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Finitely presented subgroups of systolic groups are systolic

by

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Abstract. We prove that every finitely presented subgroup of a systolic group is itself systolic.

1. Introduction. In the early eighties, Gromov deduced several properties of Riemannian manifolds of non-positive sectional curvature without using the Riemannian structure, but only a property of the induced distance function, which he called a *CAT(0) inequality* [BGS85]. Gromov proved that for a cube complex equipped with a piecewise Euclidean metric, one can locally check the CAT(0) condition in terms of the combinatorial structure of the complex [BH99, Theorem II.5.20].

In [Hag03, JŚ06] the systolic complexes, a simplicial analogue of CAT(0) spaces, were introduced. This property of complexes is also called simplicial non-positive curvature.

Definition 1. A simplicial complex is *flag* if every finite set of vertices that are pairwise connected by edges spans a simplex. A *loop of length* m in a simplicial complex X is a simplicial embedding of an m-cycle into X. An edge connecting two non-consecutive vertices of a loop is called a *diagonal*. The property that every loop of length at least four and less than m has a diagonal is called *m-largeness*. Let $m \geq 6$. A simply connected m-large flag simplicial complex is called *m-systolic*. We write just *systolic* instead of 6-systolic. A group acting properly and cocompactly by simplicial automorphisms on a systolic complex is called *systolic*.

Note that this definition of a systolic complex differs from the original one, but is equivalent to it [JŚ06, Fact 1.2(4) and Corollary 1.6].

The purpose of this note is to prove the following theorem ([1]).

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([1]) The author has learnt that the result was also proven independently in [GM13].

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THEOREM 2. *Any finitely presented subgroup of a systolic group is systolic.*

For torsion-free systolic groups, Theorem 2 was proven by Wise [Wis03, §5]. Wise considers the quotient of a systolic complex under the group action, and his proof does not generalize to groups with torsion. Note that Theorem 2 is not true if we replaces "systolic" with "CAT(0)" [BRS07, §2.3.3].

1.1. Notation and outline. All the paths in any simplicial complex are taken in its 1-skeleton. We use d_X to denote the distance in the 1-skeleton of a simplicial complex X equipped with the combinatorial metric. Given a subcomplex $Z \leq X$, the *r-neighborhood* of Z in X is defined as the simplicial span of all vertices r-close to Z, i.e.

$$N_X^r(Z) = \mathrm{Span}\{x \in X^0 \mid d_X(x, Z^0) \leq r\}.$$

A neighborhood of a single vertex will also be called a *ball* around that vertex. Let G be a group acting properly and cocompactly by automorphisms on a systolic complex X. Let $H \leq G$ have a finite presentation $\langle \mathcal{S} \mid \mathcal{R} \rangle$ with $\mathcal{S}$ symmetric. Let $\mathcal{C}_{\mathcal{S}}(H)$ be the Cayley graph of H with respect to the generating set $\mathcal{S}$. An edge connecting h and hs for $h \in H$ and a generator $s \in \mathcal{S}$ comes equipped with two orientations, one for s and one for s^{-1}, except when $s^2 = \mathbf{1}$, when there are two edges connecting h and hs. Denote by $\mathcal{C}_{\mathcal{S}}^X(H)$ a subdivision of $\mathcal{C}_{\mathcal{S}}(H)$ such that there exists a simplicial H-equivariant map $\phi : \mathcal{C}_{\mathcal{S}}^X(H) \to X$. Let e_s denote the path between $\mathbf{1}$ and $s \in \mathcal{S}$ in $\mathcal{C}_{\mathcal{S}}^X(H)$ that comes from subdivision of the edge connecting $\mathbf{1}$ and s in $\mathcal{C}_{\mathcal{S}}(H)$. Let $x_0 = \phi(\mathbf{1})$ and $\gamma_s = \phi(e_s)$ for $s \in \mathcal{S}$. Denote by L the maximum of the lengths of the γ_s. Denote also

$$\Gamma = \phi(\mathcal{C}_{\mathcal{S}}^X(H)).$$

We will frequently use $s_1, \ldots, s_m$ to denote generators from $\mathcal{S}$. We write $\gamma_{s_1 \cdots s_m}$ for the path which is the concatenation

$$\gamma_{s_1} * (s_1\gamma_{s_2}) * \cdots * (s_1 \cdots s_{m-1}\gamma_{s_m}).$$

An outline of our proof is as follows. In the first step, we find some neighborhood N of Γ in X such that every loop in Γ can be contracted in N. Since there may be homotopically nontrivial loops in N apart from Γ, we rather construct a new space Y from a disjoint union of balls around points in Hx_0 which we glue together H-equivariantly. We build Y such that Γ naturally and H-equivariantly embeds in Y, and moreover every loop in Y is homotopically equivalent to a loop in Γ. Thanks to the conclusions about N from the first step, every loop in Γ is also homotopically trivial in Y. Finally, we extend Y in an appropriate category to a maximal H-cocompact simply connected flag simplicial complex and prove that it is 6-large.

2. Proof of the theorem. We proceed in several steps as mentioned above. In the first step, we find a constant R such that loops in Γ are homotopically trivial in $N_X^R(\Gamma)$. Important properties of loops in Γ deduced in the proof of Step 1 are collected in Fact 1, since we will need them later on.

STEP 1. *There exists a constant $R < \infty$ such that every loop in Γ is homotopically trivial in $N_X^{R-L}(\Gamma)$.*

Proof. After replacing a loop with its H-translate, it is enough to consider loops in Γ containing a point at distance at most L from x_0. We distinguish two main cases. The first case deals with concatenations of H-translates of paths γ_s, $s \in \mathcal{S}$, and the second case handles general loops in Γ, where a loop can leave some $h\gamma_s$ and enter another $h'\gamma_{s'}$ at a point not belonging to Hx_0, i.e. not at the endpoint (resp. starting point) of $h\gamma_s$ (resp. $h'\gamma s'$); see Figure 1.

CASE (1): Loops of the form $\gamma_{s_1\cdots s_m}$ with $s_1 \cdots s_m x_0 = x_0$. There are three subcases:

(a) The word $s_1 \cdots s_m$ belongs to $\mathcal{R}$. Because $\mathcal{R}$ is finite, there is a number R_1 such that every such loop is homotopically trivial in $N_X^{R_1}(z)$ for every vertex $z \in \gamma_{s_1\cdots s_m}$.

(b) The word $s_1 \cdots s_m$ equals $\mathbf{1}$ but it does not belong to $\mathcal{R}$. Then $s_1 \cdots s_m$ is a concatenation of conjugates of relators from $\mathcal{R}$; each such conjugate is homotopically trivial in $N_X^{R_1}(\Gamma)$, hence so is the whole loop.

(c) The point x_0 is fixed under $s_1 \cdots s_m$ but $s_1 \cdots s_m \neq \mathbf{1}$, where $s_1 \cdots s_m$ is the shortest representative of the corresponding group element (all the other representatives differ from the shortest one by concatenation with words considered in Subcases (1.a) or (1.b)). By properness of the G-action and hence of the H-action, the number of elements $h \in H$ fixing x_0 is finite. Hence we can choose a constant R_1' such that $\gamma_{s_1\cdots s_m}$ is homotopically trivial in $N_X^{R_1'}(z)$ for every vertex $z \in \gamma_{s_1\cdots s_m}$.

CASE (2): There are two subcases:

(a) Loops coming from a path $\gamma_{s_1\cdots s_m}$ with self-intersection $x \notin Hx_0$. Without loss of generality we can assume that $x = \gamma_{s_1} \cap (s_1 \cdots s_{m-1}\gamma_{s_m})$. Figure 1 shows such a configuration. Observe that $d_X(x_0, s_1 \cdots s_m x_0) \le 2L$ in this case. By properness of the H-action, there is an upper bound N such that $s_1 \cdots s_m = p_1 \cdots p_k$, where the number k of terms $p_i \in \mathcal{S}$ is at most N. Since

$$s_1 \cdots s_m p_k^{-1} \cdots p_1^{-1} = \mathbf{1},$$

the big loop $\gamma_{s_1\cdots s_m p_k^{-1}\cdots p_1^{-1}}$ is a loop from Subcases (1.a) or (1.b), it can be contracted in $N_X^{R_1}(\Gamma)$. Thus to contract the original loop it suffices to contract

$$(\diamondsuit) \qquad \gamma_{p_1\cdots p_k} * \gamma,$$

where γ is a path in Γ from $s_1\cdots s_m x_0$ to the intersection x of γ_{s_1} and $s_1\cdots s_{m-1}\gamma_{s_m}$ concatenated with a path from x to x_0. But the total length of the loop $(\diamondsuit)$ is at most $(N+2)L$. Let R_2 be a number such that any loop $\gamma_{p_1\cdots p_k} * \gamma$ of type $(\diamondsuit)$ can be contracted in $N_X^{R_2}(z)$ for any vertex $z \in \gamma_{p_1\cdots p_k} * \gamma$.

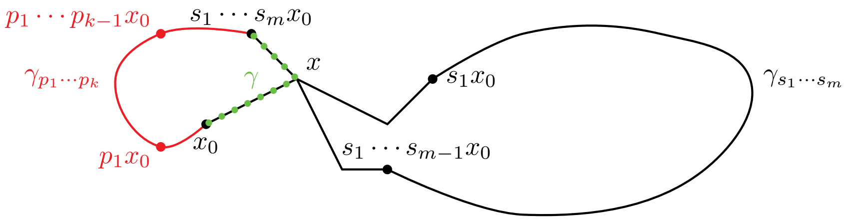

Fig. 1. Loop from Case (2); possibly $x = x_0$ or $x = s_1\cdots s_m x_0$. If both equalities hold, this example is covered by Case (1).

(b) Loops $\alpha = \alpha_1 * \cdots * \alpha_m$, where α_i may be a proper subpath of $h_i\gamma_{s_i}$ for some $h_i \in H$ and the endpoint of α_i coincides with the starting point of α_{i+1}, with indices taken modulo m. Let us denote this intersection by x_i. Let δ_i^+ be a subpath of $h_i\gamma_{s_i}$ from x_i to $h_i s_i x_0$ (the endpoint of $h_i\gamma_{s_i}$) and let δ_i^- be a subpath of $h_i\gamma_{s_i}$ from $h_i x_0$ (the starting point of $h_i\gamma_{s_i}$) to x_{i-1}.

We now proceed as in Subcase (2.a). Since $h_i\gamma_{s_i}$ and $h_{i+1}\gamma_{s_{i+1}}$ intersect at x_i, $d_X(h_i s_i x_0, h_{i+1}x_0) \le 2L$. Therefore for all $i = 1,\dots,m$ there is a path $\eta_i = h_{i+1}\gamma_{s_i^{(1)}\cdots s_i^{(k_i)}}$ in Γ between $h_i s_i x_0$ and $h_{i+1}x_0$ for some $k_i \le N$, hence the length of the loop $\delta_i^+ * \eta_i * \delta_{i+1}^-$ is at most $(N+2)L$, where the constants are as in Subcase (2.a). But this loop is homotopically trivial in $N_X^{R_2}(z)$ for any vertex z of the path $\delta_i^+ * \eta_i * \delta_{i+1}^-$, hence α is homotopic to

$$\alpha_1 * \delta_1^+ * \eta_1 * \delta_2^- * \alpha_2 * \cdots * \alpha_{n-1} * \delta_{n-1}^+ * \delta_n^- * \alpha_n * \delta_n^+ * \eta_n * \delta_1^-,$$

which is a loop from Case (1).

Altogether, the constant $R' = \max\{R_1, R_1', R_2\}$ is such that every loop in Γ is homotopic in $N_X^{R'}(\Gamma)$ to a trivial loop. Hence $R = R' + L$ works. ∎

We will call loops from Subcases (1.a) and (1.c) and loops $(\diamondsuit)$ from Case (2) *short*. From the proof of Step 1 we deduce the following.

FACT 1. *Every loop in Γ is a concatenation of conjugates of short loops. Let γ be a short loop in Γ. Then for every vertex z on γ, the loop γ is homotopically trivial in the ball $N_X^{R-L}(z)$.* ∎

We are now ready to define Y. For every $h \in H$, denote $B_h^0 = N_X^R(hx_0)^0$ and denote the copy of the vertex $v \in X$ in B_h^0 by v^h. Let $\sim$ be the equivalence relation on $\coprod_{h\in H} B_h^0$ which is the transitive closure of the relation $v^h \sim v^h s$ for $v \in B_h^0 \cap B_{hs}^0$ and $s \in \mathcal{S}$. Let

$$Y^0 = \Big(\coprod_{h\in H} B_h^0\Big)/{\sim}.$$

Note that B_h^0 injects into Y^0. For $y \in Y^0$ we write $\bar{y}$ for the vertex of X such that $y = \bar{y}^h$ for some $h \in H$. Next, we define Y^1. We connect two vertices $y, z \in Y^0$ by an edge if there exist representatives $\bar{y}^h$, $\bar{z}^g$ for y and z with $h = g$ and $\bar{y}, \bar{z}$ adjacent. Let Y be the flag completion of Y^1 and let B_h be the simplicial span of $B_h^0 \le Y$. We consider the natural action of H on Y, which is induced from the H-action on X. Note that $hB_{\mathbf{1}} = B_h$, hence the H-action on Y is proper and cocompact.

Observe that by the construction of Y, there exists a proper H-equivariant map $f : Y \to X$. It is defined by $f(y) = \bar{y}$ for $y \in Y^0$ and extends simplicially to higher-dimensional simplices. Let us define local sections

$$i_h : N_X^R(hx_0) \to B_h \tag{$\heartsuit$}$$

by $i_h(u) = u^h$ for every $u \in N_X^R(hx_0)^0$ and every $h \in H$. By definition of Y, each map i_h is bijective on 0-skeleta. Furthermore, vertices $u^h, v^h \in B_h^0$ are adjacent if and only if $u, v \in N_X^R(hx_0)^0$ are adjacent. Hence i_h is a well defined isomorphism between the 1-skeleta of $N_X^R(hx_0)$ and B_h. Since a flag complex is determined by its 1-skeleton, i_h is an isomorphism. Note that B_h might not be a ball in Y.

Observe that for any $h \in H$ and $s \in \mathcal{S}$ the two maps i_h and i_{hs} agree on $N_X^R(hx_0) \cap N_X^R(hsx_0)$ because they agree on the 0-skeleton of that intersection. Hence, there is a natural map $\varphi : \mathcal{C}_{\mathcal{S}}^X(H) \to Y$, sending the edge path he_s to $i_h(h\gamma_s)$. We can as well describe the map φ in terms of the H-action on Y, but the above definition is more useful for us. The next step ensures that φ has nice properties.

Step 2. *The map φ factors through $\phi : \mathcal{C}_{\mathcal{S}}^X(H) \to \Gamma$.*

Proof. We have to check that if two points of $\mathcal{C}_{\mathcal{S}}^X(H)$ are identified under ϕ, they are also identified under φ. To see this, observe that if two paths γ_s and $h\gamma_{s'}$ in Γ have a common point, where $s, s' \in \mathcal{S}$ and $h \in H$, then there is a sequence of generators $p_1, \ldots, p_k \in \mathcal{S}$ such that $p_1 \cdots p_k$ is the shortest word representing h. In particular $p_1 \cdots p_k e_{s'} = he_{s'}$. It follows from Fact 1 that the ball $N_X^{R-L}(p_1 \cdots p_l x_0)$ contains the whole path $\gamma_{p_1 \cdots p_k}$ for all $l = 0, 1, \ldots, k$, where the empty word represents $\mathbf{1}$. Thus if we write

$\overline{\gamma}_s$ for γ_s with opposite orientation, then $N^R_X(p_1\cdots p_l x_0)$ contains the path $\gamma = (h\gamma_{s'}) * \gamma_{p_1\cdots p_k} * \overline{\gamma}_s$ for all $l = 0, 1, \dots, k$. Hence $B_{p_1\cdots p_l}$ contains $i_{p_1\cdots p_l}(\gamma)$ for all $l = 0, 1, \dots, k$. Since two consecutive maps $i_{p_1\cdots p_{l-1}}$ and $i_{p_1\cdots p_l}$ agree on the intersection of their domains, the path $i_{\mathbf{1}}(\gamma) = i_h(\gamma)$ is contained in $\bigcap_{l=0}^{k} B_{p_1\cdots p_l}$. Thus each point on e_s is identified with the appropriate point on $he_{s'}$ under φ. This finishes the proof. ∎

By Step 2, there exists a lift $f_\Gamma : \Gamma \to Y$ of the map $f : Y \to X$. Obviously f_Γ agrees with i_h on $\Gamma \cap N^R_X(hx_0)$. From now on, we identify Γ with its f_Γ-image.

Step 3. *The complex Y is simply connected.*

As mentioned in the outline, we first prove that Y encodes an appropriate neighborhood of Γ such that loops in Γ are homotopically trivial in Y. Then we exhibit a homotopy from any loop in Y to a loop in Γ.

Proof. Take any loop γ in $\Gamma \subseteq Y$. By Fact 1, it is a concatenation of short loops. In the same way as in the proof of Step 2 one can show that each short loop γ' is fully contained in B_h for each $h \in H$ such that $d_Y(hx_0, \gamma') \le L$. Pick such an $h \in H$. We know that B_h is isomorphic to $N^R_X(hx_0)$ via the map i_h from (♡). By Fact 1 once again, the loop γ' is homotopically trivial in B_h, hence γ is homotopically trivial in Y.

Finally, we need to show that every loop in Y is homotopic to a loop in Γ. Let $\beta : S^1 \to Y^1$ be a loop in the one-skeleton of Y. We identify S^1 with $I/\partial I$, where $I = [0, 1]$. Let $0 \le t_0 < t_1 < \cdots < t_n < 1$ be cyclically ordered points on S^1 and $h_0, h_1, \dots, h_n$ elements of H such that $\beta(t_i) \in Y^0$ and $\beta([t_i, t_{i+1}]) \in B_{h_i}$ for all $i = 0, 1, \dots, n$, with indices taken modulo $n+1$. Since $B_{h_{i-1}}$ and B_{h_i} both contain $\beta(t_i)$, there is a sequence of generators $s^i_1, \dots, s^i_{n(i)} \in \mathcal{S}$ such that $h_i = h_{i-1}s^i_1 \cdots s^i_{n(i)}$ and $\beta(t_i) \in B_{h_{i-1}s^i_1\cdots s^i_l}$ for all $l = 0, 1, \dots, n(i)$. Recall that the empty word stands for $\mathbf{1}$. This means that there exist geodesics

$$\beta^i_l : (I, \partial I) \to \big(B_{h_{i-1}s_1\cdots s_l}, \{\beta(t_i), h_{i-1}s_1 \cdots s_l x_0\}\big)$$
$$\text{for all } i = 0, 1, \dots, n \text{ and } l = 0, 1, \dots, s(i).$$

Recall that $h_{i-1}s_1 \cdots s_l x_0$ also belongs to $B_{h_{i-1}s_1\cdots s_{l-1}}$. Since balls in systolic complexes are geodesically convex [HŚ08, Corollary 4.10], the image of β^i_l is contained in $B_{h_{i-1}s_1\cdots s_{l-1}}$. Next, we can find some $\varepsilon > 0$ such that $t_i + n(i)\varepsilon < t_{i+1}$ for all i. After composing β with a map $S^1 \to S^1$ homotopic to the identity, we can assume that β is constant on $[t_i, t_i + n(i)\varepsilon]$ for all $i = 0, 1, \dots, n$. Using the fact that balls in systolic complexes are contractible, we infer that

- for all $i = 0, 1, \ldots, n$ and $l = 1, \ldots, n(i)$ there is
$$H^i_l : [t_i + (l-1)\varepsilon, t_i + l\varepsilon] \times I \to B_{h_{i-1}s_1\cdots s_{l-1}} \cap B_{h_{i-1}s_1\cdots s_l}$$
with $H^i_l(t_i + (l-1)\varepsilon, t) = \beta^i_{l-1}(t)$ and $H^i_l(t_i + l\varepsilon, t) = \beta^i_l(t)$, where $H^i_l(-, 0)$ is the constant path $\beta(t_i)$, and $H^i_l(-, 1)$ is the path $h_{i-1}s_1 \cdots s_{l-1}\gamma_{s_l} \subseteq \Gamma$;
- for all $i = 0, 1, \ldots, n$ there is $H^i : [t_i + n(i)\varepsilon, t_{i+1}] \times I \to B_{h_i}$ with $H^i(t_i + n(i)\varepsilon, t) = \beta^i_{n(i)}(t)$ and $H^i(t_{i+1}, t) = \beta^{i+1}_0(t)$, where $H(-, 0) = \beta|_{[t_i+n(i)\varepsilon, t_{i+1}]}$ and $H(-, 1)$ is the constant path $h_i x_0$.

Since the above homotopies agree on the intersection of their domains, they glue together to a homotopy $H : S^1 \times I \to Y$ with $H(-, 0) = \beta$ and $H(-, 1)$ a loop in Γ. ■

In the following step, using f we extend Y to a systolic complex $\overline{Y}$ on which H still acts properly and cocompactly, and is thus a systolic group.

We say that a pair (W, f_W) is an *f-extension of Y* if the following holds. The complex W is a simply connected flag simplicial complex containing Y such that $Y^0 = W^0$ and the H-action on Y extends to an H-action on W. Furthermore, $f_W : W \to X$ is a simplicial H-equivariant map which extends f. Note that f_W maps an edge of W either to an edge or to a vertex of X.

Let $\mathcal{F}$ be the family of all f-extensions of Y. Observe that $\mathcal{F}$ is equipped with a natural partial order $\le$, where $(W_1, f_{W_1}) \le (W_2, f_{W_2})$ if there exists an H-equivariant embedding $i : W_1 \to W_2$ fixing Y such that $f_{W_2} \circ i = f_{W_1}$. The family $\mathcal{F}$ is non-empty since it contains (Y, f) by Step 3. Let $(W_\lambda, f_{W_\lambda})_{\lambda\in\Lambda}$ be an increasing chain in $\mathcal{F}$. Then $(\bigcup_\lambda W_\lambda, \bigcup_\lambda f_{W_\lambda})$ is also in $\mathcal{F}$, so it is an upper bound for $(W_\lambda, f_{W_\lambda})_{\lambda\in\Lambda}$. By the Kuratowski–Zorn Lemma, there exists a maximal element $(\overline{Y}, \overline{f}) \in \mathcal{F}$.

STEP 4. *For a maximal element $(\overline{Y}, \overline{f}) \in \mathcal{F}$, the simplicial complex $\overline{Y}$ is a systolic complex, equipped with a proper and cocompact H-action.*

Proof. We claim that the valence in $\overline{Y}^1$ of each $y \in \overline{Y}^0$ is bounded from above. Recall that $\overline{f}$ and f agree on $\overline{Y}^0 = Y^0$. Let $N_y \subseteq Y^0$ denote the set of all vertices adjacent to y in $\overline{Y}$. For every $y' \in N_y$, either $f(y') = f(y)$ or $f(y')$ is adjacent to $f(y)$. This means that $f(N_y) \subseteq N^1_X(f(y))$. In other words, $N_y \subseteq f^{-1}(N^1_X(f(y)))$. Because X is proper, the ball $N^1_X(f(y))$ is compact. But f is a proper map, hence the set $f^{-1}(N^1_X(f(y)))$ is compact and the claim is proven. In particular, $\overline{Y}$ is a proper simplicial complex. Because the vertex sets of Y and $\overline{Y}$ coincide, the action of H on $\overline{Y}$ is proper and cocompact. By definition, $\overline{Y}$ is flag and simply connected. It remains to prove 6-largeness.

Suppose for contradiction that there is some loop α of length four or five in $\overline{Y}$ without diagonals. If $\overline{f}$ maps α bijectively to $\alpha' = \overline{f}(\alpha) \subseteq X$, then there exist two non-consecutive vertices u' and v' of α' connected by a diagonal, because X is systolic. Let u and v be the vertices of α mapped to u' and v' by $\overline{f}$. For every $h \in H$, we add an edge in Y between hu and hv and extend $\overline{f}$ to the new edges naturally. Let us remind the reader that if for n different $h_1, \ldots, h_n \in H$ all the sets $\{h_i u, h_i v\}$ coincide for $i = 1, \ldots, n$, we only add one edge between $h_1 u$ and $h_1 v$ instead of n. This remark will be applied two more times without explicit mention.

If $\overline{f}$ is not bijective on α there must be two vertices u, v of α, which are mapped by $\overline{f}$ to the same vertex. If they are non-consecutive in α, we add edges between hu and hv for every $h \in H$ and extend f so that it maps any new edge to the common image of its endpoints. If u and v are consecutive, let $w \neq u$ be the other neighbor of v in α. Then we add edges between hu and hw for all $h \in H$. Note that since $\overline{f}(u) = \overline{f}(v)$, the point $\overline{f}(w)$ either is adjacent to $\overline{f}(u)$ or coincides with $\overline{f}(u)$, and hence we can extend $\overline{f}$ to the newly added edges.

In all cases, we have added the H-orbit of an edge to $\overline{Y}$. After flag completion, we obtain a flag simplicial complex $\hat{Y}$ on the set of vertices Y^0, properly containing $\overline{Y}$, together with a map $\hat{f}$ extending $\overline{f}$, and equipped with an H-action extending the H-action on $\overline{Y}$. The complex $\hat{Y}$ is also simply connected. Indeed, every edge e in $\hat{Y}^1 - \overline{Y}^1$ is a diagonal of a loop α in $\overline{Y}^1$ of length less than six. This means that e together with two consecutive edges of α form a triangle, which is filled after flag completion. Hence the path e is homotopic relative to its endpoints to a path of length two in $\overline{Y}$. In other words, any loop in $\hat{Y}$ is homotopic to a loop in $\overline{Y}$, and the latter is simply connected since it belongs to $\mathcal{F}$. Hence $(\overline{Y}, \overline{f}) \lneq (\hat{Y}, \hat{f}) \in \mathcal{F}$, which contradicts the maximality of $(\overline{Y}, \overline{f})$. ■

Remark 3. An example showing the necessity of finite presentability in Theorem 2 is due to Stallings. Denote by $\langle x, y\rangle$ the free group of rank two generated by x and y. In [Sta63] (see also [BRS07, §2.4.2]) Stallings proved that the kernel K of the homomorphism

$$\tau : \langle a, b\rangle \times \langle x, y\rangle \to \mathbb{Z}, \quad \tau(a) = \tau(b) = \tau(x) = \tau(y) = 1,$$

is finitely generated, but not finitely presentable. Hence K cannot be systolic. On the other hand, the direct product of two free groups of rank two is systolic [EP11].

Even in the case where G is hyperbolic, one cannot hope for a generalization of the theorem above. By the Rips Construction [Rip82], for each finitely presented group Q and every $\lambda > 0$, there exists a finitely presented $C'(\lambda)$ small cancellation group G and a short exact sequence

$\{\mathbf{1}\} \to N \to G \to Q \to \{\mathbf{1}\}$, where N is a finitely generated normal subgroup of G. By [Bie81], the group N is finitely presentable if and only if Q is finite. Hence, if we choose $Q = \mathbb{Z}$ and $\lambda = 1/6$, then the Rips Construction gives a finitely presented $C'(1/6)$ small cancellation group G which is hyperbolic [Gro87] and $C(7)$ [LS77, Chapter V, §2]. By [Wis03], the $C(7)$ group G is 7-systolic. But it has a finitely generated not finitely presentable subgroup N, hence a finitely generated non-systolic subgroup. In particular, systolic and even 7-systolic groups are not coherent in general.

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