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# Some pinching deformations of the Fatou function

by

**Patricia Domínguez** (Puebla) and **Guillermo Sienra** (México, D.F.)

**Abstract.** We are interested in deformations of Baker domains by a pinching process in curves. In this paper we deform the Fatou function $F(z) = z + 1 + e^{-z}$, depending on the curves selected, to any map of the form $F_{p/q}(z) = z + e^{-z} + 2\pi i p/q$, $p/q$ a rational number. This process deforms a function with a doubly parabolic Baker domain into a function with an infinite number of doubly parabolic periodic Baker domains if $p = 0$, otherwise to a function with wandering domains. Finally, we show that certain attracting domains can be deformed by a pinching process into doubly parabolic Baker domains.

**1. Introduction.** Given a holomorphic map $f$, the *Fatou set* $\Omega(f) \subset \mathbb{C}$ of $f$ is the largest open set such that the iterates $\{f^n\}_{n\geq 0}$ form a normal family. Its complement $\mathbb{C} - \Omega(f)$ is the *Julia set*, denoted by $J(f)$.

The *Fatou function* is the transcendental entire map $F(z) = z+1+e^{-z}$. In this paper we study pinching deformations of the Fatou function and the changes in their dynamics, specifically the Fatou and Julia sets. A *deformation* of an entire map $f$ is an entire map $g$ such that $q^{-1} \circ f \circ q = g$, where $q$ is a quasiconformal homeomorphism. For pinching deformations the support of $q$ is located in certain neighborhoods of invariant curves in the Fatou set of $f$. Pinching deformation was introduced in iteration theory by Makienko [M2] in the context of the dynamics of rational maps and by Petersen [P] for quadratic maps. Several interesting problems arise when deforming a given map $f$, like the convergence or divergence of certain subsequences; see for instance [H], [HT] and [T]. The pinching deformation in this paper will be carried out in Fatou domains which do not exist for rational maps, otherwise the theory extends in the same way.

The following different types of Fatou domains exist for transcendental entire maps but do not exist for rational maps.

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Definition 1.1 (Baker domain). If $V$ is a periodic component of the Fatou set of period $p$ such that for all $z \in V$, $f^{np}(z) \to \infty$ as $n \to \infty$, then $V$ is called a *periodic Baker domain.*

Definition 1.2 (Wandering domain). If $V$ is a component of the Fatou set such that $f^n(V) \not\subset V$ for all $n > 0$, then $V$ is called a *wandering domain.*

As we will explain in Section 2, the Fatou function has a completely invariant Baker domain that contains the curves $y = (1 + 2k)\pi i$, $k \in \mathbb{Z}$, which are invariant under the Fatou function $F(z)$. We will pinch the Fatou function along these curves, and in Section 3 we define the process of pinching; for completeness we state and prove an important lemma taken from Tan Lei [T] (see Lemma 3.6) which shows that the diameter of the pinching curves tends to zero under the process. We make a slight rearrangement to fit the Tan Lei Lemma into the transcendental entire framework; moreover, we introduce the concept of wandering admissible pairs that will be necessary in the proof of the main theorems.

The Teichmüller space encodes the deformation of a map $f$ by keeping the information of a quasiconformal homeomorphism $q$ compatible with the dynamics of $f$ (see Section 4). For a given transcendental entire map $f$, the framework of the Teichmüller space of $f$, $T(f)$, is taken from Harada and Taniguchi [HaTa] in which they generalize the concepts in [MS] to the case that the closure in $\mathbb{C}$ of the singular values is countable. Later on Fagella and Henriksen [FH1], [FH2] completed the general case. The Fatou function in this case was studied by Harada and Taniguchi. In Section 4, we define the set $E_f$ of all transcendental entire maps quasiconformally equivalent to $f$, and we consider the projection of the Teichmüller space into $E_f$ by means of the representation map $\rho : T(f) \to E_f$. Now, pinching deformations are paths in $T(f)$ that project to paths in the image of the map $\rho$. We ask: for a given map $f$, are limits of deformations in $E_f$? We show that for pinching deformations (in certain curves) on the Fatou domain of the Fatou function, the limits exist and we describe them. Our main results in Section 5 are stated as follows.

Theorem 5.4. *For $c = 1-t$, $t \in [0,1]$, the corresponding maps $F_c(z) = z+c+e^{-z}$ can be regarded as a path in $\rho(T(F))$ of pinching deformations of the Fatou function $F(z) = F_1(z)$ with limit the Baker–Domínguez function $G(z) = F_0(z)$.*

Theorem 5.5. *There is a pinching process from the Fatou function $F$ to any of the functions $F_{p/q}(z) = z + e^{-z} + 2\pi ip/q$, $p/q$ a rational number. The Fatou domain of such functions has a wandering domain if $p \neq 0$.*

We show in Proposition 5.6 that if $p \neq 0$, then $\dim T(F_{p/q}) = \infty$, hence none of these functions are rigid, in contrast with $G(z) = F_0$ which is rigid (see [FH1]). In Corollary 5.7, we show that the map $F_{p/q}$ with $p/q = n+1$ has $n$ wandering domains.

In the following theorem we show that there is another transcendental entire function $H(z)$ which has infinitely many attracting domains. After applying a certain pinching process, the map $H(z)$ ends in a function with doubly parabolic domains.

THEOREM 5.8. *For $c = (1/2)(t-1)$, $t \in [0,1]$, the corresponding maps $H_t(z) = z + c + e^{-z}$ can be regarded as a path in $\rho(T(H))$ of pinching deformation of the function $H_0(z)$ with limit the Baker–Domínguez function $G(z) = F_0(z) = H_1(z)$.*

The map $H(z) = H_0(z)$ is not rigid but the map $G(z)$ is rigid.

**2. The Fatou and Baker–Domínguez functions.** In this section we introduce the three main functions that will be used in Section 5. We state some of their properties as well as the notation that we will be using along the paper.

**2.1. The Fatou function.** The Fatou function $F(z) = z + 1 + e^{-z}$ is a lift (under the map $\exp(-z)$) of the map $g : \mathbb{C}^* \to \mathbb{C}^*$ given by $g_{1/e}(w) = \frac{1}{e}we^{-w}$. The map $g$ has 0 as an attractor and has a unique critical point at $w = 1$ which necessarily is in $B$, the basin of attraction of 0. See [F] and [BD, Section 7].

The lift of $B$ is the Baker domain $V$ of $F$ and contains the right half-plane. Baker proved that any multiply connected Fatou domain of a transcendental entire map is a wandering domain, therefore periodic Baker domains are simply connected (see [Ber2]). The function $F|_V$ is not univalent since $V$ contains all the critical points of $F$ which are $2\pi ik$ ($k \in \mathbb{Z}$). The function $F$ is close to the function $z \mapsto z+1$ in the right half-plane near infinity, therefore $V$ is classified as a doubly parabolic Baker domain, according to the classification given in Section 2.4 below. The complexity of this kind of domain is given by a theorem of Bergmann (see [Be]), who proved that for $\varphi : \mathbb{D} \to V$ a conformal equivalence and $\Theta$ the sets of points $\xi \in \partial\mathbb{D}$ such that $\lim_{r\to 1}(\varphi(r\xi)) = \infty$, we have $\overline{\Theta} = \partial\mathbb{D}$ provided $V$ is doubly parabolic.

**2.2. The Baker–Domínguez function.** Let us denote by $G(z) = z + e^{-z}$ a lift of the map $g_1(w) = we^{-w}$. Baker and Domínguez [BD] show that the Fatou set of $G$ has a component $V_1$ that contains the real axis and for every $z \in V_1$, $\mathrm{Re}(f^n(z)) \to \infty$ as $n \to \infty$. Moreover, since the map commutes with the translations $z + 2\pi ik$, $k \in \mathbb{Z}$, the Julia set of $G$ contains the lines $y = \pi(1+2k)$. In between these lines there is a uniquely defined component $V_k$ of the Fatou set and the set $\Theta_k$ is dense in $\partial\mathbb{D}$, with $\varphi_k : \mathbb{D} \to V_k$ a conformal equivalence for each $k$.

The critical points of $G$ are $2\pi ik$ and each of them belongs to $V_k$. Each $V_k$ is a doubly parabolic Baker domain.

**2.3. Another function.** Let us denote by $H(z) = z - 1/2 + e^{-z}$ a lift of the map $g_{e^{1/2}}(w) = e^{1/2}we^{-w}$ under $\pi(z) = \exp(-z)$. For the map $g_{e^{1/2}}$, the origin is a repelling fixed point, $w = 1/2$ is an attracting fixed point and $w = 1$ is the unique critical point which is contained in the basin of attraction. The interval $[0, 1/2]$ is in the basin of attraction and $[-\infty, 0]$ belongs to the Julia set. By [Ber2] or [M1], $\pi^{-1}J(g_{e^{1/2}}) = J(H)$, therefore $H$ has the lines $y = \pi(1+2k)$ in the Julia set. In between these lines there is an attractor $W_k$ with an attracting point at $1/2+2\pi ik$, $k \in \mathbb{Z}$. By a theorem of Kisaka, the boundary of $W_k$ is disconnected, and by a result of Baker and Domínguez [BD], $\Theta$ is dense in $\partial D$.

Clearly the Fatou map $F$ and $H$ are not topologically conjugate.

**2.4. Baker domains and their classification.** The classification of Baker domains is as follows (see [FH1], [K]).

If $f$ is transcendental entire and $U$ is a Baker domain, then $U/f$ is a Riemann surface conformally equivalent to one of the following cylinders:

(1) $\{-s < \mathrm{Im}(z) < s\}/\mathbb{Z}$ for some $s > 0$ and we call $U$ *hyperbolic.*
(2) $\{\mathrm{Im}(z) > 0\}/\mathbb{Z}$ and we call $U$ *simply parabolic.*
(3) $\mathbb{C}/\mathbb{Z}$ and we call $U$ *doubly parabolic.* In this case, either $f : U \to U$ is not proper, or it has degree at least 2.

Examples of Baker domains of higher period were given by Baker, Kotus and Lü [BKL] (see the survey [Ber2] for some references). Baker domains without singularities in their interior (univalent Baker domains) were constructed by Bergweiler [Ber1], and later a classification of this kind of domains was given by Barański and Fagella [BF] and Koning [K].

It was proved by Baker [B] that all Baker domains for transcendental entire maps are simply connected. On the other hand, transcendental entire functions of order less than 1/2 have bounded Fatou components, hence do not have Baker domains (see [HM]).

There are functions with infinitely many cycles of $p$-invariant Baker domains as proved by Rippon and Stallard [RS1]. In [RS2] it is shown that there exist functions with $p$-cycles of univalent Baker domains.

## 3. Pinching process.

For completeness we reproduce the definitions as in Tan [T] with the addition of the concept of wandering admissible pairs (see Definition 3.3). This treatment of the subject is more general than the one in [M2] and suits our needs quite well.

DEFINITION 3.1 (The model system). For $l > 0$, let $B_l$ denote the horizontal strip $\{x + iy : |y| < \pi/(2l)\}$, and $T_\sigma(z) = z + \sigma$. We consider the couple $(B_l, T_\sigma)$ as a model dynamical system, with $\mathbb{R}$ as the central line. For the system $(B_{l_0}, T_1)$ and for any $z \in \mathbb{R}$, the distance $d_{B_{l_0}}(z, T_1(z))$ coincides with the hyperbolic length of the segment $[z, T_1(z)]$ and is exactly $l_0$. Therefore $l_{B_{l_0}}(\mathbb{R}, T_1) = l_0$ and $l_{B_{l_0}}(\mathbb{R}, T_{l/l_0}) = l_0$.

Definition 3.2 (The pinching model). For $l_0 > 0$ and $t \in [0,1)$, let $t \mapsto l_t$ be a decreasing continuous function tending to 0 as $t \to 1^-$. Choose a quasiconformal map $S_t : B_{l_0} \to B_{l_0}$ such that $S_t(z)$ is a $C^1$-function of $(t,z)$, $S_t(\mathbb{R}) = \mathbb{R}$ and $S_t$ conjugates $(B_{l_0}, T_1)$ to $(B_{l_0}, T_{l/l_0})$. As $t \to 1^-$, $l_t = l_{B_{l_0}}(\mathbb{R}, T_{l/l_0}) \to 0$ and the quasiconformal constant of $S_t$ tends to $\infty$.

Definition 3.3 (Admissible pairs). For $f : \mathbb{C} \to \mathbb{C}$ an entire map and $k \in \mathbb{N}$, an open arc $\gamma \subset \mathbb{C}$ together with a neighborhood $U$ of $\gamma$ is called $f^k$-admissible if:

1. Either $\gamma$ is *periodic admissible*:
   - $f^k(\gamma) = \gamma$, $f^k(U) = U$, $f^k|_U$ is univalent and $U$, $f(U), \ldots, f^{k-1}(U)$ are mutually disjoint,

   or $\gamma$ is *wandering admissible*:
   - $f^k|_U$ is univalent and $U$, $f(U), \ldots, f^{k-1}(U), \ldots$ are mutually disjoint.
2. There is a conformal normalization $\Phi : (U, \gamma, f^k) \to (B_{l_0}, \mathbb{R}, T_1)$ for some $l_0 > 0$.

Definition 3.4 (Pinching). Let $(\gamma, U)$ be an admissible pair for $f$. Fix a choice of $S_t$ on $B_{l_0}$. Denote by $E_t|_U$ the pulled back ellipse field of the circle field in $B_{l_0}$ by $S_t \circ \Psi$. With the help of $f$, generate an invariant ellipse field $E_t$ conformal outside the grand orbit of $U$. Choose a quasiconformal map $\varphi_t : (\hat{\mathbb{C}}, E_t) \to (\hat{\mathbb{C}})$ integrating $E_t$ by the Ahlfors–Bers theorem and set $f_t = \varphi_t \circ f \circ \varphi_t^{-1}$. We call $(\varphi_t, f_t)_{t \in [0,1)}$ a *path of pinching deformations* of $f$ along $(\gamma, U)$. Note that if $\varphi_t$ is replaced by $H_t \circ \varphi_t$, where $H_t$ is a Möbius transformation, then $f_t$ is replaced by $H_t \circ f_t \circ H_t^{-1}$.

Several results related to the convergence of the pinching process can be found in [BH], [H] and [HT].

Let us denote by $U_t$ the image of $U$ under $h_t$, a family of homeomorphisms; similarly $\gamma_t$ is the image of $\gamma$ and denote by $\operatorname{diam}_\sigma(X)$ the spherical diameter of the set $X$.

Definition 3.5 (Dynamical length [T]). For a map $f$, we say that a set $\gamma$ is *$f$-invariant* if $f(\gamma) \subset \gamma$. Furthermore, if $\gamma \subset V \subset \hat{\mathbb{C}}$, with $V$ a hyperbolic open set, we define the *dynamical length* $l_V(\gamma, f)$ relative to $V$ to be

$$l_V(\gamma, f) = \sup_{z \in \gamma} d_V(z, f(z)),$$

where $d_V$ denotes the hyperbolic metric on $V$.

Remark 1 in [T] shows that for a path of pinching deformations, the path $(\gamma_t, U_t)$ is $f_t$-admissible and $\varphi_t$ shrinks the corresponding dynamical length to 0, in other words $l_{U_t}(\gamma_t, f_t) = l_t \to 0$ as $t \to 1$.

The following proof taken from [T] (Lemma 2.1 and Proposition B) shows that in the limit the diameter of the pinched curves shrinks to a point. We follow the proof of Tan Lei with small changes suited to our situation.

To begin with, we write $f_n \rightrightarrows g$ when $f_n$ converges uniformly to $g$ on compact subsets of $\mathbb{C}$ and for simplicity set $\gamma_n = h_n(\gamma)$.

Lemma 3.6. *For $f$ a transcendental entire map, let $f_n$ be a sequence of entire maps topologically conjugate to $f$ with $h_n$ as conjugacies. Assume that $l_{h_nU}(h_n(\gamma), f_n^k) \to 0$ and $f_n \rightrightarrows g$. Then $\lim_{n\to\infty} \operatorname{diam}_\sigma h_n(\bar{\gamma}) = 0$.*

*Proof.* The proof is divided into four steps. Taking a subsequence if necessary, we start by assuming that $\bar{\gamma}_n \to Y_\infty \subset \hat{\mathbb{C}}$ in the Hausdorff topology on compact sets and $\operatorname{diam}_\sigma Y_\infty > 0$; since $\gamma_n$ is connected, $Y_\infty$ is a continuum.

Normalization: Without loss of generality we can assume that all $U_n$ are in $\mathbb{C}^*$.

Inequalities (see [M, Appendix A.8]): Let us consider the following inequalities: For any simply connected domain $V \subset \mathbb{C}^*$, any $z \in V$, and for $\lambda_V(z)$ the coefficient function of the hyperbolic metric on $V$, we have

$$\lambda_V(z) \geq \frac{1}{2d_{\mathbb{C}}(z, \partial V)} \geq \frac{1}{2|z|}.$$

Set $\eta = 1/(2|z|)$. Let $d_\eta$ denote the euclidean metric on $\mathbb{C}^*$ with $\eta$ as coefficient function. Then for any arc $\kappa \subset V \subset \mathbb{C}^*$, we have $\operatorname{length}_\eta(\kappa) \leq \operatorname{length}_V(\kappa)$.

Claim. *For any $n$ and any $z \in \gamma_n$, we have $d_\eta(f_n(z), z) \leq l_{U_n}(\gamma_n, f_n)$.*

*Proof.* Let $\kappa$ be the subarc of $\gamma_n$ between $f_n(z)$ and $z$. Then

$$d_\eta(f_n(z), z) \leq \operatorname{length}_\eta(\kappa) \leq \operatorname{length}_{U_n}(\kappa) = l_{U_n}(\gamma_n, f_n)$$

by the assumption $l_{U_n}(\gamma_n, f_n) \to 0$ as $n \to \infty$.

Contradiction: Note that $f_n \rightrightarrows g$. Let $x \in Y_\infty \bigcap \mathbb{C}^*$ and $K \subset \mathbb{C}^*$ some compact neighborhood of $x$ in $\mathbb{C}^*$. Now, choose a sequence $x_n \in \gamma_n$ such that $x_n \to x$ as $n \to \infty$. Since the convergence of $f_n$ to $g$ is uniform in compact subsets of the plane, it follows that for $n$ large enough, $x_n$ is in $K$. Then

$$d_\eta(g(x), x) \leq d_\eta(g(x), g(x_n)) + d_\eta(g(x_n), f_n(x_n)) + d_\eta(f_n(x_n), x_n) + d_\eta(x_n, x).$$

The right hand side converges to 0 as $n \to \infty$, since $g$ is continuous, $f_n \rightrightarrows g$ and $d_\eta(f_n(x_n), x_n) \leq l_{U_n}(\gamma_n, f_n) \to 0$.

Thus $g(x) = x$. Moreover, the equality $g(x) = x$ holds for all $x \in Y_\infty \cap K$. This implies that $g$ is the identity in $Y_\infty \cap K$, which by hypothesis is a nonempty continuum with diameter greater than zero; this is a contradiction since $g$ is a transcendental entire function. ∎

**4. Teichmüller space.** Extending the work of Sullivan and McMullen [MS] on rational maps, the study of deformations of transcendental entire functions was carried out first by Harada and Taniguchi [HaTa] in the case that the singular values are a discrete set in $\mathbb{C}$. Then Fagella and Henriksen [FH1], [FH2] generalized the results of Harada and Taniguchi without restrictions. In this section we give the definitions of concepts that we will use afterwards.

For a given function $f$ defined on a domain $A \subset \mathbb{C}$, we say that $z, w \in A$ are *grand orbit equivalent* if there are positive integers $n, m$ such that $f^n(z) = f^m(w)$; the class of $z$ is denoted by $[z]$ and called the *grand orbit of* $z$. If $f : A \to A$ is a covering, then $A/f$ denotes the set of grand orbits of $f$, with the quotient topology. For a subset $V$ of $A$, the set $[V]_f = \{w : w \in [z] \text{ for some } z \in V\}$ is called the *grand orbit of* $V$. Typically, for a transcendental entire map the domain $A$ is taken to be equal to $\mathbb{C}$, and $V$ is a Fatou domain.

We say that two quasiconformal automorphisms $\varphi_1, \varphi_2$ of $A$ are *equivalent* if there exists a conformal automorphism $\psi$ of $A$ such that $\varphi_1 = \psi \circ \varphi_2$.

Definition 4.1. Let $\mathrm{Def}(f, A)$, the deformation space of $f$, be the set of equivalence classes of quasiconformal automorphisms $\varphi$ of $A$ which satisfy $\varphi \circ f = g \circ \varphi$ for some holomorphic map $g$ on $A$. We then say that $g$ is *quasiconformally conjugate* to $f$.

The measurable Riemann mapping theorem implies that $\mathrm{Def}(f, A)$ is identified with the unit ball of the space of all invariant Beltrami differentials for $f$ (see [MS]). Then we identify the Beltrami differentials which induce the same map:

Definition 4.2. Let $\mathrm{QC}_0(A)$ be the set of all quasiconformal automorphisms $h$ of $A$ admitting a quasiconformal isotopy to the identity compatible with $f$; that is, there is a $K$ and a homotopy $h_t$, $0 \le t \le 1$, such that

(a) $h_0(z) = z$ and $h_1 = h$.
(b) $h_t$ is a $K$-quasiconformal map of $A$ satisfying $f \circ h_t = h_t \circ f$.

The group $\mathrm{QC}_0(A)$ acts on $\mathrm{Def}(f, A)$ by $\omega_* \phi = \phi \circ \omega^{-1}$.

Definition 4.3. The *Teichmüller space* of $f$ in $V$, denoted by $T(f, V)$, is the deformation space $\mathrm{Def}(f, V)/\mathrm{QC}_0(V)$.

Theorem 4.4 ([HaTa], [MS]). *Let $f$ be an entire map and suppose that $V_\alpha$ is a family of pairwise disjoint completely invariant open subsets of $\mathbb{C}$. Then*

$$T\Big(f, \bigcup V_\alpha\Big) \simeq \prod T(f, V_\alpha).$$

We denote by $T(V)$ the space $T(\mathrm{id}, V)$.

THEOREM 4.5 ([HaTa], [MS]). *Suppose that every component of the one-dimensional manifold $V$ is hyperbolic, $f : V \to V$ is a holomorphic covering map and the grand orbit relation of $f$ is discrete. If $V/f$ is connected, then $V/f$ is a Riemann surface and*

$$T(f, V) \simeq T(V/f).$$

THEOREM 4.6 ([FH1]). *Let $V$ be a proper fixed Baker domain of the entire function $f$, and $[V]$ its grand orbit. Denote by $S$ the set of singular values of $f$ in $[V]$ and by $[\overline{S}]$ the closure of the grand orbit of $S$ taken in $[V]$. Then $T(f, V)$ is infinite-dimensional except when $V$ is doubly parabolic and the cardinality of $[\overline{S}]/f$ is finite. In that case the dimension of $T(f, V)$ is equal to $\#[\overline{S}]/f - 1$.*

The *marked points* consist of all periodic points of $f$ together with the singular values of the map. Denote by $[\overline{S}]$ the grand orbit of the closure of the marked points, and set $\overline{J} = J \cup [\overline{S}]$ and $B_1(f, \overline{J}) = \{f\text{-invariant Beltrami forms in } L^\infty(\overline{J})\}$.

THEOREM 4.7 ([FH2], [HaTa]). *Let $f$ be a transcendental entire map. Let $[V_i]$ denote the collection of pairwise disjoint grand orbits of the connected component of $\mathbb{C} - \overline{J}$. Then*

$$T(f, \mathbb{C}) = B_1(f, \overline{J}) \times \prod_i T(f, V_i).$$

The Teichmüller space of the map $f$ realizes as the set of deformations of a function in the following way:

For $f$ a transcendental entire map, with critical points $\{c_1, c_2, \ldots\}$ and critical and asymptotic values $\{v_1, v_2, \ldots\}$, let $E_f$ be the space of transcendental entire maps quasiconformally equivalent to $f$. That is, a transcendental entire map $g$ is in $E_f$ if there are quasiconformal maps $q, q_1$ on the sphere fixing three points, say $\{0, 1, \infty\}$, such that the following diagram commutes:

(1)
$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f} & \mathbb{C} \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{C} & \xrightarrow{g} & \mathbb{C} \end{array}$$

If $\{c'_1, c'_2, \ldots\}$ are the critical points of $g$, and $\{v'_1, v'_2, \ldots\}$ their critical and asymptotic values, then $q_1(c_i) = c'_i$ and $q(v_i) = v'_i$.

On the other hand (see diagram (2)), given a quasiconformal map $q$, we consider the pull back $\mu_1 = q^*(\mu_0)$ of the conformal structure $\mu_0$ on $\mathbb{C}$, and denote by $\mu_2$ the pull back $f^*(\mu_1)$. By the mesurable Ahlfors–Bers theorem (see [K]), there exists a quasiconformal map $q_1$ realizing $\mu_2$, that is, $q_1^*(\mu_0) = \mu_2$.

$$
\begin{tikzcd}
\mathbb{C}, \mu_2 \arrow[r, "f"] \arrow[d, "q_1"'] & \mathbb{C}, \mu_1 \arrow[d, "q"] \\
\mathbb{C}, \mu_0 \arrow[r, "g"] & \mathbb{C}, \mu_0
\end{tikzcd}
\tag{2}
$$

For $f$ and $q$ as above, this construction defines a map that assigns to the quasiconformal map $q$, the map $q \circ f \circ q_1^{-1} = g$; since $g$ respects the conformal structure of $\mathbb{C}$, it is a holomorphic map.

The space $E_f$ is studied in [EL] in the case that the set of critical values is finite.

PROPOSITION 4.8. *The map $\rho : T(f) \to E_f$ defined by $\rho([q]) = q \circ f \circ q^{-1}$ $= g$ is well defined.*

*Proof.* If $q_1 \in [q]$, there is a homotopy $h_t$ of quasiconformal maps, with $h_0 = \mathrm{id}$ and $h_1 = h$, such that $f \circ h_t = h_t \circ f$ and $q_1 = q \circ h^{-1}$. Therefore, $g = q \circ f \circ q^{-1} = q \circ h_t^{-1} \circ f \circ h_t \circ q^{-1}$. Hence, for $t = 1$ we have $q \circ f \circ q^{-1} = q_1 \circ f \circ q_1^{-1}$. ∎

We will use this map in what follows.

**5. The results.** The idea behind the results in this article is to consider a pinching process on maps of the punctured plane and lift these pinchings to their respective maps. The basic family of maps is $g_\lambda(w) = \lambda w e^{-w}$, $\lambda \in \mathbb{C}^*$, which is a family of self-maps of $\mathbb{C}^*$. Using $\exp(-z)$ we can lift these maps to functions of the form $f(z) = z - \log(\lambda) + 2\pi i k + e^{-z}$, for some $k \in \mathbb{Z}$. We have $g'_\lambda(w) = \lambda e^{-w}(1 - w)$, so there is a unique critical point at $w = 1$. If $|\lambda| \neq 0$, then $g_\lambda$ has a nonsuperattracting fixed point at 0. The next lemma is the observation that any dynamical deformation of $g_\lambda$ that fixes 0, 1 and $\infty$ is within the family.

LEMMA 5.1. *If $h$ is a homeomorphism that fixes $\{0, 1, \infty\}$ pointwise and $h^{-1} \circ g_\lambda \circ h = k$ is a holomorphic map, then $k(w) = \alpha w e^{-w}$ for some $\alpha \in \mathbb{C}^*$.*

*Proof.* Since $h$ fixes $\{0, 1, \infty\}$ pointwise, $k(w)$ is a self-map of $\mathbb{C}^*$ and has 0 as a fixed point as well. Since under conjugation the qualitative dynamics of the map does not change, $k(w)$ has 0 as a fixed point and also a unique critical point at 1.

Holomorphic self-maps of $\mathbb{C}^*$ were classified by Bhattacharyya [Bh] and are of the form $k(w) = \mu w^n e^{a(w)+b(1/w)}$, with $a(w), b(w)$ entire functions. Observe from the derivative of $k(w)$ that for $n \geq 2$, zero is superattracting, which is not our case; also $k(0) = 0$ implies $b(w) = b$ is a constant, therefore $k(w) = \mu w e^{a(w)+b}$.

Notice that the map $g_\lambda$ has two asymptotic values $\{\infty, 0\}$, therefore $k$ must have the same asymptotic values. A result in [S, Remark 3.3] states that if $k$ is an entire function with finite asymptotic values and finite critical points, then its derivative $k'$ is of the form $P(w)e^{Q(w)}$ with $P$ and $Q$ polynomials. This is our case. Since $k'(z) = \mu(wa'(w) + 1)e^{a(w)+b}$, we see that $a(w)$ and $a'(w)$ are polynomials. The condition that 1 is the only critical point implies that $wa'(w) + 1 = \pm(w - 1)^n$.

The fact that $g''_\lambda(1) = -\lambda e^{-1} \neq 0$ implies that 1 is a branched point of second order (like 0 in $z^2$); the same is true for $k''$ at 1. This implies that $k''(1) \neq 0$ iff $(wa'(w) + 1)'(1) \neq 0$, that is, $\pm n(w - 1)^{n-1}$ at 1 is different from zero. Therefore, $n = 1$ and $a'(w) = -1$. Hence, $k(w) = \alpha w e^{-w+c}$ for some constant $c$, which implies that $k(w) = \alpha e^c w e^{-w}$, which is what we wanted to prove. ∎

Assume now that the homeomorphism $h$ fixes $\{0, 1, \infty\}$ and it has a lift to $H$, that is, $\exp(-z) \circ H(z) = h \circ \exp(-z)$.

Corollary 5.2. *Let $g(w) = \lambda w e^{-w}$ with $|\lambda| < 1$, and suppose that $H^{-1}(z) \circ G(z) \circ H(z) = K(z)$ is holomorphic, with $G(z)$ a lift of $g(z)$ and $H$ a lift of $h$. Then $K(Z) = z + e^{-z} + a$.*

*Proof.* Since $H^{-1}(z) \circ G(z) \circ H(z) = K(z)$ is holomorphic, it follows that $h^{-1} \circ g \circ h = k$ is a holomorphic self-map of $\mathbb{C}^*$. By Lemma 5.1, $k(w) = \alpha w e^{-z}$ and $K(z)$ is a lift of $k(w)$. Then $K(z) = z + e^{-z} - \log(\alpha) + 2\pi i k$, $k \in \mathbb{Z}$, as required. ∎

In order to have a convergent sequence as in the hypothesis of Lemma 3.6, we have to show that a sequence of maps that are uniformly convergent in the punctured plane has lifts that are uniformly convergent as well; this is shown in the next lemma. Recall that we are lifting with respect to $e^{-z}$ so that $f$ is a lift of $g$ if $\exp(-f(z)) = g(\exp(-z))$; any two lifts differ by a factor $2\pi i k$.

Lemma 5.3. *Consider a sequence of maps $g_n : \mathbb{C}^* \to \mathbb{C}^*$ of the form $g_n(w) = \lambda_n w e^{-z}$ that converges uniformly on compact sets to a function of the form $g(w) = \lambda w e^{-z}$, with $\lambda \neq 0$ and the numbers $\lambda_n/\lambda$ being positive reals. Then, for each integer $k$, the family of respective lifts $f_n(z) = -\log(\lambda_n) + z + e^{-z} + 2\pi i k$ converges uniformly on compact sets to $f(z) = -\log(\lambda) + z + e^{-z} + 2\pi i k$. Here $\log(\lambda)$ is the principal branch of the logarithm function.*

*Proof.* Observe that if $K \subset \mathbb{C}$ is a compact set, then its projection under $\exp(-z)$ is a compact subset of $\mathbb{C}^*$. Now since $|f_n(z) - f(z)| = |-\log(\lambda_n) + \log(\lambda)|$ and $|g_{\lambda_n}(w) - g_\lambda(w)| = |\lambda_n - \lambda|\,|we^{-z}|$, we see that $\{f_n(z)\}$ converges uniformly on compact sets iff $\{g_{\lambda_n}\}$ does. ∎

In this section we will briefly write $T(f)$ in place of $T(f,\mathbb{C})$. For $F$ the Fatou function, observe that due to Theorem 4.6, $\dim T(F)$ is $\infty$. Also note that the families $g_\lambda(w) = \lambda w e^{-w}$ and $g_\alpha(w) = \alpha w e^{w}$ are conjugate by a rotation of $\mathbb{C}^*$.

THEOREM 5.4. *For $c = 1-t$, $t \in [0,1]$, the maps $F_c(z) = z + c + e^{-z}$ can be regarded as a path in $\rho(T(F))$ of pinching deformations of the Fatou function $F(z)$ with limit the Baker–Domínguez function $G(z) = F_0(z)$.*

*Proof.* We split the proof into two steps.

1. Let $C$ be the main cardioid of the Mandelbrot set of the quadratic family $z^2+d$, and $\Delta$ the open unit disc. The map $\Psi : \Delta \to C$ given by $\Psi(\lambda) = (1-(1-\lambda)^2)/4$ is a biholomorphism. Now, the set of $\lambda$'s for which zero is an attracting fixed point of $g_\lambda(w) = \lambda w e^{-w}$ coincides with $\Delta$. Moreover, if the multiplier of 0 is $\lambda$, then the map $f_{\Psi(\lambda)}(z) = z^2 + \Psi(\lambda)$ has an attracting fixed point $z_0$ with multiplier $\lambda$, and in addition, the attracting domain of 0 has the critical point $w = 1$ and the attracting domain of $z_0$ has the critical point $z = 0$. Since both functions have only one Fatou domain which is their respective attracting domain, $[V]_{g_\lambda}$ and $[V]_{f_{\Psi(\lambda)}}$, as defined at the beginning of Section 4, are isomorphic one-punctured tori (see [MS]).

For $\Psi(\lambda) = d$, the condition $\lambda \in [1/e, 1)$ corresponds to $d \in [(1-(1-1/e)^2)/4, 1/4) \subset [0, 1/4)$. Then, from [T, Example 1], it can be shown that for $d \in [d_0, 1/4]$, the corresponding $z^2 + d$ can be regarded as a pinching deformation for a suitable admissible pair which converges to $z^2 + 1/4$, the admissible curve being the interval $I$ in the reals from the attracting fixed point to the repelling fixed point. Notice that the interval $I$ projects to a closed curve $\sigma$ in $[V]_{f_d}$, therefore this is equivalent to pinching $[V]_{f_d}$ along $\sigma$ as explained in [T].

We translate such pinching to our case by means of the equivalence between $[V]_{g_\lambda}$ and $[V]_{f_{\Psi(\lambda)}}$. This implies that the curve $I$ above becomes the curve $\gamma(s) = (1/e)s$, $0 \le s \le 1$, in $B_{1/e}$ and corresponds to the pinching curve. Let $\Gamma$ be the grand orbit of $\gamma(s)$ under $g_{1/e}$, and let $U$ be the $\epsilon$-neighborhood of $\gamma(s)$ with respect to the hyperbolic metric in $B_{1/e}$. Then $(\gamma, U)$ is an admissible pair for $g_{1/e}$. Notice that since $B_{1/e}$ is the only Fatou component of $g_{1/e}$, the connected components of $\Gamma$ are precisely the curves in the grand orbit of $\gamma(s)$, i.e., there are no closed curves in $\Gamma$.

Then, pinching along $\Gamma$ deforms $g_{1/e}$ in such a way that the process stays within the family $g_\lambda$ by Lemma 5.1. Therefore, it converges to $g_1$ with the attracting fixed point becoming a parabolic fixed point. The fixed point $s_t$ in the boundary of $B_t$ satisfies $s_t \in [-1, 0]$, $0 \le t \le 1$.

2. Now, we lift the situation explained in 1 above. The fixed points $s_t$ lift to $r_t = -\log(s_t)$; so for $t = 0$, the fixed points are $r_0 = \pi i + 2\pi i k$. The curve $\gamma$ lifts to the family of curves $\{\tilde{\gamma}_k(s)\}$, where each curve $\tilde{\gamma}_k$ begins at

$\pi i + 2\pi ik$, ends at infinity and is parallel to the real axis. Denote by $\tilde{\Gamma}$ the grand orbit of these curves under $F$ or equivalently the lifting of all curves of $\Gamma$, and denote by $\tilde{U}_k$ the lifts of $U$, which are neighborhoods of $\tilde{\gamma}_k(s)$. Then $(\tilde{\gamma}_k(s), \tilde{U}_k)$ are admissible pairs for $F_1$.

We can proceed with the pinching process which is a lift of the one in item 1. Now, since the maps $g_\lambda$ of item 1 above converge through a path inside the family, by Corollary 5.2, the process is lifted to the family $F_c$ which by Lemma 5.3 is convergent. Therefore, pinching $F_1$ converges to $F_0$, as we wanted to prove. Moreover, under the pinching process, the points $r_0 = \pi i + 2\pi k$ move towards infinity (Lemma 3.6), and so the Baker domain is decomposed into an infinite set of Baker domains separated by the lines $\tilde{L}_k = x + \pi i + 2\pi ik$ ($x \in \mathbb{R}$), $k \in \mathbb{Z}$, which are lifts of the line $L_{-1}$ and belong to the Julia set of $F_0$. ∎

Theorem 5.5. *There is a pinching process from the Fatou function $F$ to any of the functions $F_{p/q}(z) = z + e^{-z} + 2\pi ip/q$, with $p/q$ a rational number. The Fatou domain of such functions has a wandering domain if $p \neq 0$.*

*Proof.* For $\lambda \in \Delta$, 0 is an attracting fixed point of $g_\lambda$ and the boundary points of the form $\lambda = e^{-2\pi ip/q}$ correspond to maps with a periodic parabolic domain.

It is known that for the quadratic family $f_d(z) = z^2 + d$, the parabolic points in the boundary of the main cardioid $C$ are reached by a pinching process from any map in $C - \{0\}$, therefore there are curves in the attracting domain of some $f_d$ with $d \in C$ which projects to a certain curve $\gamma$ in the torus $[V]_{f_d}$ (see [M2] or [T]). Again we find that the torus $[V]_{g_\lambda}$ is isomorphic to $[V]_{f_d}$ with $\Psi(\lambda) = d$. So we translate the curve $\gamma$ to $[V]_{g_\lambda}$ and consider the curve in the attracting domain of $g_\lambda$ that projects to $\gamma$. Choosing $\lambda = 1/e$ we obtain in the Fatou domain of the map $g_{1/e}$ the admissible pairs we need.

As in the above theorem, we lift the admissible pairs of $g_{1/e}$ to obtain the admissible pairs for the function $F$, and continue with the pinching process which deforms it to the map $F_{p/q}$. The map $F_{p/q}$ has infinitely many connected Fatou domains coming from the lifts of the parabolic basin of the map below, $e^{-2\pi ip/q}ze^{-z}$.

To show that there is a wandering domain, we follow the orbit of $i\pi$; notice that its projection is the point 1, which is the critical point of the maps $e^{-2\pi ip/q}ze^{-z}$. Its iterations move it through the different components of the parabolic domains, winding around the origin since $p/q$ is nonzero. Therefore the lift translates by $2\pi i$ the point $i\pi$ jumping from one component to another, so it is not difficult to show that $|F^n(i\pi)|$ tends to infinity; this implies that there is a wandering domain. ∎

We state the following proposition on the dimension of the space of deformations.

PROPOSITION 5.6. *$\dim(T(F_{p/q})) = \infty$ if $p \neq 0$.*

*Proof.* Notice that for $x \in \mathbb{R}$, between the lines $x + 2\pi ik$ and $x + 2\pi i(k+1)$, $k \in \mathbb{Z}$, there are the lifts of the $q$ parabolic components of $e^{-2\pi ip/q}ze^{-z}$. Choose $V_k$ to be any of those components such that $F_{p/q}(V_k) = V_{k+1}$ and denote $V = \bigcup_k V_k$. Then the grand orbit relation of $F_{p/q}$ on $V - S$ is discrete, thus the grand orbit relation of $F_{p/q}$ in $[V]$ is discrete, with the set of critical points $\{c_j\}$ belonging to different orbits as we show next.

Denote by $c$ the critical point of $e^{-2\pi ip/q}ze^{-z}$, which is in the orbit of the parabolic domain. Then the critial points of $F_{p/q}$, $\{c_j\}$ are the lifts of $c$. If we denote by $\{c, z_1, z_2, \ldots\}$ the forward orbit of $c$, we know that $z_i \neq z_j$ if $i \neq j$. Therefore if we denote by $\{z_{ik}\}_{k\in\mathbb{Z}}$ the lifts of $z_i$, they are all different and we have $z_{ik} \in V_k$. Therefore $F_{p/q}(z_{ik}) = z_{(i+1)(k+1)}$. This shows that the critical points belong to different orbits.

Now, the uniformizing chart on each $V_k$ with the map $z \mapsto z + 1$ shows that $[V]/F_{p/q}$ is biholomorphic to $\mathbb{C} - \{c_k\}_{k\in\mathbb{Z}}$, similar to Example 4 in [FH2]. Therefore, $\dim(T(g)) = \infty$. ∎

COROLLARY 5.7. *For any integer $n$, there is a path in $\rho(T(F))$ which defines a pinching process of the Fatou function that converges to a function with $n$ wandering domains.*

*Proof.* Fix $n$ and consider the function $F_{n+1}(z) = G(z) + 2\pi i(n+1)$, which is one of the functions in Theorem 5.5 with $p/q = n+1$. Moreover, it is a lift of $g_1$. The Fatou set of $F_{n+1}$ is the same as the Fatou set of $G(z)$, but now the number $2\pi i(n+1)$ moves each component $V_k$ to $V_{k+n+1}$. Therefore, there are $n$ orbits of wandering domains. ∎

Observe that by Proposition 5.6, the maps $F_{n+1}$ are nonrigid.

Finally, in the next theorem we show that the map $H$ in Section 2.3 can be deformed by a pinching process to obtain the Baker–Domínguez function $F_0$ although $F$ and $H$ are not conjugate maps. In fact this is a pinching from an attractor domain to a doubly parabolic domain.

THEOREM 5.8. *For $c = (1/2)(t-1)$, $t \in [0,1]$, the corresponding maps $H_t(z) = z + c + e^{-z}$ can be regarded as a path in $\rho(T(H))$ of pinching deformation of the function $H_0$ with limit the Baker–Domínguez function $G(z) = F_0(z) = H_1(z)$.*

*Proof.* 1. Observe that for $|\lambda| > 1$, the family $g_\lambda$ with $|1 - \log\lambda| < 1$ has the property that $w_0 = 0$ is a repelling fixed point and $w_1 = \log(\lambda)$ is an attracting fixed point; this corresponds to a second main cardioid made of quadratic like maps. Define $\gamma(s) = (1/2)(1-s)$, $0 \le s \le 1$, a curve in the basin of attraction of $g_{e^{(1/2)}}(w) = e^{(1/2)}we^{-w}$ that joins $w_1 = 1/2$ to 0. We proceed as in the above theorem to construct the admissible pair

$(\gamma, U)$ and denote by $\Gamma$ its grand orbit. By the same type of argument as in Theorem 5.4, pinching along $\Gamma$ deforms $g_{e(1/2)}$ to $g_1$.

2. Now, we lift the above situation: the lift $\overline{\gamma}$ of the curve $\gamma$ has infinitely many connected components which are curves $\overline{\gamma}_k$ that connect the attracting points $\log(2) + 2\pi ik$ of $H_0$ with infinity and are invariant under iteration. As above, there are neighborhoods of $\overline{\gamma}_k$ such that all pairs $(\overline{\gamma}_k, U_k)$ are admissible pairs and we can apply the pinching process which converges to $F_0(z)$ by the same type of argument as in Theorem 5.4. The pinching process moves the attracting points towards infinity, by Lemma 3.6. Hence, each of the attracting domains $W_k$ becomes the doubly parabolic Baker domain $V_k$ of $F_0$. ∎

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Patricia Domínguez
Facultad de Ciencias Físico Matemáticas
Benemérita Universidad Autónoma de Puebla
Puebla, Mexico
E-mail: pdsoto@fcfm.buap.mx

Guillermo Sienra
Departamento de Matemáticas
Facultad de Ciencias
UNAM
México, D.F., Mexico
E-mail: gsl@dinamica1.fciencias.unam.mx

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