpha$, $\alpha < \kappa^+$, of forcing notions. Let $(A_\alpha), \alpha < \kappa^+$, be a complete list of all sets in the Π^1_1-ideal $\mathcal{I}$ on κ such that every element of A_α has uncountable cofinality. If α is limit of cofinality $\leq \omega_1$, then $\mathbb{P}_\alpha$ is the inverse limit of all $\mathbb{P}_\beta$, $\beta < \alpha$. For other limit α, $\mathbb{P}_\alpha$ is the direct limit of $\mathbb{P}_\beta$, $\beta < \alpha$. At successor stages we let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha \star \mathbb{R}_\alpha$, where $\mathbb{R}_\alpha$ is defined as follows: $q \in \mathbb{R}_\alpha$ iff q is a bounded closed sequence of elements of κ such that $q \cap A_\alpha = \emptyset$. $\mathbb{R}_\alpha$ is ordered by the end extension relation. Thus each $\mathbb{P}_\alpha$ is countably closed. Let $\mathbb{P} = \mathbb{P}_{\kappa^+}$. Now $\mathbb{Q} \star \mathbb{P}$ satisfies the κ^+-chain condition. Note also that for all $\alpha < \kappa^+$, $\mathbb{Q} \star \mathbb{P}_\alpha$ has power κ. We prove that it is true in $V^{\mathbb{Q}}$ that $\mathbb{P}_\alpha$ does not add new subsets of κ of cardinality $\leq \aleph_1$, hence κ remains $\aleph_2$ also after forcing with $\mathbb{P}$. It also follows that $\mathbb{Q} \star \mathbb{P}$ and each $\mathbb{Q} \star \mathbb{P}_\alpha$ are countably closed.

We now show that in $V^{\mathbb{Q}\star\mathbb{P}}$ the claim is true. Suppose S and a set $\mathcal{F} = \{f_\alpha : \alpha < \kappa\}$ of regressive functions $\kappa \to \kappa$ are given in $V^{\mathbb{Q}\star\mathbb{P}}$ such that (in $V^{\mathbb{Q}\star\mathbb{P}}$) $S \subseteq S^2_1$ is ω_1-stationary. Suppose $\alpha < \kappa^+$ is such that $\widetilde{S}, \widetilde{\mathcal{F}}$ and $\widetilde{f}_i$ are $\mathbb{Q} \star \mathbb{P}_\alpha$-names for $S, \mathcal{F}$ and f_i, respectively. Since S is ω_1-stationary in $V^{\mathbb{Q}\star\mathbb{P}}$, S is not in the ideal generated by $\mathcal{I}$ in $V^{\mathbb{Q}\star\mathbb{P}_\alpha}$. Suppose $(p,q) \Vdash \widetilde{S} \notin \mathcal{I}$. For a contradiction, suppose also that (p,q) forces that $\exists$ does not have a winning strategy in the game $\mathrm{PDG}_{\omega_1}(S, \mathcal{F})$.

Let $(\mathcal{B}'_0, \in)$ be a sufficiently elementary substructure of $(V, \in)$ such that $|\mathcal{B}'_0| = \kappa$, ${\mathcal{B}'}^{<\kappa}_0 \subseteq \mathcal{B}'_0$, $\mathbb{Q}$, $\mathbb{P}_\alpha$, α, κ, $\widetilde{\mathcal{F}}$, $\widetilde{f}_i$, $\widetilde{S}$ are in $\mathcal{B}'_0$, and $\alpha \cup \kappa \subseteq \mathcal{B}'_0$. Let $\mathcal{B}_0$ be the transitive collapse of $\mathcal{B}'_0$. Thus $\mathbb{Q}, \mathbb{P}_\alpha, \alpha, \kappa \in \mathcal{B}_0$, $A_j \in \mathcal{B}_0$ for $i \leq \alpha$ and $\widetilde{f}_i \in \mathcal{B}_0$ for $i < \kappa$. Let

$$T = \{\alpha < \kappa : (\exists (p', q') \leq (p,q))((p', q') \Vdash_{\mathbb{Q}\star\mathbb{P}_\alpha} \alpha \in \widetilde{S})\}.$$

Clearly $T \in \mathcal{B}_0$ and $T \notin \mathcal{I}$. By weak compactness, there are a transitive $\mathcal{B}_1$ and an elementary embedding $j : \mathcal{B}_0 \to \mathcal{B}_1$ such that κ is the critical point of j, $\kappa \in j(T)$ and $\kappa \notin j(A_i)$ for $i \leq \alpha$. So there is some $(p', q') \in j(\mathbb{Q} \star \mathbb{P}_\alpha)$ such that $(p', q') \leq j((p,q)) = (p,q)$ and $(p', q') \Vdash_{j(\mathbb{Q}\star\mathbb{P}_\alpha)} \kappa \in j(\widetilde{S})$. Note that $\mathbb{Q}, \mathbb{P}_\alpha \in \mathcal{B}_1$ and $\widetilde{f}_i \in \mathcal{B}_1$ for $i < \kappa$.

By (the proof of) Lemma 3 in [8], there are a $\mathbb{Q} \star \mathbb{P}_\alpha$-generic G over $\mathcal{B}_1$ and a forcing notion $\mathbb{R} \in \mathcal{B}_1[G]$ such that $(p,q) \in G$, $\mathbb{R}$ is countably closed in $\mathcal{B}_1[G]$, for all $\mathbb{R}$-generic K over $\mathcal{B}_1[G]$, there is a canonical $j(\mathbb{Q} \star \mathbb{P}_\alpha)$-generic G_K over $\mathcal{B}_1$ such that $\mathcal{B}_1[G_K] = \mathcal{B}_1[G][K]$, and for some K, G_K is such that $(p', q') \in G'$. Then for every $\mathbb{Q}\star\mathbb{P}_\alpha$-name $\widetilde{X} \in \mathcal{B}_0$, there is a canonical $\mathbb{R}$-name $\widetilde{Y} \in \mathcal{B}_1[G]$ such that for all $\mathbb{R}$-generic K over $\mathcal{B}_1[G]$, $j(\widetilde{X})$ and $\widetilde{Y}$ have the same interpretation in $\mathcal{B}_1[G][K]$. We do not distinguish $j(\widetilde{X})$ and $\widetilde{Y}$. With this notation, there is $r \in \mathbb{R}$ which forces in $\mathcal{B}_1[G]$ that $\kappa \in j(\widetilde{S})$. Then there is some $(p^*, q^*) \leq (p,q)$ in G that in $\mathcal{B}_1$ forces the existence of such $\mathbb{R}$ and r. So we may assume that G is generic over V and our $V^{\mathbb{Q}\star\mathbb{P}_\alpha}$ is the same as $V[G]$.

We describe in $\mathcal{B}_1[G]$ a winning strategy of $\exists$ in the game $\mathrm{PDG}_{\omega_1}(S, \mathcal{F})$. This is a contradiction since all possible winning plays of $\forall$ are in $\mathcal{B}_1[G]$ and being unbounded is absolute in transitive models. The strategy of $\exists$ is to play on the side conditions q^i in $\mathcal{B}_1[G]$ and sets $S_i \in \mathcal{B}_0[G]$ with $\mathbb{Q}\star\mathbb{P}_\alpha$-names $\widetilde{S}_i$ in $\mathcal{B}_0$ such that:

1. $q^i \in \mathbb{R}$.
2. $q^0 \le r$.
3. $i < k < \omega_1$ implies $q^k \le q^i$.
4. $i < k < \omega_1$ implies $S_k \subseteq S_i \subseteq S$.
5. $q^i \Vdash_{\mathbb{R}} \kappa \in j(\widetilde{S}_i)$ in $\mathcal{B}_1[G]$.

Suppose $\exists$ has followed this strategy, forming conditions q^i and sets S_i for $i < k$. Let $p = \inf\{q^i : i < k\}$. If we let S be $\bigcap_{i<k} S_i$ and $\widetilde{S}$ a name for this, then in $\mathcal{B}_1[G]$,

$$p \Vdash_{\mathbb{R}} \kappa \in j(\widetilde{S}).$$

Suppose then $\forall$ moves $f_k \in \mathcal{F}$. Let $q^k \le p$ be such that for some $\delta < \kappa$ we have $q^k \Vdash_{\mathbb{R}} j(\widetilde{f}_k)(\kappa) = \delta$ in $\mathcal{B}_1[G]$ and let S_k be $\{\beta \in S : f_k(\beta) = \delta\}$ and $\widetilde{S}_k$ a name for this. Then $q^k \Vdash_{\mathbb{R}} \kappa \in j(\widetilde{S}_k)$ in $\mathcal{B}_1[G]$.

Finally, we have to prove that $\mathbb{Q} \star \mathbb{P}_\alpha$ does not add new subsets of κ of cardinality $\le \aleph_1$ over and above those added by $\mathbb{Q}$. The proof of this is, mutatis mutandis, like the proof of the Main Fact (page 761) in [8]. Here we use the assumption $\kappa \notin j(A_i)$ for $i \le \alpha$. Thus, if C is a generic sequence in the complement of $j(A_\beta)$ in $V^{j(\mathbb{Q}\star\mathbb{P}_\alpha)}$, then we can continue it to a closed condition $C \cup \{\kappa\} \in \mathbb{R}_{j(\beta)}$. ∎

Results similar to Theorem 12 have also been treated in [14] and [15].

COROLLARY 13. *It is consistent relative to the consistency of a weakly compact cardinal that the game* $\mathrm{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ *is determined for all* $\mathfrak{A}$ *and* $\mathfrak{B}$ *of cardinality* $\le \aleph_2$.

3. Non-determinacy and structure theory. In this section we prove Theorem 2, which essentially establishes, under cardinality assumptions concerning the continuum, the existence of non-determined Ehrenfeucht–Fraïssé games of length ω_1 for models of *non-classifiable* theories. This complements the observation, made in [9], that the Ehrenfeucht–Fraïssé game of length ω_1 is determined for models of *classifiable* theories.

We start by proving Theorem 2 under assumption (iii), which we consider the most interesting case. That is, we start with a countable complete stable and unsuperstable first order theory and show that, assuming $2^\omega \le \omega_3$, it has two models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\aleph_3$ for which $\mathrm{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is non-determined. Actually, we construct $\mathfrak{A}$ and $\mathfrak{B}$ so that $\exists$ does not have a winning strategy even in $\mathrm{EFG}^2_{\omega+\omega}(\mathfrak{A}, \mathfrak{B})$ and $\forall$ does not have a winning strategy even in $\mathrm{EFG}^{\omega_3}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$.

We then prove Theorem 2 under assumption (i), that is, we now start with a countable complete unstable first order theory and show that, assuming $2^\omega < 2^{\omega_3}$, it has two models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\aleph_3$ for which $\mathrm{EFG}_{\omega_1}(\mathfrak{A}, \mathfrak{B})$ is non-determined.

Theorem 2 under assumption (ii) can be dealt with in the same way as under assumption (i). The section ends with some remarks on possible improvements.

3.1. *The stable unsuperstable case.* We will prove Theorem 2, case (iii), in a series of lemmas. We assume $\omega_3^\omega = \omega_3$ all the time. Let T be a countable complete stable and unsuperstable first order theory. As usual, we work inside a large saturated model $\mathbf{M}$ of T. We start by fixing some notation. By a tree I we mean a lexicographically ordered, downwards closed subtree of $\theta^{<(\omega+1)}$ for some linear order θ, that is, $I = (I, \ll, P_\alpha, <, H)_{\alpha\le\omega} \in K^\omega_{\mathrm{tr}}(\theta)$ (see [5, Definition 8.2] or [12]). For a while, we fix a tree $I \in K^\omega_{\mathrm{tr}}(\lambda)$, where λ is some large enough cardinal, so that $(I, \ll)$ is isomorphic to $\lambda^{<(\omega+1)}$. As in [4], for $u, v \in \mathcal{P}_\omega(I)$ (= finite subsets of I), we define $r(u, v)$ to be the unique set R which satisfies:

(I) $R \subseteq X_{u,v} = \{H(\eta, \xi) : \eta \in u,\ \xi \in v\}$,
(II) for all $\nu \in X_{u,v} \setminus R$, there is $\nu' \in R$ such that $\nu \ll \nu'$,
(III) if η and ξ are distinct elements of R, then $\eta \not\ll \xi$.

We write $u \le v$ if $r(u, v) = r(u, u)$. For more on these definitions, see [4]. In [4], it is shown that there are models $\mathcal{A}$ and $\mathcal{A}_u$, $u \in \mathcal{P}_\omega(I)$, and sequences a_η from $\mathcal{A}_{\{\eta\}}$, $\eta \in I$, such that:

(i) $\mathcal{A} = \bigcup_{u\in\mathcal{P}_\omega(I)} \mathcal{A}_u \models T$,
(ii) if $u \le v$, then $\mathcal{A}_u \subseteq \mathcal{A}_v$,
(iii) for all $u, v \in \mathcal{P}_\omega(I)$, $\mathcal{A}_u \downarrow_{\mathcal{A}_{r(u,v)}} \mathcal{A}_v$,
(iv) for all $u \in \mathcal{P}_\omega(I)$, $|\mathcal{A}_u| \le \omega_3$,
(v) if $P_\omega(\eta)$ holds and $\xi \ll \eta$ is an immediate successor of ξ', then

$$a_\eta \not\downarrow_{\mathcal{A}_{\{\xi'\}}} a_\xi.$$

These models are exactly what we want except that they are too large: we want the models $\mathcal{A}_u$, $u \in \mathcal{P}_\omega(I)$, to be countable. In order to get this, we use the Ehrenfeucht–Mostowski construction.

We extend the signature L of T to L_* by adding ω_3 new function symbols, some of which will be interpreted in $\mathbf{M}$ so that they provide Skolem functions for the L-formulas. In addition we interpret the functions so that if we write $\mathrm{SH}_*(u)$ for the L_*-Skolem hull of $\{a_\eta : \{\eta\} \le u\}$ then

(vi) for all $u \in \mathcal{P}_\omega(I)$, $\mathrm{SH}_*(u) = \mathcal{A}_u$.

By the usual argument (using [12, Appendix Theorem 2.6] and compactness) we can interpret the new function symbols so that $\mathbf{M}$ remains sufficiently saturated and the following holds:

(vii) if U is a downwards closed subtree of I and f is an automorphism of U, then there is an L_*-automorphism g of $\bigcup_{u\in\mathcal{P}_\omega(U)} \mathcal{A}_u$ such that for all $\eta \in U$, $g(a_\eta) = a_{f(\eta)}$.

Finally, it is easy to see that we can choose countable $L_1 \subseteq L_*$ so that $L \subseteq L_1$, L_1 contains the Skolem functions for the L-formulas and if we write $\mathrm{SH}_1(u)$ for the L_1-Skolem hull of $\{a_\eta : \{\eta\} \le u\}$ then

(viii) for all $u, v \in \mathcal{P}_\omega(I)$, $\mathrm{SH}_1(u) \downarrow_{\mathrm{SH}_1(v)} \mathrm{SH}_*(v)$.

So we have proved the following lemma (for the notion of Φ proper for K^ω_{tr} and the Ehrenfeucht–Mostowski models $\mathrm{EM}^1(J,\Phi)$, see [5, Definition 8.1] or [12]).

Lemma 14. *There are countable $L_1 \supseteq L$ and Φ proper for K^ω_{tr} such that the following holds*:

(a) *For all $J \in K^\omega_{\mathrm{tr}}$ there are an L_1-model $\mathrm{EM}^1(J,\Phi) \models T$ and sequences $a_\eta \in \mathrm{EM}^1(J,\Phi)$, $\eta \in J$, such that $\mathrm{EM}^1(J,\Phi)$ is the L_1-Skolem hull of $\{a_\eta : \eta \in J\}$ (i.e. $\{a_\eta : \eta \in J\}$ is the skeleton of $\mathrm{EM}^1(J,\Phi)$ and as before for $u \subseteq J$, $\mathrm{SH}_1(u)$ denotes the L_1-Skolem hull of $\{a_\eta : \{\eta\} \le u\}$).*

(b) *If U is a downwards closed subtree of J and f is an automorphism of U, then there is an L_1-automorphism g of $\mathrm{SH}_1(U)$ such that for all $\eta \in U$, $g(a_\eta) = a_{f(\eta)}$.*

(c) *Assume $(s\eta_i)_{i<\omega}$ is a strictly $\ll$-increasing sequence of elements of J, η_{i+1} is an immediate successor of η_i and η_0 is the root. Then $(\eta_i)_{i<\omega}$ has an upper bound in J iff there is a sequence $a \in \mathrm{EM}(J,\Phi)$ such that for all $i < \omega$, $a \not\downarrow_{\mathrm{SH}_1}(\{\eta_i\}) a_{\eta_i+1}$.* ■

We will write $\mathrm{EM}(J,\Phi)$ for $\mathrm{EM}^1(J,\Phi){\restriction}L$.

Our next goal is to define the skeletons for the models $\mathcal{A}$ and $\mathcal{B}$ in the theorem. For this we use the weak square from [9]. We denote by S^n_m the set $\{\alpha < \omega_n : \mathrm{cf}(\alpha) = \omega_m\}$.

Theorem 15 ([9, Lemma 16]). *There are sets S, U and C_α, $\alpha \in S$, such that the following holds*:

(a) *$S \subseteq S^3_0 \cup S^3_1$ and $S \cap S^3_1$ is stationary.*
(b) *$U \subseteq S^3_0$ is stationary and $S \cap U = \emptyset$.*
(c) *For all $\alpha \in S$, $C_\alpha \subseteq \alpha \cap S$ is closed in α and of order-type $\le \omega_1$.*
(d) *For all $\alpha \in S$, if $\beta \in C_\alpha$, then $C_\beta = C_\alpha \cap \beta$.*
(e) *For all $\alpha \in S \cap S^3_1$, C_α is unbounded in α.* ■

We will construct trees I_α and J_α, $\alpha < \omega_3$, so that the following holds:

(1) If $\alpha < \beta$ then I_α is a submodel of I_β and J_α is a submodel of J_β; now for $\eta \in I_\alpha$, we will write $\mathrm{rk}(\eta)$ for the least β such that $\eta \in I_\beta$ and similarly for $\eta \in J_\alpha$.

(2) For all $\alpha \in S$, there is an isomorphism $G_\alpha : I_\alpha \to J_\alpha$.

(3) If $\alpha \in C_\beta$, then $G_\alpha \subseteq G_\beta$.

(4) For all $\alpha \le \beta$ and $\eta \in I_\alpha$, if $P_\omega(\eta)$ does not hold, then there is an immediate successor ξ of η such that $\xi \in I_{\beta+1} \setminus I_\beta$.

(5) If $(\eta_i)_{i<\omega}$ is an increasing sequence of elements of I_α (for some α) and the sequence has an upper bound ξ in I_α, then $\mathrm{rk}(\xi) = \sup_{i<\omega} \mathrm{rk}(\eta_i)$ and similarly for sequences from J_α.

(6) If $(\eta_i)_{i<\omega}$ is an increasing sequence of elements of I_α, $(\mathrm{rk}(\eta_i))_{i<\omega}$ is not eventually constant and the sequence has an upper bound ξ in I_α, then $\mathrm{rk}(\xi)$ $(= \sup_{i<\omega} \mathrm{rk}(\eta_i)) \in U$; in J_α such sequences never have an upper bound.

(7) $|I_\alpha| \le \omega_3$ and $|J_\alpha| \le \omega_3$.

(8) $I_\alpha, J_\alpha \subseteq H_\omega(\omega_3)$, where $H_\omega(\omega_3)$ is the least set H such that $\omega_3 \subseteq H$ and if $E \subseteq H$ is of power $\le \omega$, then $E \in H$.

It is easy to see that such trees can be constructed by induction on α. However, in order to get what we want we need to do a bit more work when we define I_α and J_α in the case $\alpha \in U$. In order to decide which branches like the one in (6) above we want to have an upper bound, we use a guessing machine from [13] called the black box, which we formulate so that it fits exactly our purposes.

THEOREM 16 ([1, Theorem 1.3, Chapter XIII]). ($\omega_3^\omega = \omega_3$.) *There are* $(\overline{M}^\alpha, \eta^\alpha)$, $\alpha < \omega_3$, *such that*:

(i) $\overline{M}^\alpha = (M_i^\alpha)_{i<\omega}$ *is an increasing elementary chain of elementary submodels of some* $(H_\omega(\omega_3), A, B, \sigma)$ *such that* $A, B \subseteq H_\omega(\omega_3)$ *and* σ *is a strategy of* $\exists$ *in* $\mathrm{EFG}^2_\omega(A,B)$ (*A and B can be viewed as models of empty signature*).

(ii) $M_i^\alpha = (M_i^\alpha, A_i^\alpha, B_i^\alpha, \sigma_i^\alpha) \in H_\omega(\omega_3)$.

(iii) η^α *is an increasing function from* ω *to* ω_3, $\mathbf{M}_i^\alpha \in H_\omega(\eta^\alpha(i+1))$ *and* $\sup_{i<\omega} \eta^\alpha(i) \in U$.

(iv) $(\eta^\alpha(j))_{j\le i}, (M_j^\alpha)_{j\le i} \in M_{i+1}^\alpha$.

(v) *If* $\alpha \ne \beta$, *then* $\eta^\alpha \ne \eta^\beta$.

(vi) *Player I does not have a winning strategy for the following game*: *The length of the game is* ω. *At each move* $i < \omega$, *first I chooses* M_i *and then II chooses* $\alpha_i < \omega_3$. *I must play so that in the end* (i), (ii) *and* (iv) *above are satisfied. I wins if he has played according to the rules and there is no* $\alpha < \omega_3$ *such that* $((M_i)_{i<\omega}, (\alpha_i)_{i<\omega}) = (\overline{M}^\alpha, \eta^\alpha)$.

First we (partially) uniformize the Ehrenfeucht–Mostowski construction: We assume that for all $I, I' \in K^\omega_{\mathrm{tr}}$, if I is a substructure of I' and $I' \subseteq H_\omega(\omega_3)$, then there is a unique model $\mathrm{EM}^1(I,\Phi)$, it is a substructure of $\mathrm{EM}^1(I',\Phi)$ and $\mathrm{EM}^1(I',\Phi) \subseteq H_\omega(\omega_3)$.

So let $\alpha \in U$ and assume that I_β and J_β are defined for all $\beta < \alpha$. Write $I^*_\alpha = \bigcup_{\beta<\alpha} I_\beta$ and $J^*_\alpha = \bigcup_{\beta<\alpha} I_\beta$. For $\gamma < \omega_3$, we write M^γ for $\bigcup_{i<\omega} M_i^\gamma$ and A^γ, $\mathcal{B}^\gamma$ and σ^γ are defined similarly. Let W^α be the set of all $\gamma < \omega_3$ such that:

(a) $A^\gamma = \mathrm{EM}(I^*_\alpha \cap M^\gamma, \Phi)$ and $B^\gamma = \mathrm{EM}(J^*_\alpha \cap M^\gamma, \Phi)$.

(b) $\sup_{i<\omega} \eta^\gamma(i) = \alpha$.

(c) There are $\xi^\gamma_i \in I^*_\alpha \cap M^\gamma$, $i < \omega$, such that ξ^γ_0 is the root of I^*_α, ξ^γ_{i+1} is an immediate successor of ξ^γ_i and $\xi^\gamma_i \in I_{\eta^\gamma(i)+1} - I_{\eta^\gamma(i)}$.

Notice that by Theorem 16(v), if $\gamma \neq \delta$, then $(\xi^\gamma_i)_{i<\omega} \neq (\xi^\delta_i)_{i<\omega}$. Let $C^\gamma_i = \mathrm{SH}(\{\xi^\gamma_i\})$. Then we can find a partial function $g^\gamma : A^\gamma \to \mathcal{B}^\gamma$ such that:

(d) $\mathrm{dom}(g^\gamma) = \bigcup_{i<\omega} C^\gamma_i$.

(e) g^γ is a result of a play of $\mathrm{EFG}^2_\omega(A^\gamma, B^\gamma)$ in which $\exists$ has used σ^γ.

We let W^α_J be the set of those $\gamma \in W^\alpha$ such that:

(f) g^γ is a partial isomorphism from $\mathrm{EM}(I^*_\alpha \cap M^\gamma, \Phi)$ to $\mathrm{EM}(J^\gamma_\alpha \cap M^\gamma, \Phi)$.

(g) There is J such that if we let $J_\alpha = J$, then (1), (5)–(8) above are satisfied and there is a sequence $a \in \mathrm{EM}(J, \Phi)$ such that for all $i < \omega$, $a \not\downarrow_{g^\gamma(C^\gamma_i)} g^\gamma(a_{\xi^\gamma_{i+1}})$.

We let W^α_I be the set of all $\gamma \in W^\alpha \setminus W^\alpha_J$ such that g^γ satisfies (f) above.

Now we can define I_α and J_α. First we choose I_α so that it consists of all $\eta \in I^*_\alpha$ together with the supremums for the branches $(\xi^\gamma_i)_{i<\omega}$, $\gamma \in W^\alpha_I$. J_α is chosen so that it satisfies (g) for all $\gamma \in W^\alpha_J$ (and so especially (1), (5)–(8)).

Then we let $I = \bigcup_{\alpha<\omega_3} I_\alpha$, $J = \bigcup_{\alpha<\omega_3} J_\alpha$, $\mathcal{A} = \mathrm{EM}(I, \Phi)$ and $\mathcal{B} = \mathrm{EM}(J, \Phi)$. Clearly $\mathcal{A}$ and $\mathcal{B}$ can be chosen so that $\mathcal{A}, \mathcal{B} \subseteq H_\omega(\omega_3)$.

LEMMA 17. *$\forall$ does not have a winning strategy for $\mathrm{EFG}^{\omega_3}_{\omega_1}(\mathcal{A}, \mathcal{B})$.*

Proof. For this it is enough to show that A does not have a winning strategy for $\mathrm{EFG}^{\omega_3}_{\omega_1}(I, J)$, which is clear by (2) and (3) above and Theorem 15. ∎

LEMMA 18. *$\exists$ does not have a winning strategy for $\mathrm{EFG}^2_{\omega+\omega}(\mathcal{A}, \mathcal{B})$.*

Proof. For a contradiction, assume σ is a winning strategy of $\exists$ for the game $\mathrm{EFG}^2_{\omega+\omega}(\mathcal{A}, \mathcal{B})$. We play a round of the game defined in Theorem 16(vi). We let player I play so that he follows the rules and:

(i) For all $i < \omega$, $M_i \prec (H_\omega(\omega_3), \mathcal{A}, \mathcal{B}, \sigma\restriction\omega)$,

(ii) For all $\delta, \delta' \in M_i$, if $\delta \le \delta'$, $\eta \in I_\delta \cap M_i$ and $P_\omega(\eta)$ does not hold, then there is $\xi \in (I_{\delta'+1} \setminus I_{\delta'}) \cap M_{i+1}$ such that ξ is an immediate successor of η.

(iii) The Skolem hulls of $\{a_\eta : \eta \in I \cap M_i\}$ and $\{a_\eta : \eta \in J \cap M_i\}$ are subsets of M_{i+1}.

(iv) $\mathcal{A} \cap M_i$ is a subset of the Skolem hull of $\{a_\eta : \eta \in I \cap M_{i+1}\}$ and $\mathcal{B} \cap M_i$ is a subset of the Skolem hull of $\{a_\eta : \eta \in J \cap M_{i+1}\}$.

(v) $\bigcup\{\mathrm{rk}(\eta) : \eta \in I \cap M_i\} \cup \bigcup\{\mathrm{rk}(\eta) : \eta \in J \cap M_i\} \in M_{i+1}$.

By Theorem 16(vi), the round can be played so that $\forall$ loses. Let α_i, $i<\omega$, be the choices $\exists$ made, and γ be such that $((M_i)_{i<\omega},(\alpha_i)_{i<\omega}) = (\overline{M}^\gamma,\eta^\gamma)$. Finally, let $\alpha=\bigcup_{i<\omega}\alpha_i$ ($\in U$).

Now it is easy to see that $\gamma\in W^\alpha$, in fact $\gamma\in W^\alpha_I$ or $\gamma\in W^\alpha_J$ (otherwise we have demonstrated that σ is not a winning strategy). In the first case, there is a sequence $a\in\mathcal{A}$ such that for all $i<\omega$, $a\not\downarrow_{C^\gamma_i} a_{\xi^\gamma_{i+1}}$ but in $\mathcal{B}$ there is no sequence b such that for all $i<\omega$, $b\not\downarrow_{g^\gamma(C^\gamma_i)} g^\gamma(\xi^\gamma_{i+1})$, a contradiction. In the latter case, there is a sequence $b\in\mathcal{B}$ such that for all $i<\omega$, $b\not\downarrow_{g^\gamma(C^\gamma_i)} g^\gamma(\xi^\gamma_{i+1})$ but by (the construction,) Lemma 2.3(c) and Theorem 16(v), there is no sequence $a\in\mathcal{A}$ such that for all $i<\omega$, $a\not\downarrow_{C^\gamma_i} a_{\xi^\gamma_{i+1}}$, a contradiction. ∎

Now Lemmas 2.6 and 2.7 imply Theorem 2(iii).

3.2. *The unstable case.* We will prove Theorem 2, case (i), again in a series of lemmas. We assume $\omega_3^\omega < 2^{\omega_3}$. Let T be a countable complete unstable first order theory. Let L be the signature of T.

THEOREM 19 ([12]). *Assume T is a countable unstable theory in the signature L. There are a countable signature $L_1\supseteq L$, a complete Skolem theory $T_1\supseteq T$ in the signature L_1, a first-order L-formula $\phi(x,y)$ and Φ proper for (ω,T_1)* (see [12, Definition VII 2.6]) *such that for every linear order I there is an Ehrenfeucht–Mostowski model* $\mathrm{EM}^1(I,\Phi)$ *of T_1 with a skeleton $\{a_\eta:\eta\in I\}$ such that*

$$\mathrm{EM}^1(I,\Phi)\models\phi(a_\eta,a_\xi)\quad\textit{iff}\quad I\models\eta<\xi.$$

We write $\mathrm{EM}(I,\Phi)$ for $\mathrm{EM}^1(I,\Phi)\restriction L$. Notice that by using the terminology from [13, Definition III 3.1], $\{a_\eta:\eta\in I\}$ is weakly (ω,ϕ)-skeleton-like in $\mathrm{EM}(I,\Phi)$.

In order to use Theorem 19, linear orders are needed. If A is a linear ordering, $x\in A$ and $B\subseteq A$, then by writing $x<B$ we mean that $x<y$ for every $y\in B$; $x>B$ and $C>B$ for $C\subseteq A$ are defined similarly. We denote by A^* the inverse of A. Again let S, U and C_α, $\alpha\in S$, be as in [9, Lemma 16], i.e. Theorem 15 above, with the exception that $0\in S$ and for all $\alpha\in S\setminus\{0\}$, $0\in C_\alpha$. By induction on $i<\omega_3$, we will define linear orders A^i_α and B^i_α, $\alpha<\omega_3$, and for $i\in S$, isomorphisms

$$G_i:\sum_{\beta<i+2}A^i_\beta\to\sum_{\beta<i+2}B^i_\beta.$$

We write $A^i(\beta,\alpha)$ for $\sum_{\beta\le\gamma<\alpha}A^i_\gamma$, and similarly for $B^i(\beta,\alpha)$. We will do the construction so that:

(1) $A^0_\alpha\cong\omega^*$ for all $\alpha<\omega_3$ and if $\alpha\notin U$, then $B^0_\alpha\cong\omega^*$, otherwise $B^0_\alpha\cong(\omega_1)^*$.

(2) If $i < j$, then $A^i_\alpha \subseteq A^j_\alpha$ and $B^i_\alpha \subseteq B^j_\alpha$, otherwise the sets are distinct; and if $j \in C_i$, then $G_j \subseteq G_i$.

(3) If $\mathrm{cf}(\alpha) = \omega$, then A^0_α is coinitial in A^i_α, and similarly for B.

We will do this by induction on i. However, in order to be able to show that (3) holds in each step, we need additional machinery.

Let $C \in \{A, B\}$. We say that (I, J) is a (C, i, β)-*cut* if I is an initial segment of C^i_β and $J = C^i_\beta \setminus I$. We say that the cut is *basic* if $I = \emptyset$. We define a notion of *forbidden cut* by induction on i as follows (we should talk about i-forbidden cuts, but i is always clear from the context):

(a) For all limit β, the basic $(C, 0, \beta)$-cut is forbidden.

(b) If (I, J) is a (C, i, β)-cut, $j < i$ and $(C^j_\beta \cap I, C^j_\beta \cap J)$ is forbidden, then (I, J) is forbidden.

(c) If (I, J) is a forbidden (A, i, β)-cut, $I^* = I \cup \bigcup_{\gamma<\beta} A^i_\gamma$ and $G_i(I^*)$ is not bounded by any $x \in \bigcup_{\gamma<\delta} B^i_\gamma$ but some $y \in B^i_\delta$ bounds it, then $(G_i(I^*) \cap B^i_\delta, B^i_\delta \setminus G_i(I^*))$ is forbidden, and similarly for A and B interchanged (and G_i replaced by $(G_i)^{-1}$).

Now we can state the additional properties we want our construction to have. Let $E \in \{A, B\}$, $i, \beta < \omega_3$ and (I, J) be an (E, i, β)-cut.

(4) If (I, J) is forbidden, then there are no $j < \omega_3$ and $x \in E^j_\beta$ such that $I < x < J$.

(5) Assume (I, J) is forbidden and $j \in S$ is such that $j < i$ and either $E^j_\beta \cap I$ is cofinal in I or $E^j_\beta \cap J$ is coinitial in J (we say that $\emptyset$ is both cofinal and coinitial in $\emptyset$). Then $(E^j_\beta \cap I, E^j_\beta \cap J)$ is forbidden.

(6) If β is successor, then E^0_β is coinitial in E^i_β.

LEMMA 20. *Let $E \in \{A, B\}$.*

(i) *For all $i, \beta < \omega_3$, if* (5) *holds up to stage i, then $(E^i_\beta, \emptyset)$ is not forbidden, and neither is $(\emptyset, E^i_\beta)$ if β is successor.*

(ii) *For limit β, every basic (E, i, β)-cut is forbidden.*

(iii) *Property* (4) *implies property* (3).

(iv) *If $i + 1 < \beta$ and (I, J) is a forbidden (E, i, β)-cut, then it is basic* (*and β is limit*).

Proof. Immediate. ∎

Now we are ready to do the construction: For $i = 0$, the linear orders are defined by (1) and we let G_0 be the only possible one. Clearly (1)–(6) hold. If $i \notin S$ or $\sup C_i = i$, then we let $A^i_\alpha = \bigcup_{j<i} A^j_\alpha$, $B^i_\alpha = \bigcup_{j<i} B^j_\alpha$ and if $i \in S$ (and $\sup C_i = i$), then $G_i = G \cup \bigcup_{j \in C_i} G_j$, where G is the obvious isomorphism from $A^i(i, i+2)$ to $B^i(i, i+2)$ (both are isomorphic

to $\omega^* + \omega^*$). Now (1), (2), (4) and (6) hold trivially. By Lemma 2.9(iii), (3) holds. For (5), assume that $C \in \{A, B\}$ and (I, J) is a forbidden (C, i, β)-cut. Now the reason why (I, J) is forbidden is (b) in the definition of forbidden cut (if $i \notin S$, then this is trivial and otherwise by the definition of G_i, (c) does not give forbidden cuts not forbidden by (b)). But then (5) follows immediately from the induction assumption.

We are left with the case $i \in S$ and $j = \sup C_i < i$. Notice that now $j \in C_i$. Let $\alpha < j+2$ and $A \neq \emptyset$ be an initial segment of A^j_α. Let $A^+ = A \cup \bigcup_{\gamma<\alpha} A^j_\gamma$. Then there is the least $\beta < j+2$ such that $B^+ = G_j(A^+) \cap \bigcup_{\gamma\leq\beta} B^j_\beta = G_j(A^+)$. Let $A' = (A^j_\alpha \cup A^j_{\alpha+1}) \setminus A$, $B = G_j(A) \cap B^j_\beta$ and $B' = (B^j_\beta \cup B^j_{\beta+1}) \setminus B$. Assume that at least one of $C' = \{x \in \bigcup_{k<i}(A^k_\alpha \cup A^k_{\alpha+1}) : A < x < A'\}$ and $D' = \{x \in \bigcup_{k<i}(B^k_\beta \cup B^k_{\beta+1}) : B < x < B'\}$ is nonempty. Then by the induction assumption, $B \neq \emptyset$. Let C be a copy of C' and D a copy of D'. Then we define A^i_α so that it contains $\bigcup_{k<i} A^k_\alpha$ and in each cut as above we add D so that $A < C' < D < A'$ and B^i_β is defined similarly but now $B < C < D' < B'$ (this is possible by (6) in the induction assumption). Then, by (4) in the induction assumption, we can find an isomorphism $G'_i : \bigcup_{\alpha<j+2} A^i_\alpha \to \bigcup_{\alpha<j+2} B^i_\alpha$. Notice that by (5) in the induction assumption, for all $\delta < i$, the (A, δ, α)-cut $(A^\delta_\alpha \setminus A(\delta), A(\delta))$ and (B, δ, β)-cut $(B(\delta), B^\delta_\beta \setminus B(\delta))$ are not forbidden, where $A(\delta) = \{x \in A^\delta_\alpha : x > C'\}$ and $B(\delta) = \{x \in B^\delta_\beta : x < D'\}$. So we have not violated property (4).

For all $\alpha > j+1$, we let $A^i_\alpha = \bigcup_{k<i} A^k_\alpha$, and B^i_α is defined similarly. However we will still make changes to B^i_{j+1} and A^i_{i+1}! Let A be a copy of $B^i(j+3, i+2)$ and B be a copy of $A^i(j+2, i+1)$. Furthermore, extend A^i_{i+1} so that there is an isomorphism $g : A^i_{i+1} \to B^i_{j+2}$ such that $g(A^0_{i+1}) = B^0_{j+2}$ (this is not a problem since $A^0_{i+1} = \bigcup_{k<i} A^k_{i+1} \cong \omega^* \cong B^0_{j+2}$ and by Lemma 2.9(iv), the sets A^k_{i+1}, $k < i$, do not contain forbidden $(A, k, i+1)$-cuts; so we do not violate (4)). Then we add A to (the extended) A^i_{i+1} as an end segment and B to B^i_{j+1} as an end segment. By Lemma 2.9(i), this does not violate (4). Now it is easy to extend G'_i to G_i so that $G_i(A^i(j+2, i+1)) = B$, $G_i(A^i_{i+1} - A) = B^i_{j+2}$ and $G_i(A) = B^i(j+3, i+2)$.

Now (1), (2) and (6) hold trivially, (4) is already shown to hold and by Lemma 2.9(iii), (4) implies (3). So we are left to show

LEMMA 21. (5) *holds.*

Proof. Assume (I, J) is a forbidden (E, i, β)-cut, $E \in \{A, B\}$, and $\delta \in S$ is such that $\delta < i$ and $E^\delta_\beta \cap J$ is coinitial in J; the other case is similar. If $\beta \geq j+1$ and both $J \cap \bigcup_{k<i} A^k_{j+1}$ and $J \cap \bigcup_{k<i} B^k_{j+1}$ are empty, then the claim follows easily from Lemma 2.9 and the induction assumption.

So we assume that this is not the case. If (I,J) is forbidden because of (b) in the definition of forbidden cut, the claim follows from the induction assumption. So we assume that $E=B$ and there is a forbidden (A,i,γ)-cut (C,D) such that (I,J) is forbidden by (c) applied to (C,D) (the case of A and B interchanged is symmetric). Since (I,J) is not forbidden by (b) in the definition of forbidden cut, (C,D) must be forbidden because of it, i.e. for some $\alpha<i$, $(A^\alpha_\gamma\cap C, A^\alpha_\gamma\cap D)$ is a forbidden (A,α,γ)-cut.

If

$$(\star)\qquad y<A^j_\gamma\cap D \quad \text{for no } y\in A^\alpha_\gamma\cap D,$$

then by the induction assumption, $(B^j_\beta\cap I, B^j_\beta\cap J)$ is a forbidden (B,j,β)-cut and the claim follows from the definition of forbidden cut if $\delta\ge j$ and from (5) in the induction assumption if $\delta<j$. So we assume that $(\star)$ fails. Let y be the bound. Then $\emptyset\neq D'=\{z\in A^i_\gamma\cap D: z\le y\}\subseteq A^i_\gamma\setminus \mathrm{dom}(G_j)$. So by the construction, $G_i(D')\subseteq J\setminus B^\delta_\beta$ and for all $x\in B^\delta_\beta$, either $x<G_i(D')$ or $x>G_i(D')$. By the choice of the cut (C,D), there cannot be $x\in J\cap B^\delta_\beta$ such that $x<G_i(D')$. But then $G_i(D')<J\cap B^\delta_\beta$, which contradicts the assumption that $J\cap B^\delta_\beta$ is coinitial in J. ∎

Let $A=\sum_{\alpha<\omega_3}\bigcup_{i<\omega_3}A^i_\alpha$ and $B=\sum_{\alpha<\omega_3}\bigcup_{i<\omega_3}B^i_\alpha$. Notice that by (1) and (3), $\mathrm{inv}^1_\omega(A)$ differs from $\mathrm{inv}^1_\omega(B)$ by a stationary set which consists of ordinals of cofinality ω (for the definition of inv^n_ω, see [13, Definition III 3.4]). Let $S_\alpha\subseteq S^3_0$, $i<2^{\omega_3}$, be stationary sets such that for $\alpha<\beta<2^{\omega_3}$, $S_\alpha\triangle S_\beta$ is stationary and define $\Psi_\alpha=\sum_{\alpha<\omega_3}\tau_\alpha$, where $\tau_\alpha=A^*$ if $\alpha\notin S_\alpha$ and $\tau_\alpha=B^*$ otherwise. Notice that if $\alpha\neq\beta$, then $\mathrm{inv}^2_\omega(\Psi_\alpha)$ differs from $\mathrm{inv}^2_\omega(\Psi_\beta)$ by a stationary set which consists of ordinals of cofinality ω.

Finally, let $\mathcal{A}_\alpha=\mathrm{EM}((\Psi_\alpha)^*\cdot\omega_1,\Phi)$.

LEMMA 22. *For all $\alpha,\beta<2^{\omega_3}$, $\forall$ does not have a winning strategy for $\mathrm{EFG}^{\omega_3}_{\omega_1}(\mathcal{A}_\alpha,\mathcal{A}_\beta)$.*

Proof. For this, it is enough to show that $\forall$ does not have a winning strategy for $\mathrm{EFG}^{\omega_3}_{\omega_1}((\Psi_\alpha)^*\cdot\omega_1,(\Psi_\beta)^*\cdot\omega_1)$, which follows easily from (2) in the construction of A and B and Theorem 15 (see e.g. [9, Claim 3 in the proof of Theorem 17]). ∎

LEMMA 23. *There are $\alpha<\beta<2^{\omega_3}$ such that $\exists$ does not have a winning strategy for $\mathrm{EFG}^2_{\omega_1}(\mathcal{A}_\alpha,\mathcal{A}_\beta)$.*

Proof. By using the usual forcing notion, we collapse ω_3 to an ordinal of power ω_1. Since this forcing notion does not kill those stationary subsets of ω_3 which consist of ordinals of cofinality ω and cofinalities $\le\omega_1$ are preserved, in the extension, $\mathrm{inv}^2_\omega(\Psi_\alpha)\neq\mathrm{inv}^2_\omega(\Psi_\beta)$ for all $\alpha\neq\beta$. Clearly, the skeletons of the models $\mathcal{A}_\alpha$ remain weakly (ω,ϕ)-skeleton-like in $\mathcal{A}_\alpha$. So by (the proof of) [13, Lemma III 3.15(1)], $\mathrm{inv}^2_\omega(\Psi_\alpha) \in \mathrm{INV}^2_\omega(\mathcal{A}_\alpha, \phi)$ in the extension (for the definition of INV^n_ω, see [13, Definition III 3.11] and notice that $\mathcal{A} \cong \mathcal{B}$ implies $\mathrm{INV}^2_\omega(\mathcal{A}, \phi) = \mathrm{INV}^2_\omega(\mathcal{B}, \phi)$). Also by [13, Lemma III 3.13(1)], $|\mathrm{INV}^2_\omega(\mathcal{A}_\alpha, \phi)| = \omega_1$. Since $\omega_3^\omega < 2^{\omega_3}$ in the ground model, in the generic extension, $(2^{\omega_3})^V$ is a cardinal $> \omega_1$. So there are $\alpha < \beta < (2^{\omega_3})^V$ such that $\mathcal{A}_\alpha \not\cong \mathcal{A}_\beta$ in the extension. Since countable subsets are not added, $\exists$ does not have a winning strategy for $\mathrm{EFG}^2_{\omega_1}(\mathcal{A}_\alpha, \mathcal{A}_\beta)$ (in the ground model). ∎

Now Lemmas 2.11 and 2.12 imply Theorem 2(i). ∎

Having proved the theorem, we make some remarks which follow from the proof.

REMARK 24. In many cases, the assumption on 2^ω in Theorem 2 can be removed. For example, this is true of linear orders. An easy proof for this is given in [3], alternatively this follows immediately from the proof of Theorem 2(i) by checking where the assumption $2^\omega < 2^{\omega_3}$ was needed. Another case where the assumption on 2^ω can be removed is when $\theta = \omega_3$ in the stable unsuperstable case. This follows from the proof of Theorem 2(iii) by noticing that the black box can now be replaced by an argument from [4]. Another remark is that in Theorem 2(i), (ii), ω_3 can be replaced by any cardinal $\kappa \geq \omega_3$ such that κ is a successor of a regular cardinal and $2^\kappa > \kappa^\omega$. Finally, in Theorem 2(iii), ω_3 can be replaced by any cardinal $\kappa \geq \omega_3$ such that κ is a successor of a regular cardinal and $\kappa^\omega = \kappa$.

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