FUNDAMENTA
MATHEMATICAE
170 (2001)

# Constructing $\omega$-stable structures: Computing rank

by

**John T. Baldwin** (Chicago, IL) and **Kitty Holland** (DeKalb, IL)

**Abstract.** This is a sequel to [1]. Here we give careful attention to the difficulties of calculating Morley and $U$-rank of the infinite rank $\omega$-stable theories constructed by variants of Hrushovski's methods. Sample result: For every $k < \omega$, there is an $\omega$-stable expansion of any algebraically closed field which has Morley rank $\omega \times k$. We include a corrected proof of the lemma in [1] establishing that the generic model is $\omega$-saturated in the rank 2 case.

In [1] we set up a general framework for constructing $\omega$-stable expansions of strongly minimal sets or, more generally, $\omega$-stable theories. This is one of a series of papers developing the ideas in [5, 4] where the notion of modifying the Jonnsón–Fraisse construction to obtain homogeneous-universal (generic) structures that are stable was introduced. In addition to [1], familiarity with [3], where the argument for the fusion case is expounded in a similar manner to ours, and with [2] is helpful for understanding this paper. We try to make references for specific results.

We describe in Section 1 the properties of a function $\delta$ which allow one to construct these expansions. In this paper we are considering a two-parameter family of expansions of a strongly minimal set. The first parameter $k$ determines the specific function $\delta$ of a given example (as described in Paragraph 2.1). The second parameter $\mu$ governs the algebraicity of primitive extensions (Definition 1.5). If there is no $\mu$, $\delta_k$ gives a theory of rank $\omega \times k$; if $\mu$ is finite-to-one, $\delta_k$ gives a theory of rank $k$. If the generic is $\omega$-saturated $\omega$-stability is preserved in the expansion. We established general conditions for guaranteeing $\omega$-saturation of the generic in [1]. Namely, the theory of generic must admit separation of quantifiers; we define this notion in Section 1 of this paper. The existence of an expansion of an algebraically closed field with rank $\omega \times 2$ was proved by Poizat in [7]. Extending to $\omega \times k$ for

2000 *Mathematics Subject Classification*: Primary 03C60.
Research of J. T. Baldwin partially supported by NSF grant DMS-9803496.
Research of K. Holland partially supported by NSF grant DMS-9510377.

$k > 2$ introduces further complexities for proving the lower bound. In Section 3 we concern ourselves with various values of $k$ and mention $\mu$ only in passing. We show how to calculate the $U$-rank and Morley rank of types in the general infinite rank case; a special case is the expansion of fields. The published argument for the lemma establishing separation of quantifiers in the rank two case [1] was flawed; moreover, we have since discovered a more conceptual way of organizing the proof. We thank Eric Rosen for pointing out this difficulty. In Section 2 we restrict to $k = 2$ and worry about the effect of $\mu$. For coherence, we give a proof for the separation of quantifiers result which replaces most of the third section of [1].

**1. Generalities.** In this section we summarize the salient definitions from [1] and quote some results from there which are used here. Fix a countable first order language $L$ which may have function symbols. We begin with a theory $T_{-1}$ whose class $\overline{\boldsymbol{K}}_{-1}$ of models is closed under substructure. We let $\boldsymbol{K}_{-1}$ denote the class of *finitely generated* (as structures) elements of $\overline{\boldsymbol{K}}_{-1}$. We describe the properties of a "weight" function $\delta$ on finite sequences from members of $\overline{\boldsymbol{K}}_{-1}$ which permit the construction of generic structures.

**1.1.** Notation. We will write $B \subseteq_\omega N$ to indicate $B$ is a finite subset of $N$. If $A, B \subseteq N$ we write $AB$ for $A \cup B$. For $X$ a subset of $N$, $\langle X\rangle_N$ denotes the substructure of $N$ generated by $X$. We will generally omit the subscript. For $A \in \boldsymbol{K}_{-1}$, we write $\mathrm{Diag}(A)$ for the *quantifier-free* diagram of $A$. Note that even when $A = \overline{a}$ is a finite sequence, $\mathrm{Diag}(\overline{a})$ is in general a type, not a formula.

We consider functions $\delta$ from finite sequences to the integers so that if two finite sequences $\overline{a}, \overline{b}$ have the same diagram then $\delta(\overline{a}) = \delta(\overline{b})$. Thus, in effect, $\delta$ is a function from quantifier free diagrams of finite subsets of elements of $\overline{\boldsymbol{K}}_{-1}$ into the integers. We describe below the properties of $\delta$ which are used in the proofs. Three natural examples of this framework, *ab initio*, fusion, and bicolored fields, are discussed at length in [1].

We let $\overline{\boldsymbol{K}}_0$ denote the members $A$ of $\overline{\boldsymbol{K}}_{-1}$ such that for every finite $\overline{a} \in A$, $\delta(\overline{a}) \geq 0$. The universal theory of $\overline{\boldsymbol{K}}_0$ is denoted by $T_0$; $\boldsymbol{K}_0$ denotes the finitely generated structures in $\overline{\boldsymbol{K}}_0$.

**1.2.** Definition. For $N \in \overline{\boldsymbol{K}}_{-1}$ and $X$ and $Y$ finite subsets of $N$, we write $\delta(X/Y) = \delta(XY) - \delta(Y)$. For $U$ and $V$ subsets of $N$ with $U \subseteq V$, we say that $U$ is a *strong subset* of $V$, and write $U \leq V$, if for every finite subset $X$ of $V$, $\delta(X/X \cap U) \geq 0$.

The following basic property of $\delta$ can be phrased as asserting $\delta$ is lower semi-modular [1].

**1.3.** MONOTONICITY ASSUMPTION. If $\overline{b} \cap \overline{c} \subseteq \overline{a}$ then $\delta(\overline{b}/\overline{a}\,\overline{c}) \leq \delta(\overline{b}/\overline{a})$.

We write $\delta(\overline{a}/A)$ for $\min\{\delta(\overline{a}/B) : \overline{a} \cap A \subseteq B \subseteq_\omega A\}$ with $\delta(\overline{a}/A) = -\infty$ if the minimum does not exist. This definition is coherent for finite $A$ by the monotonicity assumption and extends the notion to infinite $A$. If $A \leq A\overline{a}$, then $\delta(\overline{a}/A) \geq 0$. The condition $\delta(\overline{a}) \geq 0$ for all $A \in \overline{\boldsymbol{K}}_0$ and all finite $\overline{a} \in A$ implies $\emptyset \leq M$ for all $M \in \overline{\boldsymbol{K}}_0$. Henceforth, we work in $\overline{\boldsymbol{K}}_0$.

An arbitrary intersection of strong subsets of a set $V$ is again strong in $V$. It follows that for any $X$ and $V$ with $X \subseteq V$, there is a unique smallest subset $U$ of $V$ with $X \subseteq U \leq V$. We may therefore make the following definition.

**1.4.** DEFINITION. If $X \subseteq B$, the *intrinsic closure* of $X$ in $B$, denoted by $\mathrm{icl}_B(X)$, is the unique smallest set $C$ with $X \subseteq C \leq B$. If $A \leq B$ and $X \subseteq B$, then the *relative intrinsic closure of $X$ over $A$* denotes the excess: $\mathrm{icl}_B(X/A) = \mathrm{icl}_B(AX) - A$.

If $\delta(X/Y) < 0$ and for any proper subset $X'$ of $X$, $\delta(X'/Y) \geq 0$, we say that $X$ is a *minimal intrinsic extension* of $Y$. (By minimality of $X$, necessarily, $X \cap Y = \emptyset$.) We show in this context that, as usual (e.g. [2], Corollary 3.20), $\mathrm{icl}_B(X)$ is contained in the algebraic closure in $B$ of $X$.

Note that, since on $\overline{\boldsymbol{K}}_0$, $\delta$ takes values in the natural numbers, for $A \leq B$ and $X \subseteq B$, $\mathrm{icl}_B(X/A)$ must be finite if $X - A$ is. In particular, the intrinsic closure $\mathrm{icl}_B(X/\emptyset) = \mathrm{icl}(X)$ is finite if $X$ is. It is easily seen that if $Y, X \subseteq B$ and $X$ is a minimal intrinsic extension of $Y$, then $X \subseteq \mathrm{icl}_B(Y)$.

**1.5.** DEFINITION. If $Y \leq YX$ and for every proper, nonempty subset $X'$ of $X$, $YX' \not\leq YX$, we say that $X$ is a *minimal strong extension* of $Y$. If, in addition, $\delta(X/Y) = 0$, we say that $X$ is a *primitive* extension of $Y$.

Note that if $X$ is a minimal strong extension of $Y$, then the minimality condition on $X$ entails $Y \cap X = \emptyset$. Moreover, since $\mathrm{icl}_{XY}(a/Y)$ is finite when $Y \leq YX$ and $a \in X$, minimal strong extensions (hence, primitive extensions) are finite. If $X \subset Y$ and $Y - X$ is a minimal intrinsic (strong) extension of $X$, we will abuse the language and say that $Y$ is a minimal intrinsic (strong) extension of $X$.

We restrict to the case where $\boldsymbol{K}_0$ has the $\leq$-amalgamation property. As usual ([2]), this produces a unique countable model, denoted by $M$, which is homogeneous with respect to strong extensions of finitely generated substructures and is a union of finitely generated strong substructures. We call $M$ the *generic* model of $T_0$ and let $T$ denote the theory of $M$. We denote by $\mathcal{M}$ a large saturated model of $T$.

**1.6.** DEFINITION. A quantifier free formula $\phi(\overline{x};\overline{y})$ is a *$\delta$-formula* and $\delta_\phi = k$ if the following hold:

(1) Every sequence $\overline{a}\overline{b}$ satisfying $\phi(\overline{x};\overline{y})$ consists of distinct elements.

(2) For some $N \in \overline{\boldsymbol{K}}_0$ and $\overline{a},\overline{b} \subseteq N$ with $\delta(\overline{a}/\overline{b}) = k$, $N \models \phi(\overline{a};\overline{b})$.

(3) For any $N \in \overline{\boldsymbol{K}}_0$ and $\overline{a},\overline{b} \subseteq N$, if $N \models \phi(\overline{a};\overline{b})$, then $\delta(\overline{a}/\overline{b}) \leq k$.

(4) For any $N \in \overline{\boldsymbol{K}}_0$ and $\overline{a},\overline{a}' \in N$ with $\overline{a},\overline{a}'$ disjoint from $B$, $\overline{b} \subseteq B \subseteq N$, if $N \models \phi(\overline{a};\overline{b}) \wedge \phi(\overline{a}';\overline{b})$ and $\delta(\overline{a}/B) = \delta(\overline{a}'/B) = k$ and $B\overline{a} \leq \langle B\overline{a}\rangle$, $B\overline{a}' \leq \langle B\overline{a}'\rangle$, then $\mathrm{Diag}(B\overline{a}) = \mathrm{Diag}(B\overline{a}')$. In particular, under these conditions, $\langle B\overline{a}\rangle \simeq \langle B\overline{a}'\rangle$ via $B\overline{a} \mapsto B\overline{a}'$.

We call $\phi(\overline{x};\overline{y})$ a *complete $\delta$-formula* if in addition it satisfies the following condition:

(5) For any disjoint subtuples $\overline{x}^1,\overline{x}^2$ from $\overline{x}$, there is a $\delta$-formula $\phi'(\overline{x}^1;\overline{x}^{2\wedge}\overline{y})$ such that $T_0 \models \phi(\overline{x};\overline{y}) \to \phi'(\overline{x}^1;\overline{x}^{2\wedge}\overline{y})$ and for any $N \in \boldsymbol{K}_0$ and $\overline{a},\overline{b} \subseteq N$, $N \models \phi(\overline{a};\overline{b})$ and $\delta_\phi = \delta(\overline{a}/\overline{b})$ implies $\delta_{\phi'} = \delta(\overline{a}^1/\overline{a}^2\overline{b})$.

Note that if $\phi(\overline{x};\overline{y})$ is a complete $\delta$-formula and for some $\overline{a},\overline{b},\overline{c},\overline{d}$, both $\phi(\overline{a};\overline{b})$ and $\phi(\overline{c};\overline{d})$ hold, with $\delta(\overline{a}/\overline{b}) = \delta_\phi$, then for any disjoint subtuples $\overline{x}^1,\overline{x}^2$ from $\overline{x}$, $\delta(\overline{a}^1/\overline{a}^2\overline{b}) \geq \delta(\overline{c}^1/\overline{c}^2\overline{d})$.

(Condition (4) was stated in the weaker form where $B = \overline{b}$ in [1] so we verify here the current stronger form for bicolored fields (Paragraph 2.1).

**1.7.** Lemma. *Condition* (4) *in Definition* 1.6 *holds in bicolored fields.*

*Proof.* Since $\delta(\overline{a}/B) = \delta(\overline{a}/\overline{b})$, $\overline{a}$ must be algebraically independent of $B$ over $\overline{b}$ and similarly for $\overline{a}'$. The $\delta$-formula $\phi$ determines the field structure of $\overline{a}$ over $\overline{b}$ and the isomorphic field structure of $\overline{a}'$ over $\overline{b}$. Thus the field structure of $\langle B\overline{a}\rangle$ and that of $\langle B\overline{a}'\rangle$ are isomorphic. Also, $B\overline{a} \leq \langle B\overline{a}\rangle$ and $B\overline{a}' \leq \langle B\overline{a}'\rangle$ so the only black points in $\langle B\overline{a}\rangle$ and in $\langle B\overline{a}'\rangle$ are those in $B\overline{a}'$ and $B\overline{a}$. Thus, the $L$-structures $\langle B\overline{a}\rangle$ and $\langle B\overline{a}'\rangle$ are isomorphic. ∎

**1.8.** Definition. If $\overline{a}\overline{b} \subseteq N$ with $\overline{a} \cap \overline{b} = \emptyset$ and $\phi(\overline{x};\overline{y}) \in \mathrm{Diag}(\overline{a};\overline{b})$ is a (complete) $\delta$-formula with $\delta_\phi = \delta(\overline{a}/\overline{b})$, we say that $\phi$ is a (*complete*) *$\delta$-formula* for $\overline{a}$ *over* $\overline{b}$. If also $\overline{b} \leq B \subseteq N$, $\overline{a}\cap B = \emptyset$ and $\delta(\overline{a}/\overline{b}) = \delta(\overline{a}/B)$, we say that $\phi$ is (complete) $\delta$-formula for $\overline{a}$ over $B$ *based on* $\overline{b}$ (or with base $\overline{b}$).

By saying simply that a formula $\phi(\overline{x};\overline{y})$ is a "complete $\delta$-formula for a minimal strong extension," we mean that there is a pair $\overline{a}\overline{b}$ in some model of $T_0$ such that $\overline{a}$ is a minimal strong extension of $\overline{b}$ and $\phi(\overline{x};\overline{y})$ is a complete $\delta$-formula for $\overline{a}$ over $\overline{b}$. It is easily shown that if such $\overline{a}$ and $\overline{b}$ exist, then, in fact, for *any* $\overline{a}'$ and $\overline{b}'$ satisfying $\phi(\overline{x};\overline{y})$ for which $\delta(\overline{a}'/\overline{b}') = \delta_\phi$, $\overline{a}'$ must be a minimal strong extension of $\overline{b}'$ as well. The same terminology and remarks apply when "minimal strong extension" is replaced by "primitive extension" or "minimal intrinsic extension."

The following definability constraints on $\delta$ were invisible in the *ab initio* case but had to be introduced for the fusion and bicolored field situations.

**1.9.** *Constraints on $\delta$.* We require that $\delta$ satisfy the following conditions:

(1) For any $N \in \boldsymbol{K}_0$ and $\overline{a}, \overline{b} \subseteq N$ with $\overline{a} \cap \overline{b} = \emptyset$, there is a $\delta$-formula for $\overline{a}$ over $\overline{b}$.

(2) If $\phi$ is quantifier free, $T_0 \vdash \forall \overline{y}\ \exists^{\leq k} \overline{x}\ \phi(\overline{x}; \overline{y})$ and $\models \phi(\overline{a}; \overline{b})$, then $\delta(\overline{a}/\overline{b}) \leq 0$. In particular, for any $\overline{a} \subseteq \langle \overline{b} \rangle$, $\delta(\overline{a}/\overline{b}) \leq 0$.

Note that by iterating Condition (1) of 1.9, we easily get (for $N \in \boldsymbol{K}_0$, $\overline{a}, \overline{b} \subseteq N$ with $\overline{a} \cap \overline{b} = \emptyset$) the *a priori* stronger (than Constraint (1)) condition of existence of a *complete* $\delta$-formula for $\overline{a}$ over $\overline{b}$. Moreover, if $\overline{a} \cap B = \emptyset$ and $B \leq B\overline{a}$, then there is some $\overline{b} \leq B$ with $\delta(\overline{a}/\overline{b}) = \delta(\overline{a}/B)$ and for any such $\overline{b}$, a (complete) $\delta$-formula for $\overline{a}$ over $\overline{b}$ constitutes a (complete) $\delta$-formula for $\overline{a}$ over $B$ based on $\overline{b}$.

Constraint (2) implies that if $\overline{b} \leq A \in \boldsymbol{K}_0$ then $\langle \overline{b} \rangle \leq A$. (Constraint (2) implies that for any $\overline{a} \in \langle \overline{b} \rangle$, $\delta(\overline{a}/\overline{b}) \leq 0$. But $\overline{b} \leq \langle \overline{b} \rangle$ implies $\delta(\overline{a}/\overline{b}) = 0$. Hence, $\langle \overline{b} \rangle \leq A$.) The converse need not hold.

Since any $A \in \boldsymbol{K}_0$ has the form $\langle \overline{a} \rangle = A$ for some finite sequence $\overline{a}$ with $\overline{a} \leq A$, the isomorphism type of $A$ is determined by the diagram of the finite sequence $\overline{a}$. If $\delta(\overline{a}) = k$, there is a $\delta$-formula $\phi(\overline{x})$ with $\delta_\phi = k$ such that $\phi(\overline{a})$ holds. For any $\overline{a}'$, if $\delta(\overline{a}') = k$, $\overline{a}' \leq \langle \overline{a}' \rangle$ and $\phi(\overline{a}')$ holds, then clause (4) of Definition 1.6 implies $\langle \overline{a}' \rangle \approx \langle \overline{a} \rangle$. Thus, since there are only countably many $\delta$-formulas, there are only countably many isomorphism types in $\boldsymbol{K}_0$. Moreover, since for any finite $\overline{a}$ there are only countably many possibilities for $\mathrm{icl}_{\langle \overline{a} \rangle}(\overline{a}) \subset \langle \overline{a} \rangle \in \boldsymbol{K}_0$, there are only countably many quantifier free types of finite sets realized in members of $\overline{\boldsymbol{K}}_0$.

**1.10.** Definition. We denote by $I(\overline{y})$ the collection of all formulas $\forall \overline{x}\ \neg\phi(\overline{x}; \overline{y})$, where for some $\overline{a}$ and $\overline{b}$, $\overline{a}$ is a minimal intrinsic extension of $\overline{b}$ and $\phi(\overline{x}; \overline{y})$ is a complete $\delta$-formula for $\overline{a}$ over $\overline{b}$.

Note that if $\overline{b} \subseteq B$, then $B \models I(\overline{b})$ if and only if $\overline{b} \leq B$. This yields immediately

**1.11.** Lemma. *If $A, B \in \boldsymbol{K}_0$ and $A$ is an elementary submodel of $B$, then $A \leq B$.*

Thus, $I(\overline{x}) \cup \mathrm{Diag}(\overline{c})$ is realized by $\overline{b}$ in a model $N$ just if $\langle \overline{c} \rangle \simeq \langle \overline{b} \rangle$ and $\overline{b} \leq N$.

In [1], we referred to the following notion as *separation of quantifiers* in analogy with the notion in [6]. However, in view of the theorem below, this notion is strictly stronger than the direct translation of Hrushovski's notion to this specialized context.

**1.12.** Definition. We say $(\boldsymbol{K}_0, \delta)$ *admits strong separation of quantifiers* if for any $\overline{b} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b} \rangle \in \boldsymbol{K}_0$ with $\overline{a}$ minimal strong over $\langle \overline{b} \rangle$, the following holds: For any formula $\tau(\overline{x}; \overline{y})$ in $I(\overline{x}, \overline{y}) \cup \mathrm{Diag}(\overline{a}, \overline{b})$ there are formulas $\sigma(\overline{y}) \in I(\overline{y})$ and $\alpha(\overline{y}) \in \mathrm{Diag}(\overline{b})$ such that whenever $\overline{b}' \subseteq C \in \boldsymbol{K}_0$ and $C \models (\sigma \wedge \alpha)(\overline{b}')$, there is $D \in \boldsymbol{K}_0$ with $C \leq D$ and $\overline{a}' \in D$ such that $D \models \tau(\overline{a}'; \overline{b}')$.

The following crucial result is proved in [1].

**1.13.** Theorem. *If $(\boldsymbol{K}_0, \delta)$ admits strong separation of quantifiers and has the $\leq$-amalgamation property then the generic $M$ is $\omega$-saturated.*

**2. Rank 2 fields.** In this section we consider only bicolored fields with the function $\delta$ specified in the next paragraph. The analysis works for an arbitrary strongly minimal theory with the definable multiplicity property, elimination of quantifiers and of imaginaries in a countable language $L_f$ with dimension function $d_f$ given by Morley rank, but we have written it as a description of an expansion of an algebraically closed field.

**2.1.** *Bicolored fields.* $T_f$ is a theory of algebraically closed field of a fixed characteristic. The function $d_f(X)$ denotes the transcendence degree over the prime field of a set $X$. Form $L$ by adjoining a unary predicate $P$ (for "black" points); all other points are "white". Let $T_f^{\forall}$ be the $L$-theory axiomatized by the universal sentences of $T_f$; the models of $T_f^{\forall}$ are the *bicolored fields.* For $T_f^{\forall}$ and $X \subset_\omega N$, $\delta_k(X) = k \cdot d_f(X) - |X \cap P(N)|$. In this section we restrict to the case $k = 2$. Let $T_0$ be $T_f^{\forall}$ along with the requirement that for each finite $X$ contained in a model of $T_f$, $\delta(X) \geq 0$.

To avoid simple repetition we refer the reader to Section 2 of [1] for the precise notion of a code. Informally, a code is a complete description of a primitive extension. The function $\mu$ is a map from codes to natural numbers. The theory $T_0^\mu$ guarantees that if the code $\boldsymbol{c}$ is exemplified by $X/Y$ in a model $M$ then there are at most $\mu(\boldsymbol{c})$ independent copies of $X$ over $Y$ in $M$. This intuition is expressed by the axiom

$$\neg \exists \overline{x}_1, \ldots, \overline{x}_{\mu(c)+1}\ E_{\boldsymbol{c}}(\overline{x}_1, \ldots, \overline{x}_{\mu(c)+1}),$$

where $E_{\boldsymbol{c}}$ is from Definition 2.8 of [1]. Formally, $T_0^\mu$ is the extension of $T_0$ by all these axioms. We write $\overline{\boldsymbol{K}}^\mu_{-1}$ for the class of models of $T_0^\mu$.

We prove that if $\mu$ is finite-to-one then the class $\boldsymbol{K}_0^\mu$ (the finitely generated models of $\overline{\boldsymbol{K}}^\mu_{-1}$) admits separation of quantifiers in the sense of Definition 1.12. As noted in the introduction, this proof is a reformulation and correction of the one occurring in [1]. By Theorem 1.13 this implies the theory of the generic is $\omega$-saturated and finiteness of Morley rank follows. We begin with some technical results that underly the proof of separation of quantifiers.

**2.2.** Definition. We say that a tuple $\overline{g}$ *splits over* a set $X$ if $\overline{g}$ lies neither entirely inside, nor entirely outside $X$.

Note that if $\varrho(\overline{x};\overline{y})$ is a complete $\delta$-formula for a minimal strong extension and $\overline{b} \subseteq X \leq M$, $\overline{a} \subseteq M$ with $\varrho(\overline{a};\overline{b})$, then $\overline{a}$ cannot split over $X$. The next lemma depends and expands upon this basic fact.

**2.3.** Lemma. *Let $\boldsymbol{c}$ be a primitive code and suppose that $M \leq N$ and $\overline{e}_1,\dots,\overline{e}_r \in N$ satisfy $E_{\boldsymbol{c}}(\overline{e}_1,\dots,\overline{e}_r)$ with each $\overline{e}_i$ lying in or splitting over $M$. Then either all of the $\overline{e}_i$ lie in $M$, or at most $m(\boldsymbol{c})-1$ of them lie in $M$ and $r \leq q(\boldsymbol{c})$.*

*Proof.* Write $m = m(\boldsymbol{c})$ and $\overline{b} = F_{\boldsymbol{c}}(\overline{e}_1,\dots,\overline{e}_m)$. If at least $m$ of the $\overline{e}_i$ lie in $M$, then $\overline{b} \subseteq M$. Since $M \leq N$, no $\overline{e}_i$ can split over $M$, and we are done. So suppose that fewer than $m$ lie in $M$.

If $r \leq m$, we are done. Otherwise, without loss of generality, suppose that the $\overline{e}_i$ lying in $M$ are among $\overline{e}_1,\dots,\overline{e}_m$. Let $\overline{g}_1,\dots,\overline{g}_{r-m}$ list the remainder of the $\overline{e}_i$. Let $G$ be the union of the ranges of $\overline{g}_1,\dots,\overline{g}_{r-m}$, $G_0 = G\cap M$ and $G_1 = G-M$. Let $E$ be the union of the ranges of $\overline{e}_1,\dots,\overline{e}_m$, and $E_0 = E\cap M$ and $E_1 = E - M$.

For each $i \leq r-m$, since $\overline{g}_i$ satisfies $\phi_{\boldsymbol{c}}(\overline{x};\overline{b})$ and splits over $M$, we have $\delta(\overline{g}_i/(\overline{g}_i\cap M)\overline{b}) \leq -1$, whence, since $d_f(\overline{b}/E) = 0$ and $E\cap \overline{g}_i = \emptyset$, $\delta(\overline{g}_i/E(G\cap M)) \leq \delta(\overline{g}_i/(\overline{g}_i\cap M)E) \leq -1$. Thus, since $M \leq N$, basic properties of the dimension $\delta$ yield

$$\begin{aligned}0&\leq\delta(GE/(GE)\cap M) = \delta(G/E(G\cap M)) + \delta(E/(GE)\cap M)\\ &\leq\sum_i \delta(\overline{g}_i/E(G\cap M)) + \delta(E/(GE)\cap M) \leq -(r-m) + m\cdot n(\boldsymbol{c}),\end{aligned}$$

giving $r \leq m\cdot n(c) + m$, as required. ∎

We now describe an extension $C[\overline{g}]$ of a structure $C$.

**2.4.** Notation. Let $\overline{b} \in C \in \overline{\boldsymbol{K}}^{\mu}_{-1}$. Suppose $\overline{g}$ is an $L_f$-generic realization of $\varrho^f(\overline{x},\overline{b})$ over $C$, where $\varrho^f$ is the $L_f$-part of a complete $\delta$-formula for a minimal strong extension of $\overline{b}$ and $D$ is the $L_f$ model of $T_f^{\forall}$ generated by $C\overline{g}$ expanded by making all elements of $D - C\overline{g}$ white. We write $C[\overline{g}]$ for $D$.

We list some elementary properties of $C[\overline{g}]$.

**2.5.** Lemma. *Let $C$, $\overline{b}$ and $\overline{g}$ be as above.*

(1) *$C \leq C[\overline{g}]$.*

(2) *If $\overline{e} \subseteq C[\overline{g}]$, $\overline{e} \not\subseteq C$ with $\overline{e}$ and $\overline{g}$ primitive over $C$, then $\overline{e} = \overline{g}$, up to reordering.*

(3) *If $\overline{b} \subseteq X \subseteq C$ and $\overline{e} \subseteq C$, then $\delta(\overline{e}/X) = \delta(\overline{e}/X\overline{g})$ (since $\delta(\overline{g}/X) = \delta(\overline{g}/X\overline{e})$). In particular, a complete $\delta$-formula for $\overline{e}$ over $\langle X\rangle$ serves as a complete $\delta$-formula for $\overline{e}$ over $\langle X\overline{g}\rangle$.*

If $\overline{g}$ is, in fact, primitive over $\overline{b}$, then by Lemma 2.2 and Definition 2.7 of [1] there are $\overline{c} \subseteq \langle\overline{b}\rangle$ and a primitive code $\boldsymbol{c}$ with $\phi_{\boldsymbol{c}}(\overline{g};\overline{c})$. As the example in

Section 4 of that paper shows, even if $C[\overline{g}]$ respects $\mu(\boldsymbol{c})$, it may yet violate $\mu(\boldsymbol{d})$ for some other $\boldsymbol{d}$. Key to the proof of separation of quantifiers is that there be only finitely many such $\boldsymbol{d}$. The following lemma, together with $\mu$ being finite-to-one, will guarantee this.

**2.6.** Lemma. *Suppose $\boldsymbol{d}$ is a primitive code, $\overline{e}_1, \ldots, \overline{e}_H \subseteq C[\overline{g}]$ with $E_{\boldsymbol{d}}(\overline{e}_1, \ldots, \overline{e}_H)$, and $H > \mu(\boldsymbol{d}) \geq q(\boldsymbol{d}) + m(\boldsymbol{d})$. Then either $\overline{e}_i = \overline{g}$ for some $i$, up to a possible reordering of variables, or $H \leq 3l(\overline{g})$.*

*Proof.* First, suppose that at least $m(\boldsymbol{d})$ of the $\overline{e}_i$ lie in $C$. Then $F_{\boldsymbol{d}}(\overline{e}_1, \ldots, \overline{e}_m)$ lies in $C$. Thus, as $C \leq C[\overline{g}]$, none of the $\overline{e}_i$ splits over $C$. Since $H > \mu(\boldsymbol{d})$, $C$ does not contain all of the $\overline{e}_i$, and every element of $C[\overline{g}] - C\overline{g}$ is white, so at least one, call it $\overline{e}_l$, lies inside $\overline{g}$. But $\overline{e}_l$ is then primitive over $C$ and is contained in the primitive $\overline{g}$ over $C$, so that (setwise) $\overline{g} = \overline{e}_i$.

We can thus assume that fewer than $m(\boldsymbol{d})$ of the $\overline{e}_i$ lie in $C$. By Lemma 2.3, at most $q(\boldsymbol{d})$ of them lie in or split over $C$. By assumption, $H - q(\boldsymbol{d}) \geq m(\boldsymbol{d})$, so at least $m(\boldsymbol{d})$ of the (pairwise disjoint) $\overline{e}_i$ lie inside $\overline{g}$, making $m(\boldsymbol{d}) \leq m(\boldsymbol{d})n(\boldsymbol{d}) \leq l(\overline{g})$, whence $q(\boldsymbol{d}) \leq 2l(\overline{g})$. Thus, since $\overline{e}_1, \ldots, \overline{e}_H$ either lie in or split over $C$ (at most $q(\boldsymbol{d})$ of them), or lie in $\overline{g}$ (at most $l(\overline{g})$ of them), $H \leq q(bd) + l(\overline{g}) \leq 3l(\overline{g})$, as desired. ∎

**2.7.** Theorem. *If $\mu$ is finite-to-one then $K^{\mu}$ admits separation of quantifiers.*

*Proof.* We verify Definition 1.12. Fix $\overline{b} \leq \overline{a}\overline{b} \leq \langle \overline{a}\overline{b} \rangle \in \overline{\boldsymbol{K}}^{\mu}_{-1}$ with $\overline{a}$ minimal strong over $\langle \overline{b} \rangle$ and $\tau(\overline{x}; \overline{y}) \in I^*(\overline{a}\overline{b})$. To simplify notation we rename $I^*(\overline{b})$ as the set of all finite conjunctions of formulas in the original $I^*(\overline{b})$. (We suppress the splitting of $\beta$ into an $\alpha$ and $\sigma$ which we used in definition.) We must show that there is $\beta(\overline{y})$ in the new $I^*(\overline{b})$ such that whenever $\overline{b}' \subseteq C \in \overline{\boldsymbol{K}}^{\mu}_{-1}$, and $C \models \beta(\overline{b}')$, there is $D \in \overline{\boldsymbol{K}}^{\mu}_{-1}$ with $C \leq D$ and $D \models \exists \overline{x}\ \tau(\overline{x}; \overline{b}')$.

Let $\varrho(\overline{x}; \overline{y})$ be a complete $\delta$-formula for $\overline{a}$ over $\overline{b}$. Choose $\overline{g}$ so that $\varrho(\overline{x}; \overline{y})$ is a complete $\delta$-formula for $\overline{g}$ over $\overline{b}'$ with $\delta(\overline{g}/C) = \delta(\overline{g}/\overline{b}')$. We will find $\beta$ as above so that if $C \models \beta(\overline{b}')$, then either $C[\overline{g}] \models \tau(\overline{g}; \overline{b}')$ and $C[\overline{g}] \in \overline{\boldsymbol{K}}^{\mu}_{-1}$ (so that we may take $D = C[\overline{g}]$ for some such $\overline{g}$), or $C$ itself models $\exists \overline{x}\ \tau(\overline{x}; \overline{b}')$ (so that we may take $D = C$).

We may assume, without loss of generality, that $C$ is algebraically closed, since if we set $C'$ to be the field algebraic closure of $C$ with all new points colored white, then $C \models \psi(\overline{b}')$ if and only if $C' \models \psi(\overline{b}')$ for all $\psi \in I^*(\overline{b})$, $C \models \exists \overline{x}\ \tau(\overline{x}; \overline{b}')$ if and only if $C' \models \exists \overline{x}\ \tau(\overline{x}; \overline{b}')$, and $C[\overline{g}] \simeq C'[\overline{g}]$.

We must find a $\beta(\overline{y})$ such that if $C \models \beta(\overline{b}')$, then either $C[\overline{g}] \in \overline{\boldsymbol{K}}^{\mu}_{-1}$ (and we may take $D = C[\overline{g}]$) or $C \models \exists \overline{x}\ \tau(\overline{x}; \overline{b}')$.

The minimal strong extension $\overline{a}$ of $\langle\overline{b}\rangle$ is of one of three types: It may be a single white point; it may be a single black point $d_f$-independent of $\overline{b}$; and it may be a black primitive.

We consider the first two cases together. Set $a = \overline{a}$. Let $\boldsymbol{d}$ be any primitive code and suppose that $C[g] \models E_{\boldsymbol{d}}(\overline{e}_1, \dots, \overline{e}_r)$ for some $r > m(\boldsymbol{d})$. As $C \in \overline{\boldsymbol{K}}^{\mu}_{-1}$, to show $C[\overline{g}] \in \overline{\boldsymbol{K}}^{\mu}_{-1}$, it is enough to show that $\overline{e}_1, \dots, \overline{e}_r$ must all lie in $C$. Since $C[g] - C$ contains at most one black point and $l(\overline{e}_i) \geq 2$ for each $i$, at most one of the $\overline{e}_i$ does not lie entirely inside $C$ and none can lie completely outside $C$. In particular, at least $m(\boldsymbol{d})$ of the $\overline{e}_i$ do lie entirely inside $C$, so that $F_{\boldsymbol{d}}(\overline{e}_1, \dots, \overline{e}_m) \subseteq C$. Thus, since $C \leq C[g]$, none of the $\overline{e}_i$ can split over $C$, giving $\overline{e}_1, \dots, \overline{e}_r \subseteq C$, as desired.

Now we must guarantee $C[g] \models \tau(g; \overline{b}')$. If $g$ is white and field algebraic over $C$ then $g \in C$ and there is nothing to show. Otherwise, $g$ must be independent of $C$. Suppose that $\tau(x, \overline{y})$ is a formula from $I^*(a, \overline{b})$ that we must satisfy. (If we guarantee each such formula is satisfied by $g, \overline{b}'$, then the finite conjunction is as well.) If $\tau$ is in $\mathrm{Diag}(\overline{a}, \overline{b})$, this is easy. So we concentrate on the case that $\tau$ is $(\forall \overline{z})\neg\tau''(x, \overline{y}, \overline{z})$ where $\tau''$ has the form

$$\tau^f(x, \overline{y}, \overline{z}) \wedge \bigwedge_{i<\lg(\overline{z})} P(z_i) \wedge \overline{z} \cap \overline{x}\overline{y} = \emptyset$$

and is satisfied by $\overline{a}\overline{b}$ in (the white algebraic closure of) $\langle\overline{a}\overline{b}\rangle$. Moreover, for any $a', \overline{b}', \overline{c}'$, if $\tau^f(a', \overline{b}', \overline{c}')$, $d_f(\overline{c}'/a'\overline{b}') \leq h = \lg(\overline{z})/2 - 1$; $\delta(\overline{c}/a'\overline{b}') < 0$. Let $\tau'(\overline{z}, \overline{y})$ be the $L_f$-formula which holds of $\overline{c}'\overline{b}'$ (in any $M$ containing $\overline{c}'\overline{b}'$) if for generic $g$, $\tau^f(g, \overline{b}', \overline{c}')$ holds. Then

$$T^{\mu}_0 \cup I^*(\overline{b}) \vdash \neg(\exists \overline{z})\tau'(\overline{z}, \overline{y}) \wedge \bigwedge_{i<\lg(\overline{z})} P(z_i).$$

To see this note that if there is black $\overline{c}'$ and $g', \overline{b}'$ with $g'$ independent of $\overline{c}'$ over $\overline{b}'$ and $\tau^f(g', \overline{b}', \overline{c}')$, then in fact $\delta(\overline{c}'/\overline{b}) < 0$. Choose $\beta(\overline{y})$ from $I^*(\overline{y})$ so that

$$T^{\mu}_0 \vdash \beta(\overline{y}) \rightarrow \neg(\exists \overline{z})\tau'(\overline{z}, \overline{y}) \wedge \bigwedge_{i<\lg(\overline{z})} P(z_i).$$

Suppose $C \models \beta(\overline{b}')$ and some $\overline{c}' \in C[g]$ satisfies $\tau''(g, \overline{b}', \overline{c}')$ where $\tau''$ was described above. Then the members of $\overline{c}'$ are black (and so in $C$). So $C \models \tau'(\overline{b}', \overline{c}') \wedge \bigwedge_{i<\lg(\overline{z})} P(c'_i)$, which contradicts the fact that $C \models \neg(\exists \overline{z})\tau'(\overline{z}, \overline{y}) \wedge \bigwedge_{i<\lg(\overline{z})} P(z_i)$.

Now, we consider the less trivial case in which $\overline{a}$ is a black primitive. By part 4 of Definition 2.1 of [1] (and since $\delta(\overline{a}/\overline{b}) = 0$),

$$T^{\mu}_0 \cup I^*(\overline{b}) \cup \{\varrho(\overline{x}; \overline{y})\} \vdash I^*(\overline{a}; \overline{b}).$$

Thus, for any consequence $\omega(\overline{x};\overline{y})$ of $T_0^\mu \cup I^*(\overline{a};\overline{b})$, there is $\beta(\overline{y}) \in \bigwedge I^*(\overline{b})$ such that

$$T_0^\mu \vdash \beta(\overline{y}) \wedge \varrho(\overline{x};\overline{y}) \to \omega(\overline{x};\overline{y}).$$

We use this fact to construct, by stages, a formula $\beta(\overline{y}) \in I^*(\overline{b})$ such that if $C \models \beta(\overline{b}')$ then either $C[\overline{g}] \in \overline{\boldsymbol{K}}^\mu_{-1}$ or $C \models \exists \overline{x}\ \tau(\overline{x};\overline{b}')$.

Find a primitive code $\boldsymbol{d}_0$ and $\overline{c} \subseteq \mathrm{acl}_f(\overline{b})$ such that $\overline{a}$ is a generic solution of $\phi^f_{\boldsymbol{d}_0}(\overline{x};\overline{c})$. Let $\gamma(\overline{u};\overline{y})$ isolate the $L_f$-type of $\overline{c}$ over $\overline{b}$. Let $\chi(\overline{u},\overline{y})$ be the formulas such that $\chi(\overline{c}\overline{d})$ holds if and only if the Morley rank of $\varrho_f(\overline{x};\overline{d}) \wedge \phi^f_{\boldsymbol{d}_0}(\overline{x};\overline{c})$ equals the Morley rank of $\varrho_f(\overline{x};\overline{d})$. Then $T_0^\mu \cup \mathrm{Diag}(\overline{b})$ proves

$$\exists \overline{u}\ \gamma(\overline{u};\overline{y}) \wedge \chi(\overline{u},\overline{y}),$$

and $T_0^\mu \cup I^*(\overline{b})$ proves

$$\forall \overline{u}\ [\phi^f_{\boldsymbol{d}_0}(\overline{x};\overline{u}) \wedge \gamma(\overline{u};\overline{y}) \to \tau(\overline{x};\overline{y})].$$

Next, by our choice of $\mu$ finite-to-one, we may list all primitive codes $\boldsymbol{d}_1,\ldots,\boldsymbol{d}_l$ with $\mu(\boldsymbol{d}_i) \le 3l(\overline{g})$. Then $T_0^\mu \cup I^*(\overline{a};\overline{b})$ proves

$$\bigwedge \neg E_{\boldsymbol{d}_i}(\overline{w}_1,\ldots,\overline{w}_{\mu(\boldsymbol{d}_i)+1}),$$

where the conjunction ranges over $i = 0,\ldots,l$ and all choices of $\overline{w}_1,\ldots \ldots,\overline{w}_{\mu(\boldsymbol{d}_i)+1}$ from among $\overline{x}\,\overline{y}$, and

$$\bigwedge \forall \overline{v}_1,\ldots,\overline{v}_{m(\boldsymbol{d})},\overline{z}\ (E_{\boldsymbol{d}_i}(\overline{z},\overline{v}_1,\ldots,\overline{v}_m) \wedge\ \overline{v}_1,\ldots,\overline{v}_m \subseteq \overline{x}\,\overline{y} \\ \to \overline{z} \cap \overline{x}\,\overline{y} = \emptyset \vee \overline{z} \subseteq \overline{x}\,\overline{y}),$$

where the conjunction ranges over $i = 0,\ldots,l$.

We will use the following notation. Fix a code $\boldsymbol{d}$. For each sequence $\sigma$ of $m(\boldsymbol{d})$, $n(\boldsymbol{d})$ tuples from $\{1,\ldots,p\}$ and sequence $\overline{g}$ of constants or $\overline{x}$ of variables of length $p$, $\overline{g}^\sigma$ or $\overline{x}^\sigma$ denotes the subsequence of $\overline{g},\overline{x}$ respectively indexed by $\sigma$. This will be used with $p = \lg(\overline{g})$.

For each such $\boldsymbol{d}$ and $\sigma$, if $\neg E_{\boldsymbol{d}}(\overline{a}^\sigma)$ holds in the white algebraic closure of $\langle \overline{a}\overline{b}\rangle$ then $T_0^\mu \cup I^*(\overline{a};\overline{b})$ proves

$$\neg E_{\boldsymbol{d}}(\overline{x}^\sigma).$$

By the remark above, we may find $\beta_1(\overline{y}) \in I^*(\overline{y})$ such that $T_0^\mu \cup \{\beta_1(\overline{y}) \wedge \varrho(\overline{x};\overline{y})\}$ proves each of the last five displayed formulas. Immediately, if $C \models \beta_1(\overline{b}')$, then also $C[\overline{g}] \models \beta_1(\overline{b}')$ (since $C \le C[\overline{g}]$). Thus:

(1) $\overline{g}$ is a generic solution of $\phi^f_{\boldsymbol{d}_0}(\overline{x};\overline{c}')$ for some $\overline{c}' \subseteq \mathrm{acl}_f(\overline{b}')$;

(2) $C \models \forall \overline{u}\,\overline{z}\ [\gamma(\overline{u};\overline{b}') \wedge \phi^f_{\boldsymbol{d}_0}(\overline{z};\overline{u}) \to \tau(\overline{z};\overline{y})]$;

(3) $E_{\boldsymbol{d}_i}(\overline{y}_1,\ldots \overline{y}_{\mu(\boldsymbol{d}_i)+1})$ is not realized in $\overline{g}\overline{b}'$ for $i = 0,\ldots,l$;

(4) if $\overline{e}'_1,\ldots,\overline{e}'_{m(\boldsymbol{d}_i)} \subseteq \overline{g}\overline{b}'$ and $\overline{e}' \subseteq C[\overline{g}]$ with $E_{\boldsymbol{d}_i}(\overline{e}',\overline{e}'_1,\ldots,\overline{e}'_{m(\bar{d}_i)})$, then $\overline{e}'$ does not split over $\overline{g}\overline{b}'$, for $i = 0,\ldots,l$;

(5) if $\neg E_{\boldsymbol{d}_i}(\overline{a}^\sigma)$ then $\neg E_{\boldsymbol{d}_i}(\overline{g}^\sigma)$;
(6) $C[\overline{g}] \models \tau(\overline{g}; \overline{b}')$.

For each code $\boldsymbol{d}$ with $m(\boldsymbol{d}) \cdot n(\boldsymbol{d}) \leq l(\overline{x})$ and sequence $\sigma$ as described above such that $E^f_{\boldsymbol{d}}(\overline{a}^\sigma)$ holds in $\mathrm{acl}_f(\overline{a}\overline{b})$, we will define a formula $\beta^2_{\boldsymbol{d},\sigma}(\overline{y})$; $\beta_2(\overline{y})$ is the conjunction of all these formulas. Write $m = m(\boldsymbol{d})$.

Case 1: $F_{\boldsymbol{d}}(\overline{a}^\sigma) \subseteq \mathrm{acl}_f(\overline{b})$. We choose $\beta^2_{\boldsymbol{d},\sigma}(\overline{y}) \in I^*(\overline{y})$ such that if $C \models \beta_1(\overline{b}') \wedge \beta^2_{\boldsymbol{d},\sigma}(\overline{b}')$ and $C[\overline{g}] \models E^f_{\boldsymbol{d}}(\overline{g}^\sigma)$, then $F_{\boldsymbol{d}}(\overline{g}^\sigma) \subseteq \mathrm{acl}_f(\overline{b}')$.

Since $F_{\boldsymbol{d}}(\overline{a}^\sigma) \subseteq \mathrm{acl}_f(\overline{b})$, there are an $L_f$-formula $\lambda(\overline{u}; \overline{y})$ and an integer $l$ such that $T^f \vdash \forall \overline{y}\ \exists^{\leq l} \overline{u}\ \lambda(\overline{u}; \overline{y})$ and $T^\mu_0 \cup I^*(\overline{a}\overline{b}) \vdash \lambda(F_{\boldsymbol{d}}(\overline{x}^\sigma); \overline{y})$. Thus, there is $\beta^2_{\boldsymbol{d},\sigma}(\overline{y}) \in I^*(\overline{b})$ such that $T^\mu_0 \vdash \beta^2_{\boldsymbol{d},\sigma}(\overline{y}) \wedge \varrho(\overline{x}; \overline{y}) \to \lambda(F_{\boldsymbol{d}}(\overline{x}^\sigma); \overline{y})$. This choice of $\beta^2_{\boldsymbol{d},\sigma}(\overline{y})$ immediately satisfies our requirements.

Case 2: $F_{\boldsymbol{d}}(\overline{a}^\sigma) \not\subseteq \mathrm{acl}_f(\overline{b})$. In this case we choose the formula $\beta^2_{\boldsymbol{d},\sigma}(\overline{y}) \in I^*(\overline{y})$ so that if $C \models \beta_1(\overline{b}') \wedge \beta^2_{\boldsymbol{d},\sigma}(\overline{b}')$ then all solutions of $E_{\boldsymbol{d}}(\overline{z}, \overline{g}^\sigma)$ in $C$ lie in $\overline{b}'$.

Let $\overline{e}$ satisfy the unique $L$-type $q \in S(\mathrm{acl}_f(\overline{a}\overline{b}))$ of a realization of $E^f_{\boldsymbol{d}}(\overline{z}, \overline{a}^\sigma)$ outside $\mathrm{acl}_f(\overline{a}\overline{b})$ in a strong extension of $\mathrm{acl}_f(\overline{a}\overline{b})$. Then $\overline{e}$ depends on $\overline{a}$ over $\overline{b}$, because if not, we could successively realize $q$ by $\overline{f}_1, \ldots, \overline{f}_{m(\boldsymbol{d})}$, each independent of its predecessors over $\overline{a}\overline{b}$, whence $\overline{f}_1, \ldots, \overline{f}_{m(\boldsymbol{d})}$ would be independent of $\overline{a}$ over $\overline{b}$, giving $F_{\boldsymbol{d}}(\overline{a}^\sigma) = F_{\boldsymbol{d}}(\overline{f}_1, \ldots, \overline{f}_m) \subseteq \mathrm{acl}_f(\overline{f}_1, \ldots, \overline{f}_m)$, and $\mathrm{acl}_f(\overline{f}_1, \ldots, \overline{f}_m) \cap \mathrm{acl}_f(\overline{a}) \subseteq \mathrm{acl}_f(\overline{b})$, whence $F_{\boldsymbol{d}}(\overline{a}^\sigma) \subseteq \mathrm{acl}_f(\overline{b})$.

Since $\overline{e}$ is not independent of $\overline{a}$ over $\overline{b}$, there are $l$, strictly less than the rank of $\varrho(\overline{x}; \overline{b})$, and an $L_f$-formula $\psi(\overline{x}; \overline{y}\overline{z})$, satisfied by $\overline{a}\overline{b}\overline{e}$, such that for any (by the definability of rank) $\overline{c}\overline{d}$ the rank of $\psi(\overline{x}; \overline{c}\overline{d})$ is $l$. Then

$$T^\mu_0 \cup I^*(\overline{a}\overline{b}) \vdash \forall \overline{z} \subseteq P\ [E^f_{\boldsymbol{d}}(\overline{z}; \overline{x}^\sigma) \wedge \overline{z} \not\subseteq \overline{x}\,\overline{y} \to \psi(\overline{x}; \overline{y}\,\overline{z})].$$

To see this, note that if $E^f_{\boldsymbol{d}}(\overline{e}; \overline{a}^\sigma)$ holds then $\delta(\overline{e}/\overline{a}\overline{b}) < 0$. So $T^\mu_0 \cup I^*(\overline{a}\overline{b})$ implies that $\overline{e}$ is not both black and contained in $\mathrm{acl}_f(\overline{a}\overline{b})$. The uniqueness of $q$ allows us to fix $\psi$ and $l$. Find $\beta^2_{\boldsymbol{d},\sigma}(\overline{y}) \in I^*(\overline{b})$ such that $T^\mu_0 \cup \{\beta^2_{\boldsymbol{d},\sigma}(\overline{y}) \wedge \varrho(\overline{x}; \overline{y})\}$ proves this displayed consequence of $T^\mu_0 \cup I^*(\overline{a}\overline{b})$.

Now suppose that $C \models \beta_1(\overline{b}') \wedge \beta^2_{\boldsymbol{d},\sigma}(\overline{b}')$, and $\overline{e}' \subseteq C$ with $C[\overline{g}] \models E^f_{\boldsymbol{d}}(\overline{e}', \overline{g}^\sigma)$. Since $\psi(\overline{g}; \overline{b}'\overline{e})$ must fail (as $\overline{g}$ is independent of $C$ over $\overline{b}'$), it follows that $\overline{e}' \subseteq \overline{b}'$.

Finally, set $\beta(\overline{y}) = \beta_1(\overline{y}) \wedge \beta_2(\overline{y})$. (Recall that $\beta_2(\overline{y})$ is a conjunction over the $\beta^2_{\boldsymbol{d},\sigma}$.) Note that $T^\mu_0 \cup I^*(\overline{b}) \vdash \beta(\overline{y})$.

Now suppose that $C \models \beta(\overline{b}')$. As $C[\overline{g}]$ satisfies $\exists \overline{x}\ \tau(\overline{x}; \overline{b}')$, if $C[\overline{g}] \in \overline{\boldsymbol{K}}^\mu_{-1}$, we finish. So, suppose that $C[\overline{g}] \notin \overline{\boldsymbol{K}}^\mu_{-1}$, so that there are a primitive code $\boldsymbol{d}$ and $\overline{e}'_1, \ldots, \overline{e}'_{\mu(\boldsymbol{d})+1} \subseteq C[\overline{g}]$ with $E_{\boldsymbol{d}}(\overline{e}'_1, \ldots, \overline{e}'_{\mu(\boldsymbol{d})+1})$. We argue that $C \models$

$\exists \overline{x}\ \tau(\overline{x}; \overline{b}')$. We first show that $F_{\boldsymbol{d}}(\overline{e}'_1, \ldots, \overline{e}'_{m(\boldsymbol{d})}) \subseteq C$. By Lemma 2.6, either $\overline{e}'_{i_0} = \overline{g}$, up to reordering, for some $i_0$, or $\mu(\boldsymbol{d}) + 1 \leq 3l(\overline{g})$.

In the first case, all of the other $\overline{e}'_i$ must lie in $C$, so $F_{\boldsymbol{d}}(\overline{e}'_i, \ldots, \overline{e}'_{m(\boldsymbol{d})}) \subseteq C$ is immediate.

In the second case, we have $\boldsymbol{d} = \boldsymbol{d}_i$ for some $i = \{1, \ldots, l\}$. If $F_{\boldsymbol{d}}(\overline{e}'_1, \ldots$ $\ldots, \overline{e}'_{m(\boldsymbol{d})}) \not\subseteq C$, then fewer than $m(\boldsymbol{d}_i)$ lie in $C$ and by Lemma 2.3, either none splits over $C$, or at most $q(\boldsymbol{d})$ lie in or split over $C$. In the first subcase, all lie in $C$ or lie in $\overline{g}$, so that, since fewer than $m(\boldsymbol{d})$ lie in $C$ and $\mu(\boldsymbol{d}) + 1 - (m(\boldsymbol{d}) - 1) = q(\boldsymbol{d}) + m(\boldsymbol{d}) + 2 > m(\boldsymbol{d})$, at least $m(\boldsymbol{d})$ lie in $\overline{g}$. In the second subcase, since $\mu(\boldsymbol{d}) + 1 - q(\boldsymbol{d}) = m(\boldsymbol{d}) + 1 > m(\boldsymbol{d})$, again, at least $m(\boldsymbol{d})$, say $\hbar = \langle \overline{e}_{j_1} \ldots \overline{e}_{j_m} \rangle$ lie in $\overline{g}$. Choose $\sigma$ so that $\overline{g}^\sigma = \hbar$. We have $C[\overline{g}] \models E_{\boldsymbol{d}}(\overline{g}^\sigma)$. Thus, by condition (5), $E_{\boldsymbol{d}}(\overline{a}^\sigma)$ holds in $\langle \overline{a}\overline{b} \rangle$. Since $C \models \beta_1(\overline{b}')$, the $\overline{e}'_i$ do not all lie in $\overline{b}'\overline{g}$, and no $\overline{e}'_i$ splits over $\overline{b}'\overline{g}$, so at least one, call it $\overline{e}'$, lies in $C - \{\overline{b}'\}$. Our choice of $\beta^2_{\boldsymbol{d},\sigma}$ depended on whether $F_{\boldsymbol{d}}(\overline{a}^\sigma)$ were in $\mathrm{acl}_f(\overline{b})$. If it was not, since $C \models \beta^2_{\boldsymbol{d},\sigma}(\overline{b}')$ we could not have such an $\overline{e}'$. So $F_{\boldsymbol{d}}(\overline{a}^\sigma) \subseteq \mathrm{acl}_f(\overline{b})$. Then $C \models \beta^2_{\boldsymbol{d},\sigma}(\overline{b}')$ immediately gives $F_{\boldsymbol{d}}(\overline{g}^\sigma) \subseteq \mathrm{acl}_f(\overline{b}')$.

So in either case, we have $F_{\boldsymbol{d}}(\overline{e}'_1, \ldots, \overline{e}'_{m(\boldsymbol{d})}) \subseteq C$ and some $\overline{e}'_i \in \overline{g}$. Since $C \leq C[\overline{g}]$, this $\overline{e}'_i$ is a primitive over $C$, contained in the primitive $\overline{g}$ over $C$, so $\overline{e}'_i = \overline{g}$, up to reordering. Now $C \models \beta_1(\overline{b}')$ gives us that for some $\overline{c}'$ satisfying $\gamma(\overline{u}; \overline{b}')$, $\overline{g}$ is a generic solution of $\phi_{\boldsymbol{d}_0}(\overline{x}; \overline{c}')$. This forces $\boldsymbol{d} = \boldsymbol{d}_0$, up to reordering, and $F_{\boldsymbol{d}}(\overline{e}'_1, \ldots, \overline{e}'_m) \subseteq \mathrm{acl}_f(\overline{c}')$. Then again from $C \models \beta_1(\overline{b}')$ (condition (2)), we have $C \models \tau(\overline{e_i}'; \overline{b}')$. ∎

It now follows that this theory has Morley rank 2; see Section 3 of [1].

**3. Morley rank and $U$-rank.** In this section we deal with the theory of a generic $M$ built as described in Section 1. We assume that $\delta$ satisfies the constraints described in Definition 1.9, that $\delta$-formulas exist, and that strong separation of quantifiers holds so that the generic $M$ is saturated, by Theorem 1.13. We first note that the theory $T$ of a (saturated) generic is $\omega$-stable. We then compute an upper bound on the Morley rank of types in $T$. We then exhibit some rather *ad hoc* conditions to give a lower bound on $U$-rank and then note that under these conditions the maximal $U$-rank and Morley rank of a 1-type are equal. The main aim of this section is to show in detail that the theory obtained by expanding a strongly minimal set (which has elimination of imaginaries and the definable multiplicity property) in the style of [7, 1] with no $\mu$ and with dimension function $\delta(\overline{a}) = k \cdot d_f(\overline{a}) - |\overline{a} \cap P|$ has Morley rank $\omega \cdot k$ ($P$ is the set of "black" points).

Note that if $\delta(\overline{a}/B) = \delta(\overline{a}/\overline{n})$ where $\overline{n} \subset B$, then for any $\overline{m}$ with $\overline{n} \subset \overline{m} \leq B$, $\overline{a}\overline{m} \leq \overline{a}B$.

**3.1.** Lemma. *Let $B \le N \models \mathrm{Th}(M)$. If $\overline{a}$ and $\overline{a}'$ are disjoint from $B$, $B\overline{a} \le N$, $B\overline{a}' \le N$, $\overline{a}'$ satisfies a complete $\delta$-formula, $\phi(\overline{x}, \overline{n})$, for $\overline{a}$ over $B$ with base $\overline{n}$ and $\delta(\overline{a}/B) = \delta(\overline{a}'/B)$ then $\mathrm{tp}(\overline{a}/B) = \mathrm{tp}(\overline{a}'/B)$.*

*Proof.* Without loss of generality $N$ is countable. It suffices to prove that for any finitely generated strong substructure $A$ of $B$ containing $\overline{n}$, $\mathrm{tp}(\overline{a}/A) = \mathrm{tp}(\overline{a}'/A)$. We can enlarge $A$ so that $A\overline{a} \le N$ and $A\overline{a}' \le N$. For any such $A$, by Definition 1.6(4), $\mathrm{tp}_{qf}(\overline{a}/A) = \mathrm{tp}_{qf}(\overline{a}'/A)$. Since the generic $M$ is $\omega$-saturated, there is an elementary embedding of $N$ into $M$. Thus, we may assume $A\overline{a} \le M$ and $A\overline{a}' \le M$. By genericity there is an automorphism of $M$ fixing $A$ and mapping $\overline{a}$ to $\overline{a}'$. So $\mathrm{tp}(\overline{a}/A) = \mathrm{tp}(\overline{a}'/A)$ (in the sense of both $M$ and $N$ since $N \prec M$), as required. ∎

**3.2.** Corollary. *If $T$ is the theory of the generic, then $T$ is $\omega$-stable.*

*Proof.* It suffices to show that there are only countably many types, $\mathrm{tp}(\overline{a}/X)$, over each countable set $X$ with $X \le \mathcal{M}$. Moreover, since for any $\overline{a}$, $\mathrm{icl}_{\mathcal{M}}(X\overline{a}) - X$ is finite, we may assume that $X\overline{a} \le \mathcal{M}$. Now choose $\overline{n} \le X$ and a $\delta$-formula $\phi(\overline{x}, \overline{n})$ so that $\phi$ is a complete $\delta$-formula for $\overline{a}$ based on $\overline{n}$. By Lemma 3.1, this determines the type over $X$. Since there are only countably many choices for $\phi$ and $\overline{n}$, we have the result. ∎

**3.3.** Definition. Let $A, B \subseteq N \in \boldsymbol{K}_0$ with $B - A$ finite and suppose $A \subseteq B$, $A \le N$ and $\delta(B/A) = 0$. Then there exist $A = A_0, A_1, \ldots, A_n = B$ such that for $i = 0, \ldots, n-1$, $A_{i+1} - A_i$ is primitive over $A_i$. We call $A_1, \ldots, A_n$ a *primitive decomposition* of $B$ over $A$. We denote $A_i - A_{i-1}$ by $A_i'$. We say the step $A_{i+1}/A_i$ is *algebraic* if the type of $A_{i+1}'$ over $A_i$ is algebraic. More generally, if $A, C \subseteq N \in \boldsymbol{K}_0$ with $C - A$ finite and $A \le C$, a *primitive decomposition* of $C$ over $A$ is a primitive decomposition of $B$ over $A$, where $B$ is maximal with $A \subseteq B \subseteq C$ and $\delta(B/A) = 0$.

This analysis is more complicated than, e.g., that in [7], because we consider the case where there are primitives which are algebraic; that is, our analysis covers the case where there is a function $\mu$ enforcing algebraicity.

Suppose that $A \le N$, $A \subseteq B \subseteq N$ and $\delta(B/A) = 0$. Then $B \le N$ as well, and whenever $P$ is primitive over $A$, either $P \subseteq B$ or $PB - B$ is primitive over $B$. It follows easily by induction that in the latter case, if $B_1, \ldots, B_n$ is a primitive decomposition of $B$ over $A$, then $B_1P, \ldots, B_nP$ is a primitive decomposition of $BP$ over $AP$. We now easily get the following result, which was long ago explicitly pointed out to one of the authors by Shelah.

**3.4.** Lemma. *If $\delta(B/A) = 0$ and $A_1, \ldots, A_k$ and $B_1, \ldots, B_m$ are two primitive decompositions of $B$ over $A$, then $k = m$ and the sets $\{A_1', \ldots, A_k'\}$ and $\{B_1', \ldots, B_m'\}$ are equal.*

*Proof.* If $B$ is primitive over $A$ there is nothing to prove. Suppose the result holds for any pair $B, A$ which has a decomposition of length less than $k$ and suppose $A_1, \ldots, A_k$ and $B_1, \ldots, B_m$ are two primitive decompositions of $B$. If $A'_1 = B'_1$, then we are finished, by induction, since then $A_2, \ldots, A_k$ and $B_2, \ldots, B_m$ are two primitive decompositions of $B$ over $A_1 = B_1$. Otherwise, choose maximal $j$, necessarily less than $k$, such that $A_j \cap B'_1 = \emptyset$. Then by the remarks preceding the lemma, we must have $B'_1 = A'_{j+1}$ and $B_2, \ldots, B_m$ and $A_1 B'_1, \ldots, A_j B'_1 = A_{j+1}, A_{j+2}, \ldots, A_k$ are two primitive decompositions of $B$ over $B_1$, whence, by induction, we are done. ∎

The following fact about primitive decompositions is easily proved by induction on the length of the decomposition using the monotonicity of $\delta$.

**3.5.** Lemma. *Let $N \prec \mathcal{M}$, $\overline{a} \cap N = \emptyset$ and suppose that $\phi(\overline{x}; \overline{m})$ is a complete $\delta$-formula for $\overline{a}$ over $N$ based on $\overline{m}$.*

(1) *Then $\overline{a}_0, \ldots, \overline{a}_k$ is a primitive decomposition of $\overline{a}$ over $N$ if and only if $\overline{a}_0, \ldots, \overline{a}_k$ is a primitive decomposition of $\overline{a}$ over $\overline{m}$. Moreover, the same steps in the two primitive decompositions are algebraic.*

(2) *If $\models \phi(\overline{a}'; \overline{m})$ and $\overline{a}'_0, \ldots, \overline{a}'_k$ is a primitive decomposition of $\overline{a}'$ over $\overline{m}$ with $\overline{a}'_0, \ldots, \overline{a}'_i \in N$, $\overline{a}'_{i+1}, \ldots, \overline{a}'_k$ do not intersect $N$, and $\delta(\overline{a}'_{i+1}, \ldots, \overline{a}'_k / N) = \delta(\overline{a}'_{i+1}, \ldots, \overline{a}'_k / \overline{n})$ then $\overline{a}'_{i+1}, \ldots, \overline{a}'_k$ is a primitive decomposition of $\overline{a}'$ over $N$.*

*Proof.* The only difficult point is to show for (1) that if $\mathrm{tp}(\overline{a}_{i+1}/\overline{a}_i N)$ is algebraic then so is $\mathrm{tp}(\overline{a}_{i+1}/\overline{a}_i \overline{m})$. If $\mathrm{tp}(\overline{a}_{i+1}/\overline{a}_i N)$ is algebraic then $\mathrm{tp}(\overline{a}_{i+1}/\overline{a}_i \overline{n})$ is algebraic for some finite $\overline{n}$ with $\overline{m} \subset \overline{n} \subset N$. Since $\phi(\overline{x}; \overline{m})$ is a complete $\delta$-formula for $\overline{a}$ over $N$ based on $\overline{m}$, Definitions 1.6 and 1.8 show there is $\phi_i(\overline{x}_{i+1}, \overline{a}_i, \overline{m})$ which is a substitution instance of a subformula of $\phi(\overline{x}, \overline{m})$ and is a complete $\delta$-formula for $\overline{a}_{i+1}$ over $N\overline{a}_i$ based on $\overline{m}\,\overline{a}_i$. Thus, by Lemma 3.1 (applied with $B$ as $\overline{n}\,\overline{a}_i$), if $\phi_i(\overline{c}, \overline{a}_i, \overline{m})$ and $\delta(\overline{c}/\overline{n}\,\overline{a}_i) = 0$, then $\mathrm{tp}(\overline{a}_{i+1}/\overline{n}\,\overline{a}_i) = tp(\overline{c}/\overline{n}\,\overline{a}_i)$. So, if $\mathrm{tp}(\overline{a}_{i+1}/\overline{a}_i \overline{m})$ is not algebraic, there are infinitely many $\overline{c}$ realizing $\mathrm{tp}(\overline{a}_{i+1}/\overline{a}_i \overline{m})$ with $\delta(\overline{c}/\overline{n}\,\overline{a}_i) < 0$. This contradicts $N \in \overline{\boldsymbol{K}}_0$. (Cf. Lemma 3.19 of [2].) ∎

**3.6.** Lemma. *Let $N \prec \mathcal{M}$, $\overline{a} \cap N = \emptyset$ and suppose that $\phi(\overline{x}; \overline{m})$ is a complete $\delta$-formula for $\overline{a}$ over $N$ based on $\overline{m}$. If $k$ is the number of nonalgebraic steps in a primitive decomposition of $\overline{a}$ over $N$, then*

$$R_M(\phi(\overline{x}; \overline{m})) \leq \omega \cdot \delta(\overline{a}/\overline{m}) + k.$$

*Proof.* We proceed by induction on $\delta(\overline{a}/\overline{m})$ and $k$. Fix $\phi, \overline{m}, \overline{a}, k$ and assume that for all $\overline{m}' \in N$ and all $\phi', \overline{a}', k'$ for which $\delta(\overline{a}'/\overline{m}') < \delta(\overline{a}/\overline{m})$ or $\delta(\overline{a}/\overline{m}) = \delta(\overline{a}'/\overline{m}')$ and $k' < k$, we have $R_M(\phi'(\overline{x}'; \overline{m}')) \leq \omega \cdot \delta(\overline{a}'/\overline{m}') + k'$.

By Lemma 3.1 there is at most one type $q(\overline{x}) \in S(N)$ such that $\phi(\overline{x}; \overline{m}) \in q$, and for some (hence any) $\overline{b}$ realizing $q$, $\delta(\overline{b}/N) = \delta(\overline{a}/\overline{m})$, $N\overline{b} \leq \mathcal{M}$, and $\overline{b} \cap N = \emptyset$. We will show all other complete types over $N$ containing $\phi(\overline{x}; \overline{m})$

also contain a formula of rank strictly less than $\omega \cdot \delta(\overline{a}/\overline{m}) + k$. The lemma follows immediately.

Fix $\overline{a}'$ satisfying $\phi(\overline{x}; \overline{m})$ and suppose that the type of $\overline{a}'$ over $N$ is not of the form $q$ described above. Set $\overline{b}' = \overline{a}' - N$. Now $R_M(\mathrm{tp}(\overline{a}'/N)) = R_M(\mathrm{tp}(\overline{b}'/N))$, so it suffices to show that $\mathrm{tp}(\overline{b}'/N)$ contains a formula of rank strictly less than $\omega \cdot \delta(\overline{a}/\overline{m}) + k$.

If $\delta(\overline{b}'/N) < \delta(\overline{a}/\overline{m})$, we are done by the inductive hypothesis. If $N\overline{b}' \not\leq \mathcal{M}$, let $\overline{d}$ denote $\mathrm{icl}(\overline{b}'/N)$, whence, since $\overline{d} \subseteq \mathrm{acl}(\overline{b}'N)$, $R_M(\mathrm{tp}(\overline{d}/N)) = R_M(\mathrm{tp}(\overline{b}'/N))$. But $\delta(\overline{d}/N) < \delta(\overline{b}'/N) \leq \delta(\overline{a}/\overline{m})$, so we may again appeal to the inductive hypothesis.

In the remaining case, $\delta(\overline{b}'/N) = \delta(\overline{a}/N)$ and $N\overline{b}' \leq \mathcal{M}$ but $\overline{b}' \neq \overline{a}'$. Write $\overline{a}' = \overline{b}'\overline{c}'$, so that $\overline{c}' = \overline{a}' \cap N$, and let $\overline{a} = \overline{b}\overline{c}$ be the corresponding partition of $\overline{a}$.

Since $\overline{a}'$ satisfies the *complete* $\delta$-formula $\phi(\overline{x}; \overline{m})$ for $\overline{a}$ over $\overline{m} \leq N$, we get the second inequality in the following expression:

$$\begin{aligned} (1) \qquad \delta(\overline{a}/N) = \delta(\overline{b}'/N) &\leq \delta(\overline{b}'/\overline{c}'\overline{m}) \leq \delta(\overline{b}/\overline{c}\overline{m}) \\ &\leq \delta(\overline{b}/\overline{c}\overline{m}) + \delta(\overline{c}/\overline{m}) = \delta(\overline{a}/N), \end{aligned}$$

so $\delta(\overline{c}/\overline{m}) = 0$. Note that any primitive decomposition of $\overline{c}$ must contain a nonalgebraic step, since $\overline{c}$ algebraic over $N$ and $N \prec \mathcal{M}$ would imply $\overline{c} \subseteq N$. Since also $\delta(\overline{c}'/\overline{m}) \leq \delta(\overline{c}/\overline{m})$ and $\overline{m} \leq \overline{m}\overline{c}'$, $\delta(\overline{c}'/\overline{m}) = 0$ as well. Combined with Equation (1) this yields

$$\delta(\overline{a}'/\overline{m}) = \delta(\overline{b}'/\overline{c}'\overline{m}) = \delta(\overline{b}/\overline{c}\overline{m}) = \delta(\overline{a}/\overline{m}).$$

Thus, by (4) of Definition 1.6 of a complete $\delta$-formula, $\overline{a}$ and $\overline{a}'$ have the same diagram over $\overline{m}$ and isomorphic primitive decompositions over $\overline{m}$. By Lemma 3.5(1) these decompositions have the same length and same number of algebraic steps as the decomposition of $\overline{a}$ over $N$. Since $\overline{c}' \in N$ and $\delta(\overline{c}'/\overline{m}) = 0$, Lemma 3.5(2) implies the number $k'$ of nonalgebraic steps in a primitive decomposition of $\overline{a}'$ over $\overline{m}$ is at most the number of such in a primitive decomposition for $\overline{a}'$ over $\overline{m}\overline{c}'$, which, as we have just seen, is the same as their number in a primitive decomposition of $\overline{a}$ over $\overline{c}N$. Since $\overline{c}$ has a nonalgebraic step, $k'$ is less than $k$. Thus, by the inductive hypothesis, $\mathrm{tp}(\overline{a}'/N)$ contains a formula of rank at most $\omega \cdot \delta(\overline{a}'/N) + k'$, which, as desired, is strictly smaller than $\omega \cdot \delta(\overline{a}/\overline{m}) + k$. ∎

The key idea for Corollary 3.7 is hidden in the induction step of Lemma 3.6. When $Na$ is not a strong substructure of the universe, $\mathrm{icl}(a/N)$ will have smaller value of $\delta$ while the primitive decomposition can have arbitrary length. For the lower bound argument we analyze a primitive decomposition of $\mathrm{icl}(a/N)$ in Theorem 3.13.

**3.7.** Corollary. *If for some $n \geq 1$, and every $a$, $\delta(a) \leq n$, then every 1-type in $T$, the theory of the generic, has Morley rank at most $\omega \cdot n$.*

*Proof.* It suffices to compute the ranks of types over models $M$ which are elementary in the universe. But given any $a$ we can find $\overline{m} \in M$ and $\phi(x;\overline{m})$ such that $\phi(x;\overline{m})$ is a complete $\delta$-formula for $a$ over $M$ based on $\overline{m}$. So

$$R_M(\mathrm{tp}(a/M)) \leq R_M(\phi(x;\overline{m})),$$

which by Lemma 3.6 is at most $\omega \cdot \delta(a/\overline{m}) + k \leq \omega \cdot n + k$, where $k$ is the number of nonalgebraic steps in a primitive decomposition of $a$ over $\overline{m}$. Since $a$ is a single point, either $k = 0$, and $R_M(\mathrm{tp}(a/M)) \leq \omega \cdot n$, or $k = 1$, i.e. $\delta(a/\overline{m}) = 0$, whence $\omega \cdot \delta(a/\overline{m}) + k = 1$. ■

**3.8.** Lemma. *If $\varphi(\overline{x};\overline{y})$ is a complete $\delta$-formula for some primitive $\overline{a}$ over $\overline{m}$, then $\varphi(\overline{x};\overline{m})$ is strongly minimal or algebraic.*

*Proof.* Suppose $\models \varphi(\overline{a};\overline{m})$. Let $\overline{m}_1 \supset \overline{m}$ be finite and let $\overline{m}'_1$ be the intrinsic closure of $\overline{m}_1$ in $\mathcal{M}$. Since $\overline{a}$ is primitive over $\overline{m}$, the definition of $\varphi$ shows that $\overline{a} \cap \overline{m}'_1 \neq \emptyset$ implies $\overline{a} \subseteq \overline{m}'_1$ (otherwise, $\delta(\overline{a}/\overline{a} \cap \overline{m}'_1) < 0$). So, in this case, $\mathrm{tp}(\overline{a}/\overline{m}_1)$ is algebraic (since, on general grounds [2], $\mathrm{icl}(\overline{m}_1)$ is algebraic over $\overline{m}_1$). If, on the other hand, $\overline{a} \cap \overline{m}'_1 = \emptyset$, then $\overline{m}'_1\overline{a}$ is strong in $\mathcal{M}$ since $\delta(\overline{a}/\overline{m}'_1) \leq \delta(\overline{a}/\overline{m}) = 0$ and $\overline{m}'_1$ is strong in $\mathcal{M}$. By Lemma 3.1 all $\overline{a}$ satisfying $\varphi(\overline{x};\overline{m})$ with $\overline{a} \cap \overline{m}'_1 = \emptyset$ are automorphic over $\overline{m}'_1$. Thus, $\varphi(\overline{x};\overline{m})$ is strongly minimal or algebraic. ■

**3.9.** Lemma. *For any $A \leq \mathcal{M}$ and $\overline{c} \in \mathcal{M}$, if $\delta(\overline{c}/A) = 0$ and $\overline{c}$ has a primitive decomposition in which exactly $m$ steps are nonalgebraic, then $U(\overline{c}/A) = m$.*

*Proof.* The proof is by induction on $m$. If $m = 0$, the result is trivial. If $m = 1$, apply Lemma 3.8. Suppose $m = k + 1$ and $k \geq 1$. Fix $\overline{c}$. If $\overline{c} = \overline{c}'\overline{a}$, where $\delta(\overline{c}'/A) = 0$, and $\overline{a}$ is algebraic over $\overline{c}'A$, we could replace $\overline{c}$ by $\overline{c}'$, without changing either $U$-rank or $m$, so suppose we cannot do so. Write $\overline{c} = \overline{b}\overline{d}$ where $\overline{d}$ is primitive over $A\overline{b}$. By assumption, $\overline{d}$ is nonalgebraic over $\overline{b}A$. By the Lascar inequality,

$$U(\overline{d}/A\overline{b}) + U(\overline{b}/A) \leq U(\overline{c}/A) \leq U(\overline{d}/A\overline{b}) \oplus U(\overline{b}/A).$$

By induction $U(\overline{b}/A) = k$, so, by the definition of $\oplus$, the two end terms of the inequality are equal. By Lemma 3.8, $U(\overline{d}/A\overline{b}) = 1$ and we finish. ■

**3.10.** Lemma. *For any $A \leq \mathcal{M}$ and $\overline{a}$, if $\overline{a} = \overline{b}\overline{c}$ where $\overline{c}$ is maximal with $\delta(\overline{c}/A) = 0$ and $\overline{c}$ has a primitive decomposition with $m$ nonalgebraic steps, then $U(\overline{a}/A) = U(\overline{b}/A\overline{c}) + m$.*

*Proof.* By the Lascar inequality,

$$U(\overline{b}/A\overline{c}) + U(\overline{c}/A) \leq U(\overline{a}/A) \leq U(\overline{b}/A\overline{c}) \oplus U(\overline{c}/A).$$

By Lemma 3.9, $U(\overline{c}/A) = m$ and as in the proof of Lemma 3.9 we have the required equality. ∎

Our goal now is to provide a general method for establishing the exact rank of theories constructed in this way. We introduce an *ad hoc* condition which is sufficient for the lower bound and provide examples where it can be applied. We need one technical condition on 1-types.

**3.11.** DEFINITION. Fix $B \subseteq \mathcal{M}$, $b \in \mathcal{M}$ and let $\overline{a} = \mathrm{icl}(b/B)$. We say $\mathrm{tp}(b/B)$ is a $(j, m)$-*type* if $B \leq B\overline{a}$, and for $\overline{a} = \overline{b\overline{c}}$, where $\overline{c}$ is maximal in $\overline{a}$ with $\delta(\overline{c}/B) = 0$,

(1) $\overline{c}$ has a primitive decomposition with $m$ nonalgebraic steps;
(2) $\delta(\overline{a}/\overline{c}B) = j$;
(3) $\overline{a}$ is minimal strong over $\overline{c}B$.

**3.12.** DEFINITION. We say $\boldsymbol{K}_0$ (or the theory $T$ of the generic of $\boldsymbol{K}_0$) is *ample* if for every $\mathrm{tp}(b/B)$, if $\overline{a} = \mathrm{icl}(b/B)$ is minimal strong with $\delta(\overline{a}/B) = j$, then for every $m < \omega$, there are $B_m \supset B$ and $b_m$ such that $b_m$ realizes $\mathrm{tp}(b/B)$ and $\mathrm{tp}(b_m/B_m)$ is a $(j-1, m)$-type.

The key fact in the definition of ample is that if $\mathrm{tp}(b/B)$ is a $(j, m)$-type then for any $C \supseteq B$ such that $b$ and $C$ are independent over $B$, $\mathrm{tp}(b/C)$ is also a $(j, m)$-type. Thus, $\mathrm{tp}(b_m/B_m)$ must be a forking extension of $\mathrm{tp}(b/B)$ and without loss of generality we could take $B_m$ to be elementary in the universe. Now we show that the existence of such types is sufficient to calculate the Morley and $U$-rank in various bicolored fields.

**3.13.** THEOREM. *Suppose $\boldsymbol{K}_0$ is ample, $n = \max\{\delta(a) : a \in N \in \boldsymbol{K}_0\}$, and $T$ is the theory of the generic model $M$.*

(1) *Let $N \prec \mathcal{M}$. If $\mathrm{tp}(e/N)$ is a $(j, m)$-type then $U(e/N) = \omega \cdot j + m = R_M(e/N)$.*
(2) *In $T$, $R_M(x = x) = \omega \cdot n$ is the maximal $U$-rank of a 1-type.*

*Proof.* (1) We show by induction that if $\mathrm{tp}(b/N)$ is a $(j, m)$-type then $U(\mathrm{tp}(b/N)) \geq \omega \cdot j + m$. For $j = 0$ and any $m$, the result follows by Lemma 3.9. Now suppose we have the result for $j' < j$ and any $m$. Consider $\mathrm{tp}(b/N)$ where $\overline{a} = \mathrm{icl}(b/N)$, $\overline{a} = \overline{b\overline{c}}$, $\overline{c}$ is maximal in $\overline{a}$ with $\delta(\overline{c}/N) = 0$, $\overline{a}$ is minimal strong over $N\overline{c}$ and $\delta(\overline{a}/N\overline{c}) = j$. Apply the fact that $\boldsymbol{K}_0$ is ample to $p = \mathrm{tp}(b/N\overline{c})$; for every $m < \omega$, there exist $N_m \supset N\overline{c}$ and $b_m$ such that $b_m$ realizes $\mathrm{tp}(b/N\overline{c})$ and $p_m = \mathrm{tp}(b_m/N_m)$ has type $(j-1, m)$. Then $p_m$ is a forking extension of $p$ and by induction $U(p_m) = \omega \cdot (j-1) + m$. Thus, $U(p) \geq \omega \cdot j$. Note $U(b/N\overline{c}) = U(\overline{a}/N\overline{c})$. By Lemma 3.10, $U(b/N)) = U(\overline{a}/N) = U(\overline{a}/N\overline{c}) + m \geq \omega \cdot j + m$. But by Lemma 3.6 this is also an upper bound on the Morley rank and, in general, $U$-rank is at most Morley rank so we finish.

(2) The upper bound is $\omega \cdot n$ by Corollary 3.7. Let $p = \mathrm{tp}(c/\emptyset)$ be the type of a point with $\delta(c) = n$ and $\mathrm{icl}(c) = c$; choose $N \prec \mathcal{M}$ which is independent of $c$ over the empty set. So $\mathrm{icl}(c/N) = Nc$ and $\mathrm{tp}(c/N)$ is an $(n, 0)$-type. By part (1) we finish. ∎

We have shown the equality of Morley rank and $U$-rank for $(j, m)$-types when the theory is ample. At this point we restrict to the consideration of bicolored fields. We give examples of $(j, m)$-types for enough $j$ and $m$ to indicate the general idea of proving the lower bounds. These examples have infinite Morley rank and there are no algebraic primitives since we are not considering a $\mu$-function. We say a point is a *black* (or *white*) *transcendental* if it is algebraically independent of the subfield being considered and has the appropriate color. Our discussion of bicolored fields in Section 2 focused on the case $k = 2$. We consider several values of $k$ below. That argument and most of our analysis works for suitable (as specified in Section 2) strongly minimal sets. But the examples require calculations which are specifically about fields.

**3.14.** Example. We show that for several choices of $\delta$, the class of bicolored fields $\boldsymbol{K}_0$ determined by $\delta$ is ample. More precisely, we sketch the argument that if $\delta(X) = k \cdot d_f(X) - |X \cap P(N)|$, then $\boldsymbol{K}_0$ is $k$-ample.

1. First consider the case of a bicolored field with the dimension function $\delta(\overline{a}) = 2d_f(\overline{a}) - |\overline{a} \cap P|$, where $P$ is the collection of black points. Now $\mathrm{tp}(b/B)$ is a $(j, m)$-type for the following $j$ and $m$:

• $(0, 1)$-type: $\overline{a} = (b, b + d)$, which are algebraically independent black points and no other point in their algebraic closure is black and $B = \{d\}$.

• $(0, 2)$: $\overline{a} = (b_1, b_1 + d, b_2 + d, b_1 + b_2 + d)$, $b = b_1 + b_2 + d$ and $B = \{d\}$. Again, the $b_i$ and the elements of $\overline{a}$ are black but no other point in their algebraic closure is black.

• $(0, m)$: $\overline{a}$ enumerates $A_m$ as defined below, $b = e^m_{m-1}$ and $B = \{d\}$. For $m \geq 1$, to construct a $(0, m)$-type let $c^m_i$ for $i < m$ be algebraically independent black points, fix an element $d$ and let $e^m_p = d + \sum_{i \leq p} c^m_i$ also be black. Further, no other elements in the algebraically closed field generated by these is black. If $p = \mathrm{tp}(e^m_{m-1}/\emptyset)$ for $p < n$, the type of a transcendental black element, then $\mathrm{tp}(e^m_{m-1}/d)$ is a $(0, m)$-type since $A_m = \mathrm{icl}(e^m_{m-1}/d) = \{c^m_p, e^m_p : p < m\}$. This set has $2m$ elements, all black, and dimension $m$ so $\delta(A_m) = 0$. Moreover, each proper subset $X$ which contains $e^m_{m-1}$ has $\delta(X) > 0$. Moreover $A_m$ can be decomposed as a sequence of $m$ primitives, the $p$th is $\{c^m_p, e^m_p\}$.

• $(1, m - 1)$: $\overline{a}$ enumerates $A_m$ as defined below, $b = e^m_{m-1}$ and $B = \{d\}$. For $m \geq 1$, to construct a $(1, m - 1)$-type let $c^m_i$ for $i < m$ be algebraically independent black points, fix an element $d$ and let $e^m_p = d + \sum_{i \leq p} c^m_i$ also be black for $p < m-1$; but $e^m_{m-1}$ is white. Further, no other elements in the algebraically closed field generated by these is black. If $p = \mathrm{tp}(e^m_{m-1}/\emptyset)$, the type of a transcendental white element, then $\mathrm{tp}(e^m_{m-1}/d)$ is a $(1, m-1)$-type since $A_m = \mathrm{icl}(e^m_{m-1}/d) = \{c^m_p, e^m_p : p < m\}$. This set has $2m$ elements, all but one black, and dimension $m$ so $\delta(A_m) = 1$. Moreover, each proper subset $X$ which contains $e^m_m$ has $\delta(X) > 1$. Moreover $A_m$ can be decomposed as a sequence of $m-1$ primitives, the $p$th is $\{c^m_p, e^m_p\}$, followed by a minimal strong extension $\{c^m_{m-1}, e^m_{m-1}\}$ which has dimension 1 over the predecessors.

With this tool we are able to find the lower bounds on the $U$-rank of the theories in question by finding a sufficient supply of $(j, m)$-types.

2. Consider the case of a bicolored field with the dimension function $\delta(\overline{a}) = 3d_f(\overline{a}) - |\overline{a} \cap P|$. The type of a white transcendental point, which is strong in the universe, witnesses that $T$ is ample. For this, the required $b_{m-1}$ is the $f^m_{m-1}$ defined below, and $B_{m-1} = \{d_1, d_2\}$.

For $m \geq 1$, to construct a $(2, m-1)$-type let $c^m_i$ for $i < m$ be algebraically independent black points, fix elements $d_1, d_2$ and let $e^m_p = d_1 + \sum_{i \leq p} c^m_i$ and $f^m_p = d_1 + d_2 + \sum_{i \leq p} c^m_i$ also be black for $p < m$; but $e^m_{m-1}$ and $f^m_{m-1}$ are white. Further, no other elements in the algebraically closed field generated by these is black. If $p = \mathrm{tp}(f^m_{m-1}/\emptyset)$, the type of a transcendental, white element, which sits strongly in the universe, then $\mathrm{tp}(f^m_{m-1}/d_1, d_2)$ is a $(2, m-1)$-type. For, $A_m = \mathrm{icl}(e^m_{m-1}/d_1, d_2) = \{c^m_p, e^m_p, f^m_p : p < m\}$. This set has $3m$ elements, all but two black, and dimension $m$ so $\delta(A_m/d_1, d_2) = 2$. Moreover, each proper subset $X$ of $A_m$ which contains $f^m_{m-1}$ has $\delta(X) > 2$. Finally, $A_m$ can be decomposed as a sequence of $m-1$ primitives, the $p$th is $\{c^m_p, e^m_p, f^m_p\}$, followed by a minimal strong extension $\{c^m_{m-1}, e^m_{m-1}, f^m_{m-1}\}$ which has dimension 2 over the predecessors.

For $m \geq 1$, to construct a $(1, m-1)$-type let $c^m_i$ for $i < m$ be algebraically independent black points, fix elements $d_1, d_2$ and let $e^m_p = d_1 + \sum_{i \leq p} c^m_i$ and $f^m_p = d_1 + d_2 + \sum_{i \leq p} c^m_i$ also be black for $p < m$; but $e^m_{m-1}$ is black and $f^m_{m-1}$ is white. Further, no other elements in the algebraically closed field generated by these is black. If $p = \mathrm{tp}(f^m_{m-1}/\emptyset)$, the type of a transcendental white element, then $\mathrm{tp}(f^m_{m-1}/d_1, d_2)$ is a $(2, m-1)$-type. For, $A_m = \mathrm{icl}(f^m_{m-1}/d) = \{c^m_p, e^m_p, f^m_p : p < m\}$. This set has $3m$ elements, all but one black, and dimension $m$ so $\delta(A_m) = 1$. Moreover, each proper subset $X$ which contains $f^m_{m-1}$ has $\delta(X) > 1$. Moreover $A_m$ can be decomposed as a sequence of $m-1$ primitives, the $p$th is $\{c^m_p, e^m_p, f^m_p\}$, followed by a minimal strong extension $\{c^m_{m-1}, e^m_{m-1}, f^m_{m-1}\}$ which has dimension 1 over the predecessors.

Note that there are distinct specializations of the type of a white transcendental to a $(2, m)$-type and to a $(1, m)$-type rather than an inductive procedure.

3. A similar argument will show that if $\delta(\overline{a}) = kd_f(\overline{a}) - |\overline{a} \cap P|$ then $\boldsymbol{K}_0$ is $k$-ample.

**3.15.** Summary. If a strongly minimal theory (with elimination of imaginaries and the definable multiplicity property) is expanded by a unary predicate with dimension function $\delta(\overline{a})=kd_f(\overline{a})-|\overline{a}\cap P|$, then $R_M(x = x) = \omega \cdot k$ is the maximal $U$-rank of a 1-type. For, our examples have indicated how to show that the theories are ample and the result follows by Theorem 3.13. Our treatment of algebraic primitives means that this analysis also includes the exact calculation of the finite rank if we introduce as in [1, 5] a function $\mu$ bounding the number of instances of primitives.

## References

[1] J. T. Baldwin and K. Holland, *Constructing $\omega$-stable structures: Rank 2 fields*, J. Symbolic Logic 65 (2000), 371–391.

[2] J. T. Baldwin and N. Shi, *Stable generic structures*, Ann. Pure Appl. Logic 79 (1966), 1–35.

[3] K. Holland, *Model completeness of the new strongly minimal sets*, J. Symbolic Logic 64 (1999), 946–962.

[4] E. Hrushovski, *Strongly minimal expansions of algebraically closed fields*, Israel J. Math. 79 (1992), 129–151.

[5] —, *A new strongly minimal set*, Ann. Pure Appl. Logic 62 (1993), 147–166.

[6] —, *Simplicity and the Lascar group*, preprint, 199x.

[7] B. Poizat, *Le carré de l'égalité*, J. Symbolic Logic 64 (1999), 1339–1356.

Department of Mathematics, Statistics
and Computer Science
University of Illinois at Chicago
M/C 249
851, S. Morgan St.
Chicago, IL 60607, U.S.A.
E-mail: jbaldwin@uic.edu

Department of Mathematics
Northern Illinois University
DeKalb, IL 60115, U.S.A.
E-mail: kholland@math.niu.edu

*Received 14 May 1999;*
*in revised form 4 January 2001*