FUNDAMENTA
MATHEMATICAE
227 (2014)

Consistency of the Silver dichotomy in generalised Baire space

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Abstract. Silver's fundamental dichotomy in the classical theory of Borel reducibility states that any Borel (or even co-analytic) equivalence relation with uncountably many classes has a perfect set of classes. The natural generalisation of this to the generalised Baire space κ^κ for a regular uncountable κ fails in Gödel's L, even for κ-Borel equivalence relations. We show here that Silver's dichotomy for κ-Borel equivalence relations in κ^κ for uncountable regular κ is however consistent (with GCH), assuming the existence of $0^\#$.

A fundamental result in classical descriptive set theory is *Silver's dichotomy*:

Theorem 1 (Silver [4]). *If a Borel (or even co-analytic) equivalence relation on the reals has uncountably many classes then it has a perfect set of classes, i.e., there is a perfect set of reals any two distinct elements of which belong to different classes.*

It is convenient to express the conclusion of Silver's theorem in terms of the *continuous reducibility* of equivalence relations. Let id denote the equivalence relation of equality on Cantor space 2^ω. If E, F are equivalence relations on Polish spaces then we say that E is *continuously reducible* to F (written $E \leq_c F$) if there is a continuous function f such that $E(x,y)$ iff $F(f(x), f(y))$. Then Silver's theorem says that if E is a Borel equivalence relation on the reals with uncountably many classes then id is continuously reducible to E. A more generous notion is *Borel reducibility*, where the "reduction" f is allowed to be Borel (we then write $E \leq_B F$).

In this article we look at Silver's dichotomy in *generalised Baire space.* Let κ be an infinite cardinal such that $\kappa^{<\kappa} = \kappa$. Then the *generalised Baire space* κ^κ associated to κ is the space of functions from κ to κ topologised

2010 *Mathematics Subject Classification*: 03E15, 03E35, 03E45, 03E55.
Key words and phrases: Silver dichotomy, Silver indiscernibles, Borel reducibility, generalised Baire space.

DOI: 10.4064/fm227-2-4

with basic open sets of the form

$$N_\sigma = \{f : \kappa \to \kappa \mid f \text{ extends } \sigma\}$$

where σ belongs to $\kappa^{<\kappa}$. Our hypothesis on κ implies that this gives a basis for the topology of size κ. Borel sets in this space, the *κ-Borel sets*, are obtained by closing the collection of basic open sets under unions and intersections of size κ. We get closure under complements using the fact that the complement of a basic open set is the union of at most κ basic open sets. A function from generalised Baire space to itself is *κ-Borel* if the pre-image under this function of any basic open set is κ-Borel. And the *generalised Cantor space* 2^κ associated to κ is the closed subspace of κ^κ consisting of those functions which map κ to 2. As in the classical setting we have the corresponding notions of κ-continuous and κ-Borel reducibility (again written as $\leq_c$, $\leq_B$, respectively) of equivalence relations on spaces like 2^κ or κ^κ which are equipped with a notion of Borel set.

As in the classical case, the reducibility of id_κ (the equality relation on 2^κ) to an equivalence relation E on κ^κ can be reformulated as a statement about perfect sets. We say that $X \subseteq \kappa^\kappa$ is *κ-perfect* if X consists of the κ-branches $[T]$ through a subtree T of $\kappa^{<\kappa}$ which is $<\kappa$-closed and has the property that every node can be extended to a splitting node.

PROPOSITION 2. *Suppose that E is an equivalence relation on κ^κ. Then id_κ is κ-Borel reducible to E iff id_κ is κ-continuously reducible to E iff there is a κ-perfect set $X \subseteq \kappa^\kappa$ any two distinct elements of which belong to different classes of E.*

Proof. Given $X = [T]$ as above we obtain an order-preserving injection from $2^{<\kappa}$ into the set of splitting nodes of T; this induces a κ-continuous $\sigma : 2^\kappa \to [T]$ which reduces id_κ to E. Conversely, if f is a κ-Borel function that reduces id_κ to E then f is κ-continuous on a κ-comeager set (i.e. a set whose complement is contained in the union of κ closed sets, each with an empty interior) and this κ-comeager set contains a κ-perfect set; we can thin out this κ-perfect set to a κ-perfect subset whose f-image is the desired κ-perfect set X. ■

Now we ask:

PROBLEM 3. *Does Silver's dichotomy hold for generalised Baire space κ^κ? That is, if a κ-Borel equivalence relation E on κ^κ has more than κ classes, is there a κ-continuous reduction of id_κ on 2^κ to E?*

The answer is negative in Gödel's L, in a strong sense.

THEOREM 4 (SDF–Hyttinen–Kulikov [1, 2]). *Assume that $V = L$. Then Silver's dichotomy fails in generalised Baire space for all uncountable regular κ: There are κ-Borel equivalence relations with more than κ classes* *which lie strictly below* id_κ *as well as a family of* 2^κ *κ-Borel equivalence relations including* id_κ *which are pairwise* $\le_B$*-incomparable. If* κ *is inaccessible then there is a family of* 2^κ *κ-Borel equivalence relations which are pairwise* $\le_B$*-incomparable and* $\le_B$*-below* id_κ*.*

The problem with Silver's dichotomy in L derives from the existence of *weak κ-Kurepa trees* on a regular cardinal κ. These are trees T of height κ with more than κ branches of length κ such that every node of T splits and the αth level of T has size at most $\mathrm{card}(\alpha)$ for stationary-many ordinals $\alpha < \kappa$. We say that T is *κ-Kurepa* if "stationary-many ordinals" can be replaced by "all infinite ordinals".

LEMMA 5. *Suppose* $V = L$ *and* κ *is regular and uncountable. Then there exists a weak* κ*-Kurepa tree on* κ*. If* κ *is a successor cardinal then there is a* κ*-Kurepa tree on* κ*.*

Proof. Our tree will be a subtree of the binary tree $2^{<\kappa}$. For singular $\alpha < \kappa$ let $\beta(\alpha)$ be the least limit ordinal $\beta > \alpha$ such that α is singular in L_β.

First assume that κ is inaccessible. Then T consists of all $\sigma \in 2^{<\kappa}$ such that:

(∗) For singular cardinals $\alpha \le |\sigma|$ of cofinality ω, $\sigma|\alpha$ belongs to $L_{\beta(\alpha)}$.

Any node of T can be extended to nodes in T of any greater length (just add 0's). And any node of T of length α splits into two nodes in T of length $\alpha + 1$, so the αth splitting level consists of nodes of length α. It follows that the αth splitting level of T has size at most $\mathrm{card}(\alpha)$ for α a singular cardinal of cofinality ω.

MAIN CLAIM. The tree T has κ^+ branches.

Proof. For a limit ordinal β between κ and κ^+ we say that β is *critical* if some subset of κ is definable over L_β but not an element of L_β. The set of critical ordinals is cofinal in κ^+, and for critical β the Skolem hull of κ in L_β is all of L_β.

Now for each critical β define

$$C_\beta = \{\alpha < \kappa \mid \text{the Skolem hull of } \alpha \text{ in } L_\beta \text{ contains no ordinals between } \alpha \text{ and } \kappa\}.$$

Then C_β is a club in κ for each critical β and moreover if $\beta_0 < \beta_1$ are both critical then sufficiently large elements of C_{β_1} are limit points of C_{β_0}; this is because β_0 is an element of the Skolem hull of α in L_{β_1} for a large enough α and therefore so is C_{β_0}.

In particular the C_β's for critical β are distinct. Now we claim that each C_β is a branch through T. For this we need only check that if $\alpha < \kappa$ is a singular cardinal of cofinality ω then $C_\beta \cap \alpha$ belongs to $L_{\beta(\alpha)}$. This is clear if α does not belong to C_β, for then $C_\beta \cap \alpha$ is bounded in α and therefore is an element of L_α. Otherwise note that $C_\beta \cap \alpha$ is definable over $L_{\bar\beta+1}$ where $L_{\bar\beta}$ is the transitive collapse of the Skolem hull of α in L_β; as α is regular in $L_{\bar\beta}$, it follows that $\bar\beta$ is less than $\beta(\alpha)$, so $C_\beta \cap \alpha$ is an element of $L_{\beta(\alpha)}$, as desired.

The case of a successor cardinal κ is similar, except one can now obtain a κ-Kurepa tree on κ as all sufficiently large $\alpha < \kappa$ are singular. This proves the Main Claim and therefore Lemma 5. ■

Now note that if T is weak κ-Kurepa then there can be no κ-continuous injection from 2^κ into $[T]$, the set of κ-branches through T: If κ is inaccessible then this would yield a club of $\alpha < \kappa$ such that the αth level of T has 2^α nodes, and if $\kappa = \gamma^+$, it would yield an $\alpha < \kappa$ such that T has $2^\gamma = \kappa$ nodes on level α. In fact there cannot be such an injection which is κ-Borel by Proposition 2.

Now to give an indication why Theorem 4 is true, define xE_Ty iff x, y are not branches through T or $x = y$. Then E_T is a κ-Borel equivalence relation with κ^+ classes, yet id_κ cannot κ-Borel reduce to E_T for the reasons given above. Thus Silver's dichotomy fails at all uncountable regular cardinals in L. And E_T is κ-Borel reducible to id_κ via the reduction that sends each branch of T to itself and the non-branches of T to some fixed non-branch of T.

On the other hand, Silver [3] also showed that it is possible to get rid of κ-Kurepa trees on a regular cardinal κ using an inaccessible above κ: If $\lambda > \kappa$ is inaccessible and a Lévy collapse is performed to make λ into κ^+ (where the conditions are partial functions p of size less than κ from $\lambda \times \kappa$ to λ so that $p(\alpha, \beta) < \alpha$ for (α, β) in $\mathrm{Dom}(p)$) then in the generic extension there are no κ-Kurepa trees on κ. In fact, there are not even any weak κ-Kurepa trees on κ in Silver's model. This suggests that a model like Silver's may obey the Silver dichotomy for κ^κ, provided λ is chosen appropriately. Our main theorem states that this is indeed the case.

To gain further insight into the problem we next consider the following ZFC-provable negative result. A relation on $(\kappa^\kappa)^n$ is Σ^1_1 if it is the projection of a κ-Borel relation B on $(\kappa^\kappa)^n \times \kappa^\kappa$, i.e. equal to $\{(x_0, \ldots, x_{n-1}) \mid (x_0, \ldots, x_{n-1}, x_n) \in B \text{ for some } x_n\}$; it is Δ^1_1 if both itself and its complement are Σ^1_1.

THEOREM 6. *Let κ be regular and uncountable. Then there is a Δ^1_1 equivalence relation E with κ^+ classes such that id_κ is not κ-Borel reducible to E. So the Silver dichotomy provably fails for Δ^1_1 equivalence relations on κ^κ.*

Proof. The relation is $xE^{\mathrm{rank}}y$ iff x, y do not code wellorders or x, y code wellorders of the same length. This has exactly κ^+ classes. It is Δ^1_1 because the assumption that κ is uncountable and regular implies that wellfoundedness for linear orders of κ is Δ^1_1 (it is even closed). Suppose T were a κ-perfect tree whose distinct κ-branches were $E^{\rm rank}$-inequivalent. Now let x be a generic branch through T (treating T as a version of κ-Cohen forcing), and let $p \in T$ be a condition forcing that x codes a wellorder of some rank $\alpha < \kappa^+$. Then any sufficiently generic branch through T extending p codes a wellorder of rank α, which contradicts the fact that there are distinct such branches in V. ■

So a first step toward obtaining the consistency of Silver's dichotomy for κ^κ is the following:

THEOREM 7. *Assume $\kappa^{<\kappa} = \kappa$. Then the relation $E^{\rm rank}$ of the previous theorem is not κ-Borel.*

Proof. For $\alpha < \kappa^+$ let $\mathcal{L}_\alpha$ denote the forcing to Lévy collapse α to κ (a condition is a function from an ordinal less than κ into α). If $g : \kappa \to \alpha$ is $\mathcal{L}_\alpha$-generic then g^* denotes the subset of κ defined by $i \in g^*$ iff $g((i)_0) \le g((i)_1)$ where $i \mapsto ((i)_0, (i)_1)$ is a bijection between κ and $\kappa \times \kappa$.

By induction on κ-Borel rank we show that if B is κ-Borel then there is a club C in κ^+ such that

- $(*)$ For $\alpha \le \beta$ in C of cofinality κ and (p_0, p_1) a condition in $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$, (p_0, p_1) forces that (g_0^*, g_1^*) belongs to B in the forcing $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ iff it forces this in the forcing $\mathcal{L}_\alpha \times \mathcal{L}_\beta$.

If $B = U(\sigma_0) \times U(\sigma_1)$ is a basic open set then we may take C to consist of all ordinals greater than κ in κ^+. This is because for any $\alpha \le \beta$, if (p_0, p_1) belongs to $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ then (p_0, p_1) $\mathcal{L}_\alpha \times \mathcal{L}_\beta$-forces $(g_0^*, g_1^*) \in B$ exactly if (p_0^*, p_1^*) extends (σ_0, σ_1) where p_0^* is the set of i such that $(i)_0, (i)_1$ are in the domain of p_0 and $p_0((i)_0) \le p_0((i)_1)$ (similarly for p_1^*); this is independent of the pair α, β.

Inductively, suppose that B is the intersection of κ-Borel sets B_i, $i < \kappa$, of smaller κ-Borel rank. By intersecting clubs obtained by applying $(*)$ to the B_i's we obtain a club C ensuring the desired conclusion for B, as (p_0, p_1) forces $(g_0^*, g_1^*) \in B$ iff for each $i < \kappa$ it forces $(g_0^*, g_1^*) \in B_i$.

Finally if B is the complement of the κ-Borel set B_0 then by induction we have a club C_0 such that for $\alpha \le \beta$ in C_0 of cofinality κ and (p_0, p_1) in $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$, (p_0, p_1) $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$-forces $(g_0^*, g_1^*) \in B_0$ iff it $\mathcal{L}_\alpha \times \mathcal{L}_\beta$-forces it. Now thin out the club C_0 to a club C so that for α in C of cofinality κ, if (p_0, p_1) is in $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ and there is some $\beta \ge \alpha$ in C_0 of cofinality κ and some (q_0, q_1) in $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ below (p_0, p_1) which $\mathcal{L}_\alpha \times \mathcal{L}_\beta$-forces (g_0^*, g_1^*) in B_0, then there is such a (q_0, q_1) in $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ (which then $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$-forces (g_0^*, g_1^*) in B_0). So for $\alpha \le \beta$ of cofinality κ in this thinner club C, (p_0, p_1) $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$-forces (g_0^*, g_1^*) in B iff none of its extensions in $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ forces (g_0^*, g_1^*) in B_0 in the forcing $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ iff none of its extensions in $\mathcal{L}_\alpha \times \mathcal{L}_\alpha$ forces (g_0^*, g_1^*) in B_0 in the forcing

$\mathcal{L}_\alpha \times \mathcal{L}_\beta$ iff none of its extensions in $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ forces (g_0^*, g_1^*) in B_0 in the forcing $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ iff (p_0, p_1) $\mathcal{L}_\alpha \times \mathcal{L}_\beta$-forces (g_0^*, g_1^*) in B, completing the induction.

It follows that E^{rank} is not κ-Borel, as otherwise we have $g_0^* E^{\mathrm{rank}} g_1^*$ where g_0 and g_1 are sufficiently generic for $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ with $\alpha < \beta$. ∎

Now using an analogous argument we have:

THEOREM 8. *Suppose that $0^\#$ exists, κ is regular in L, and λ is the κ^+ of V. Then after forcing over L with the Lévy collapse turning λ into κ^+, the Silver dichotomy holds for κ^κ.*

Proof. Suppose that E is a κ-Borel equivalence relation in the Lévy collapse extension $L[G]$. For simplicity we assume that E has a κ-Borel code in L and therefore has κ-Borel rank less than $(\kappa^+)^L$. Suppose that E has more than κ classes in $L[G]$ and let p be a Lévy collapse condition forcing that the Lévy collapse names $(\sigma_\alpha \mid \alpha < \lambda)$ are pairwise E-inequivalent. We can assume that the σ_α's are of size less than λ and choose $f : \lambda \to \lambda$ in L so that for each $\alpha < \lambda$, σ_α is an $\mathcal{L}_{f(\alpha)}$-name where $\mathcal{L}_\beta$ denotes the part of the Lévy collapse forcing which collapses ordinals less than β to κ. We may assume that for each α, the E-equivalence class of σ_α does not depend on the choice of $\mathcal{L}_{f(\alpha)}$-generic, as otherwise this would fail for a pair of mutual $\mathcal{L}_{f(\alpha)}$-generics and by building a κ-perfect set of mutual $\mathcal{L}_{f(\alpha)}$-generics we obtain a κ-perfect set of distinct E-equivalence classes. It follows that if $\alpha < \beta$ and p belongs to $\mathcal{L}_{f(\alpha)}$ then (p, p) forces in $\mathcal{L}_{f(\alpha)} \times \mathcal{L}_{f(\beta)}$ that σ_α and σ_β are E-inequivalent.

Let I consist of the Silver indiscernibles between κ and λ, and for $i < j$ in I let π_{ij} be an elementary embedding from L to L with critical point i, sending i to j. As p, the sequence $(\sigma_\alpha \mid \alpha < \lambda)$ and the function f defined above are constructible, they are L-definable from parameters less than some $i \in I$ together with indiscernibles $\geq \lambda$. Then for $j < k$ in I above i, we have $\sigma_k = \pi_{jk}(\sigma_j)$ and $f(k) = \pi_{jk}(f(j))$. Let I_0 be the final segment of I consisting of all elements of I greater than i.

In analogy to the previous proof we show that for each κ-Borel B there is a club C contained in I_0 such that:

(∗) Suppose that $i_0 < i_1 < \cdots < i_n = j < i_{n+1} = k$ belong to C, $(p_0, p_1) \leq (p, p)$ belongs to $\mathcal{L}_{f(j)} \times \mathcal{L}_{f(j)}$ and is L-definable from the parameters in $i_0 \cup \{i_0, i_1, \ldots, i_n\}$ together with indiscernibles $> j$. Then (p_0, p_1) forces that $(\sigma_j^{g_0}, \sigma_j^{g_1})$ belongs to B in the forcing $\mathcal{L}_{f(j)} \times \mathcal{L}_{f(j)}$ iff $(p_0, \pi_{i_0 i_1} \pi_{i_1 i_2} \cdots\ \pi_{i_{n-1} i_n} \pi_{i_n i_{n+1}}(p_1))$ forces that $(\sigma_j^{g_0}, \sigma_k^{g_1})$ belongs to B in the forcing $\mathcal{L}_{f(j)} \times \mathcal{L}_{f(k)}$.

Note that the composition $\pi_{i_0 i_1} \pi_{i_1 i_2} \cdots \pi_{i_{n-1} i_n} \pi_{i_n k}$ sends $(i_0, i_1, \ldots, i_n)$ to $(i_1, i_2, \ldots, i_{n+1})$.

We now prove $(*)$ (with an appropriate choice of C) by induction on the κ-Borel rank of B. If $B = U(\tau_0) \times U(\tau_1)$ is a basic open set then (p_0, p_1) forces that $(\sigma_j^{g_0}, \sigma_j^{g_1})$ belongs to B iff both p_0 forces that $\sigma_j^{g_0}$ belongs to $U(\tau_0)$ and p_1 forces that $\sigma_j^{g_1}$ belongs to $U(\tau_1)$; as the latter is equivalent to $\pi_{i_0 i_1}\pi_{i_1 i_2}\cdots\pi_{i_{n-1} i_n}\pi_{i_n i_{n+1}}(p_1)$ forcing that $\sigma_k^{g_1}$ belongs to $U(\tau_1)$, the conclusion of $(*)$ follows, where we can take C to be the entire club I_0.

Inductively, suppose that B is the intersection of κ-Borel sets B_α, $\alpha < \kappa$, of smaller κ-Borel rank. Then $(*)$ for the B_α's implies $(*)$ for B by intersecting κ clubs.

Finally, suppose that B is the complement of the κ-Borel set B_0 and the club C_0 witnesses $(*)$ for B_0. Let C consist of all limit points of C_0; we show that C witnesses $(*)$ for B. Suppose that $i_0 < i_1 < \cdots < i_n = j < i_{n+1} = k$ and (p_0, p_1) are as in the hypothesis of $(*)$ where $i_0, i_1, \ldots, i_{n+1}$ belong to C. Let π denote the composition $\pi_{i_0 i_1}\pi_{i_1 i_2}\cdots\pi_{i_{n-1} i_n}\pi_{i_n i_{n+1}}$.

If $(p_0, \pi(p_1))$ does not force that $(\sigma_j^{g_0}, \sigma_k^{g_1})$ belongs to B then there is an extension (q_0, q_1) of $(p_0, \pi(p_1))$ in $\mathcal{L}_{f(j)} \times \mathcal{L}_{f(k)}$ which forces that $(\sigma_j^{g_0}, \sigma_k^{g_1})$ belongs to B_0. As i_0 is a limit point of C_0, we can choose $i_{-1} < i_0$ in C_0 (greater than the parameters used in the definition of (p_0, p_1)) so that (q_0, q_1) is L-definable from parameters in $i_0 \cup \{i_0, i_1, \ldots, i_{n+1}\}$ together with parameters $< i_{-1}$ and indiscernibles $> k$. Now we consider the condition (q_0, q_1^*) in $\mathcal{L}_{f(j)} \times \mathcal{L}_{f(j)}$, where q_1^* is defined in L from $i_{-1} < i_0 < \cdots < i_n$ (together with indiscernibles greater than $i_n = j$) just like q_1 is defined from $i_0 < i_1 < \cdots < i_{n+1}$ (together with the same parameters $< i_{-1}$ and indiscernibles greater than $i_{n+1} = k$). By induction, (q_0, q_1^*) forces that $(\sigma_j^{g_0}, \sigma_j^{g_1})$ belongs to B_0. Moreover (q_0, q_1^*) is an extension of (p_0, p_1) as (q_0, q_1) is an extension of $(p_0, \pi(p_1))$ (this implies that q_1^* is an extension of p_1). So (p_0, p_1) does not force that $(\sigma_j^{g_0}, \sigma_j^{g_1})$ belongs to B.

Conversely, suppose that (p_0, p_1) does not force that $(\sigma_j^{g_0}, \sigma_j^{g_1})$ belongs to B. Then there is an extension (q_0, q_1) of (p_0, p_1) which forces that $(\sigma_j^{g_0}, \sigma_j^{g_1})$ belongs to B_0. We may assume that (q_0, q_1) is definable in L from parameters in $i_0 \cup \{i_0, i_1, \ldots, i_n\}$ together with indiscernibles greater than i_n. By induction $(q_0, \pi(q_1))$ forces that $(\sigma_j^{g_0}, \sigma_k^{g_1})$ belongs to B_0 where π is the composition $\pi_{i_0, i_1}\pi_{i_1 i_2}\cdots\pi_{i_{n-1} i_n}\pi_{i_n i_{n+1}}$. This condition extends the condition $(p_0, \pi(p_1))$ and therefore establishes that $(p_0, \pi(p_1))$ does not force that $(\sigma_j^{g_0}, \sigma_j^{g_1})$ belongs to B.

Now apply $(*)$ to the κ-Borel set E, producing a club C. As mentioned before, we can assume that (p, p) does $\mathcal{L}_{f(i)} \times \mathcal{L}_{f(i)}$-force $\sigma_i^{\dot g_0} E \sigma_i^{\dot g_1}$. It follows that for $i < j$ in C, (p, p) also $\mathcal{L}_{f(i)} \times \mathcal{L}_{f(j)}$-forces $\sigma_i E \sigma_j$, as p is not moved by any elementary embedding which is the identity below an element of I_0. But this contradicts our assumption that σ_α, σ_β are forced by (p, p) in $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ to be E-inequivalent when p belongs to $\mathcal{L}_\alpha$ and $\alpha < \beta$. ∎

I close with two remarks. The first is that if $0^{\#}$ exists and κ is an L-cardinal which is countable in V then the Silver dichotomy holds for κ^κ in some *inner model* with the same cardinals as L up to κ. This is because the Lévy collapse forcing which turns the κ^+ of $V = \omega_1$ of V into the κ^+ of the generic extension has a generic in V (it is built as the limit of countable generics along the indiscernibles less than ω_1 of V). The second remark is that I do not know if the above use of $0^{\#}$ is necessary. Surely one needs to start with an inaccessible $\lambda > \kappa$ to obtain the Silver dichotomy by forcing over L (preserving cardinals up to κ) but as far as I know it is indeed possible that inaccessibility is sufficient:

QUESTION. Does the consistency of ZFC plus an inaccessible suffice for the consistency of ZFC plus the Silver dichotomy for the generalised Baire space $\omega_1^{\omega_1}$?

Acknowledgements. This research was supported by the FWF (Austrian Science Foundation) Einzelprojekt P 23316 N13.

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Received 27 November 2013;
in revised form 24 April 2014